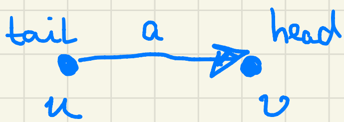
Define: A directed simple graph: (V, A)

V — set of vertices (finite),

A — set of arcs — ordered pairs (2-tuples) of vertices:

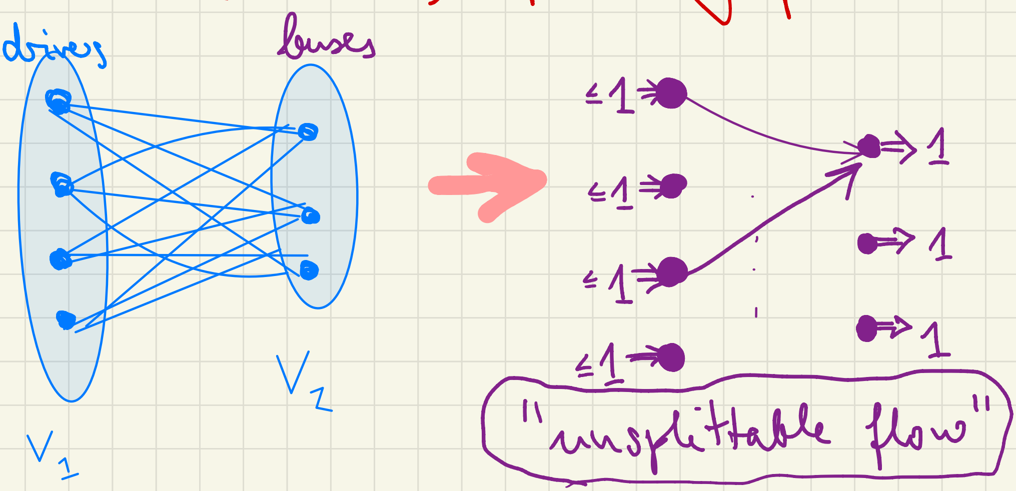
$$\subseteq \{ (u,v) : u, v \in V, \boxed{u \neq v} \}$$



$a = (u, v)$

arc from u to v.

Take an (undirected) bipartite graph



Interpret the assignment model as a transportation

Introduce a good "driving".
Send 1 unit of "driving"
from supply (set of drivers)
to each of the buses.

Transportation model:

Suppliers ~ Consumers

Suppliers each supply the same good ("single-commodity") in quantities, up to a given capacity.

Have a need for the good in some given quantities.

(In assignment model, supplies & demands are all 1.)

Given:

$G = (V, A)$ directed graph (network)

with $A \subseteq \{ (v_1, v_2) : v_1 \in V_1,\ v_2 \in V_2 \}$

$$\underset{\text{supply}}{\overset{v_1}{\bullet}} \longrightarrow \underset{\text{demand}}{\overset{v_2}{\bullet}}$$

Supplies: $b_v = \begin{cases} \text{demand of } v_2 = v \in V_2 \\ -\text{ supply of } v_1 = v \in V_1 \end{cases}$

Modeling trick in preparation for generalization

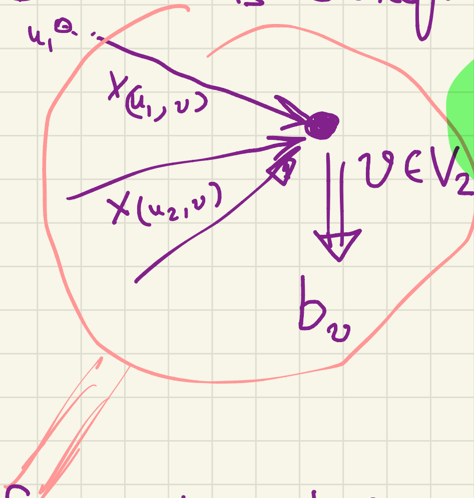
Decision variables:

