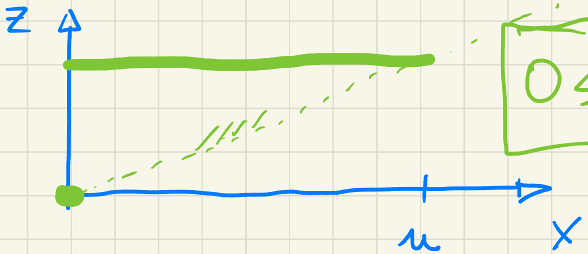
# Forcing constraint:

If binary variable $z = 0$,
then the continuous variable $x$ $(x \leq u)$
must be zero.

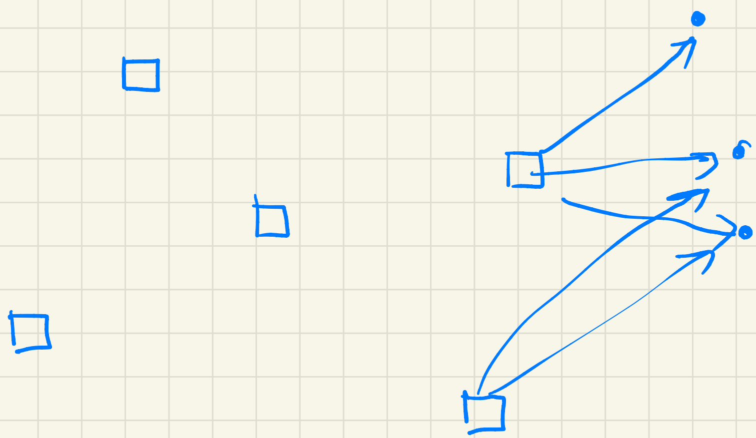



$$\boxed{0 \leq x \leq u \cdot z.}$$

$(z \in \{0, 1\})$

## $\boxed{\text{Facility location:}}$

Given sites for potential facilities
to serve customers



Graph $(V, A)$ $\quad V = V_1 \sqcup V_2$

facilities $\qquad$ customers

Demands $b_j$ for customer $j \in V_2$
(for a single good)

Each facility has a capacity
(potential supply) $u_i$ for $i \in V_1$.

Costs per unit of good $c_{ij}$

Amount of good shipped from
facility $i$ to customer $j$ : $x_{ij}$.

Cost of opening facility $i$ : $d_i$.

Logic constraint :

A facility can only serve a
customer if it is open.

Decision variable:

$$z_i = \begin{cases} 1 & \text{if we open facility } i \\ 0 & \text{o/w.} \end{cases}$$

$$\min \sum_{i \in V_1} d_i z_i + \sum_{(i,j) \in A} c_{(i,j)} x_{(i,j)}$$

s.t.

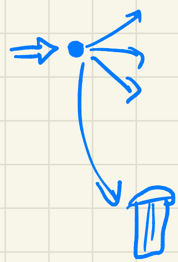
$$\sum_{\substack{i \in V_1 \\ (i,j) \in A}} x_{(i,j)} = b_j \quad \forall j \in V_2$$

(disaggregated) forcing constraint

$$x_{(i,j)} \leq \boxed{u_i} \cdot z_i \quad \forall i \quad \forall (i,j) \in A$$

$$\boxed{\sum_{\substack{j \in V_2 \\ (i,j) \in A}} x_{(i,j)} \leq u_i}$$

total amount of goods supplied by facility $i$.



Alternative:

Combine them

$$\sum_{\substack{j \in V_2 \\ (i,j) \in A}} x_{(i,j)} \leq u_i \cdot z_i$$

(aggregated forcing constraint)

- if $z_i = 1$, then this is the facility capacity constraint.

- if $z_i = 0$, then it forces the aggregated flow to 0, together with $x_{(i,j)} \geq 0$ it implies that $x_{(i,j)} = 0 \quad \forall j$.

