1 Exercises

1.1

Verify that 1184 and 1210 are amicable.

$$\sigma(1184) = \sigma(2^5)\sigma(37) = 2394 = 1184 + 1210$$
$$\sigma(1210) = \sigma(2)\sigma(5)\sigma(11^2) = 2394 = 1184 + 1210$$

2 Problems

2.1

Verify that 2620, 2924 and 17296, 18416 are amicable pairs.

$$\sigma(2620) = \sigma(2^2)\sigma(5)\sigma(131) = 5544 = 2620 + 2924$$
$$\sigma(2924) = \sigma(2^2)\sigma(17)\sigma(43) = 5544 = 2620 + 2924$$

$$\sigma(17296) = \sigma(2^4)\sigma(23)\sigma(47) = 35712 = 17296 + 18416$$
$$\sigma(18416) = \sigma(2^4)\sigma(1151) = 35712 = 17296 + 18416$$

2.2

It was long thought that even perfect numbers ended alternately in 6 and 8. Show that this is wrong by verifying that the prefect numbers corresponding to the primes $2^{13} - 1$ and $2^{17} - 1$ both end in 6.

The perfect numbers corresponding to these primes are $2^{12}(2^{13} - 1) = 2^{25} - 2^{12}$ and $2^{16}(2^{17} - 1) = 2^{33} - 2^{16}$

$$\begin{aligned}
2 &\equiv 2 \pmod{10} \\
2^2 &\equiv 4 \pmod{10} \\
2^4 &\equiv 6 \pmod{10} \\
2^8 &\equiv 6 \pmod{10} \\
2^{16} &\equiv 6 \pmod{10} \\
2^{32} &\equiv 6 \pmod{10} \\
2^{25} &\equiv 2 \pmod{10} \\
2^{12} &\equiv 6 \pmod{10} \\
2^{25} - 2^{12} &\equiv -4 \equiv 6 \pmod{10} \\
2^{33} &\equiv 2 \pmod{10} \\
2^{16} &\equiv 6 \pmod{10} \\
2^{33} - 2^{16} &\equiv -4 \equiv 6 \pmod{10}
\end{aligned}$$

2.3

Classify the integers $2, 3, ..., 21$ as abundant, deficient, or perfect.

$$\begin{aligned} \sigma(2) = 3 \Rightarrow deficient, &\quad \sigma(3) = 4 \Rightarrow deficient, \\ \sigma(4) = 7 \Rightarrow deficient, &\quad \sigma(5) = 6 \Rightarrow deficient, \\ \sigma(6) = 12 \Rightarrow perfect, &\quad \sigma(7) = 8 \Rightarrow deficient, \\ \sigma(8) = 15 \Rightarrow deficient, &\quad \sigma(9) = 13 \Rightarrow deficient, \\ \sigma(10) = 18 \Rightarrow deficient, &\quad \sigma(11) = 12 \Rightarrow deficient, \\ \sigma(12) = 28 \Rightarrow abundant, &\quad \sigma(13) = 14 \Rightarrow deficient, \\ \sigma(14) = 24 \Rightarrow deficient, &\quad \sigma(15) = 24 \Rightarrow deficient, \\ \sigma(16) = 31 \Rightarrow deficient, &\quad \sigma(17) = 18 \Rightarrow deficient, \\ \sigma(18) = 39 \Rightarrow abundant, &\quad \sigma(19) = 20 \Rightarrow deficient, \\ \sigma(20) = 42 \Rightarrow abundant, &\quad \sigma(21) = 32 \Rightarrow deficient. \end{aligned}$$

2.4

Classify the integers $402, 403, ..., 421$ as abundant, deficient, or perfect.

TODO sounds boring...

