Constant term:

$$0 = \zeta^* - \sum_{i \in N} y_i^* b_i .$$

coefficient of x_j:

$$c_j = \underbrace{c_j^*}_{\leq 0} + \sum_{i \in N} y_i^* a_{ij} .$$

$\square$

Dual pairing of variables:

original variables of (P) $\longleftrightarrow$ slack vars of (D)

$x_1, \dots, x_n$ $\qquad$ $z_1, \dots, z_n$

slack variables of (P) $\longleftrightarrow$ orig. vars of (D)

$w_1, \dots, w_m$ $\qquad$ $y_1, \dots, y_m$

$(x_{n+1}, \dots, x_{n+m})$ $\qquad$ $(z_{n+1}, \dots, z_{n+m})$

$$x \in \mathbb{R}^{n+m} \longleftrightarrow z \in \mathbb{R}^{n+m}$$

Dual (complementary) basis:

If a primal variable (orig. or slack) is a basic variable,

then we make the corresponding dual variable (slack or orig.) nonbasic.

Transform the dual into standard form

$$- \max \quad -\sum b_i y_i$$

$$\text{s.t.} \quad \sum_{i=1}^{m} (-a_{ij}) y_i \leq -c_j$$

$$y_i \geq 0$$

dual slacks z_j.

Write a dictionary of the dual w.r.t. the complementary basis.

|| basic variables in dual indexed by N

nonbasic variables in dual indexed by $\mathcal{B}$.

$$\max \quad \bar{\eta} + \sum_{i \in \mathcal{B}} \bar{d}_i z_i \qquad -\bar{b}_i$$

$$\text{s.t.} \quad z_j = \bar{z}_j + \sum_{i \in \mathcal{B}} \bar{a}_{ji} z_i \qquad j \in N$$

$$z_j \geq 0$$

same coefficients as in the primal dictionary

negatives of obj coeff. in the primal dictionary

$\bar{c}_j$!

- Both dictionaries are related by the negative transpose property!

- Basic solutions of primal & dual dictionary have "complementarity" by definition:
  - For a pair of primal variable x_j and dual slack z_j:

    at most one of the two is nonzero b/c exactly one of the two is basic.
  - For a pair of primal slack w_i and dual variable y_i:

    at most one of the two is nonzero b/c exactly one of the two is basic.

notation: $x_j \perp z_j$, $w_i \perp y_i$

or $x_j \cdot z_j = 0$, $w_i \cdot y_i = 0$

or $\langle x, z \rangle + \langle w, y \rangle = 0$.

New notation for the primal dictionary:

dual solution.

$$\max \ \bar{\xi} - \sum_{j \in N} \bar{z}_j x_j$$

$$\text{s.t. } x_i = \bar{x}_i - \sum_{j \in N} \bar{a}_{ij} x_j \qquad i \in \mathcal{B}.$$

The objective function coefficients in the primal dictionary are negatives of the basic dual solution.

$\Rightarrow$ The optimality criterion for the primal problem (nonpositivity of $\bar{c}_j$) is feasibility of the dual solution!

Summary:

A basis $\mathcal{B}$ for the primal problem

- (primal) feasible $\Leftrightarrow$ $\bar{b}_i \geq 0$
- dual feasible $\Leftrightarrow$ $\bar{z}_j \geq 0$

feasible basis w/ optimality certificate

primal-dual feasible basis.

New variant of the simplex method:

$$\begin{array}{ccc}
(P) & \xrightarrow{\text{dualize}} & (D) \\
\text{steps of the "dual simplex method"} \big\downarrow & & \big\downarrow \text{sequence of bases of } (D) \text{ — simplex method} \\
\text{basis / dictionary for } (P) \text{ that is primal/dual feasible} & \xleftarrow[\text{basis}]{\text{complementary}} & \text{basis / dictionary for } (D) \text{ that is primal \& dual feasible}
\end{array}$$

Dual simplex method —

get started if we have a basis with a primal infeasible but dual feasible solution.

iterate simplex steps, preserving dual feasibility until we achieve primal feasibility!

