Polyhedral Geometry of simplex method.

"Extreme point" — notion from convex geometry.

vertices — as 0-dimensional faces of a polyhedron.

In the context of feasible region of ~~a~~ problem $x \in \mathbb{R}^{m+n}$

in std. equation form: $\begin{array}{cc} & m+n \\ m\{ & \boxed{A} \end{array} x = b$ $\boxed{A \text{ full row rank}}$ (rank m)

max $c^T x$ $\qquad x \geq 0.$

In any basic solution: Choose a basis $\mathcal{B} \subseteq \{1, \ldots, m+n\}$

a subset of card. m
so that the columns of A indexed by $\mathcal{B}$
are $\begin{cases} \text{linearly indep.} \\ \text{Spanning} \\ \text{a basis of } \mathbb{R}^m \end{cases}$ equivalent properties
form an invertible sq. matrix

$\longrightarrow$ determines nonbasis $\mathcal{N} = \{1, \ldots, m+n\} \setminus \mathcal{B}$.

Rewrite the std. equation form:

$$\left(A_{\mathcal{B}} \mid A_{\mathcal{N}} \right) \begin{pmatrix} x_{\mathcal{B}} \\ x_{\mathcal{N}} \end{pmatrix} = b, \qquad \begin{pmatrix} x_{\mathcal{B}} \\ x_{\mathcal{N}} \end{pmatrix} \geq 0.$$

submatrices formed by columns indexed
$= (B \mid N)$

$$A_{\mathcal{B}} x_{\mathcal{B}} + A_{\mathcal{N}} x_{\mathcal{N}} = b$$

$$A_{\mathcal{B}} x_{\mathcal{B}} = b - A_{\mathcal{N}} x_{\mathcal{N}}$$

"equation solving = rewriting as a dictionary"

$$x_{\mathcal{B}} = A_{\mathcal{B}}^{-1}(b - A_N x_N)$$

$$x_{\mathcal{B}} = A_{\mathcal{B}}^{-1} b - A_{\mathcal{B}}^{-1} A_N x_N$$

$$\max \quad c^T x = c_{\mathcal{B}}^T x_{\mathcal{B}} + c_N^T x_N$$

basic

$$= c_{\mathcal{B}}^T \left(A_{\mathcal{B}}^{-1} b - A_{\mathcal{B}}^{-1} A_N x_N \right) + c_N^T x_N$$

... a function of x_N

$$= c_{\mathcal{B}}^T A_{\mathcal{B}}^{-1} b - \underbrace{c_{\mathcal{B}}^T A_{\mathcal{B}}^{-1} A_N}_{= \left((A_{\mathcal{B}}^{-1} A_N)^T c_{\mathcal{B}} \right)^T} \underline{x_N} + c_N^T \underline{x_N}$$

$$\max = \underbrace{c_{\mathcal{B}}^T A_{\mathcal{B}}^{-1} b}_{\bar{\xi}} + \underbrace{\left(c_N - (A_{\mathcal{B}}^{-1} A_N)^T c_{\mathcal{B}} \right)^T}_{\bar{c}} x_N$$

$$\text{s.t.} \quad x_{\mathcal{B}} = \underbrace{A_{\mathcal{B}}^{-1} b}_{\bar{b}} - \underbrace{A_{\mathcal{B}}^{-1} A_N}_{\bar{A}} x_N$$

Primal dictionary in matrix form.

Take a feasible basis $\mathcal{B}$:

$\bar{b} \geq 0$.

$\Rightarrow$ The basic feasible solution (BFS) of $\mathcal{B}$

is $x := \begin{pmatrix} x_{\mathcal{B}} \\ x_{\mathcal{N}} \end{pmatrix} = \begin{pmatrix} \bar{b} \\ 0 \end{pmatrix}$.

Count inequalities that are "tight" at this point... in addition to all the equations.

$\longrightarrow$ In the dictionary, have the nonnegativity constraints: $\boxed{x \geq 0.}$

All the nonnegativity constraints $x_j \geq 0$, $j \in \mathcal{N}$ are tight at this point (BFS)

$\boxed{\text{Nonnegativity constraints}}$ $x_i \geq 0$, $i \in \mathcal{B}$

either tight or loose at this point (BFS).

If $\bar{b}_i > 0$ for all $i \in \mathcal{B}$

$\longrightarrow$ BFS is (primal) "nondegenerate"

$\Rightarrow$ exactly $m+n$ (indep.) tight constraints

$\Rightarrow$ BFS uniquely determined by these equations

If some $\bar{b}_i = 0$ for some $i \in \mathcal{B}$.

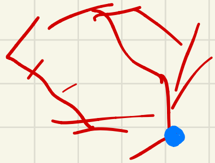
$\longrightarrow$ BFS is (primal) "degenerate"

"too many tight inequalities"

$\Rightarrow$ more than $m+n$ tight constraints.

$\Rightarrow$ BFS is overdetermined

$\mathbb{R}^{m+n}$

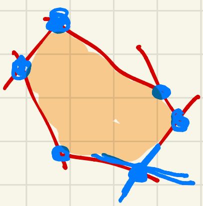


tight inequalities (≥ 0)
+ equations of the dictionary

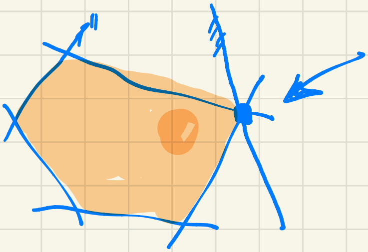
forget about basic variables

$$\mathbb{R}^n = \mathbb{R}^{|N|}$$

If $n=2$



2 tight inequalities

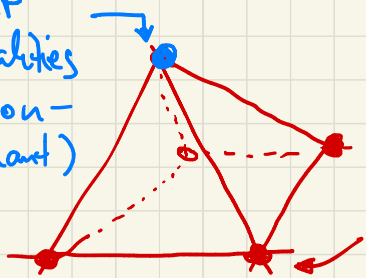


overdetermined point
(3 tight inequalities in dimension 2)

If $n>2$, e.g., $n=3$:

4 tight inequalities (all non-redundant)

Square pyramid



3 tight inequalities.
(nondegenerate)

but the apex is overdetermined:

Any 3 of the 4 equations determining the triangular sides determine it.

$\exists$ only finitely many vertices

$$\Uparrow$$

$\exists$ only finitely many BFS

$$\Uparrow$$

$\exists$ only finitely many bases

$$\Uparrow$$

$\exists$ only finitely many m-subsets of $\{1, \dots, m+n\}$, $\binom{m+n}{m}$.

$\boxed{\text{If}}$ we never see the same basis again in simplex method, it has to terminate in finitely many steps.