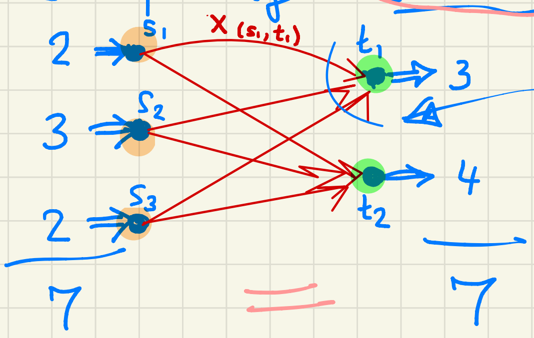
Replace inequalities on the supply side by equations.

$$\Longrightarrow \quad \overbrace{\sum_{v:\,(u,v)\in A} x_{(u,v)}}^{\text{outflow from vertex } u} = -b_u \quad \longleftrightarrow \quad \sum_{v:\,(u,v)\in A} (-1)\cdot x_{(u,v)} = b_u$$

$\Longrightarrow$ Get equation system + nonnegativities (almost in standard equation form)

Example of a balanced transport problem



1st row · x = inflow to node w/ demand t_1.

Write the equations in matrix form:

$$\begin{array}{c} \\ t_1 \\ t_2 \\ s_1 \\ s_2 \\ s_3 \end{array}
\begin{array}{c}
\begin{array}{cccccc} x_{(s_1,t_1)} & x_{(s_1,t_2)} & x_{(s_2,t_1)} & x_{(s_2,t_2)} & x_{(s_3,t_1)} & x_{(s_3,t_2)} \end{array} \\
\begin{pmatrix}
+1 & & +1 & & +1 & \\
 & +1 & & +1 & & +1 \\
-1 & -1 & & & & \\
 & & -1 & -1 & & \\
 & & & & -1 & -1
\end{pmatrix}
\end{array}
\begin{pmatrix} x_{(s_1,t_1)} \\ x_{(s_1,t_2)} \\ x_{(s_2,t_1)} \\ x_{(s_2,t_2)} \\ x_{(s_3,t_1)} \\ x_{(s_3,t_2)} \end{pmatrix}
= \begin{pmatrix} b_{t_1} \\ b_{t_2} \\ b_{s_1} \\ b_{s_2} \\ b_{s_3} \end{pmatrix}$$

each column: exactly one +1, −1.

Writing the constraint matrix row by row:

+1 for "demand" rows, arcs entering a demand vertex

−1 for "supply" rows, arcs leaving a supply vertex.

Read the matrix column by column

+1 : head of arc

−1 : tail of arc.

"Network matrix" / vertex-arc incidence matrix of a directed graph:

$$\begin{array}{cc} \text{vertices} \updownarrow & \begin{pmatrix} & & \\ & N & \\ & & \end{pmatrix} \\ & \underset{\text{arcs}}{\longleftrightarrow} \end{array}$$

$$\min c^T x \quad \longleftrightarrow \quad -\max -c^T x$$

$$\text{s.t.} \quad Nx = b$$

$$x \geq 0. \quad x \in \mathbb{R}^A$$

Missing: Full rank.

Claim: $\boxed{\text{Matrix has rank exactly } \#\text{rows} - 1.}$

Claim: $\exists$ vector of multipliers λ_v for $v \in V$,

so that $\sum \lambda_v \underbrace{N_{v,\bullet}}_{\substack{v\text{-th} \\ \text{row of } N}} = 0^T.$

$\lambda_v = 1 \quad \forall v \in V.$ satisfies this system.

$\Rightarrow$ not full row rank.

Deleting 1 (arbitrary) row gives a full-rank system.

$\rightsquigarrow$ Simplex method can be implemented very efficiently for network matrices b/c combinatorial interpretation.

