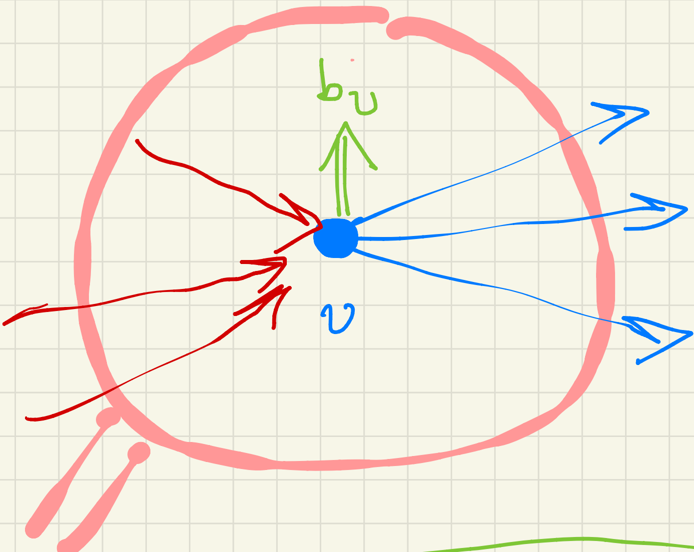
(excess) = inflow − outflow = demand

$$\boxed{\sum_{\substack{u \in V: \\ (u,v) \in A}} x_{(u,v)} - \sum_{\substack{w \in V: \\ (v,w) \in A}} x_{(v,w)} = b_v}$$



General network flow problem
(min-cost flow)

$$\min \quad c^T x$$
$$\text{s.t.} \quad Nx = b$$
$$x \geq 0$$

prescribed flow excesses

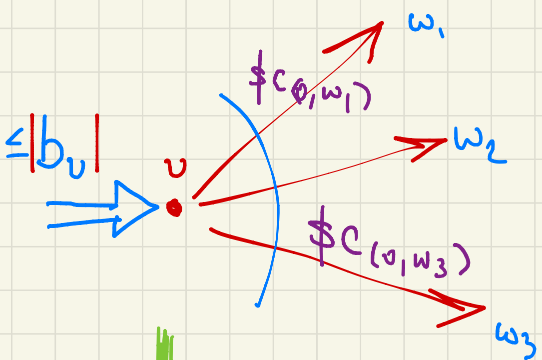
Unbalanced case? $\sum_{v \in V} b_v \neq 0$

1 row of $Nx = b$

encodes flow conservation constraint

of a node v.

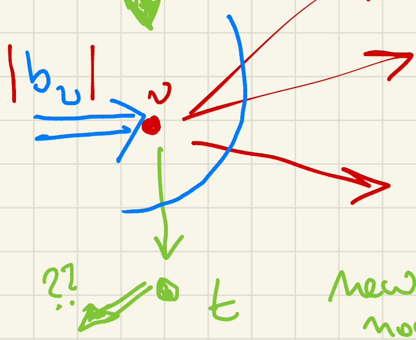
Supply node
that supplies
"up to" given
$|b_v|$?



$$\sum_{(v,w) \in A} x_{(v,w)} \le |b_v|.$$

introduce
slack variable ~~s~~ $x_{(v,t)}$

$$\boxed{\sum_{(v,w) \in A} x_{(v,w)} + \underbrace{\cancel{s}}_{x_{(v,t)}} = |b_v|}$$

