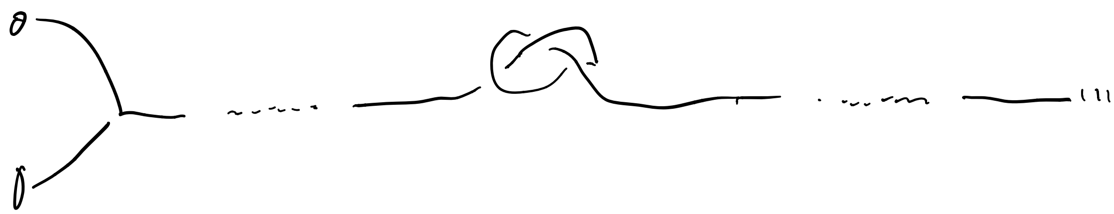
Knot Theory

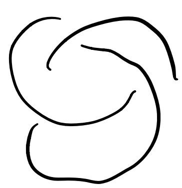
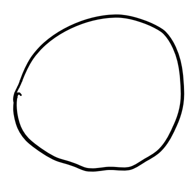
Motivating Problem: You pull your headphones out of your pocket and see this:



Of course, you can get rid of this knot by threading one end all the way through.

Can you get rid of it without threading one end through? Intuitively, it seems impossible, but how can we prove it?

Mathematicians formalize "without threading one through" by connecting the two loose ends, so they would phrase the question as:

Is  isotopic to  ?

where two things are isotopic if one can be deformed to the other through pushing, pulling, stretching, and squeezing, but not tearing.

History

(sort of)

Knot theory actually got started in 1867 when Lord Kelvin came up with the idea that atoms were knots in the luminiferous aether: atoms were indivisible, and could be neither created nor destroyed (he thought) so it might make sense that they were knots.

The first knot tables were made by Tait (who later enlisted a couple other people) There was no theory behind it, just intuition

1880's — Tait et al

⋮

1920's — Alexander (used Alexander polynomial)

⋮

1970's — John H. Conway (used tangles)
60's?

As a curious side note, various people made knot tables w/ more and more accuracy over the years but they all made one mistake.

Perko Pair

They all thought these two were distinct.

This was finally noticed by a lawyer named Perko in 1974

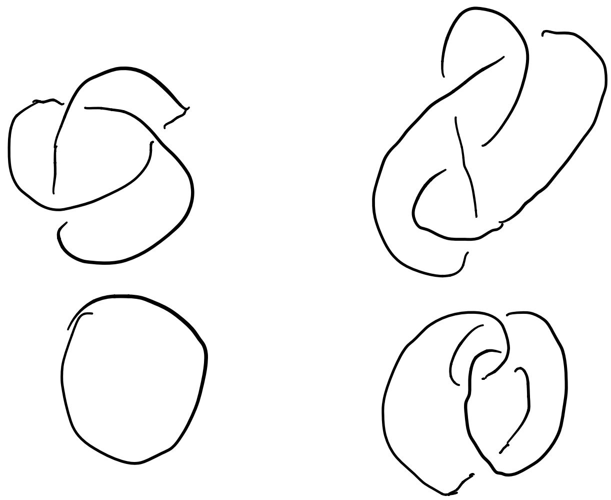
Formal (ish) Definitions

Def. A *knot* is a circle embedded in $\mathbb{R}^3$, considered up to isotopy

What I've been drawing so far have been *knot diagrams*

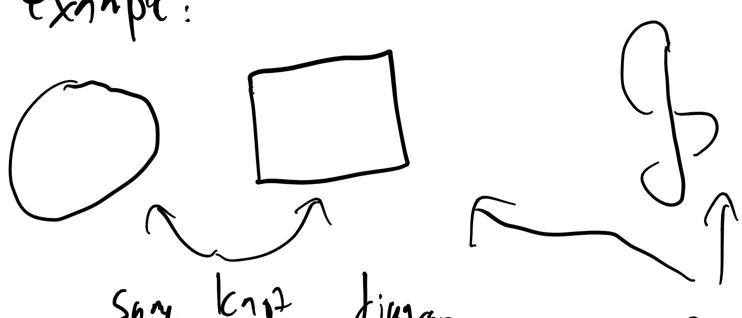
Def. A <u>knot</u> <u>diagram</u> (or <u>projection</u>) is a plane graph where 4 edges meet at each vertex, together w/ "appropriate" overcrossing / undercrossing information.

