$$(x_1,\ x_2,\ x_3,\ x_4,\ x_5)$$

- $x^0 = (1 \quad \cdot \quad 1 \quad \cdot \quad \cdot)$

$x^1 = (1 \quad \cdot \quad \cdot \quad 1 \quad \cdot)$ $\qquad x^1 - x^0 = (0 \quad 0 \quad -1 \quad 1 \quad 0)$

$x^2 = (\cdot \quad 1 \quad \cdot \quad 1 \quad \cdot)$ $\qquad x^2 - x^0 = (-1 \quad 1 \quad -1 \quad 1 \quad 0)$

$x^3 = (\cdot \quad 1 \quad \cdot \quad \cdot \quad 1)$ $\qquad x^3 - x^0 = (-1 \quad 1 \quad -1 \quad 0 \quad 1)$

$x^4 = (\cdot \quad \cdot \quad 1 \quad \cdot \quad 1)$ $\qquad x^4 - x^0 = (-1 \quad 0 \quad 0 \quad 0 \quad \boxed{1})$

Dimension of $\mathrm{Span}\{x^1 - x^0, \ldots, x^4 - x^0\}$

= rank of the matrix formed by these vectors.

In [9]:
```
G = matrix(QQ, [[ 0, 0, -1, 1, 0],
                [-1, 1, -1, 1, 0],
                [-1, 1, -1, 0, 1],
                [-1, 0,  0, 0, 1]]); G
```

Out[9]:

$$\begin{pmatrix} 0 & 0 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In [11]:
```
G.echelon_form()
```

Out[11]:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

In [12]:
```
G.rank()
```

Out[12]: 4

Dimension of the convex hull of <u>all</u> solutions? Consider the solutions

$$\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \end{array}$$

$$\begin{array}{l} \text{selecting} \\ \text{one} \\ \text{vertex} \end{array} \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & 0 & 0 & 1 \end{array} \right. \rightsquigarrow \begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & 0 & 0 & 1 \end{array}$$

dimension = rank = 5

Found a valid inequality such that the convex hull of all the tight solutions has dimension 4.

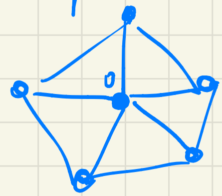
This is a <u>facet</u> = codimension 1 face.

The valid inequality is a facet-defining inequality.

Same question for the second example:

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 2$$

also valid for this wheel.



Dimensions ?

$$\begin{array}{c} \\ \bullet\ x^0 = \\ x^1 = \\ x^2 = \\ x^3 = \\ x^4 = \end{array} \begin{array}{cccccc} (x_0, & x_1, & x_2, & x_3, & x_4, & x_5) \\ (0 & 1 & \cdot & 1 & \cdot & \cdot) \\ (0 & 1 & \cdot & \cdot & 1 & \cdot) \\ (0 & \cdot & 1 & \cdot & 1 & \cdot) \\ (0 & \cdot & 1 & \cdot & \cdot & 1) \\ (0 & \cdot & \cdot & 1 & \cdot & 1) \end{array}$$

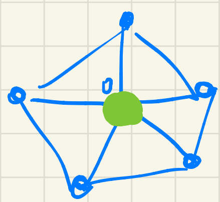
still has rank 4.

$$\begin{array}{l} x^1 - x^0 = (0\ \ 0\ \ 0\ \ -1\ \ 1\ \ 0) \\ x^2 - x^0 = (0\ \ -1\ \ 1\ \ -1\ \ 1\ \ 0) \\ x^3 - x^0 = (0\ \ -1\ \ 1\ \ -1\ \ 0\ \ 1) \\ 0\ \ 1\ \ -1\ \ 0\ \ 0 \\ x^4 - x^0 = (0\ \ -1\ \ 0\ \ 0\ \ 0\ \ \boxed{1}) \end{array}$$

Is there a tight solution with $x_0 = 1$?

$x_0 = 1 \Rightarrow x_i = 0$

for $i = 1, \ldots, 5$.



Unique solution : $(1, 0, 0, 0, 0, 0)$.

LHS of the ineq. is $0 \Longrightarrow$ not tight.

