We have exponentially many (strengthened) cutset inequalities.
Need to solve (in a constraint generation scheme) the subproblem of finding for a solution $\tilde{x}$ (to a relaxation) a violated cutset inequality! Again: an optimization problem:

$$\min \sum_{\{i,j\} \in \delta(S)} \tilde{x}_{\{i,j\}}$$

parameters, not decision variables

$$\text{s.t.} \quad \emptyset \neq S \subsetneq V$$

(a combinatorial optimization problem: "undirected min cut").

Observation:

- Feasible solutions to these subproblems can already provide violated inequalities!

- Often (primal) heuristics for solving the subproblem are used:
- "Separation subproblems" "flip" primal heuristics to dual, proof devices.

Observation:

$\tilde{x}$ can be any vector — it does not have to be an integer solution.

2 A different algorithm:

formulation (~~relaxation~~) $\longrightarrow$ ~~Solver (IP)~~ $\longrightarrow$ ~~integer~~ solution

further LP relaxation $\longrightarrow$ Solver (LP) $\longrightarrow$ $\tilde{x}$ real sol

Run the same subproblem, find an inequality from the exponential-sized formulation that is violated by $\tilde{x}$.

Add this inequality to the LP and re-run the solver.

(−) weaker relaxation
(further relaxation by forgetting integrality)

(+) much faster to solve LP

"Solve the "root relaxation" of the problem using cutting planes."

LP-based cutting plane algorithm - for solving a linear optimization problem with exponentially many constraints.

2(b) Observation: We can implement this cutting plane algorithm using the dual simplex method.

- Formulation (LP) was solved using the simplex method
  → have optimal dictionary:

$$\max \ \bar{\xi} - \sum_{j \in N} \bar{z}_j x_j \qquad \bar{z}_j \geq 0 \text{ dual feasible}$$

$$\text{s.t.} \quad x_i = \bar{b}_i + \sum_{j \in N} \bar{a}_{ij} x_j \qquad i \in \mathcal{B}$$

primal feasible $\bar{b}_i \geq 0$

$$x_i, \ x_j \geq 0$$

Add a constraint:

$$\boxed{\sum_{j=1}^{n} d_j x_j \leq d_0}$$

original variables.

new slack

$$\sum_{j=1}^{n} d_j x_j + \underbrace{s}_{\geq 0} = d_0$$

current basic & nonbasic

Idea: Make the slack variable s basic.

Eliminate the basic variables in

$$s = d_0 - \sum_{\substack{j=1 \\ j\in\mathcal{N}}}^{n} d_j x_j - \sum_{\substack{i=1 \\ i\in\mathcal{B}}}^{n} d_i x_i$$

by using the eqns of the dictionary!

$$x_i = \bar{b}_i + \sum_{j\in\mathcal{N}} \bar{a}_{ij} x_j$$

$$s = d_0 - \sum_{\substack{j=1 \\ j\in\mathcal{N}}}^{n} d_j x_j - \sum_{\substack{i=1 \\ i\in\mathcal{B}}}^{n} d_i \left(\bar{b}_i + \sum_{j\in\mathcal{N}} \bar{a}_{ij} x_j\right)$$

$$= \underbrace{\left(d_0 - \sum_{\substack{i=1 \\ i\in\mathcal{B}}}^{n} d_i \bar{b}_i\right)}_{=:\,\bar{s}} - \sum_{j\in\mathcal{N}} \underbrace{\left(\cdots\right)}_{=:\,\bar{d}_j} x_j$$

If we added an inequality that was violated by the current basic solution $\tilde{x}$, then the value of s for $x = \tilde{x}$ is the right-hand side constant

$$\bar{s} = \left(d_0 - \sum_{i=1,\, i \in \mathcal{B}}^{n} d_i \bar{b}_i \right)$$

So $\bar{s} < 0$.

$\Rightarrow$ New dictionary with the additional basic variable $\bar{s}$

- is primal infeasible (one negative RHS const.)
- still dual feasible

$\Rightarrow$ Use the dual simplex method starting from this dictionary.

In practice, "a few" dual simplex steps suffice to restore primal feasibility

— fast update of the optimal dictionary after adding a violated constraint !!

Refinement:

When a constraint becomes loose (some later optimal basic solution has its slack in

the basis, positive value)),
then drop the constraint
from the dictionary.

⟹ Method to keep the size of a problem during a cutting plane method small!