Given feasible y for (D),

the primal (P) cannot be unbounded.

Possibilities acc. to fund. theorem:

(P) \ (D)	has an opt. sol.	infeasible	unbounded
has an optimal sol.	Strong duality ✓	✗	✗
infeasible	✗	✓	✓
unbounded	✗	✓	✗

weak duality

$$\max \sum_{j=1}^{n} c_j x_j \qquad (P)$$

$$\text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1, \dots, m$$

$$x_j \geq 0, \quad j = 1, \dots, n.$$

$$\min \sum_{i=1}^{m} y_i b_i \qquad (D)$$

$$\text{s.t.} \quad \sum_{i=1}^{m} y_i a_{ij} \geq c_j, \quad j = 1, \dots, n$$

$$y_i \geq 0 \quad i = 1, \dots, m$$

Rewrite it: (matrix form)

$$-\max \; -\sum b_i y_i \qquad = -b^T y$$

$$\text{s.t.} \quad \sum_{i=1}^{m} (-a_{ij}) y_i \le -c_j \qquad -A^T y \le -c$$

$$y_i \geq 0 \qquad y \geq 0$$

Primal $\qquad\qquad\qquad\qquad$ Dual.

$$\left(\begin{array}{c|c} 0 & c^T \\ \hline b & A \end{array}\right) \underset{-\,\cdot^T}{\overset{-\,\cdot^T}{\rightleftarrows}} \left(\begin{array}{c|c} 0 & -b^T \\ \hline -c & -A^T \end{array}\right)$$

"negative transpose property"

Theorem: The dual of the dual LP is the primal LP.

(LP has a "symmetric" duality theory) $\square$

Theorem (Strong duality theorem):

If (P) has an optimal solution x^*, then the dual (D) also has an optimal solution y^* of the same value:

$$\zeta^* = c^T x^* = {y^*}^T b.$$

Proof. B/c (P) has an opt. sol. x^* by fundamental theorem $\exists$ primal feas. basis with basic solution x^* that satisfies the optimality criterion.

$$\max \sum_{j=1}^{n} c_j x_j$$

Slack variable $w_i = b_i - \sum a_{ij} x_j$

$$\text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1, \dots, m$$

$$x_j \ge 0, \quad j = 1, \dots, n.$$

(P)

$$\min \sum_{i=1}^{m} y_i b_i$$

$$\text{s.t.} \quad \sum_{i=1}^{m} y_i a_{ij} \ge c_j, \quad j = 1, \dots, n$$

$$y_i \ge 0 \quad i = 1, \dots, m$$

(D)

$$-\sum y_i a_{ij} \le -c_j$$

slack variable z_j.

$$z_j = \sum y_i a_{ij} - c_j$$

Write objective function as in the <u>optimal</u> primal dictionary:

obj coeffs ≤ 0

$$\zeta = \sum_{j=1}^{n} c_j x_j = \zeta^* + \sum_{j \in N} c_j^* x_j + \sum_{i \in N} d_i^* w_i$$

x_j is a decision variable

w_i is a slack variable

define $c_j^* = 0$ for all other $j \in N$

define $d_i^* = 0$ for others

Claim: $\boxed{y_i^* := -d_i^* \text{ for all } i \in N,\ w_i \text{ is a slack variable}}$

$$y_i^* := 0 \quad \text{for all other } i$$

this solution is feasible for (D).

& $\sum c_j x_j^* = \sum y_i^* b_i$. ✓

- $y_i^* \geq 0$ b/c optimality condition.
- Check $\sum y_i^* a_{ij} \geq c_j$:
- Manipulate objective function (for all x) : $\underbrace{b_i - \sum_j a_{ij} x_j}_{\overline{w_i}}$

$$\zeta = \underbrace{\sum_{j=1}^{n} c_j x_j}_{\text{linear fxn of } x} = \zeta^* + \sum_{j \in N} c_j^* x_j + \sum_{i \in N} \underbrace{d_i^*}_{\underbrace{-y_i^*}_{\geq 0}} w_i$$

$$= \underbrace{\underbrace{\zeta^* - \sum_{i \in N} y_i^* b_i}_{\text{indep. of } x} + \sum_{j \in N} \left(c_j^* + \underbrace{\sum y_i^* a_{ij}}\right) x_j}_{\text{linear fxn of } x}$$

Comparison of coefficients :

Constant term:

$$0 = \zeta^* - \sum_{i \in N} y_i^* b_i .$$

coefficient of x_j:

$$c_j = \underbrace{c_j^*}_{\leq 0} + \sum_{i \in N} y_i^* a_{ij} .$$

$\square$

Dual pairing of variables:

original variables of (P) $\longleftrightarrow$ slack vars of (D)

$x_1, \dots, x_n$ $z_1, \dots, z_n$

slack variables of (P) $\longleftrightarrow$ orig. vars of (D)

$w_1, \dots, w_m$ $y_1, \dots, y_m$

$(x_{n+1}, \dots, x_{n+m})$ $(z_{n+1}, \dots, z_{n+m})$

$$x \in \mathbb{R}^{n+m} \longleftrightarrow z \in \mathbb{R}^{n+m}$$