The convex hull of tight solutions still has dimension 4 but space is now $\mathbb{R}^6$ (and convex hull of all solns. has dim. 6)

In [18]:
```
Gw = matrix(QQ, 4, 1).augment(G, subdivide=True); Gw
```

Out[18]:

$$\left(\begin{array}{r|rrrrr} 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right)$$

In [19]:
```
Gw.rank()
```

Out[19]: 4

<u>Not</u> a facet-defining inequality.

Make it facet-defining by finding a coefficient for x_0 so that the solution $(1,0,0,0,0,0)$ becomes tight but all previously tight solutions remain tight.

Pick coefficient 2:

"Lifted" inequality:

$$2x_0 + x_1 + x_2 + x_3 + x_4 + x_5 \leq 2.$$

$\rightarrow$ facet-defining valid inequality! because the convex hull of the tight solutions has dimension 5 (codimension 1)

```
In [21]: x0 = vector(QQ, (0, 1, 0, 1, 0, 0))
         d  = vector(QQ, (1,0,0,0,0,0)) - x0
```

Out[21]: $(1, -1, 0, -1, 0, 0)$

```
In [22]: Gl = Gw.stack(d, subdivide=True); Gl
```

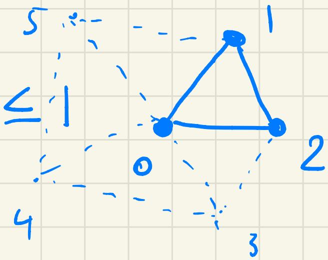
Out[22]:

$$\left(\begin{array}{cccccc} 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ \hline 1 & -1 & 0 & -1 & 0 & 0 \end{array}\right)$$

```
In [23]: Gl.rank()
```

Out[23]: 5

Triangle: $x_0 + x_1 + x_2 \leq 1$



List the tight solutions:

	x_0	x_1	x_2	x_3	x_4	x_5
x^0	1	0	0	0	0	0
x^1	0	1	0	1	0	0
x^2	0	1	0	0	1	0
x^3	0	0	1	0	1	0
x^4	0	0	1	0	0	1
x^5	0	0	0	1	0	1

$x^1 - x^0 = (-1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0)$

$x^2 - x^0 = (-1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0)$

$x^3 - x^0 = (-1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0)$

$x^4 - x^0 = (-1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1)$

$x^5 - x^0 = (-1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1)$

In [27]:
```
Gt = matrix(QQ, [[-1, 1, 0, 1, 0, 0],
                 [-1, 1, 0, 0, 1, 0],
                 [-1, 0, 1, 0, 1, 0],
                 [-1, 0, 1, 0, 0, 1],
                 [-1, 0, 0, 1, 0, 1]]); Gt
```

Out[27]:
$$\begin{pmatrix} -1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

In [28]:
```
Gt.echelon_form()
```

Out[28]:
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

In [29]:
```
Gt.rank()
```

Out[29]: 5

Computer calculation shows:

This has rank 5.

So this is again a facet-defining inequality.

(Contrast to the original inequalities, edge inequalities — clique inequalities corresponding to non-maximal cliques.)

Example:

$$x_1 + x_2 \leq 1.$$

Tight solutions:

$$x_0 \quad \overbrace{x_1 - x_2 - x_3 - x_4 - x_5}$$

	x_0	x_1	x_2	x_3	x_4	x_5
x^0	0	1	0	1	0	0
x^1	0	1	0	0	1	0
x^2	0	0	1	0	1	0
x^3	0	0	1	0	0	1

$$\begin{array}{lcccccc}
x^1 - x^0 & 0 & 0 & 0 & \boxed{-1} & 1 & 0 \\
x^2 - x^0 & 0 & -1 & 1 & -1 & \boxed{1} & 0 \\
x^3 - x^0 & 0 & -1 & 1 & -1 & 0 & \boxed{1}
\end{array}$$

3 vectors $\Rightarrow$ rank ≤ 3

$\Rightarrow$ not facet-defining.

In [31]:
```
Ge = matrix(QQ, [[0,  0, 0, -1, 1, 0],
                 [0, -1, 1, -1, 1, 0],
                 [0, -1, 1, -1, 0, 1]]); Ge
```

Out[31]:
$$\begin{pmatrix} 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{pmatrix}$$

In [32]:
```
Ge.echelon_form()
```

Out[32]:
$$\begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

In [33]:
```
Ge.rank()
```

Out[33]: 3