dual feasible

Current Dictionary

$$
\begin{array}{llcrcrcrcrc}
\text{maximize} & \zeta & = & & & \boxed{-2}\,x_1 & + & \boxed{-2}\,x_2 & + & \boxed{-12}\,x_3 & + \ \boxed{-8}\,x_4 \\
\text{subject to:} & w_1 & = & \boxed{-8} & - & \boxed{0}\,x_1 & - & \boxed{-5}\,x_2 & - & \boxed{4}\,x_3 & - \ \boxed{-3}\,x_4 \\
 & w_2 & = & \boxed{-15} & - & \boxed{3}\,x_1 & - & \boxed{2}\,x_2 & - & \boxed{-2}\,x_3 & - \ \boxed{-3}\,x_4 \\
 & w_3 & = & \boxed{11} & - & \boxed{2}\,x_1 & - & \boxed{-2}\,x_2 & - & \boxed{-6}\,x_3 & - \ \boxed{0}\,x_4
\end{array}
$$

$x_1,\ x_2,\ x_3,\ x_4,\ \ w_1,\ w_2,\ w_3 \ \geq 0$

1 Select row ⇒

primal infeasible

does not participate in ratio test b/c sign

2 Ratio test

$$\frac{12}{2} > \frac{8}{3}$$

3 Pivot

Current Dictionary

still dual feasible

$$
\begin{array}{llcrcrcrcrc}
\text{maximize} & \zeta & = & \boxed{-40} & + & \boxed{-10}\,x_1 & + & \boxed{-22/3}\,x_2 & + & \boxed{-20/3}\,x_3 & + \ \boxed{-8/3}\,w_2 \\
\text{subject to:} & w_1 & = & \boxed{7} & - & \boxed{-3}\,x_1 & - & \boxed{-7}\,x_2 & - & \boxed{6}\,x_3 & - \ \boxed{-1}\,w_2 \\
 & x_4 & = & \boxed{5} & - & \boxed{-1}\,x_1 & - & \boxed{-2/3}\,x_2 & - & \boxed{2/3}\,x_3 & - \ \boxed{-1/3}\,w_2 \\
 & w_3 & = & \boxed{11} & - & \boxed{2}\,x_1 & - & \boxed{-2}\,x_2 & - & \boxed{-6}\,x_3 & - \ \boxed{0}\,w_2
\end{array}
$$

$x_1,\ x_2,\ x_3,\ x_4,\ \ w_1,\ w_2,\ w_3 \ \geq 0$

primal feasible :)

Complementarity slackness:

Theorem: Take a feasible solution $\bar{x}$ for (P) and a feasible solution $\bar{y}$ for (D).

Then both $\bar{x}$ and $\bar{y}$ are optimal solutions if and only if $\bar{x}$ and $\bar{y}$ are complementary:

$$\bar{x}_j \perp \bar{z}_j, \qquad \bar{w}_i \perp \bar{y}_i.$$

(where $\bar{z}, \bar{w}$ are slacks).

Proof: Revisit the weak duality theorem.

difference is $\sum \bar{z}_j x_j$

$$c^T x = \sum_{j=1}^{n} \underbrace{\underbrace{c_j}_{\overset{(D)}{\leq} \sum_{i=1}^{m} y_i a_{ij}} x_j}_{\geq 0 \ (P)} \leq \sum_{j=1}^{n} \underbrace{\Big(\sum_{i=1}^{m} y_i a_{ij}\Big) x_j}_{\text{bilinear term}} = \sum_{i=1}^{m} \underbrace{y_i}_{\substack{\geq 0 \\ (D)}} \Big(\underbrace{\sum_{j=1}^{n} a_{ij} x_j}_{\substack{\leq b_i \\ (P)}}\Big) \leq \sum y_i b_i$$

difference is $\sum \bar{y}_i \bar{w}_i$

$\square$

Remark:

Theorem (strict complementary slackness): If (P), (D) have optimal solutions, then $\exists$ a primal-dual optimal pair

with "strict complementary slackness":
Exactly one of the two variables
in each complementary pair is zero.
(Cannot be achieved in general with
basic solutions!)

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