For example:



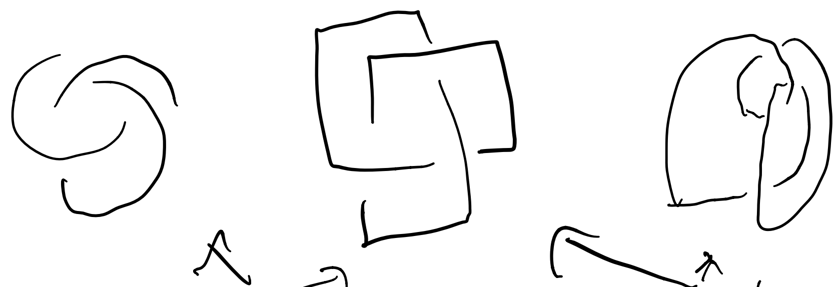
Every knot diagram <u>represents</u> a unique knot, but a knot can have many knot diagrams represent it.

For example:



same knot diagram

• same knot, diff't diagram



Same knot diagram

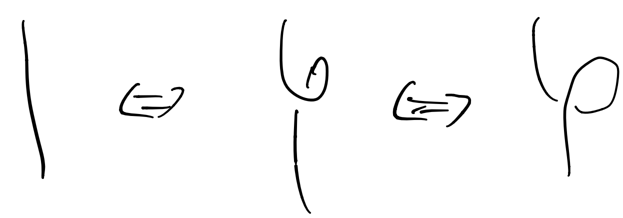
Same knot, diff't diagram

Question: How can we tell if two knot diagrams represent the same knot?

Answer: It's very hard!

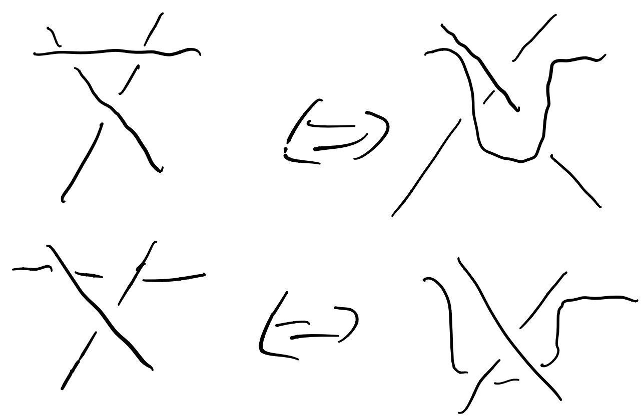
Half the answer:

Thm (Reidemeister): Two knot diagrams rep't the same knot iff one can be obtained from the other by a fin. seq. of Reidemeister moves:

R_1: 

R_2: 

R_3



This is half an answer bc it allows us to show that two knots diagrams are equal mechanically: just find the sequence of Reidemeister moves turning one to the other.

But how to show that two knots aren't equal? We have to be more clever.

General plan to show that D_1 and D_2 don't rep't same knot:

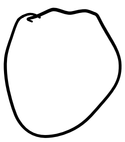
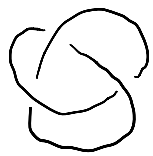
1. Define a function f w/ domain set of knot diagrams.
2. Show that f is invariant under Reidemeister moves.
3. Show that $f(D_1) \neq f(D_2)$

A fn satisfying 1 & 2 is called a

Knot invariants

Two Easy-to-Define but Hard-to-Compute Knot Invariants

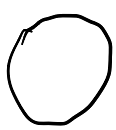
Note: these will not help us prove that

 $\neq$ 

Def: The crossing number of a knot K is the minimum number of crossings in any diagram rep'ing K.

