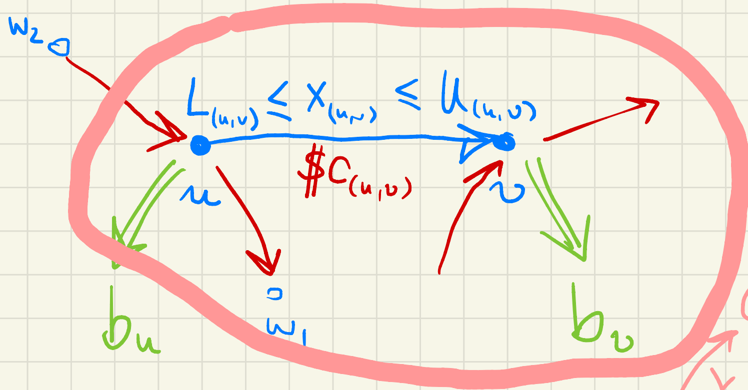
Lower bounds:

$$L \le x \le U \qquad L \in \mathbb{R}^A_{\ge 0}, \quad U \in \mathbb{R}^A$$



$$0 \le x'_{(u,v)} \le U_{(u,v)} - L_{(u,v)}$$

$$x_{(u,v)} = L_{(u,v)} + x'_{(u,v)}$$

Flow conservation:

$$\sum_{\substack{w:\\(w,u)\in A}} x_{(w,u)} - \sum_{\substack{w:\\(u,w)\in A\\ \ne (u,v)}} x_{(u,w)} - \overbrace{x_{(u,v)}}^{= L_{(u,v)} + x'_{(u,v)}} = b_u$$

$$\sum_{\substack{w:\\(w,v)\in A\\ \ne (u,v)}} x_{(w,v)} - \sum_{\substack{w:\\(v,w)\in A}} x_{(v,w)} + x_{(u,v)} = b_v$$

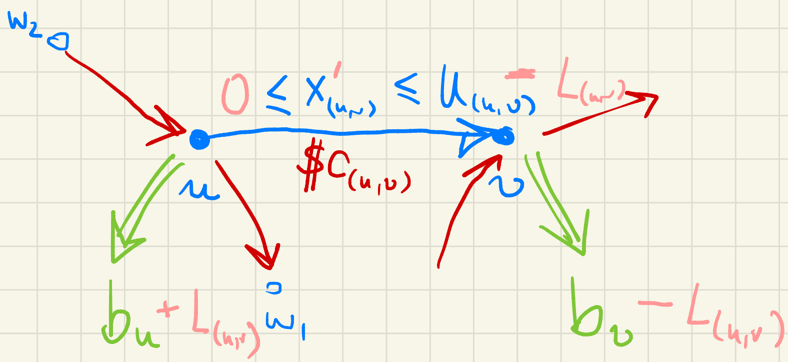
$$L_{(u,v)} \le x_{(u,v)} \le U_{(u,v)}.$$

$$\sum_{\substack{w:\\(w,u)\in A}} x_{(w,u)} - \sum_{\substack{w:\\(u,w)\in A\\ \neq (u,v)}} x_{(u,w)} - x'_{(u,v)} = \underbrace{b_u + L_{(u,v)}}_{b'_u}$$

$$\sum_{\substack{w:\\(w,v)\in A\\ \neq (u,v)}} x_{(w,v)} - \sum_{\substack{w:\\(v,w)\in A}} x_{(v,w)} + x'_{(u,v)} = \underbrace{b_v - L_{(u,v)}}_{b'_v}.$$

$\Longrightarrow$ Flow conservation constraints with a changed exceedance vector b':

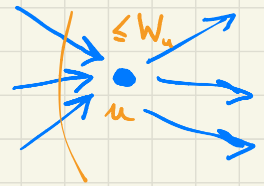
$$b'_w = \begin{cases} b_u + L_{(u,v)} & \\ b_w & \text{if } w \neq u,v \\ b_v - L_{(u,v)} & \end{cases}$$



$$\min \sum_{\substack{a\in A\\ a\neq (u,v)}} c_a x_a + c_{(u,v)} \overbrace{x_{(u,v)}}^{= L_{uv} + x'_{(u,v)}}$$

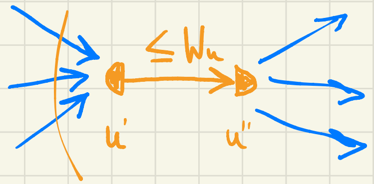
$$= \underbrace{c_{(u,v)} L_{(u,v)}}_{\text{const.}} + \underbrace{\sum_{\substack{a\in A\\ a\neq (u,v)}} c_a x_a + c_{u,v} x'_{(u,v)}}_{\text{flow cost}}$$

Node capacities W_u :

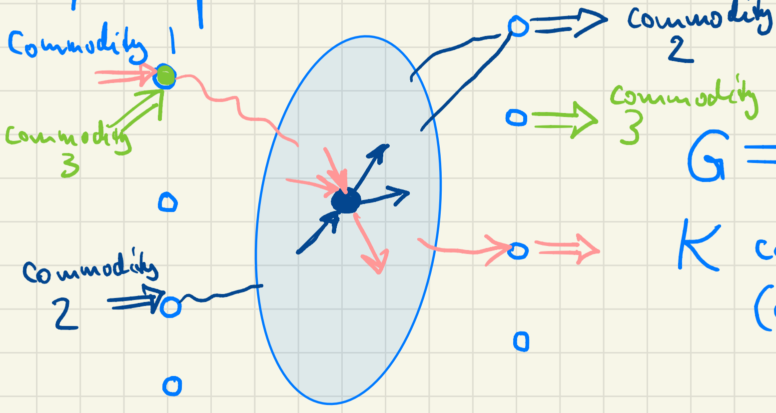


inflows $= \sum_{\substack{w: \\ (w,u) \in A}} x_{(w,u)} \leq W_u$

outflows $= \sum_{\substack{v: \\ (u,v) \in A}} x_{(u,v)} \leq W_u$



▶ Multiple commodities :



$G = (V, A)$

K commodities

(e.g. origin - destination pairs in a telecom network)

For each commodity $k = 1, \ldots, K$,

we have separate arc flow variables:

$$x^k_{(u,v)} \quad (u,v) \in A \quad \mapsto \quad x^k \in \mathbb{R}^A_{\geq 0}$$

that satisfy individually flow conserv. constraints:

$$\boxed{\begin{array}{c} N x^k = b^k \\ x^k \geq 0 \end{array}} \leftarrow \text{flow excedance for commodity } k$$

with $\boxed{\text{shared}}$ capacities:

total arc flow $= \sum_{k=1}^{K} x^k_{(u,v)} \leq U_{(u,v)}$

$$u \longrightarrow v$$

Block structure:

$$\begin{array}{ccccc}
[\;x^1\;] & [\;x^2\;] & \cdots & [\;x^K\;] & \\
\boxed{N} & & & & = b^1 \\
& \boxed{N} & & & = b^2 \\
& & \ddots & & \\
& & & [\;N\;] & = b^K \\
\boxed{I} & \boxed{I} & \cdots & \boxed{I} & \leq U
\end{array}$$