with "strict complementary slackness":
Exactly one of the two variables in each complementary pair is zero.
(Cannot be achieved in general with basic solutions!)

(Symmetric) (metric)

- Traveling salesperson problem.

satisfy triangle ineq. $c_{\{i,j\}} + c_{\{j,k\}} \geq c_{\{i,k\}}$ $\forall i,j,k$

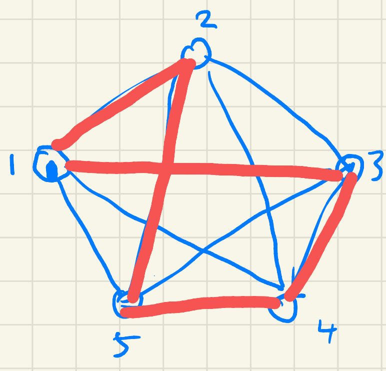
So symmetric distances

Given n cities, distances $c_{\{i,j\}} \geq 0$ for $i,j \in [n]$. Find a "tour" (Hamiltonian circuit) in the graph (V, E) of minimum cost.

simple cycle (no vertex repetition)

complete

$V = [n]$



Combinatorial optimization problem!

How to model this problem as an (integer) linear optimization problem?

"Natural formulation":

Ignore starting city & direction in which we follow the tour.

→ Use variables that encode which edges belong to the tour.

encode $T \subseteq E = \binom{V}{2}$

$$x_{\{i,j\}} = \begin{cases} 1 & \text{if } \{i,j\} \in T \\ 0 & \text{o/w} \end{cases} \qquad x \in \mathbb{R}^E$$

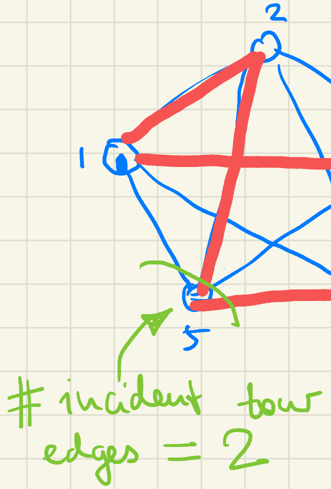
Objective function:

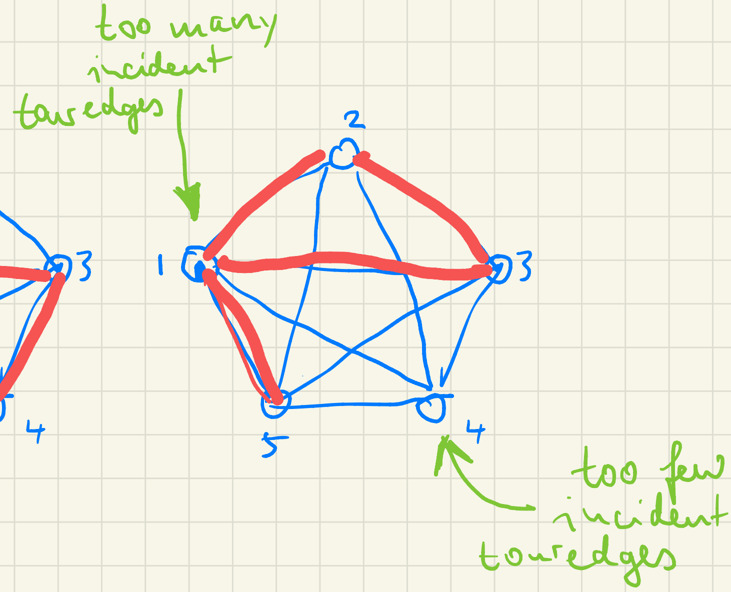
min total distance traveled

$$= \sum_{\{i,j\} \in T} c_{\{i,j\}} = \sum_{\{i,j\} \in E} c_{\{i,j\}} x_{\{i,j\}}$$

linear fxn in x.

Constraints:



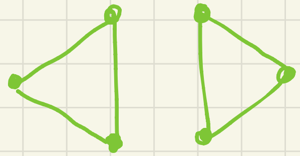


$$\#\text{ incident tour edges of vertex } i = \sum_{\substack{j \in V: \\ \{i,j\} \in E}} x_{\{i,j\}} = 2 \quad \forall i \in V$$

"degree constraints".

Is this list of constraints sufficient to give a correct IP formulation of TSP?

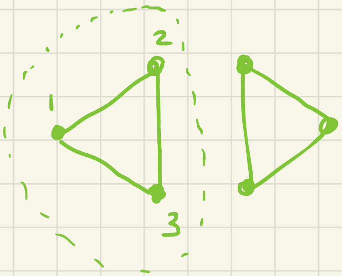
No! Observe: for $n \geq 6$, get as 0/1 solutions of the formulation w/ degree constraints:



disjoint unions of simple cycles!

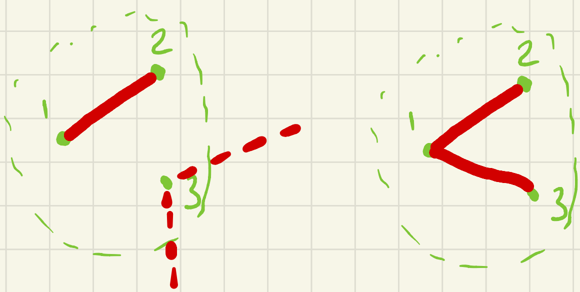
Additional constraints to eliminate the "short cycles".

S



How many edges can a tour have within a nonempty proper subset $\emptyset \neq S \subsetneq V$ of the vertices

S



↓

$\leq |S| - 1$ edges!

Subtour elimination constraint (short cycle constraint):

$$\#\text{ tour edges} = \sum_{\{i,j\} \in E(S)} x_{\{i,j\}} \leq |S| - 1 \quad \forall\, \emptyset \neq S \subsetneq V$$

on graph induced by vertex set S

w/ induced edge set $E(S) = \{\, \{i,j\} \in E : i,j \in S \,\}$.

Theorem: Degree constraints + subtour elim. form a correct IP formulation.

Problem: These are exponentially many constraints:

$$2^n - 2.$$

(actually enough to write short-cycle constraints for $|S| \le \left\lfloor \frac{n}{2} \right\rfloor$

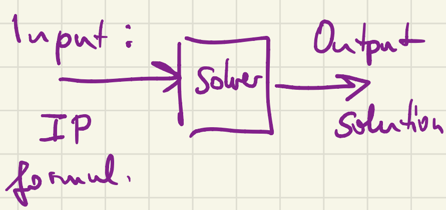
$$\Longrightarrow \quad \approx 2^{n-1} \quad)$$

$\Rightarrow$ Writing down the problem formulation takes exponential time!

Idea: Constraint generation:

Generate constraints "on the fly" during the solution process.

1. Constraint generation using an IP solver (e.g. Gurobi) as a "black box".



Write down a problem formulation with a subset of the known constraints.

Check whether solution from solver is an actual solution.

If not, find a constraint violated by the solution. Add this constraint to the formulation & repeat.