2.5

If $\sigma(n) = kn$, then n is called a *k-perfect number*. Verify that 672 is 3-perfect and $2,178,540 = 2^2 * 3^2 * 5 * 7^2 * 13 * 19$ is 4-perfect.

$$\sigma(672) = \sigma(2^5)\sigma(3)\sigma(7) = 2016 = 3 * 672$$
$$\sigma(2178540) = \sigma(2^2)\sigma(3^2)\sigma(5)\sigma(7^2)\sigma(13)\sigma(19) = 8714160 = 4 * 2178540$$

2.6

Show that no number of the form $2^a 3^b$ is 3-perfect.

In order for this to work, would need $\sigma(2^a 3^b) = 2^a 3^{b+1}$. Note that $\sigma(2^a 3^b) = \sigma(2^a)\sigma(3^b) = \frac{1}{2}(2^{a+1} - 1)(3^{b+1} - 1) = \frac{1}{2}(2^{a+1}3^{b+1} - 2^{a+1} - 3^{b+1} + 1) = 2^a 3^{b+1} - \frac{1}{2}(2^{a+1} + 3^{b+1} - 1)$. This would only equal $2^a 3^{b+1}$ if $2^{a+1} + 3^{b+1} = 1$, but since the term is at minimum $4 + 9 = 13$ with $a, b = 1$, this cannot be the case.

2.7

Let us say that n is *superperfect* if and only if $\sigma(\sigma(n)) = 2n$. Show that if $n = 2^k$ and $2^{k+1} - 1$ is prime, then n is superperfect.

If $n = 2^k$, then $\sigma(n) = \sigma(2^k) = 2^{k+1} - 1$. Since $2^{k+1} - 1$ is prime, we know that $\sigma(2^{k+1} - 1) = 2^{k+1} = 2 * 2^k$.

2.8

It was long thought that every abundant number was even. Show that 945 is abundant, and find another abundant number of the form $3^a * 5 * 7$.

$\sigma(945) = \sigma(3^3)\sigma(5)\sigma(7) = 1200 > 945$.
Try $n = 3^4 * 5 * 7 = 2835$. $\sigma(2835) = \sigma(3^4)\sigma(5)\sigma(7) = 4235 > 2835$.

2.9

In 1575, it was observed that every even perfect number is a triangular number. Show that this is so.

If n is an even perfect number, then we know by Euler's theorem that $n = 2^{p-1}(2^p - 1)$. We can rewrite this, multiplying by $\frac{2}{2}$, as $\frac{2^p(2^p-1)}{2}$. So, setting k as $2^p - 1$, this fits the formula for a triangular number, namely $n = k(k+1)/2$.

2.10

In 1652, it was observed that

$$\begin{aligned}6 &= 1 + 2 + 3,\\ 28 &= 1 + 2 + 3 + 4 + 5 + 6 + 7,\\ 496 &= 1 + 2 + 3 + ... + 31.\end{aligned}$$

Can this go on?

A sum of consecutive integers $1 + ... + k = \frac{k(k+1)}{2}$, and in the previous exercise we've shown that every perfect even number is a triangular number, so sure.

2.11

Let

$$\begin{aligned}p &= 3 * 2^e - 1,\\ q &= 3 * 2^{e-1} - 1,\\ r &= 3^2 * 2^{2e-1} - 1,\end{aligned}$$

where e is a positive integer. If p, q, and r are all prime, show that $2^e pq$ and $2^e r$ are amicable.

Assume all p, q, $r > 2$, which is the case for $e > 1$. know that $\sigma(2^e pq) = \sigma(2^e)\sigma(p)\sigma(q)$ and $\sigma(2^e r) = \sigma(2^e)\sigma(r)$. In order for the two numbers to be amicable, need $\sigma(r) = 1 + r = 3^2 * 2^{2e-1} = (p+1)(q+1)$:

$$\begin{aligned}(p+1)(q+1) &= (3 * 2^e)(3 * 2^{e-1})\\ &= 3^2 * 2^{2e-1}\\ &= r + 1\end{aligned}$$

For $e = 1$, have $2^e pq = 20$ with $\sigma(2^e pq) = 42$ and $2^e r = 34$, with $\sigma(2^e r) = 54$?

