1 Exercises

1.1

Verify that the table is correct as far as it goes, and complete it.

$$\begin{aligned} d(11) &= 2 \\ d(12) &= 6 \\ d(13) &= 2 \\ d(14) &= 4 \\ d(15) &= 4 \\ d(16) &= 5 \end{aligned}$$

1.2

What is $d(p^3)$ Generalize to $d(p^n), n = 4, 5, ...$

$p^3 = p * p * p$ must be divisible by 1, p, p^2 and p^3, so 4. $d(p^n) = n + 1$.

1.3

What is $d(p^3 q)$? What is $d(p^n q)$ for any positive n?

$p^3 q$ is divisble by 1, p, p^2, p^3, q, pq, $p^2 q$, $p^3 q$, so $d(p^3 q) = 7$. For $p^n q$, we'll have divisors of 1, q, n terms of p^k, and n terms of $p^k q$, so $d(p^n q) = 2(n + 1)$.

1.4

Calculate $d(240)$.

$d(240) = d(2^4) * d(3) * d(5) = 5 * 2 * 2 = 20$.

1.5

Verify that the following table is correct as far as it goes, and complete it.

$$\begin{aligned} \sigma(9) &= 13 \\ \sigma(10) &= 18 \\ \sigma(11) &= 12 \\ \sigma(12) &= 28 \\ \sigma(13) &= 14 \\ \sigma(14) &= 24 \end{aligned}$$

1.6

What is $\sigma(p^3)$? $\sigma(pq)$, where p and q are different primes?

$\sigma(p^3) = 1 + p + p^2 + p^3$. $\sigma(pq) = 1 + p + q + pq$.

1.7

Show that $\sigma(2^n) = 2^{n+1} - 1$.

$\sigma(2^n) = 1 + 2 + ... + 2^n$, so $2\sigma(2^n) - \sigma(2^n) = \sigma(2^n) = 2^{n+1} - 1$.

1.8

What is $\sigma(p^n)$, $n = 1, 2, ...$?

$\sigma(p^n) = 1 + p + ... + p^n$, or, using a well-known alternate defintion for a geometric series, $\sigma(p^n) = \frac{p^{n+1}-1}{p-1}$.

1.9

Calculate $\sigma(240)$.

$\sigma(240) = \sigma(2^4) * \sigma(3) * \sigma(5) = (1 + 2 + 4 + 8 + 16)(1 + 3)(1 + 5) = 744$.

1.10

Compute $f(n)$ for $n = 13, 14, ..., 24$.

$$\begin{array}{cc} f(13) = 1, & f(14) = 1 \\ f(15) = 1, & f(16) = 32 \\ f(17) = 1, & f(18) = 6 \\ f(19) = 1, & f(20) = 4 \\ f(21) = 1, & f(22) = 1 \\ f(23) = 1, & f(24) = 12 \end{array}$$

2 Problems

2.1

Calculate $d(42)$, $\sigma(42)$, $d(420)$, and $\sigma(420)$.

$d(42) = d(2) * d(3) * d(7) = 8$.
$\sigma(42) = \sigma(2) * \sigma(3) * \sigma(7) = 3 * 4 * 8 = 96$.
$d(420) = d(2^2) * d(3) * d(5) * d(7) = 24$.
$\sigma(420) = \sigma(2^2) * \sigma(3) * \sigma(5) * \sigma(7) = 7 * 4 * 6 * 8 = 1344$.

2.2

Calculate $d(540)$, $\sigma(540)$, $d(5400)$, and $\sigma(540)$.

$d(540) = d(2^2) * d(3^3) * d(5) = 3 * 4 * 2 = 24$.
$\sigma(540) = \sigma(2^2) * \sigma(3^3) * \sigma(5) = 7 * 34 * 6 = 1428$.
$d(5400) = d(2^3) * d(3^3) * d(5^2) = 4 * 4 * 3 = 48$.
$\sigma(5400) = \sigma(2^3) * \sigma(3^3) * \sigma(5^2) = 15 * 34 * 31 = 15810$.

2.3

Calculate d and σ of $10115 = 5 * 7 * 17^2$ and $100115 = 5 * 20023$.

