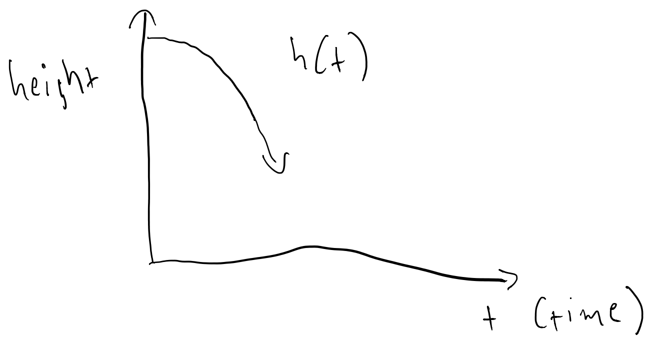
Continuous Functions

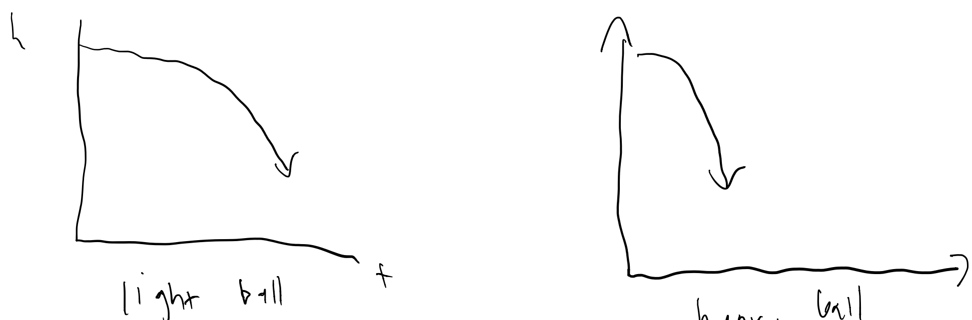
In some sense, being able to define continuous functions is the point of topology:

Suppose you're modeling nature,
Say you drop a ball and you want to consider it's height as a function of time



If you're interested in the laws of reality, you're interested in what properties this function has.

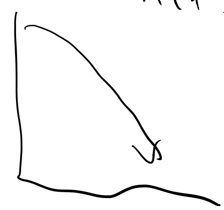
For example, Aristotle thought this function would be steeper for heavier balls



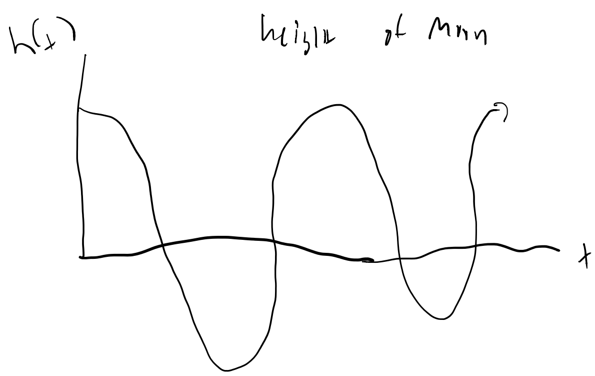
(he was wrong)

Galileo figured out that the shape of this curve is quadratic

$$h(t) = -at^2 + c$$



Newton figured out that you can generalize this to "balls" whose mass is significant wrt to the earth



..... and all sorts of complications involving air resistance, etc., ...

Topology backs up from all this & asks a more fundamental question: what property must h have just by virtue of being a function found in nature

If t_0 is "close" to t_1, then $h(t_0)$ is "close" to $h(t_1)$

h is a continuous function

In some sense, the point of topology is to make this precise.

First attempt from calculus (Bolzano, 1817, Weierstrass):

$f : \mathbb{R} \to \mathbb{R}$ is continuous if

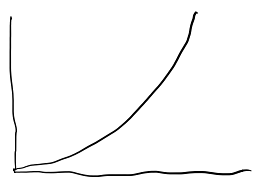
$\forall x \in \mathbb{R}\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall x' \in \mathbb{R}$

if $|x - x'| < \delta$, then $|f(x) - f(x')| < \varepsilon$

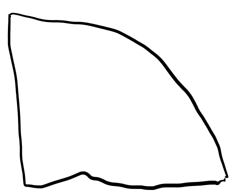
This works for functions from $\mathbb{R} \to \mathbb{R}$, so why do we need any other notion of continuity?

There are more things in nature than functions from $\mathbb{R}$ to $\mathbb{R}$. For example, consider the area functional: This takes a function from $[0,1]$ to $[0,1]$ and returns its area (integral)

$$\int_0^1 x^2\,dx = \frac{1}{3}x^3 \Big|_0^1 = \frac{1}{3}$$

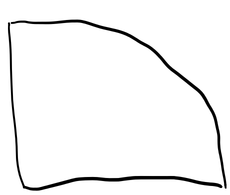


$x \mapsto x^2$ $\longmapsto$ $1/3$



$x \mapsto \sqrt{1-x^2}$ $\longmapsto$ $\pi/4 \approx .785398$

This is still continuous intuitively: a small change in the function results in a small change in the output.



$x \mapsto \sqrt{1 - .999x^2}$ $\longmapsto$ $.785781$

This is also continuity, but not

Clear how to formalize this w/ $\varepsilon - \delta$!

In fact you still can use $\varepsilon - \delta$ with this example, but it takes some thinking. Here's an example where you can't!

Let $\mathbb{N}_\perp = \{0, 1, 2, \ldots\} \cup \{\perp\}$

Think of a computer program which can either output a natural number, or never terminate.

There is notion of continuous function $\mathbb{N}_\perp \to \mathbb{N}_\perp$ which corresponds to anything you could ever observe a "computer" to do: namely, it has to satisfy

$f(\perp) = \perp$ or, if $f(\perp) \in \mathbb{N}$ then f must be constant.

This is also a notion of continuity when $\perp$ is close to every natural number, but the natural numbers aren't close to one another.

This can't be formalized with $\varepsilon - \delta$'s, but all these examples can be formalized w/

topological spaces.

Let's recall the definition of topological space. It's a set X together with a collection $\mathcal{O}(X)$ called the "open sets" of X. The intended interpretation is that a set A is open if $\forall x$ in A, all points close to x are also in A. $\mathcal{O}(X)$ must satisfy

1. $\emptyset, X \in \mathcal{O}(X)$
2. $A, B \in \mathcal{O}(X) \Rightarrow A \cap B \in \mathcal{O}(X)$
3. $A_i \in \mathcal{O}(X) \Rightarrow \bigcup A_i \in \mathcal{O}(X)$

Given that, our intuitive definition of continuity

If x is close to x' then $f(x)$ is close to $f(x')$

becomes $f^{-1}($