Hard to compute — have to look at all diagrams rep'ing K

But important conceptually: measures complexity of knots, knot tables often arranged by crossing num

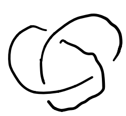
0 

0_1

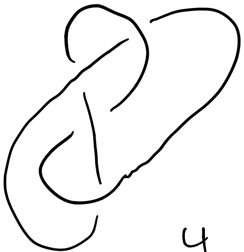
3 

3_1

For example,  has 4 crossings, but crossing number is 3:

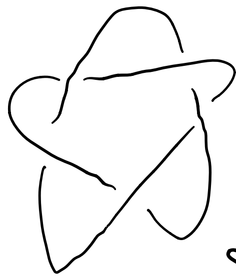


4

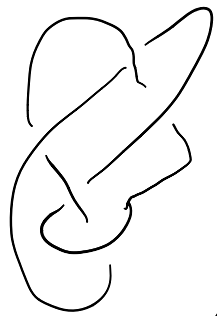


4_1

5



5_1

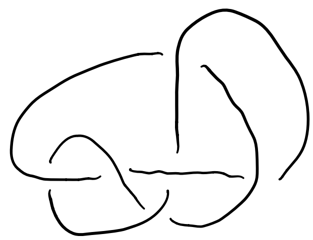


5_2

6

6_1

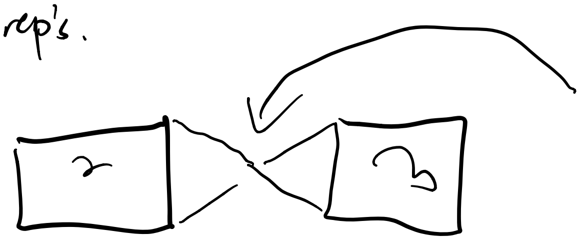
6_2



6_3

Many open problems

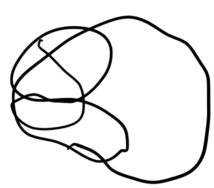
1. (Tait) If a knot diagram D is alternating and has no nugatory crossings, then the crossing number of D is the crossing number of the knot it rep's.



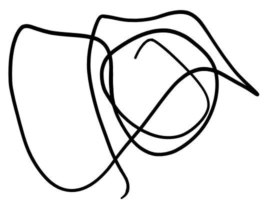
nugatory crossing

This was proved in the affirmative in the 90's using Jones polynomial

2. (Still open)



Crossing number c_1



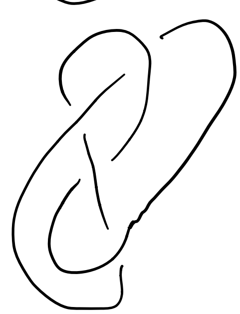
Crossing number c_2

Crossing number of $c_1 \# c_2 = c_1 + c_2$?

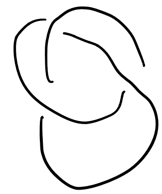
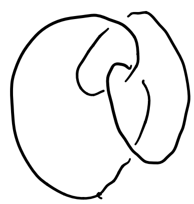
Second Hard-to-Compute Invariant

Def A bridge of a knot diagram is an arc that goes over at least one undercrossing

Each arc is a bridge



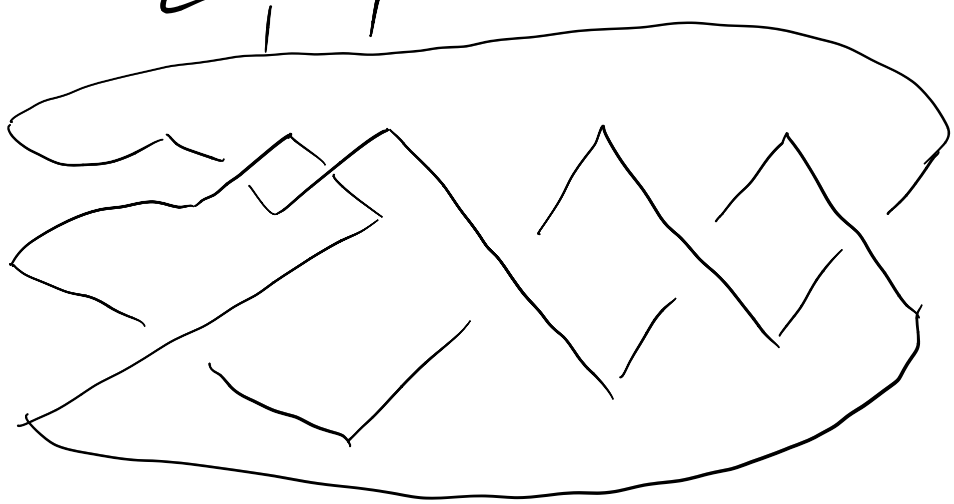
Def. Bridge number is minimum # of bridges in any knot diagram

