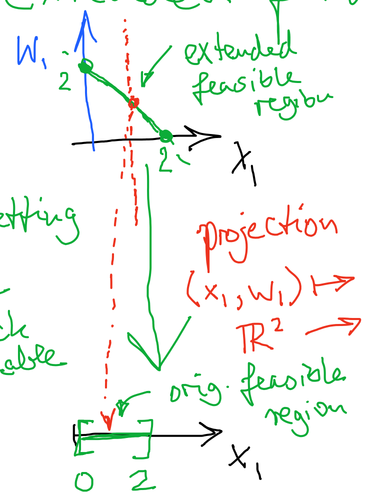
Claim: Steps of Phase II of the simplex method advance through a sequence of vertices (extreme points) of the convex polyhedron that is the feasible region.

"Extended formulation"



forgetting the slack variable

projection

$(x_1, w_1) \mapsto x_1$

$\mathbb{R}^2 \to \mathbb{R}^1$

$x_1 + w_1 =$

x_1, w_1

injection (1:1)

$x_1 \leq 2$

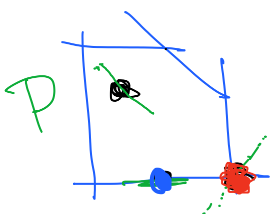
$x_1 \geq 0$

linear bijection (1

from original feasi

to extended

General topology tells us the distinction of:

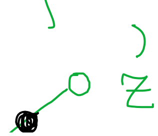


• interior point

•• boundary points

Convex geometry defines:

Extreme point: An element $x \in P$ of a convex set is not an extreme point if $\exists\, y, z \in P$, $y \neq z$ so that $x \in (y, z)$ (open line segment



" y o x 'btw. y and z'.

$\boxed{\text{Every basic feasible solution is an extreme point!}}$

Detour: An extended formulation in a modeling technique. $\in \mathbb{R}$ rather in $\mathbb{R}_{\geq 0}$

A problem with a free variable:

$$\boxed{\begin{array}{lrcl} \max & x_1 & & \\ \text{s.t.} & x_1 + x_2 & \leq & 3 \\ & x_1 - x_2 & \leq & 4 \\ & x_1 \geq 0 & & x_2 \in \mathbb{R} \end{array}}$$

Not in standard (inequality) form

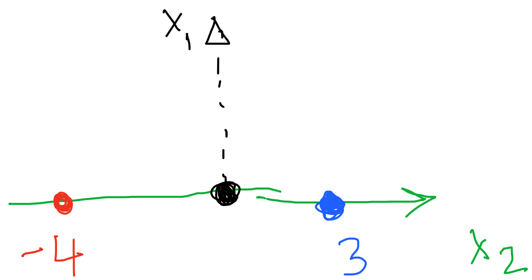
Bring it to standard form:

Idea: $\boxed{\text{Every real number is the difference of 2 nonneg. variables.}}$

$|x_1|$ nonlinear (x_1) 3 ... $= \max\{a, \dots$

$$\mathbb{R}_{\geq 0} \times \mathbb{R} \ni \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \xmapsto{\text{injection}} \begin{pmatrix} \\ y_2 \\ z_2 \end{pmatrix} \in \mathbb{R}_{\geq 0}$$

$$\boxed{\begin{array}{l} y_2 = \max\{0, x \\ z_2 = \max\{0, - \end{array}}$$



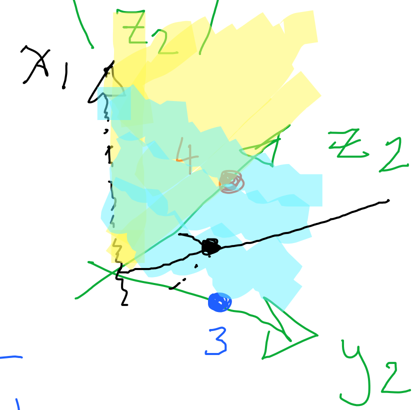


Image is a non-convex set:

Union of 2 pla

(x_1, z_2) - plane

$\cup$ (x_1, y_2) - plane

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \simeq \begin{pmatrix} x_1 \\ y_2 - z_2 \end{pmatrix} \xleftarrow[\text{projection}]{\text{linear}}$$

also makes sense for all points in $\mathbb{R}^3$.

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \xleftarrow{\text{For example}} \begin{pmatrix} x_1 = 0 \\ y_2 = 2 \\ z_2 = 2 \end{pmatrix} \in \mathbb{R}^3_{\geq 0}$$

$$\begin{array}{ll} \max & x_1 \\ \text{s.t.} & x_1 + x_2 \leq 3 \\ & x_1 - x_2 \leq 4 \end{array} \quad \rightsquigarrow \quad \begin{array}{ll} \max & x_1 \\ \text{s.t.} & x_1 + y_2 - z_2 \\ & x_1 - (y_2 - z_2 \end{array}$$

$x_1 \geq 0$ $\cancel{x_2} \in \mathbb{R}$ $x_1 \geq 0$, $y_2 \geq z$:

linear optimization problem in std. form

1) Every $\boxed{\text{feas.}}$ sol. in orig. problem maps to a $\boxed{\text{feas}}$ solution in ext. problem of the same objective value.

2) Every $\boxed{\text{feas.}}$ sol. in ext. problem maps to a $\boxed{\text{feasible}}$ solution in orig. problem of the same objective value

$\Rightarrow$ Same holds for optimal instead of $\boxed{\text{feasible}}$

Given any dictionary:

$$\max \bar{z} + \sum_{j \in N} \bar{c}_j x_j$$

$$\text{s.t.} \quad x_i = \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j \qquad i \in B$$

can consider the projection that forgets about basic variables:

$$\underbrace{\left(x_1, x_2, x_3, \ldots, x_9, x_{10}\right)}_{\substack{\mathcal{B} \qquad \in N \qquad \mathcal{B}}} \longmapsto \left(x_1, x_3, \ldots, x_9\right)$$

$$\mathbb{R}^{|\mathcal{B} \sqcup N|} \ni x = \left(x_{\mathcal{B}} \mid x_N\right) \xmapsto[\text{projection}]{} x_N \in \mathbb{R}^{|N|}$$

b/c dictionary:

$$\left(x_{\mathcal{B}} = \bar{b} - N x_N\right) \xleftarrow[\text{injection}]{} \!\!\shortmid$$

$$\downarrow$$

$$\uparrow$$

In the space of the nonbasic variables, ~~the~~ feasible set is subset of $\mathbb{R}^{|N|}_{\geq 0}$.

$x_N = 0$ is an extreme point
