Recap from Friday: A subproblem in public transport

Monday, October 5, 2020 8:50 AM

Variables:

Drivers & buses as variables?

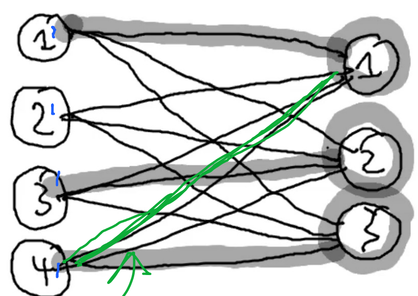
4 drivers : $\{1, \dots, 4\} = I$

3 buses : $\{1, \dots, 3\} = J$

??

Simplification:
Fixed shift 9–12.
• 1 driver drives
• 1 bus for entire shift

Drivers Buses



$\{4', 1\} \in E$

matching

graph theoretic view.

Bipartite graph:

$\boxed{V = \{1, 2, 3\} \sqcup \{1', 2', 3', 4'\}}$

E

Def. An undirected simple graph : $G = (V, E)$

V — a finite set of "vertices"

$E \subseteq \binom{V}{2} = \{ \underbrace{\{i, j\}}_{\text{2-elt sets (unordered pairs)}}, \ i, j \in V, \ i \neq j \}$

Bipartite: $V = I \sqcup J$ (disjoint union).

elements of E are of the form $\{i,j\}$, $i \in I$, $j \in J$
"edges between vertices in I & vertices in J only".

An assignment of drivers to buses —
a matching M, i.e. a <u>special</u> subset of E: $M \subseteq E$.

Skip graph-theoretic def'n of matching.

Next step: Combinatorial object M is modeled (represented) using finitely many variables.

General technique: Encode subset $M \subseteq E$ using "binary" (0/1) variables:
$$X_e = \begin{cases} 1 & \text{if } e \in M \\ 0 & \text{o/w.} \end{cases} \quad \text{for } e \in E$$

$$\hookrightarrow (X_e)_{e \in E} \in \{0,1\}^{|E|} \subseteq \mathbb{R}^{|E|}$$

Bijection between subsets of E and the 0/1-vectors with $|E|$ entries.

In example: 4 drivers, 3 buses;

$$E = \Big\{ \{1', 1\}, \{1', 2\}, \{1', 3\}, \{2', 1\}, \ldots, \{4', 1\}, \ldots \{4', 3\} \Big\}.$$

drivers | buses

Cardinality 12

We will work in high dimension! $\{0,1\}^{12} \subseteq \mathbb{R}^{12}$.

A particular valid assignment (matching) that <u>covers</u> every bus:

$$M = \{\{1', 2\}, \{2', 1\}, \{4', 3\}\} \subseteq E$$

encoded by a vector in $\mathbb{R}^{12}$.

$$(x_e)_{e \in E} = (\underbrace{x_{\{1',1\}}}_{=0}, \underbrace{x_{\{1',2\}}}_{=1}, \underbrace{x_{\{1',3\}}}_{=0}, \underbrace{x_{\{2',1\}}}_{=1}, 0, 0, 0, 0, 0, \ldots, =1 \ldots$$

0 $\qquad$ $, \cdot 0 \cdot$ $\qquad$ $x_{\{4',3\}}$)

Observation: • Every assignment of drives to buses can be represented as vector $x \in \{0,1\}^E$.

• If we are given a vector $x \in \{0,1\}^E$ that represents an assignment, then the assignment is determined uniquely.

Question: • Does every vector $x \in \{0,1\}^E$ represent an assignment?

• If yes, why?

• If no, how do we recognize vectors $x \in \{0,1\}^E$ that do represent an assignment?

Concerns about overlap:

$x_{e \in E}$ What condition do the variables have to satisfy for an assignment to be "valid."

|| If we write variables in an array as above, we can have at most one "1" in a row row and

every column.

Express as an inequality.

"Every column": Fix a column index j,

At most 1 of the variables $x_{\{i,j\}}$ for $i \in I$, $\{i,j\} \in E$ is 1.

Express the number of these variables set 1 as a sum: $\sum_{i \in I} x_{\{i,j\}}$.

Express the constraint "at most 1 set to 1" as an inequality: $\boxed{\underbrace{\sum_{i \in I} x_{\{i,j\}} \leq 1.}_{\text{a linear inequality}}}$ for fixed $j \in J$

fixed index set of summation.

$\longrightarrow$ of the form $g(x) \leq 1$

where $g : \mathbb{R}^{|E|} \longrightarrow \mathbb{R}$

is a linear function.

Same for the rows: $\boxed{\sum_{j \in J} x_{\{i,j\}} \leq 1}$ for fixed $i \in I$.

"No more than 1 driver can drive a given bus. for all buses.

No more than 1 bus can be driven by a given driver. for all drivers.

Consider the vector $\underset{(0,\ldots,0)}{0} \in \{0,1\}^E$.

— Does it satisfy the inequalities? ✓✓

Receive clarification from operators:

Every bus needs a driver. ("Hard" constraint)

We need an add'l (or modified) constraint to enforce this... (in particular rule out sol'n $x = 0$.)