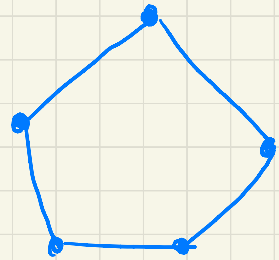
- Stable set problem (independent set problem)

Given an undirected graph $G=(V,E)$.
A set $S \subseteq V$ is called "stable" ("independent") if no two $u,v \in S$ are connected by $\{u,v\} \in E$.



5-cycle

$$\max |S| \quad \text{s.t. } S \subseteq V \text{ is stable.}$$

max-cardinality stable set

$$\max \sum_{v \in S} w_v \quad \text{s.t. } S \subseteq V \text{ stable}$$

given weight

max (weighted) stable set.

Introduce variables

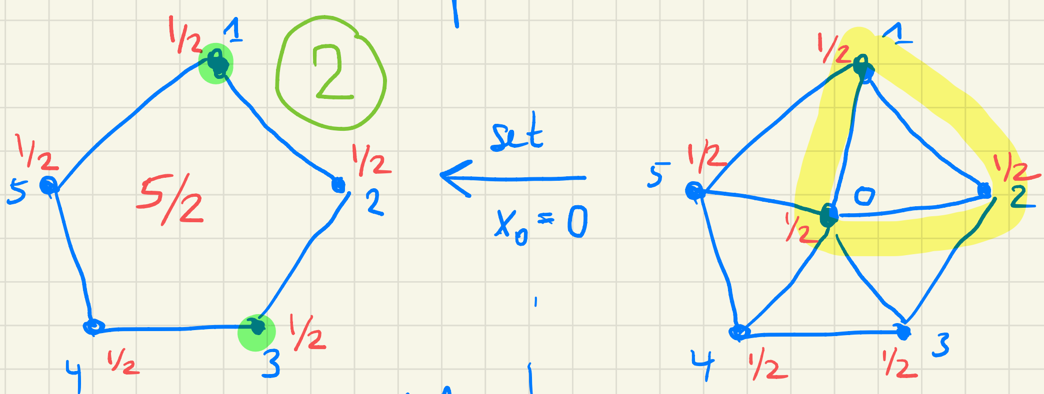
$$x_v = \begin{cases} 1 & \text{if } v \in S \\ 0 & \text{o/w} \end{cases}$$

$$\max \sum_{v \in V} w_v x_v \qquad > 0$$

$$\text{s.t.} \quad \underbrace{x_u + x_v}_{\text{how many of the two endpoints of the edge are taken}} \le 1 \quad \forall \{u,v\} \in E$$

$$x \in \{0,1\}^V.$$

This is a correct formulation as an IP.



$$\begin{aligned}
x_1 + x_2 &\le 1 \\
x_2 + x_3 &\le 1 \\
x_3 + x_4 &\le 1 \\
x_4 + x_5 &\le 1 \\
x_1 \qquad + x_5 &\le 1
\end{aligned}$$

Formulation has:

$$\begin{aligned}
x_0 + x_1 &\le 1 \\
x_1 + x_2 &\le 1 \\
x_0 + x_2 &\le 1
\end{aligned}$$

Feasible for LP: $x = (\frac{1}{2}, \dots \frac{1}{2})$

But this solution is actually not a convex combination of feasible integer solutions!

Optimal integer solutions to max-cardinality stable set have exactly 2 vertices in S

$\Rightarrow$ optimal value $= 2$.

$\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$ is an optimal vertex solution of the LP. (Unique opt. soln.)

Q: Improve the formulation to avoid the fractional solutions.

For the 5-cycle, the maximal cliques are the edges. So edge formulation is the same as the clique formulation.

Combinatorial reasoning:

From each triangle, can choose at most one vertex.

$x_0 + x_1 + x_2 \leq 1.$

$$
\begin{array}{rllllll}
\frac{1}{2}(x_1 & + x_2 & & & & \le 1) \\
+\frac{1}{2}(& x_2 & + x_3 & & & \le 1) \\
+\frac{1}{2}(& & x_3 & + x_4 & & \le 1) \\
+\frac{1}{2}(& & & x_4 & + x_5 & \le 1) \\
+\frac{1}{2}(x_1 & & & & + x_5 & \le 1) \\
\hline
\end{array}
$$

$$\underbrace{x_1 + x_2 + x_3 + x_4 + x_5}_{\in \mathbb{Z}} \le \frac{5}{2}.$$

Chvátal-G.

$$\underbrace{\boxed{x_1 + x_2 + x_3 + x_4 + x_5 \le 2}}_{= \text{objective function}}$$

$\Rightarrow$ fract'l solution $(\frac{1}{2}, \dots, \frac{1}{2})$ is cut off by this new inequality!

This inequality does <u>not</u> imply the original edge inequality!

Q: Which solutions satisfy this new inequality with equality (are tight)?

All optimal integer solutions do.

$$
\begin{array}{rllll}
\frac{1}{2}(x_0 & + x_1 & & \le 1) \\
+\frac{1}{2}(& x_1 & + x_2 & \le 1) \\
+\frac{1}{2}(x_0 & & + x_2 & \le 1) \\
\hline
\end{array}
$$

$$\underbrace{x_0 + x_1 + x_2}_{\in \mathbb{Z}} \le \frac{3}{2}$$

Chvátal-G

$$\boxed{x_0 + x_1 + x_2 \le 1}$$

More generally:

If Q is vertex set of a <u>clique</u> (complete subgraph: $\forall u, v \in Q$, $\{u, v\} \in E$), then valid inequality

$$\sum_{v \in Q} x_v \le 1.$$

The edge inequalities $x_u + x_v \le 1$ are dominated by cliques.

x_1	x_2	x_3	x_4	x_5
1	·	1	·	·
1	·	·	1	·
·	1	·	1	·
·	1	·	·	1
·	·	1	·	1

5 solutions are tight.

Convex hull of these solutions and its dimension.

$\Rightarrow$ Another correct IP formulation:

$$\max \sum w_v x_v$$

$$\text{s.t.} \sum_{v \in Q} x_v \leq 1. \quad \forall \text{ maximal cliques } Q.$$

$$x_v \in \{0,1\} \quad \forall v \in V.$$