Aggregated formulation has fewer constraints, but it is a "weaker" formulation.

Relationship between strong and weak formulations of the same problem.

Revisit modeling of logical AND.

Correct formulation:

$0 \quad 0 \quad 1 \qquad x_i + x_j + (1 - y_{ij}) \geq 1$

$\Leftrightarrow \qquad x_i + x_j \geq y_{ij}$

$0 \quad 1 \quad 1 \qquad x_i + (1 - x_j) + (1 - y_{ij}) \geq 1$

$\Leftrightarrow \qquad x_i - x_j - y_{ij} \geq -1$

$1 \quad 0 \quad 1 \qquad (1 - x_i) + x_j + (1 - y_{ij}) \geq 1$

$1 \quad 1 \quad 0 \qquad (1 - x_i) + (1 - x_j) + y_{ij} \geq 1$

$\Leftrightarrow \quad y_{ij} \geq x_i + x_j - 1$ ✓

1st constraint and 2nd constraint together:

Sum these constraints:

$$\boxed{\begin{aligned} x_i + x_j &\geq y_{ij} \\ x_i - x_j - y_{ij} &\geq -1 \end{aligned}}$$

$$2x_i - y_{ij} \geq y_{ij} - 1$$

$$2x_i - 2y_{ij} \geq -1$$

$$x_i - y_{ij} \geq -\frac{1}{2}$$

valid inequality for all points that satisfy the given ineqs.

$$x_i \in \mathbb{Z},\ y_{ij} \in \mathbb{Z}$$

$$\Rightarrow x_i - y_{ij} \in \mathbb{Z}.$$

uses integrality

$$x_i - y_{ij} \geq 0$$

$$\Longleftrightarrow \boxed{y_{ij} \leq x_i.}$$

(by combining 1st & 3rd ineq., get $y_{ij} \leq x_j$.)

## More general: Chvátal–Gomory cut.

Take <u>integer</u> variables $x \in \mathbb{Z}^n_{\geq 0}$

and an inequality system

$$Ax \leq b. \quad \left( \sum_{j=1}^{n} a_{ij} x_j \leq b_i \right) \qquad m\,\overbrace{\boxed{\;A\;}}^{n}$$

Any nonnegative vector of multipliers $\lambda \in \mathbb{R}^m_{\geq 0}$ gives a valid inequality

$$\lambda^T A x \leq \lambda^T b$$

$$\sum_{j=1}^{n} \Big( \underbrace{\sum_{i=1}^{m} \lambda_i a_{ij}} \Big) x_j = \sum_{i=1}^{m} \lambda_i \sum_{j=1}^{n} a_{ij} x_j \leq \sum \lambda_i b_i$$

If $\lambda^T A \in \mathbb{Z}^n$ $\left( \forall j, \ \sum_{i=1}^{m} \lambda_i a_{ij} \in \mathbb{Z} \right)$,

then for $x \in \mathbb{Z}^n_{\geq 0}$

$\lambda^T A x \in \mathbb{Z}$, so $\lambda^T A x \leq \lambda^T b$

implies $\lambda^T A x \leq \lfloor \lambda^T b \rfloor$.

a new valid inequality.

If $\lambda^T A \notin \mathbb{Z}^n$, then b/c $x \geq 0$

$$\underbrace{\lfloor \lambda^T A \rfloor}_{\in \mathbb{Z}^n} x \leq \lambda^T A x \leq \lambda^T b$$

implies $\lfloor \lambda^T A \rfloor x \leq \lfloor \lambda^T b \rfloor$.

Chvátal-Gomory cut.

There are infinitely many of these cuts but the set

$$\left\{ x \in \mathbb{R}^n_{\geq 0} : \quad Ax \leq b, \quad \forall \lambda \in \mathbb{R}^m_{\geq 0} : \lfloor \lambda^T A \rfloor x \leq \lfloor \lambda^T b \rfloor \right\}$$

is a polyhedron.

"1 st Chvátal closure".