$$\min \sum_{\{i,j\} \in \delta(S)} \tilde{x}_{\{i,j\}}$$

undirected min cut

$$\text{s.t.} \quad \emptyset \neq S \subsetneq V$$

$$\sum_{i \in V} x_i \geq 1$$

$$\sum_{i \in V} (1 - x_i) \geq 1$$

Variables

$$x_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{o/w.} \end{cases} \quad \text{for } i \in V$$

To make the objective function linear, introduce more variables:

$$y_{\{i,j\}} = \begin{cases} 1 & \text{if } \{i,j\} \in \delta(S) \\ 0 & \text{o/w.} \end{cases}$$

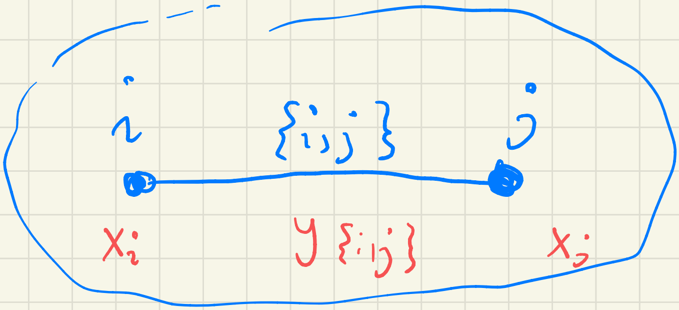
Rewrite:

linear in y.

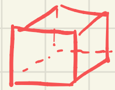
$$\min \sum_{\{i,j\} \in E} \tilde{x}_{\{i,j\}} \, y_{\{i,j\}}$$

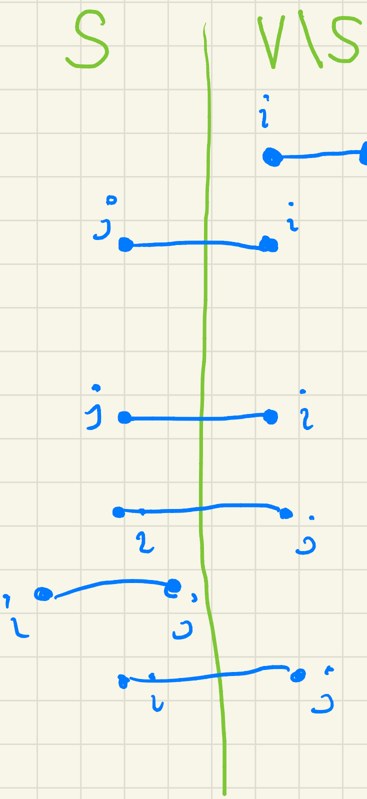
constant index set for summation

Need to encode the relation btw. x_i and $y_{\{i,j\}}$ using constraints.



Tabulate possible values: (truth table)



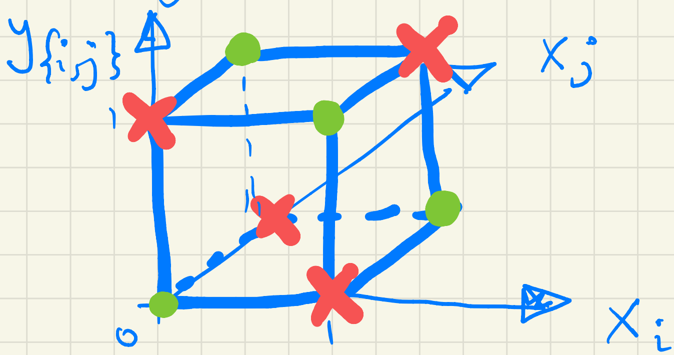


	x_i	$y_{\{i,j\}}$	x_j
●	0	0	0
X	~~0~~	~~0~~	~~1~~
X	~~0~~	~~1~~	~~0~~
●	0	1	1
X	~~1~~	~~0~~	~~0~~
●	1	0	1
●	1	1	0
X	~~1~~	~~1~~	~~1~~

$2^3 = 8$

Claim: If we enforce this truth table for each $\{i,j\} \in E$, then x, y variables correctly model the cut $\delta(S)$ induced by S.

Fix i,j:



Idea is now to find linear inequalities that describe the set of <u>solutions</u>, cutting off the non-solutions.

Many inequalities can do this but there is a "best" choice:

Take the inequalities that are tight at the adjacent solutions.

Cut off the non-solution

$\times \quad (\bar{x}_i, \bar{y}_{\{i,j\}}, \bar{x}_j) = (0, 0, 1)$

Given a point $(x_i, y_{\{i,j\}}, x_j) \in [0,1]^3$
measure how far it is from the point
$(0,0,1)$: (ℓ_1-norm distance)

$$0 < |\underbrace{x_i}_{\geq 0} - 0| + |\underbrace{y_{\{i,j\}}}_{\geq 0} - 0| + |\underbrace{x_j}_{\leq 1} - 1|.$$

$$= x_i + y_{\{i,j\}} + (1 - x_j)$$

$$\boxed{\text{The inequality} \quad x_i + y_{\{i,j\}} + (1 - x_j) \geq 1}$$

cuts off the point $(0,0,1)$
but is valid for all solutions
and tight (satisfied w/ equality) for adj. sol.

Such inequalities are "facet-defining inequalities" (tight at dimension many affinely independent solutions).

This inequality, cutting off 1 vertex of cube and tight at the adjacent vertices, is called "no-good inequality":

To cut off a point $(\bar{x}_i, \bar{y}_{\{i,j\}}, \bar{x}_j) \in \{0,1\}^3$ from the cube $[0,1]^3$, use the inequality:

$$\underbrace{|x_i - \bar{x}_i| + |y_{\{i,j\}} - \bar{y}_{\{i,j\}}| + |x_j - \bar{x}_j|}_{\text{rewrite equivalently as a linear inequality using } (x_i, y_{\{i,j\}}, x_j) \in [0,1]^3.} \geq 1$$

Need a total of 4 inequalities:

0,0,1 $\quad x_i + y_{\{i,j\}} + (1 - x_j) \geq 1$

0,1,0 $\quad x_i + (1 - y_{\{i,j\}}) + x_j \geq 1$

1,0,0 $\quad (1 - x_i) + y_{\{i,j\}} + x_j \geq 1$

1,1,1 $\quad (1 - x_i) + (1 - y_{\{i,j\}}) + (1 - x_j) \geq 1.$

This is a correct formulation of the defined relationship of x_i, x_j, $y_{\{i,j\}}$.

NB: These inequalities define the polyhedron that is the convex hull, i.e., smallest convex set containing the solutions.