(Consider transportation problems with single divisible commodity & splittable flows)

$\underbrace{X_{(u,v)}}_{\text{arc flow}} \in \mathbb{R}_{\geq 0}$ for $(u,v) \in A$

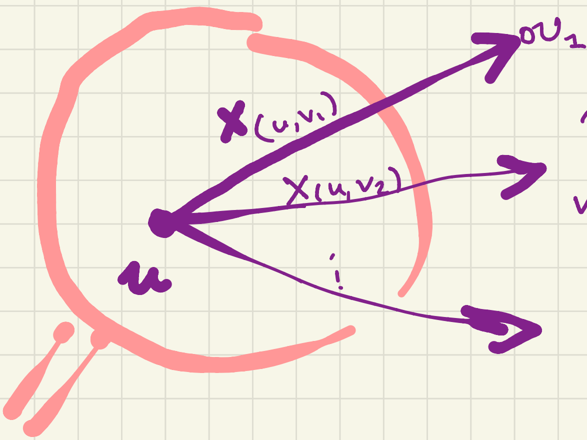
Constraints:

- Demand is satisfied:



$$\sum_{u:(u,v)\in A} x_{(u,v)} = b_v .$$

- Supply is not exceeded:



$$\overbrace{\sum_{v:(u,v)\in A} x_{(u,v)}}^{\text{outflow from vertex } u} \leq -b_u$$

for $u \in V_1$.

If we have transport capacities:

$$x_{(u,v)} \leq U_{(u,v)}$$

given arc capacity

If cost per unit of flow on an arc:

given $c_{(u,v)} \in \mathbb{R}_{\geq 0}$

write the total cost

as $\sum_{(u,v)\in A} c_{(u,v)}\, x_{(u,v)}$,

to be minimized.

Easy special case: (assume from now!)

Supplies & demands are "balanced":

$$\sum_{v \in V} b_v = 0$$

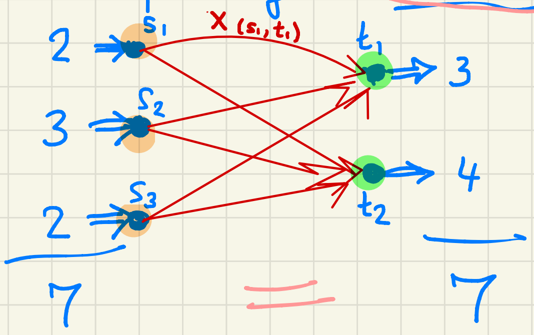
$$\left(\Longleftrightarrow \underbrace{\sum_{v \in V_1} |b_v|}_{\text{total supply}} = \underbrace{\sum_{v \in V_2} b_v}_{\text{total demand}} \right)$$

Replace inequalities on the supply side by equations.

$$\Longrightarrow \quad \sum_{\substack{v:\\ (u,v)\in A}} \overbrace{x_{(u,v)}}^{\text{outflow from vertex } u} = -b_u$$

$\Longrightarrow$ Get equation system + nonnegativities (almost in standard equation form)

Example of a balanced transport problem



Write the equations in matrix form:

$$\begin{array}{c} \begin{array}{cccccc} x_{(s_1,t_1)} & x_{(s_1,t_2)} & x_{(s_2,t_1)} & x_{(s_2,t_2)} & x_{(s_3,t_1)} & x_{(s_3,t_2)} \end{array} \\ \left(\begin{array}{c} \\ \\ \text{(to be continued)} \\ \\ \\ \end{array} \right) \end{array} \begin{pmatrix} x_{(s_1,t_1)} \\ x_{(s_1,t_2)} \\ x_{(s_2,t_1)} \\ x_{(s_2,t_2)} \\ x_{(s_3,t_1)} \\ x_{(s_3,t_2)} \end{pmatrix} = \begin{pmatrix} \\ \\ \\ \\ \\ \end{pmatrix}$$