$\exists$ only finitely many vertices

$\Uparrow$

$\exists$ only finitely many BFS

$\Uparrow$ $\qquad$ $\exists$ only finitely many feasible bases.

$\exists$ only finitely many bases

$\Uparrow$

$\exists$ only finitely many m-subsets of $\{1, \dots, m+n\}$, $\binom{m+n}{m}$.

$$\{\text{vertices}\} = \{\text{BFS}\} \xleftarrow{\text{many-to-1}} \{\text{feasible bases}\} \subseteq \{\text{bases}\} \subseteq \{m\text{-subsets of } [m+n]\} = \binom{[m+n]}{m}.$$

$\boxed{\text{If}}$ We never see the same basis again in simplex method, it has to terminate in finitely many steps.

Consider the objective function.

Its value during Phase-II of simplex method is nondecreasing. (Strictly increasing) in every pivot ~~when the~~

when we make a nondegenerate step.

(step length > 0).

$$\boxed{\lambda = \frac{\bar{b}_i}{\bar{a}_{ij}} > 0}$$

$\mathcal{N} \ni j$ index of the entering variable

If every dictionary visited in the simplex method is nondegenerate, then every step is nondegenerate!
(Converse does not hold)

then the objective value $\bar{\zeta}$ strictly increases:

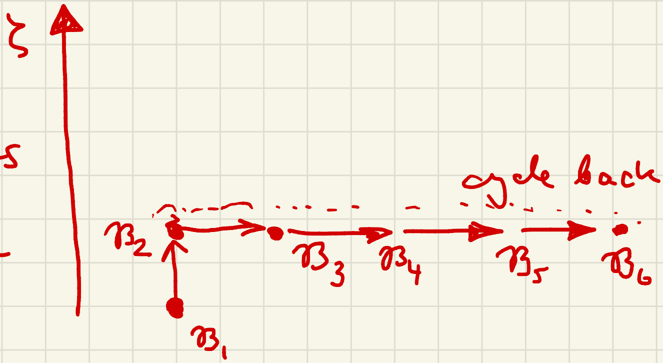
We have $\bar{c}_j > 0$ (by definition of procedure)

ζ increases by $\bar{c}_j \cdot \lambda$,

If no dictionary in the simplex method is (primal) degenerate, then it is impossible to visit the same basis twice.

$\Longrightarrow$ finite termination of Phase II.

Problems with degeneracy <u>may</u> admit correct sequence of (primal) simplex steps that are infinite ("cycling".)



There always exists a finite sequence leading to a dictionary satisfying the optimality condition.

Two ways to make simplex method finite:

1) <u>Perturbation.</u> / lexicographic

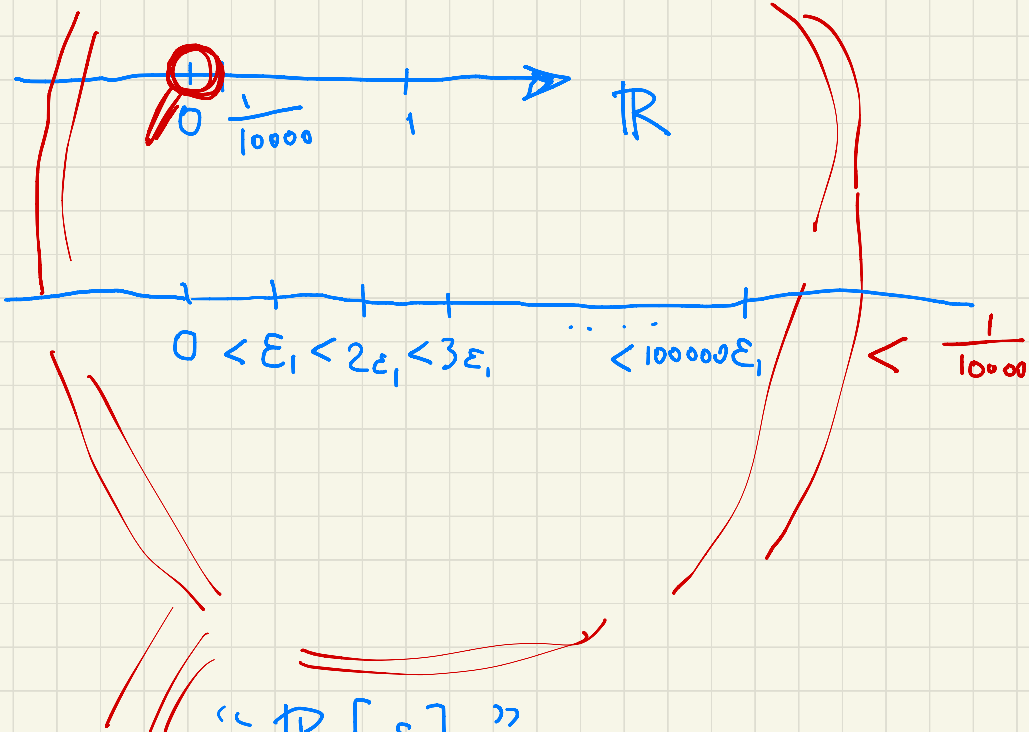
Replace problematic 0s by "very small numbers"

→ done in solvers numerically

→ here: "symbolically"

Introduce symbolic parameters

$\varepsilon_1, \dots, \varepsilon_m$ "positive but smaller than any positive real number"



"$\mathbb{R}[\varepsilon]$"

element are written as

$$\boxed{\alpha_0 + \alpha_1 \varepsilon_1}$$

e.g. $2 < 2 + 2\varepsilon_1 < 2.0000001$

$$\wedge$$

$$2 + 5\varepsilon_1$$

Compare as the tuple (α_0, α_1) compares lexicographically.

Example degenerate dictionary

$$\begin{aligned} \max \quad \zeta &= x_1 + 2x_2 \\ \text{s.t.} \quad x_3 &= \cancel{0}\ (0+\varepsilon_1) + x_1 - \boxed{x_2} \\ x_4 &= 2 + x_1 - 2x_2 \end{aligned}$$

primal degeneracy.

~~step length 0~~

$$\frac{0 + \varepsilon_1}{1} = \varepsilon_1 > 0.$$

non-degenerate step.

Update the dictionary:

Eliminate x_2 from objective function (add $2\times$ eqn. of entering variable)

$$\max \quad \zeta = \boxed{2\varepsilon_1} + 3x_1 - 2x_3$$

new $\bar{\zeta} > 0$.

Refinement:

Instead of just perturbing the "problematic" right-hand side constants, perturb all $\bar{b}_i$ in initial dictionary using independent ε_i.

The row operations (equivalent rewrites)

preserve independence of the ε's
in the right-hand side constants.

$\Longrightarrow$ Claim: In any dictionary
of the perturbed problem,
all $\bar{b}_i$ are nonzero:
Every $\bar{b}_i$ takes the form

$$\boxed{\beta_{i0} + \beta_{i1}\varepsilon_1 + \cdots + \beta_{im}\varepsilon_m}$$

$= 0$

$\Longleftrightarrow (\beta_{i0}, \beta_{i1}, \ldots, \beta_{im})$
$= (0, 0, \ldots, 0).$

$\Longrightarrow$ Theorem: Simplex method with
lexicographic perturbation
terminates finitely.