What if we have inequalities?

nonnegative RHS

$$\begin{array}{llcl}
\text{max} & 5x_1 + 4x_2 + 3x_3 & & \\
\text{s.t.} & 2x_1 + 3x_2 + x_3 & \le & 5 \\
 & 4x_1 + x_2 + 2x_3 & \le & 11 \\
 & 3x_1 + 4x_2 + 2x_3 & \le & 8. \\
 & x_1, \; x_2, \; x_3 \ge 0. & &
\end{array}$$

"subject to (constraints)"

Standard form LP.

(≥ 0 variables, $\le$ constraints, $(\in \mathbb{R})$)

Idea: Introduce a variable for the "slack" in each $\le$ constraint. "a slack variable".

Names: $x_4 \ge 0$ — slack in 1st constraint

$x_5 \ge 0$ 2nd

$x_6 \ge 0$ 3rd

Equivalent formulation with slack variables added:

Alternative notation:

$$\begin{array}{lll}
\text{max} & 5x_1 + 4x_2 + 3x_3 & \\
\text{s.t.} & 2x_1 + 3x_2 + x_3 + \boxed{x_4} & = 5 \\
 & 4x_1 + x_2 + 2x_3 + \boxed{x_5} & = 11
\end{array}$$

w_1 w_2

$$x_1 + x_2 + 2x_3 \qquad + \boxed{x_5} \qquad = 11$$
$$3x_1 + 4x_2 + 2x_3 \qquad + \overset{w_3}{\boxed{x_6}} = 8$$

$$x_1, \; x_2, \; x_3, \; x_4, \; x_5, \; x_6 \geq 0.$$

This is an LP in standard equation form.

$$\begin{array}{cc} & \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ A = & \left(\begin{array}{ccc} 2 & 3 & 1 \\ 4 & 1 & 2 \\ 3 & 4 & 2 \end{array} \underbrace{\begin{array}{ccc} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{array}}_{\text{identity matrix}}\right) \end{array} \Longrightarrow \text{full row rank } (3).$$

Can set all "original variables" x_1, x_2, x_3 to 0.

$\Longrightarrow$ feasible solution. b/c rhs vector $b \geq 0$.

▷ Initial assumption (to get started w/ algorithm)!

After introducing slack variables:

x_1, x_2, x_3 — set to 0.

x_4, x_5, x_6 — set them to "right values"

... by writing x_4, x_5, x_6 as functions of x_1, x_2, x_3.

positive

$$\max \quad 0 + 5x_1 + 4x_2 + 3x_3$$
$$x_4 = 5 - 2x_1 - 3x_2 - x_3$$
$$x_5 = 11 - 4x_1 - x_2 - 2x_3$$
$$x_6 = 8 - 3x_1 - 4x_2 - 2x_3$$

The "dictionary" corresponding to the "basis" with variables x_4, x_5, x_6

"basic" variables x_i for $i \in \mathcal{B}$

"nonbasic variables" x_j for $j \in \mathcal{N}$

Claim / Invariant: The system of

A dictionary defines a particular solution "the basic solution"

— set nonbasic variables

linear equations is equivalent to the original system.

to 0

set basic variables to right-hand side constant

In notation with variables

$\underbrace{x_1, \ldots, x_n}_{\text{decision variables}}, \underbrace{x_{n+1}, \ldots, x_{n+m}}_{\text{slack variables}}$: Use index set for all variables:

$\{1, \ldots, n+m\}$

Index sets for basic variables: $\mathcal{B} \subseteq \{1, \ldots, n+m\}$

nonbasic variables: $N \subseteq \{1, \ldots, n+m\}$

Partition of index set: $\{1, \ldots, n+m\} = \mathcal{B} \sqcup N$ disjoint.

By def. of "basic solution", it satisfies the equations.

s it always feasible? (The one in example is.)

· Feasible if right-hand side constants are ≥ 0

• "Feasible dictionary" — "feasible basic solution".

Is the solution in the example optimal?

Idea: Find a nonbasic variable with positive obj coeff in the dictionary.

Try to increase it from 0.

Adjust the basic variables:

$x_1 = \lambda \geq 0$

param. eqn. of a ray!

$$\left\{\begin{array}{l} \cancel{x_1 = 0} \quad x_2 = 0 \checkmark \quad x_3 = 0 \checkmark \\ x_4 = 5 - 2\lambda \Rightarrow \lambda \leq \frac{5}{2} \\ x_5 = 11 - 4\lambda \Rightarrow \lambda \leq \frac{11}{4} \\ x_6 = 8 - 3\lambda \Rightarrow \lambda \leq \frac{8}{3} \end{array}\right.$$

$$x(\lambda) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 5 \\ 11 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \\ -4 \\ -3 \end{pmatrix}$$

Walk along the ray.

How far? As far as possible — until leaving the feasible region. (Nonnegativities of the basic variables.)

Take minimum of ratios $\Rightarrow$ maximum λ.

$\min\left\{\frac{5}{2}, \frac{11}{4}, \frac{8}{3}\right\} = \frac{5}{2}$. basic variable x_4 attains the min. ratio.

By choosing λ as max. step length that keeps a point in the ray feasible, we guarantee: 1 basic variable will change to 0.

Idea: Make this basic variable non-basic ("leaving") & the chosen non-basic variable basic ("entering"). ("pivot")

Use Gaussian elimination to rewrite the dictionary.

nonbasic

leaving, entering

$$\max \quad 0 + 5x_1 + 4x_2 + 3x_3$$

basic

$$x_4 = 5 - 2x_1 - 3x_2 - x_3$$
$$x_5 = 11 - 4x_1 - x_2 - 2x_3$$
$$x_6 = 8 - 3x_1 - 4x_2 - 2x_3$$

$$2x_1 = 5 - 3x_2 - x_3 - x_4$$
$$x_1 = \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4$$

Rewrite so that basic variables are on LHS, nonbasic RHS.
isolated with coefficient 1

$x_2 \quad x_3 \quad x_4$

$$x_5 = 1 + 5x_2 + 0x_3 + 2x_4$$
$$x_6 = \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4$$
$$x_1 = \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4$$

Use equation for x_1 to remove x_1 from the equation of x_5:

$$x_5 = 11 - 4\cdot\left(\frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4\right) - x_2 - 2x_3$$

$$x_6 = 8 - 3\cdot\left(\frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4\right) - 4x_2 - 2x_3$$

Also rewrite objective function:

$$\max \quad \zeta = 5\left(\frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4\right) + 4x_2 + 3x_3$$

$$= \boxed{\frac{25}{2}} - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4 \quad \checkmark$$

function of nonbasic variables.

New dictionary — corresponding to a new basis

$$\mathcal{B} = \{5, 6, 1\}. \Rightarrow \mathcal{N} = \{2, 3, 4\}$$

$\Longrightarrow$ Read off a basic solution:

$$\begin{aligned} x_5 &= 0 \\ x_6 &= \frac{1}{2} \\ x_1 &= \frac{5}{2} \end{aligned} \qquad \begin{aligned} x_2 &= 0 \\ x_3 &= 0 \\ x_4 &= 0. \end{aligned} \qquad x = \begin{pmatrix} 5/2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1/2 \end{pmatrix}.$$

Read off obj. value: $\zeta = \dfrac{25}{2}.$

This new feasible basic solution — optimal?

Repeat the same procedure.