Core technical topic:

Thursday, October 8, 2020 10:59 AM

Simplex method for linear optimization.

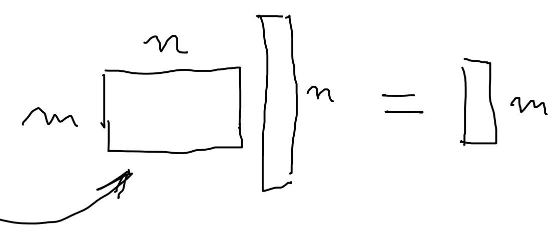
<u>Definition:</u> A linear optimization problem in standard form:

Variables $x_1, \dots, x_n \geq 0$ $\qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n_{\geq 0}.$

Constraints: Linear equations

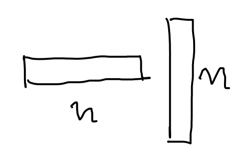
$$\left.\sum_{j=1}^{n} a_{ij} x_j = b_i \right] \; i = 1, \dots, m \qquad Ax = b$$

wide matrix



We assume: matrix A has full row rank.
(no redundant equation)

Objective function: $\max \sum c_j x_j \qquad \max c^T x$



What if system does not have full row rank:

$$\left.\begin{array}{lrcl} (1) & x_1 + 2x_2 & = & 3 \\ (2) & 2x_1 + 3x_2 & = & 4 \\ (3) & 3x_1 + 5x_2 & = & 7. \end{array}\right]$$

3rd eqn $=$ 1st $+$ 2nd eqn.

$\Longrightarrow$ 3rd eqn is redundant

$\Longrightarrow$ remove it.

3 eqns in 2 variables

Maximum rank of system is $2 = \min\{2, 3\}$

(Rank of a matrix
$=$ Row rank $=$ # linearly indep. rows $\leq$ # rows
$=$ Column rank $=$ # linearly indep columns $\leq$ # columns)

Why is the subsystem (1, 2) of full row rank (rank 2)?

Eliminate x_1 by subtracting it from (2) 2 times.

$(2) - 2\cdot(1)$: $\quad -x_2 = -2.$

The equivalent system

$$\begin{array}{ll} (1) & \boxed{x_1} + 2x_2 = 3 \\ (2') & \quad\quad\; -\boxed{x_2} = -2 \end{array}$$

echelon form

full row rank.

What if we have inequalities?

nonnegative RHS

$$
\begin{aligned}
\max \quad & 5x_1 + 4x_2 + 3x_3 \\
\text{s.t.} \quad & 2x_1 + 3x_2 + x_3 \le 5 \\
& 4x_1 + x_2 + 2x_3 \le 11 \\
& 3x_1 + 4x_2 + 2x_3 \le 8 \\
& x_1, \; x_2, \; x_3 \ge 0 \quad (\in \mathbb{R})
\end{aligned}
$$

"subject to (constraints)"

Idea: Introduce a variable for the "slack" in each $\le$ constraint. "a slack variable"

Names: $x_4 \ge 0$ — slack in 1st constraint

$x_5 \ge 0$ 2nd

$x_6 \ge 0$ 3rd

Equivalent formulation with slack variables added:

$$
\begin{array}{ll}
\max & 5x_1 + 4x_2 + 3x_3 \\
\text{s.t.} & 2x_1 + 3x_2 + x_3 + x_4 \qquad\qquad\quad = 5 \\
& 4x_1 + x_2 + 2x_3 \qquad + x_5 \qquad = 11 \\
& 3x_1 + 4x_2 + 2x_3 \qquad\qquad + x_6 = 8 \\
& x_1, \; x_2, \; x_3, \; x_4, \; x_5, \; x_6 \geq 0.
\end{array}
$$

This is an LP in standard form.

$$
A = \begin{array}{cccccc}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6
\end{array}
$$

$$
A = \left(\begin{array}{ccc}
2 & 3 & 1 \\
4 & 1 & 2 \\
3 & 4 & 2
\end{array}\;\underbrace{\begin{array}{ccc}
1 & \cdot & \cdot \\
\cdot & 1 & \cdot \\
\cdot & \cdot & 1
\end{array}}_{\text{identity matrix}}\right) \Rightarrow \text{full row rank } (3).
$$

Can set all "original variables" x_1 x_2 x_3 to 0

$\Longrightarrow$ feasible solution. b/c rhs vector $b \geq 0$.

▷ Initial assumption (to get started w/ algorithm)!

After introducing slack variables:

x_1, x_2, x_3 —— set to 0.

x_4, x_5, x_6 —— set them to "right values" ... by writing x_4, x_5, x_6 as functions of x_1, x_2, x_3.

$$\max \quad 0 + 5x_1 + 4x_2 + 3x_3$$

$$\left.\begin{aligned} x_4 &= 5 - 2x_1 - 3x_2 - x_3 \\ x_5 &= 11 - 4x_1 - x_2 - 2x_3 \\ x_6 &= 8 - 3x_1 - 4x_2 - 2x_3 \end{aligned}\right]$$

The "dictionary" corresponding to the "basis" with variables x_4, x_5, x_6

x_4, x_5, x_6

"basic variables"

"nonbasic variables"

Claim / Invariant: The system of linear equations is equivalent to the original system.

"A dictionary defines a particular solution "the basic solution"

- set nonbasic variables to 0
- set basic variables to right-hand side constant