$d(10115) = d(5) * d(7) * d(17^2) = 2 * 2 * 3 = 12$.
$\sigma(10115) = \sigma(5) * \sigma(7) * \sigma(17^2) = 6 * 8 * 307 = 14736$.
$d(100115) = d(5) * d(20023) = 2 * 2 = 4$.
$\sigma(100115) = \sigma(5) * \sigma(20023) = 6 * 20024 = 120144$.

2.4

Calculate d and σ of $10116 = 2^2 * 3^2 * 281$ and $100116 = 2^2 * 3^5 * 103$.

$d(10116) = d(2^2) * d(3^2) * d(281) = 3 * 3 * 2 = 18$.
$\sigma(10116) = \sigma(2^2) * \sigma(3^2) * \sigma(281) = 7 * 13 * 282 = 25662$.
$d(100116) = d(2^2) * d(3^5) * d(103) = 3 * 6 * 2 = 36$.
$\sigma(100116) = \sigma(2^2) * \sigma(3^5) * \sigma(103) = 7 * 286 * 104 = 208208$.

2.5

Show that $\sigma(n)$ is odd if n is a power of two.

Can write $\sigma(n) = \sigma(k^2) = 1 + k + k^2$. If k is even, then k^2 is even, and $1 + k + k^2$ is odd. If k is odd, then k^2 is odd, $k + k^2$ is even, and $1 + k + k^2$ is odd.

2.6

Prove that if $f(n)$ is multiplicative, then so is $f(n)/n$.

If $f(n)$ is multiplicative, then $f(n) = f(p_1^{e_1})f(p_1^{e_1})...f(p_1^{e_1})$ with $n = p_1^{e_1} * p_2^{e_2} * ... * p_k^{e_k}$. So, $f(n)/n = \frac{f(p_1^{e_1})f(p_2^{e_2})...f(p_k^{e_k})}{p_1^{e_1} * p_2^{e_2} * ... * p_k^{e_k}} = \frac{f(p_1^{e_1})}{p_1^{e_1}} * \frac{f(p_2^{e_2})}{p_2^{e_2}} * ... * \frac{f(p_k^{e_k})}{p_k^{e_k}}$. This implies $f(n)/n$ is multiplicative.

2.7

What is the smallest integer n such that $d(n) = 8$? Such that $d(n) = 10$?

Consider that $8 = 2^3$, so we could have the product of any 3 primes, for example $d(2) * d(3) * d(5)$. We also can't have any divisor that is a square prime, because $d(p^2) = 3$ and $3 \nmid 8$. But, we can have a cubic divisor, since $d(p^3) = 4$, and as it turns out, $2 * 2^2 < 2 * 5$, so $24 = 2^3 * 3$ is the smallest such n.
For $d(n) = 10$, note that $10 = 2 * 5$. To $d(n) = 10$, have to have a $d(k^4)$ somewhere, and it might as well be 2^4. So, the smallest such n is $48 = 2^4 * 3$.

2.8

Does $d(n) = k$ have a solution n for each k?

Any k would have at least the solution $n = p^{k-1}$ for any prime p.

2.9

In 1644, Mersenne asked for a number with 60 divisors. Find one smaller than 10,000.

Consider that $60 = 2^2 * 3 * 5$. So in our number we need at one divisor that is a prime raised to the fourth power, and one that is a square. So, it would be best to allocate these to the lowest primes, 2 and 3 respectively. We could take care of the 2^2 term with a prime raised to the third power or two distinct prime powers. Since 5 is the next power and $5^3 > 5 * 7$, we opt for the latter. $5040 = 2^4 * 3^2 * 5 * 7$ is such a number.

2.10

Find infinitely many n such that $d(n) = 60$.

For any $n = p^{59}$, $d(n) = 60$. Since there are infinitely many primes p, there are infinitely many $n = p^{59}$. Or you could have infinitely many permutations of the forms $p_1^3 * p_2^2 * p_3^4$ or $p_1 * p_2 * p_3^2 * p_4^4$.

2.11

If p is an odd prime, for which k is $1 + p + ... + p^k$ odd?