2.12

Show that if $p > 3$ and $2p + 1$ is prime, then $2p(2p + 1)$ is deficient.

$\sigma(2p(2p+1)) = \sigma(2)\sigma(p)\sigma(2p+1) = 6 * (p+1)^2 = 6p^2 + 12p + 6$. We want to compare this with $2 * 2p(2p+1) = 8p^2 + 4p$. For $p = 5$, the smallest acceptable p, 216 ¡ 300. Consider the difference between the two, $2p^2 - 8p - 6$. The derivative of this with respect to p is $4p - 8$, which is > 0 for $p > 3$. So, $2 * 2p(2p+1) > \sigma(2p(2p+1))$, and $2p(2p+1)$ is deficient.

2.13

Show that all even perfect numbers end in 6 or 8.

By Euler's theorem, if n is an even perfect number, then it can be written $n = 2^{p-1}(2^p - 1)$. Consider the sequence of powers of 2 (mod 10):

$$2^1 \equiv 2 \pmod{10}$$
$$2^2 \equiv 4 \pmod{10}$$
$$2^3 \equiv 8 \pmod{10}$$
$$2^4 \equiv 6 \pmod{10}$$
$$2^5 \equiv 2 \pmod{10}$$
$$2^6 \equiv 4 \pmod{10}$$
$$2^7 \equiv 8 \pmod{10}$$
$$2^8 \equiv 6 \pmod{10}$$
$$...$$

Subsequent powers of 2 will follow this cycle of 2, 4, 8, 6 (mod 10). 2^{p-1} will have an even exponent when $p > 2$, so it will be will be $\equiv$ to either 4 or 6 (mod 10). If it's $\equiv 4$, then $2^p - 1 \equiv 7$ and $2^{p-1}(2^p - 1) \equiv 4 * 7 \equiv 8 \pmod{10}$. If it's $\equiv 6$, then $2^p - 1 \equiv 1$ and $2^{p-1}(2^p - 1) \equiv 1 * 6 \equiv 6 \pmod{10}$. For $p = 2$, $2^{p-1}(2^p - 1) = 6$, which ends with a 6.

2.14

If n is an even perfect number and $n > 6$, show that the sum of its digits is congruent to 1 (mod 9).

n can be written $d_k 10^k + d_{k-1} 10^{k-1} + ... + d_1 10 + d_0$. Note that $10^k \equiv 1 \pmod 9$. So $n \equiv d_k + d_{k-1} + ... + d_1 + d_0 \pmod 9$. So if the sum of digits $\Sigma d_i \equiv 1 \pmod 9$, then $n \equiv 1 \pmod 9$. If n is an even perfect number, can write $n = 2^{p-1}(2^p - 1)$. Consider the sequence of powers of 2 (mod 9):

$$2^1 \equiv 2 \pmod 9$$
$$2^2 \equiv 4 \pmod 9$$
$$2^3 \equiv 8 \pmod 9$$
$$2^4 \equiv 7 \pmod 9$$
$$2^5 \equiv 5 \pmod 9$$
$$2^6 \equiv 1 \pmod 9$$
$$2^7 \equiv 2 \pmod 9$$
$$2^8 \equiv 4 \pmod 9$$
$$2^8 \equiv 8 \pmod 9$$
$$2^4 \equiv 7 \pmod 9$$
$$2^5 \equiv 5 \pmod 9$$
$$...$$

By a similar argument as the previous exercise, $2^{p-1} \equiv 4, 7,$ or 1 and $2^p - 1 \equiv 7, 4,$ or 1, respectively. Then $4 * 7 \equiv 7 * 4 \equiv 1 \pmod 9$, and $1 * 1 \equiv 1 \pmod 9$.

2.15

If p is odd, show that $2^{p-1}(2^p - 1) \equiv 1 + 9p(p-1)/2 \pmod{81}$.

TODO