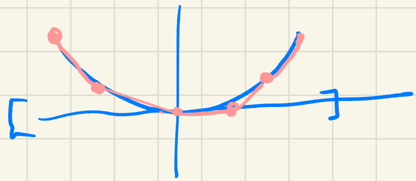
What if we have a convex function that is not piecewise linear?

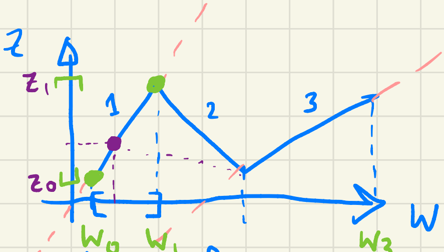
MICO

Approximate using convex piecewise linear functions



$z = f_1(w) = m_1 w + n_1$

What can we do if we have piecewise linear but nonconvex functions?



Use MILP modeling.

i-th piece $[w_{i-1}, w_i]$ domain

Introduce binary variables to select the piece in which w lies.

$$b_i = \begin{cases} 1 & \text{if } w \text{ lies in } i\text{-th interval} \\ 0 & \text{otherwise} \end{cases}$$

"w should lie in exactly one of the pieces"

$$b_1 + b_2 + b_3 = 1$$

Need to link the variables y_i, z, w.

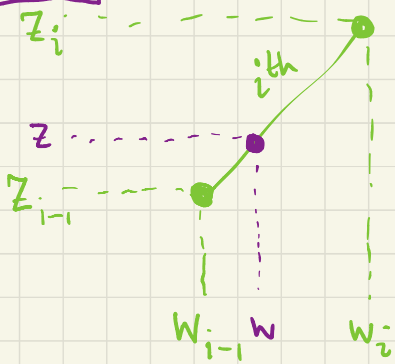
Piecewise linear function f:

k pieces with breakpoints $w_0, w_1, \dots, w_k$

function takes values $z_0, z_1, \dots, z_k$

$$\begin{array}{ccc} z_0, & z_1, \dots, & z_k \\ \| & & \| \\ f(w_0) & \dots & f(w_k) \end{array}$$

Fix i:



Introduce convex multipliers

$$\lambda_{i-1}, \lambda_i \geq 0$$
$$\lambda_{i-1} + \lambda_i = 1$$
$$w = \lambda_{i-1} w_{i-1} + \lambda_i w_i$$
$$z = \lambda_{i-1} z_{i-1} + \lambda_i z_i.$$

Join this for all $i = 1, \dots, k$.

Introduce convex multipliers

$$\lambda_0, \dots, \lambda_{k-1}, \lambda_k \geq 0$$
$$\lambda_0 + \dots + \lambda_k = 1.$$
$$w = \lambda_0 w_0 + \dots + \lambda_k w_k$$
$$z = \lambda_0 z_0 + \dots + \lambda_k z_k.$$

All choices of piece i "inject" into this model

Injection:

A fixed choice of i
corresponds to a choice of $\lambda_{i-1}, \lambda_i \geq 0$,
setting all other λ_j, $j \neq i-1, i$,
to 0.

Issue: The larger model is a relaxation of the disjunction (union) of these k submodels:

It also allows solutions with more than 2 nonzero λ_i's, or solutions with 2 non-consecutive nonzero λ_i's.

For $k=3$:

	λ_0	λ_1	λ_2	λ_3
$i=1$	*	*	0	0
$i=2$	0	*	*	0
$i=3$	0	0	*	*
general model	*	*	*	*

Use mixed integer modeling:

Express: → If $i=2$, then $\lambda_0, \lambda_3 = 0$.
→ If $i=3$, then $\lambda_0, \lambda_1 = 0$.

Have binary variables $b_j = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{o/w} \end{cases}$.

Columnwise: λ_1 can be nonzero only if $i = 1$ or $i = 2$.

$$\lambda_1 \leq \underbrace{b_1 + b_2}_{= \begin{cases} 1 & \text{if } i=1 \text{ or } i=2 \\ 0 & \text{o/w}. \end{cases}};$$

General:

$$\lambda_i \leq b_i + b_{i+1}$$

(for boundary:

$$\lambda_0 \leq b_1$$

$$\lambda_k \leq b_k$$

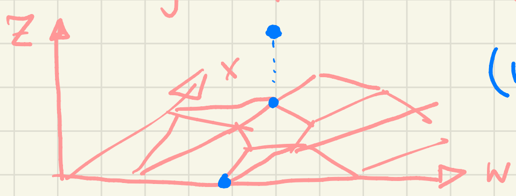
).

These types of model are called "SOS 2".

"special ordered set"

Generalize to higher dimension:

functions $f : \mathbb{R}^2 \to \mathbb{R}$ $\quad (w, x) \mapsto f(w,x) = z$



(w^i, x^i, z^i)

$i = 0, \dots, k$

indexes vertices

$$\begin{pmatrix} w \\ x \\ \cdots \\ z \end{pmatrix} = \sum_{i=0}^{k} \lambda_i \begin{pmatrix} w^i \\ x^i \\ \cdots \\ z^i \end{pmatrix}$$

$$\sum \lambda_i = 1$$

$$\lambda_i \geqslant 0$$

Include combinatorial conditions to ensure that we pick one of the pieces $j = 1, \ldots, \ell$

and pick nonzero convex multipliers <u>only</u> for vertices that belong to the j-th piece P_j.

$$\lambda_i \leq \sum_{\substack{j=1 \\ (w^i, x^i, z^i) \\ \in P_j}}^{\ell} b_j$$

$\square$