For example, bridge # of  is 2, bc it's equivalent to 

Bridge # of independent invariant, but bring it up bc of slide show.

Knots w/ bridge number 2 are also called rational tangles, b/c they can be associated w/ lists of #s

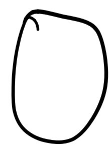
2, 1, -3



John Conway proved that $[a_1, \ldots, a_m]$ is an equivalent knot to $[b_1, \ldots, b_n]$ iff

$$a_m + \cfrac{1}{a_{m-1} + \cfrac{1}{\cdots + \cfrac{1}{a_1}}} = b_n + \cfrac{1}{b_{n-1} + \cfrac{1}{\cdots + \cfrac{1}{b_1}}}$$

Tri-colorability

Let's <u>actually</u> prove that  $\neq$ 

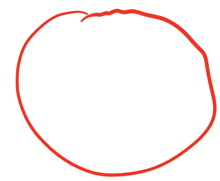
1.

Def. A knot is called tricolorable if there's an assignment of one of 3 colors to each of its arcs such that

1. At least two colors are used
2. At each crossing, either 1 or 3 colors are used

Example:

Unknot not three colorable



Trefoil is

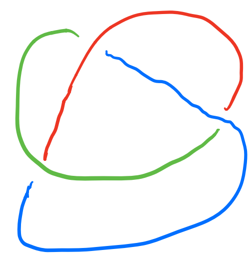
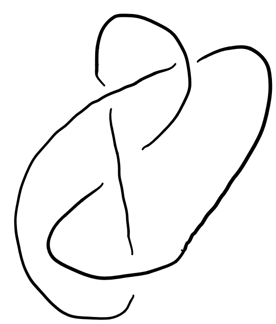
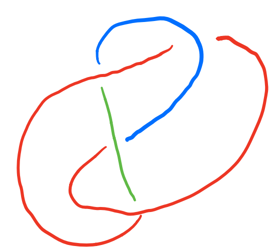


