Duality theory for linear optimization
— theory of upper bounds.

General: Optimization problems are hard.

If we can't solve a problem to optimality:

- try to find some feasible solution
- try to improve known feasible solutions...

When should we be satisfied with a known feasible solution?

We would want to say:

The known solution x^1 has value $c^T x^1 = \zeta_1$ and I can prove that <u>no</u> feasible solution has value greater than $\zeta_1 + \varepsilon$, where ε is a small constant.

Consider:

$$\max \quad 4x_1 + x_2 + 3x_3$$
$$\text{s.t.} \quad x_1 + 4x_2 \le 1 \qquad (\mathrm{I})$$
$$3x_1 - x_2 + x_3 \le 3 \qquad (\mathrm{II})$$
$$x_1,\ x_2,\ x_3 \ge 0$$

Proof device:

Take some <u>nonnegative</u> multiplier & form linear combinations (conic combinations) of given valid inequalities $\Longrightarrow$ get new valid inequality.

Example: Use multipliers 3 for (II) and 2 for (I)

$$
\begin{array}{rl}
& 3 \cdot (3x_1 - x_2 + x_3 \le 3) \\
+ & 2 \cdot (x_1 + 4x_2 \le 1) \\
\hline
& 11x_1 + 5x_2 + 3x_3 \le 11
\end{array}
$$

valid inequality

$$\zeta = \underbrace{4x_1}_{\leq 11x_1} + \underbrace{1x_2}_{\leq 5x_2} + \underbrace{3x_3}_{\leq 3x_3} \leq 11x_1 + 5x_2 + 3x_3$$

b/c $x_1 \geq 0$ b/c $x_2 \geq 0$

$\leq 11.$

Q: Is there a better choice for multipliers.

lower ~~upper~~ bound

This is an optimization problem.

Write it:

Introduce variables for the multipliers:

$y_1 \geq 0$ for ineq. (I)

$y_2 \geq 0$ for ineq. (II)

$$\begin{aligned} \min \quad & \text{upper bound} = y_1 \cdot 1 + y_2 \cdot 3 \\ \text{s.t.} \quad & y_1 \cdot 1 + y_2 \cdot 3 \geq 4 \\ & y_1 \cdot 4 + y_2 \cdot (-1) \geq 1 \\ & y_1 \cdot 0 + y_2 \cdot 1 \geq 3 \\ & y_1 \geq 0, \quad y_2 \geq 0 \end{aligned}$$

This is a linear optimization problem, the "dual" problem.
(The original LP is now called "primal".)

More generally for an LP in standard ineq. form:

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m \\ & x_j \geq 0, \quad j = 1, \ldots, n. \end{aligned} \qquad (P)$$

$$\begin{aligned} \min \quad & \sum_{i=1}^{m} y_i b_i \\ \text{s.t.} \quad & \sum_{i=1}^{m} y_i a_{ij} \geq c_j, \quad j = 1, \ldots, n \\ & y_i \geq 0 \quad i = 1, \ldots, m \end{aligned} \qquad (D)$$

In matrix form:

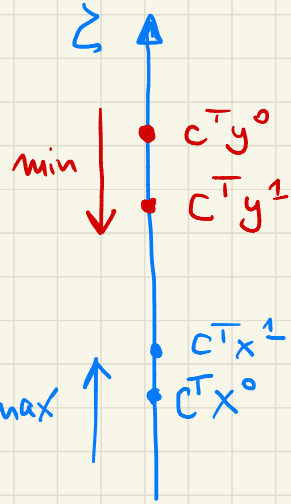
$$\max\ c^T x \qquad\qquad \min\ y^T b \qquad\qquad \min\ b^T y$$
$$\text{s.t. } Ax \le b \qquad \text{s.t. } y^T A \ge c^T \qquad \text{s.t. } A^T y \ge c$$
$$x \ge 0 \qquad\qquad y \ge 0 \qquad\qquad y \ge 0$$

- Weak duality theorem:

  Let x be any feasible solution of (P),
  let y be any feasible solution of (D).
  Then we have: $c^T x \le y^T b$.

Proof:

$$c^T x = \sum_{j=1}^{n} \underbrace{c_j}_{\le \sum_{i=1}^{m} y_i a_{ij}\ (D)} \underbrace{x_j}_{\ge 0\ (P)} \le \sum_{j=1}^{n} \underbrace{\left(\sum_{i=1}^{m} y_i a_{ij}\right) x_j}_{\text{bilinear term}} = \sum_{i=1}^{m} \underbrace{y_i}_{\ge 0\ (D)} \underbrace{\left(\sum_{j=1}^{n} a_{ij} x_j\right)}_{\le b_i\ (P)} \le \sum y_i b_i. \qquad \square$$



for feasible y^i for (D)

for feasible x^0 for (P)

Given feasible y for (D),

the primal (P) cannot be unbounded.

Possibilities acc. to fund. theorem:

(P) \ (D)	has an opt. sol.	infeasible	unbounded
has an optimal sol.			X
infeasible			
unbounded	X		X