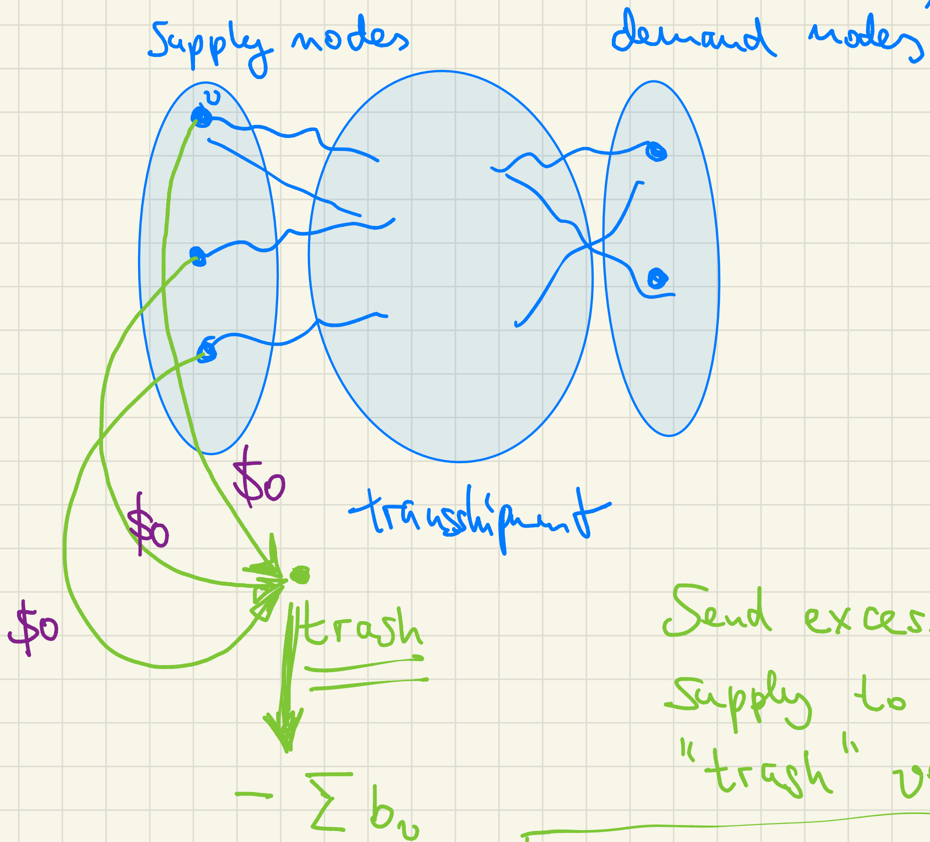


new auxiliary node t, new arc (v, t)

Suppose overall too much supply

$$\sum b_v < 0$$

(supplies have neg. sign)



Send excess supply to "trash" vertex t.

$$\boxed{\text{total prescribed flow excess of trash vertex } = -\sum b_v.}$$

This is an <u>extended formulation</u> of a generalized network flow problem as an ordinary network flow problem.

Costs: $c_{(v,t)} = 0$ 0 cost of not supplying

- Capacities & <u>lower bounds</u> for flows

capacitated network:

$$u \xrightarrow{\le U_{u,v}} v \qquad (u,v) \in A$$

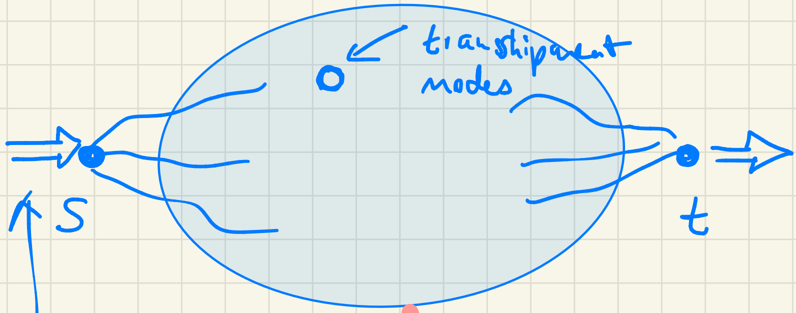
$U \in \mathbb{R}^A$ given upper bd. parameters

- min-cost flow in cap. networks

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Nx = b \\ & x \le U \\ & x \geq 0 \end{aligned}$$

- max flow:

Given special vertices $s, t \in V$
"source" "sink"