Figure-8 isn't



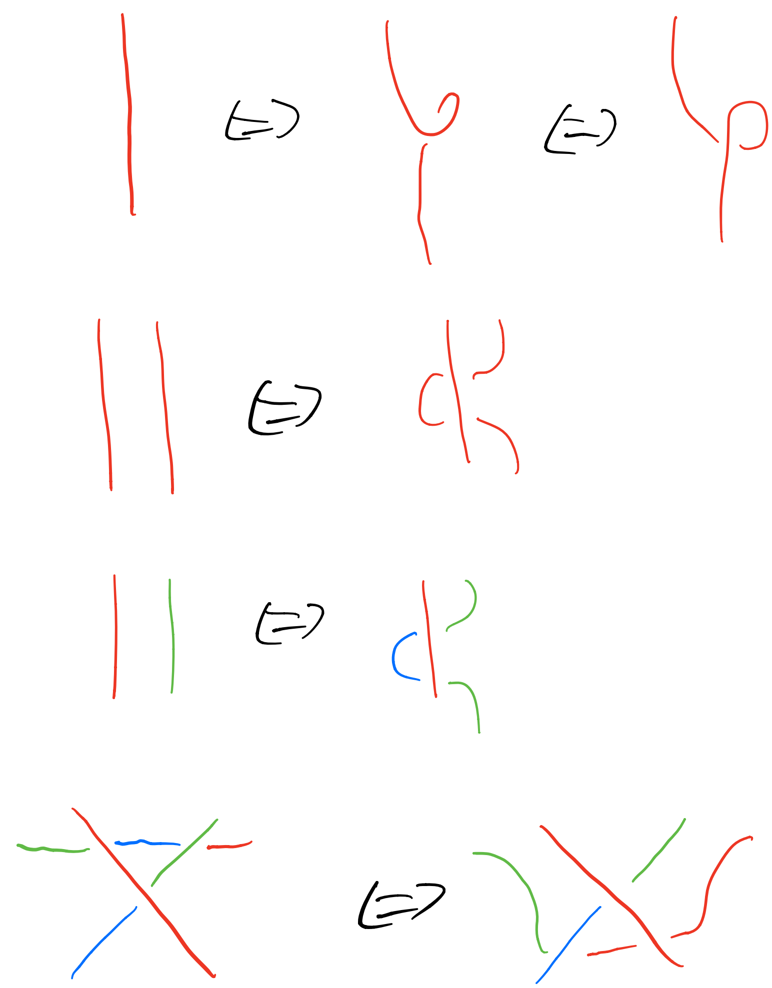
One attempt

For example:



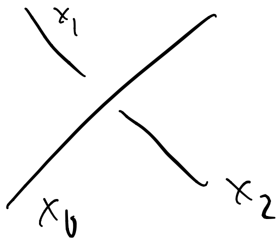
So, if we can prove that tricolorability is a knot invariant, we will have finally proved that the unknot & the trefoil are different. (But not the figure 8 knot!)

Pf.



Generalizations of 3-coloring

Notice if you label the colors 0, 1, 2 then the tricolorability condition becomes



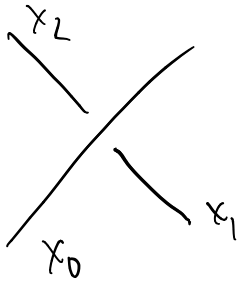
$$x_0 + x_1 + x_2 \equiv 0 \pmod 3$$

(e.g. $1+1+1 \equiv 3 \equiv 0 \pmod 3$

$0+1+2 \equiv 3 \equiv \pmod 3$

$1+1+2 \equiv 4 \equiv 1 \not\equiv 0 \pmod 3$)

Generalize to n colors w/



$$-2x_0 + x_1 + x_2 \equiv 0 \pmod n$$

<u>Still</u> a knot invariant

Further generalization: Pick t coprime to n and require $(1-t)x_0 + t x_1 = x_2$

Biggest generalization: colors aren't integers, but a general structure called a <u>quandle</u>

Polynomial Knot Invariants

Tricolorability assigns to every knot a Yes/No value

Number of 3 colorings a number to every knot.

This doesn't distinguish every knot though.

Could get more power by considering knot invariant that assigns

$$K \longmapsto (\#\text{ of 3 colorings of } K,\ \#\text{ of 4 colorings of } K,\ \#\text{ of 5 colorings of } K,\ \dots)$$

But this is no easier than working with n-colorings separately.

Polynomial Knot Invariants both

- Combine the power of infinitely many integer-valued knot invariants
- Are easy to compute

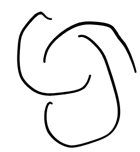
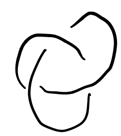
How are they easy to compute?

They are generally defined by skein relations

$$\langle \times \rangle = P \langle)(\rangle + Q \langle \asymp \rangle$$

$$\langle O \rangle = R$$

Important Polynomials are

- Alexander polynomial (1920's, used in Alexander's table of knots)
- Jones polynomial (1980's, opened up applications to physics, can prove  not equal to , won 2 Fields medals)
- Homfly polynomial (generalizes both Alexander & Jones)

<u>Applications of Knot theory</u>

- DNA
- Chemistry (knotted molecules)
- Fluid dynamics
- Statistical Mechanics
- Quantum Gravity