$$\sum_{\substack{\{i,j\}\\ \in E(S)}} x_{\{i,j\}} \leq |S|-1$$

is not satisfied for $x = \tilde{x}$.

(Search problem)

Stronger: Find a subset $\emptyset \neq S \subsetneq V$

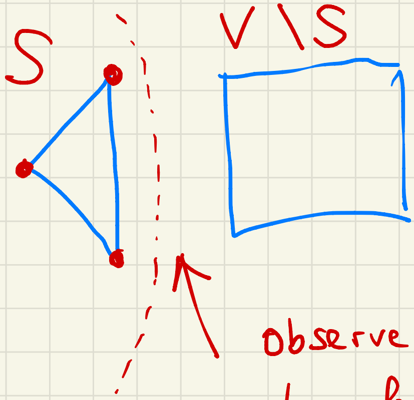
that maximizes

$$\boxed{\sum_{\substack{\{i,j\}\\ \in E(S)}} \tilde{x}_{\{i,j\}} - (|S|-1)}.$$

(optimization problem).

Q: IP formulation for this optimization problem? (Later.)

Next: An alternative approach to subtour elimination.



$S \subseteq V.$

observe that there are too few edges...

tour edges form a disconnected graph.

- Disconnectedness is certified by subset S so that the

  cutset induced by S $= \delta(S)$

  graph edges

  $= \{ \{s,t\} \in E :$
  $s \in S,\ t \notin S \}$

  contains no tour edge.

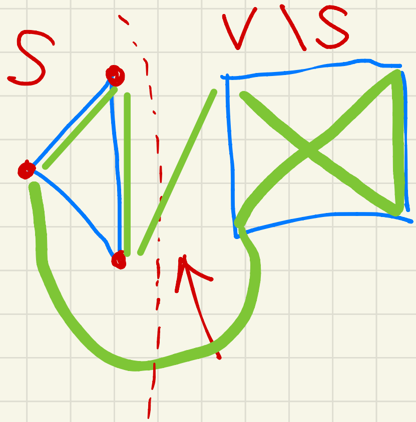
- The graph formed by the tour edges is connected $\iff$ $\forall\, \emptyset \neq S \subsetneq V$, the cutset $\delta(S)$ contains a tour edge.

Cutset formulation for TSP:

degree: $\sum_{\substack{w \in V \\ \{v,w\} \in E}} x_{\{v,w\}} = 2 \quad \forall v \in V$

cutset ineq: $\boxed{\sum_{\{u,v\} \in \delta(S)} x_{\{u,v\}} \geq 1} \quad \forall \emptyset \neq S \subsetneq V$

$x_{\{u,v\}} \in \{0,1\} \quad \forall \{u,v\} \in E$



The number of tour edges crossing each cut is even!

$$\boxed{\sum_{\{u,v\} \in \delta(S)} x_{\{u,v\}} \geq 2 \quad \forall \emptyset \neq S \subsetneq V}$$

strengthened cutset inequality.

"Strength" of the two correct IP formulations (comparing versions with ≥ 1, ≥ 2 cutset inequalities)

can be discussed using the "linear (continuous) relaxation" ("LP")

Ask about the real solutions to the inequality systems (without considering integrality constraint:

replacing $x_{\{u,v\}} \in \{0, 1\}$

by $0 \leq x_{\{u,v\}} \leq 1$.)

Relevant b/c IP solvers solve IPs by going through the linear optimization relaxation, applying additional techniques "branch and bound" & "cutting planes".

First formulation using subtour elim. constraints is equivalent (equal strength) to the formulation using strengthened cutset inequalities:

The polyhedra:

$$P_{subtour} = \left\{ x \in \mathbb{R}^E : 0 \le x_{\{i,j\}} \le 1 \quad \forall \{i,j\} \in E \right.$$
$$\sum_{\substack{j: \\ \{i,j\} \in E}} x_{\{i,j\}} = 2 \quad \forall i \in V$$
$$\left. \sum_{\substack{\{i,j\} \in E \\ i,j \in S}} x_{\{i,j\}} \le |S| - 1 \quad \forall \emptyset \ne S \subsetneq V \right\}$$

$$P_{\substack{strengthened \\ cutset}} = \left\{ x \in \mathbb{R}^E : 0 \le x_{\{i,j\}} \le 1 \quad \forall \{i,j\} \in E \right.$$
$$\sum_{\substack{j: \\ \{i,j\} \in E}} x_{\{i,j\}} = 2 \quad \forall i \in V$$

$\{i,j\} \in \delta(\{i\}) \longleftrightarrow$

$$\left. \sum_{\substack{\{i,j\} \\ \in \delta(S)}} x_{\{i,j\}} \ge 2 \quad \forall \emptyset \ne S \subsetneq V \right\}$$

$$\subsetneq \left\{ x \in \mathbb{R}^E : \sum_{\substack{\{i,j\} \\ \in \delta(S)}} x_{\{i,j\}} \ge 1 \quad \forall \emptyset \ne S \subsetneq V \right\} = P_{cutset}$$

We have exponentially many (strengthened) cutset inequalities.
Need to solve (in a constraint generation scheme) the subproblem of finding for a solution $\tilde{x}$ (to a relaxation) a violated cutset inequality! Again: an optimization problem:

$$\min \sum_{\{i,j\} \in \delta(S)} \tilde{x}_{\{i,j\}}$$

$$\text{s.t. } \emptyset \neq S \subsetneq V$$

(a combinatorial optimization problem: "undirected min cut").