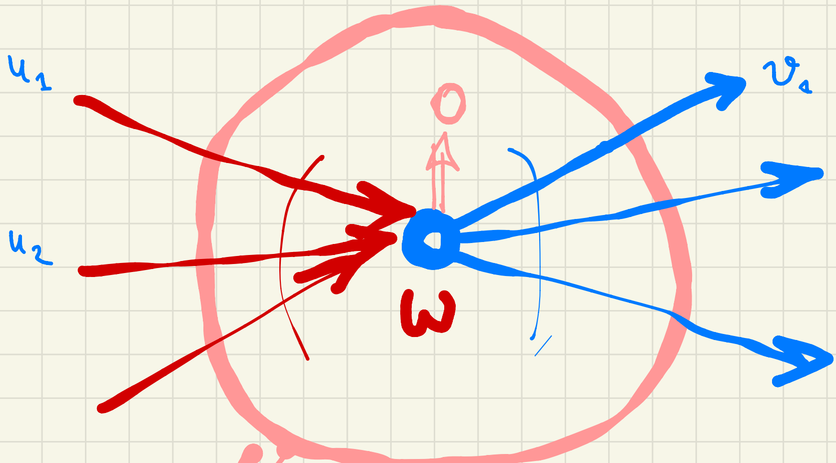
$\Rightarrow$ network simplex method

Modeling with more general networks.

for now: single commodities.

1st generalization: $\Rightarrow$ Suppliers $\cdots$ warehouses $\cdots$ Consumers $\Rightarrow$

demand: $\quad -\quad\quad 0 \quad\quad +$

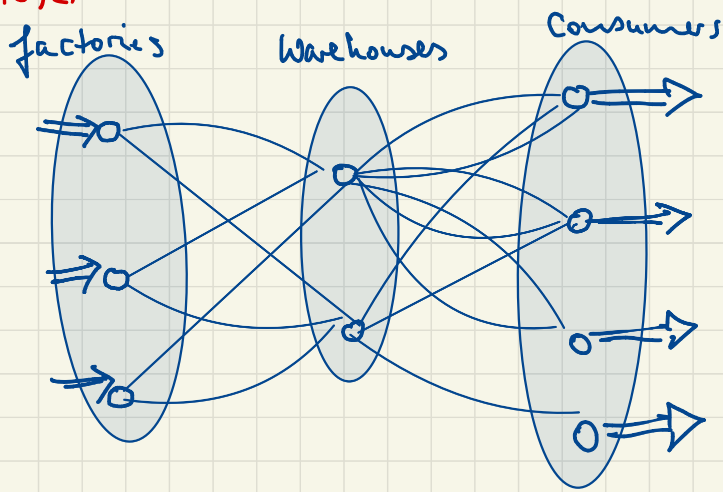


"flow conservation at vertex w"

inflow = outflow

$$\sum_{\substack{u \in V: \\ (u,v) \in A}} x_{(u,v)} = \sum_{\substack{v \in V: \\ (w,v) \in A}} x_{(w,v)}$$

transhipment balance

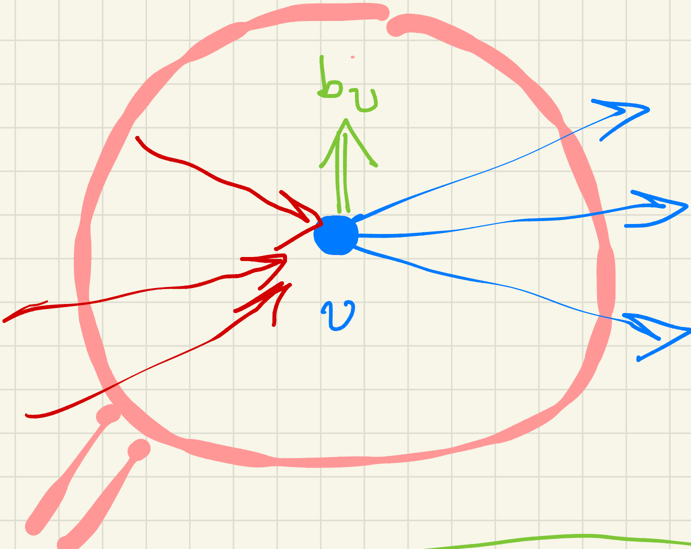


all arcs: left to right.

All 3 types of constraints generalize to a single flow conservation constraint:

(excess) = inflow − outflow = demand

$$\boxed{\sum_{\substack{u \in V:\\ (u,v)\in A}} x_{(u,v)} - \sum_{\substack{w \in V:\\ (v,w)\in A}} x_{(v,w)} = b_v}$$



General network flow problem
(min-cost flow)

$$\min \ c^T x$$
$$\text{s.t.} \ Nx = b$$
$$x \geq 0$$

prescribed flow excesses