If p is odd, then any p^i is odd as well. Also, any sums pairs like $p^i + p^j$ are even. Since we have the 1 term, we want $p + ... + p^k$ to be even. So, we would want to be able to group it into pairs. This is possible if k is even.

2.12

For which n is $\sigma(n)$ odd?

If $n = p^k$ and k even, we have odd $\sigma(n)$, like in the previous exercise. The same could be said for any k if $p = 2$. If n cannot be written as a power of a single prime p, it can still be written as $n = p_1^{e_1} * ... * p_k^{e_k}$, and $\sigma(n) = \sigma(p_1^{e_1}) * *... * \sigma(p_k^{e_k})$. In order for that product to be odd, each term would have to be odd, i.e. each term $p_i^{e_i}$ in the prime-power decomposition of n would either require $p_i = 2$ or $2|e_i$.

2.13

If n is a square, show that $d(n)$ is odd.

Write $n = k^2$, then $d(n) = d(k^2) = d((p_1^{e_1} * ... * p_k^{e_k})^2) = d(p_1^{2e_1} * ... * p_k^{2e_k}) = d(p_1^{2e_1}) * ... * d(p_k^{2e_k}) = (1 + 2e_1) * ... * (1 + 2e_k)$. Since each term of this product is odd, the whole product is odd.

2.14

If $d(n)$ is odd, show that n is a square.

$d(n) = d(p_1^{e_1} * ... * p_k^{e_k}) = d(p_1^{e_1}) * ... * d(p_k^{e_k}) = (1 + e_1) * ... * (1 + e_k)$. Since we know $d(n)$ is odd, we know that none of the terms of the product can be even, and e_i must be even. But, if e_i is even, then it can be written as $2e'_i$, and n can be written $p_1^{2e'_1} * ... * p_k^{2e'_k} = (p_1^{e'_1} * ... * p_k^{e'_k})^2$.

2.15

Observe that $1 + 1/3 = 4/3$; $1 + 1/2 + 1/4 = 7/4$; $1 + 1/5 = 6/5$; $1 + 1/2 + 1/3 + 1/6 = 15/6$; $1 + 1/7 = 8/7$; and $1 + 1/2 + 1/4 + 1/8 = 15/8$. Guess and prove a theorem.

$\Sigma_{d|n} 1/d = \sigma(n)/n$.
Consider the sum over the divisors of n: $1/1 + 1/d_1 + ... + 1/d_k + 1/n$. To perform the addition, we want to have all fractions have the same denominator. n is the natural candidate for this, as by definition $\exists d'_i \ni d_i d'_i = n$. Furthermore, the set of these d'_is (call it D') is equivalent to the set of d_is (D): If there were a d'_i that is not in D, it would contradict the construction of D as the set of all divisors of n. For d_is, we still know that $d'_i = n/d_i \in D$, and thus $\exists d'_j \in D' \ni \frac{n}{d_i} d'_j = n$, namely $d'_j = d_i$. So all d_is must be in D' too. So, we now have $n/n + d'_1/n + ... + d'_k/n + 1/n = \frac{1}{n}(1 + d'_k + ... + d'_1 + n) = \sigma(n)/n$.

2.16

Find infinitely many n such that $\sigma(n) \leq \sigma(n-1)$.

Consider n that are odd primes. By the definition of prime numbers, $\sigma(n) = 1 + n$. Since n is odd, then we know that $n - 1$ must be even, thus it must have have 2 among its divisors. So, $\sigma(n-1) = 1 + 2 + (n-1) + r$, where r represents the remaining terms in the summation and $r > 0$. $1 + 2 + (n-1) + r = 2 + n + r \geq 1 + n$. Since there are infinitely many odd primes, there are infinitely many such n.

2.17

If N is odd, how many solutions does $x^2 - y^2 = N$ have?

Write $x^2 - y^2 = (x+y)(x-y) = ab = N$. The number of divisors of N is $d(N)$, so there are $d(N)$ possible positive values for a (with appropriate bs) and $d(N)$ negative ones (with appropriate negative bs). The solutions to $x + y = a$, $x - y = b$ are $x = \frac{a+b}{2}$, $y = \frac{a-b}{2}$. So, should have $2d(N)$ such (x, y). The comments say something about a, b needing to be odd to ensure unique (x, y) pairs, but I'm not really seeing it!