exceedance at source & sink

problem variable f

$$\max f$$

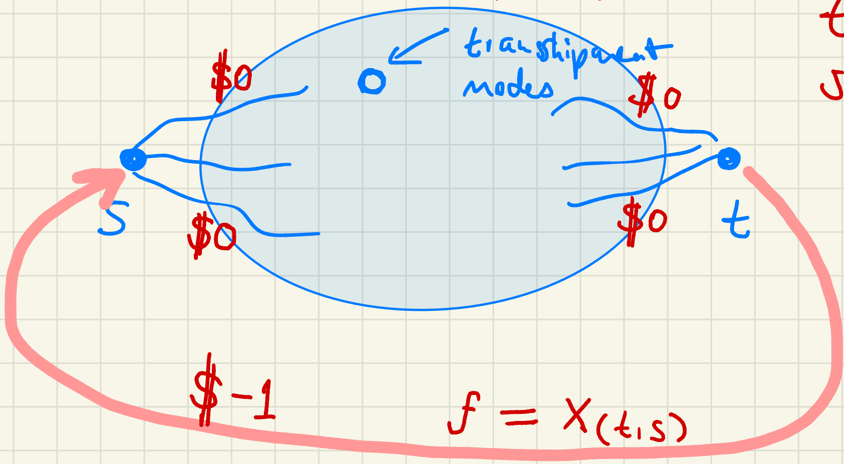
$$\text{s.t. } Nx = b \text{ where } b_v = \begin{cases} f & \text{for } v = t \\ 0 & \text{for } v \neq s,t \\ -f & \text{for } v = s. \end{cases}$$

$$x \leq u$$

$$x \geq 0.$$

convert flow problem to a circulation problem

New arc (t,s) ships flow back to the source.



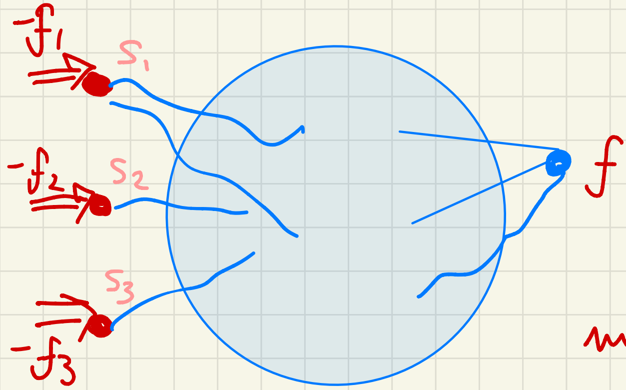
$\le \cancel{\infty}$ or $\le \sum_{(u,v) \in A} u_{(u,v)}$

$\max f$ s.t. flow conservation in augmented network $\hat{x} \le \hat{u}$ $\hat{x} \ge 0$	$\longleftrightarrow$	$-\min -f = -x_{(t,s)}$ s.t. flow conservation in augm. network $\hat{x} \le \hat{u}$ $\hat{x} \ge 0$

Max flow reduces to min-cost circulations in capacitated networks.

$\hat{b} = 0$

Max flow with several suppliers.

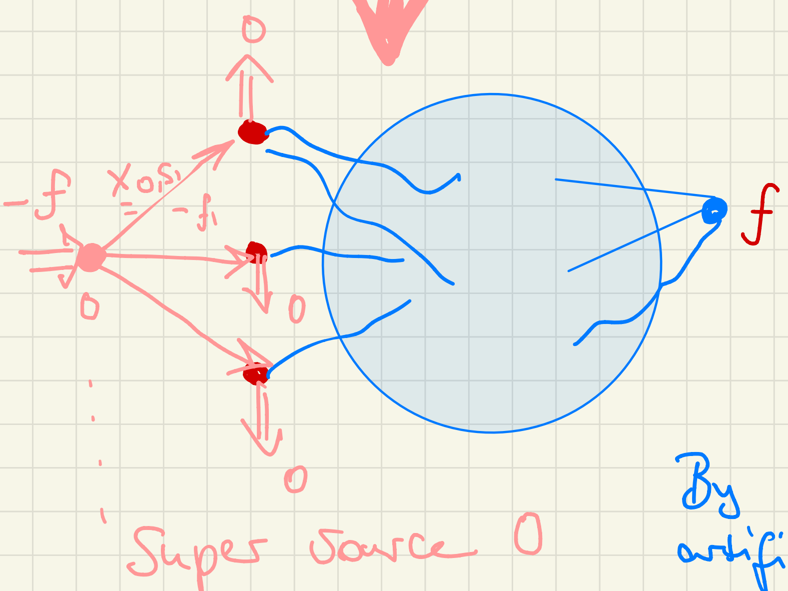


$\max f \; (= -\sum f_i)$

s.t. $Nx = b$

flow conservation at non-source, sink nodes

+ source & sink nodes



By introducing artificial "super" sources, sinks, problems with several sources or sinks can be transformed to single-source/sink problems.