This is the "Phase I" of the simplex method.

Idea: The problem of finding a feasible solution is an optimization problem.

$$\boxed{\begin{array}{ll} \max & \sum \cancel{c_j}\, x_j \\ \text{s.t.} & \sum a_{ij} x_j \le b_i \\ & x_j \ge 0 \end{array}}$$

replace it by a new objective function . . . that drives to feasibility !

!!

$$\max \quad -3x_1 + 4x_2$$

$$\begin{aligned} \text{s.t.} \quad -x_0 - 4x_1 - 2x_2 &\le -8 \\ -x_0 - 2x_1 \phantom{{}-2x_2} &\le -2 \\ -x_0 + 3x_1 + 2x_2 &\le 10 \\ -x_0 - x_1 + 3x_2 &\le 1 \\ -x_0 \phantom{{}-x_1} - 3x_2 &\le -2 \\ x_0,\ x_1,\ x_2 &\ge 0 \end{aligned}$$

Lift this problem into higher dim. by introducing artificial variable $x_0$

Large values of $x_0$: feasible

As usual, introduce slacks for all the inequalities.

If $x_0 = 0$, $(x_1, x_2)$ feasible for orig. LP.

$\min x_0 \iff \boxed{\max \eta = -x_0}$

1 pivot of $x_0$ against most neg.

$\Rightarrow$ Basis $\{x_0, w_2, w_3, w_4, w_5\}$

primal feasible.

does not satisfy optimality condition for $\max \eta$.

$\Rightarrow$ Choose entering $x_1$

(pos. coeff. in $\eta$)

Ratio test:

$x_0 : \quad \le 2$

$w_2 : \quad \le 3$

$w_3 : \quad \le \frac{18}{7} \ (>2)$

$w_4 : \quad \le 3$

$w_5 : \quad \le \frac{3}{2}$ ||| Step length

$\Longrightarrow$ New basis

$\{x_0, w_2, w_3, w_4, x_1\}$

Using the "advanced pivot tool" with the setting "primal with $x_0$"

$(x_0, w_2, w_3, w_4, x_1)$.

$\eta = -2$

Entering: $x_2$ ($>0$ coeff in $\eta$)

Ratio test:

$x_0$: $\leq \frac{2}{3}$ ! step length.

$w_2$: $\leq 3 \cdot \frac{2}{5}$ ($>1$)

$w_3$: $\leq \frac{15}{2} \cdot \frac{4}{23}$ ($>1$)

$w_4$: $\leq \frac{5}{2} \cdot \frac{4}{23} = \frac{18}{23}$

$x_1$ does not give step length

Pivot $x_0$ leaving!

New basis

$\{x_2, w_2, w_3, w_4, x_1\}$

$\rightarrow x_0 = 0$.

$\rightarrow$ is also ... (feasible)

to get a (feasible)
basis for the original problem. Forget about $x_0$, $z$.

Continue with Phase $\underline{\text{II}}$
(pivoting with orig. objective $z$)

$w_5$ entering variable.

Ratio test:

$x_2$: X

$w_2$: $\leq 4$

$w_3$: $\leq \frac{11}{3} \cdot 6 = 22$

$w_4$: $\leq \frac{2}{3} \cdot \frac{6}{7}$ = steplength

$x_1$: $\leq \frac{5}{3} \cdot 6$

Pivot with $w_4$ leaving

Basis

Basis

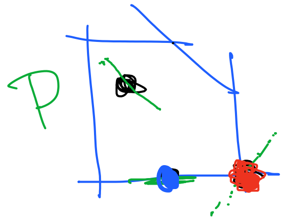
$x_2, w_2, w_3, w_5, x_1$.

Primal feasible
& satisfies optimality condition
$\Rightarrow$ Optimal solution

Geometry of the simplex method.

Claim: Steps of Phase II of the simplex method advance through a sequence of vertices (extreme points) of the convex polyhedron that is the feasible region.

General topology tells us the distinction of:




• interior point

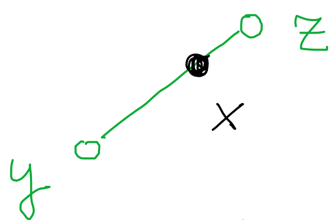
•• boundary points

Convex geometry defines:

**Extreme point:** An element $x \in P$ of a convex set is not an extreme point if

$\exists\, y, z \in P$, $y \neq z$ so that $x \in (y, z)$

(open line segment btw. $y$ and $z$).




Every basic feasible solution is an extreme point!