2.18

Develop a formula for $\sigma_2(n)$, the sum of the squares of the positive divisors of n.

Observe that for prime p, $\sigma_2(p^k) = 1 + p^2 + p^4 + ... + p^{2k} = 1 + (p^2)^1 + (p^2)^2 + ... + (p^2)^k = \frac{(p^2)^{k+1}-1}{p^2-1}$. For primes p, q, $\sigma_2(p^k q^j) = 1 + p^2 + ... + p^{2k} + p^2q^2 + ... + p^2q^{2j} + ... + p^{2k}q^2 + ... + p^{2k}q^{2j} + q^2 + ... + q^{2j} = (1 + p^2 + ... + p^{2k})(1 + q^2 + ... + q^{2j}) = \frac{(p^2)^{k+1}-1}{p^2-1}\frac{(q^2)^{q+1}-1}{q^2-1}$. So, since any number n can be represented as a prime-power decomposition $p_1^{e_1}...p_k^{e_k}$, $\sigma_2(n) = \frac{(p_1^2)^{e_1+1}-1}{p_1^2-1}...\frac{(p_k^2)^{e_k+1}-1}{p_k^2-1}$.

2.19

Guess a formula for

$$\sigma_k(n) = \Sigma_{d|n} d^k,$$

where k is a positive integer.

By the same logic as before, $\sigma_k(n) = \frac{(p_1^k)^{e_1+1}-1}{p_1^k-1}...\frac{(p_k^k)^{e_k+1}-1}{p_k^k-1}$.

2.20

Show that the product of the positive divisors of n is $n^{d(n)/2}$.

Consider the easy case where $n = p^k$ and denote our product function as f. We have $f(p^k) = p * p^2 * ... * p^k = p^{1+2+...+k}$. We are trying to show, for this special case, that $p^{1+2+...+k} = p^{\frac{k}{2}d(p^k)}$, or equivalently, $1+2+...+k = \frac{k}{2}d(p^k)$. The sum of consecutive integers on the left side can be written as $\frac{k(k+1)}{2}$. We also know that $d(p^k) = k+1$, making the right side $\frac{k(k+1)}{2}$, so we've shown that the relation holds for the special case of $n = p^k$. Now consider $f(p^kq^j) = p...p^k * pq...pq^j * p^kq...p^kq^j * q...q^j$. In this product we have 1 term of $p...p^k$ with no q terms, and j terms of $p...p^k$ with terms $\in \{q, ..., q^j\}$. So we have $j+1$ terms of $p...p^k$ and, symmetrically, $k+1$ terms of $q...q^j$. Knowing that $p...p^k = f(p^k) = (p^k)^{d(p^k)/2}$ and that $k+1 = d(p^k), j+1 = d(q^j)$, can write $f(p^kq^j) = (p^k)^{\frac{1}{2}d(q^j)d(p^k)}(q^j)^{\frac{1}{2}d(q^j)d(p^k)} = (p^kq^j)^{\frac{1}{2}d(q^j)d(p^k)} = (p^kq^j)^{\frac{1}{2}d(p^kq^j)}$ Assume that f is multiplicative up to r for composite numbers that can be written $p_1^{e_1}...p_k^{e_r}$. Observe that for $f(p_1^{e_1}...p_k^{e_r}p_{r+1}^{e_{r+1}})$, we can apply similar logic as we did for p^kq^j, namely that there will be $e_{r+1}+1 = d(p_{r+1}^{e_{r+1}})$ terms of $f(p_1^{e_1}...p_k^{e_r})$ and $d(p_1^{e_1}...p_k^{e_r})$ terms of $f(p_{r+1}^{e_{r+1}})$, and we can extract the common exponent $d(p_1^{e_1}...p_k^{e_r}p_{r+1}^{e_{r+1}})/2$. Since any number can be written as such a prime-power decomposition, $f(n) = n^{d(n)/2}$.