Lecture 14
Enumeration with Deduction.
Type Systems.

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Deductive reasoning for synthesis

Main idea: Look for the proof to find the program

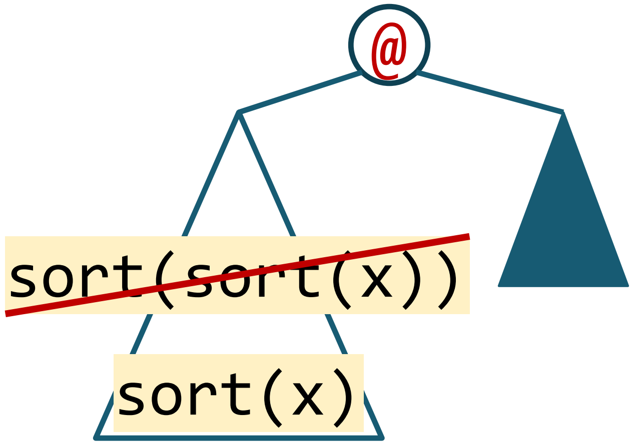
- The space of valid program derivations is smaller than the space of all programs
- The result is provably correct!

Applications:

- Constraint-based search: use loop invariants to encode the space of correct looping programs
- Enumerative search: prune unverifiable candidates early
- Deductive search: search in the space of provably correct transformations / decompositions

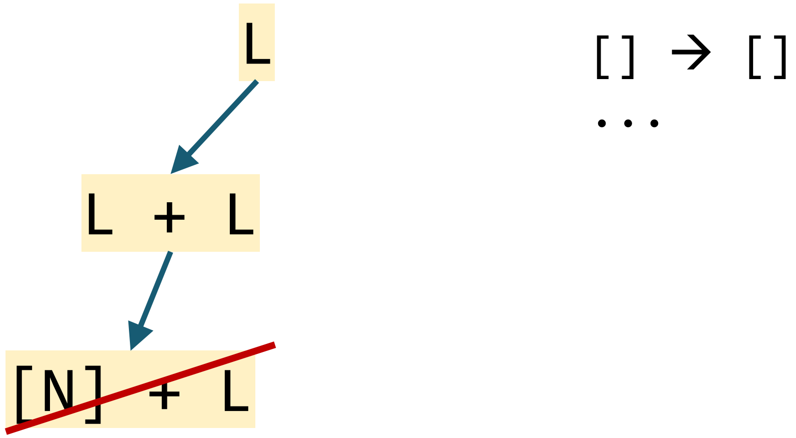
When can we discard a subprogram?

It's equivalent to something we have already explored



Equivalence reduction

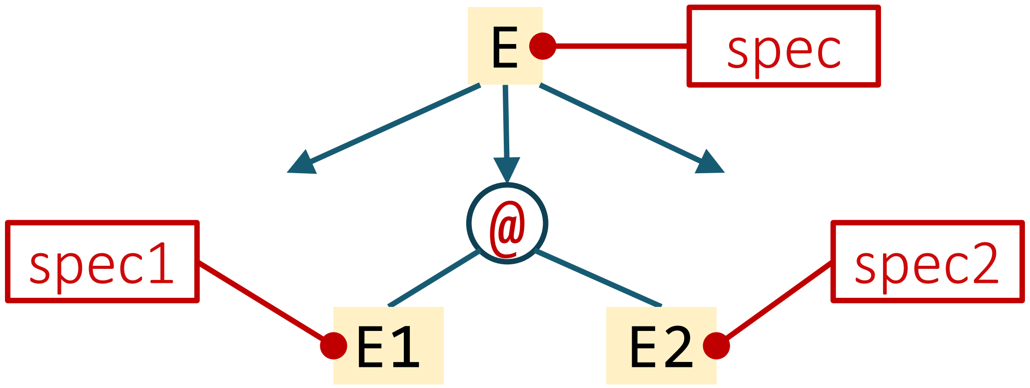
No matter what we combine it with, it cannot fit the spec



Top-down propagation

Top-down propagation

Idea: once we pick the production, infer specs for subprograms

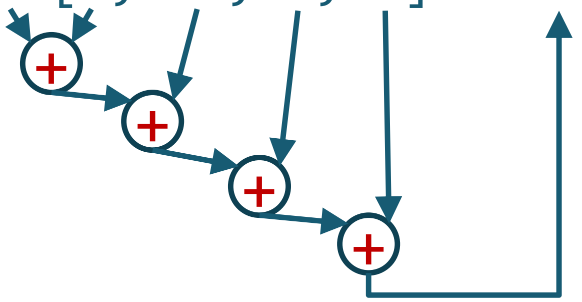


If `spec1 = ⊥`, discard `E1 @ E2` altogether!

λ²: TDP for list combinators

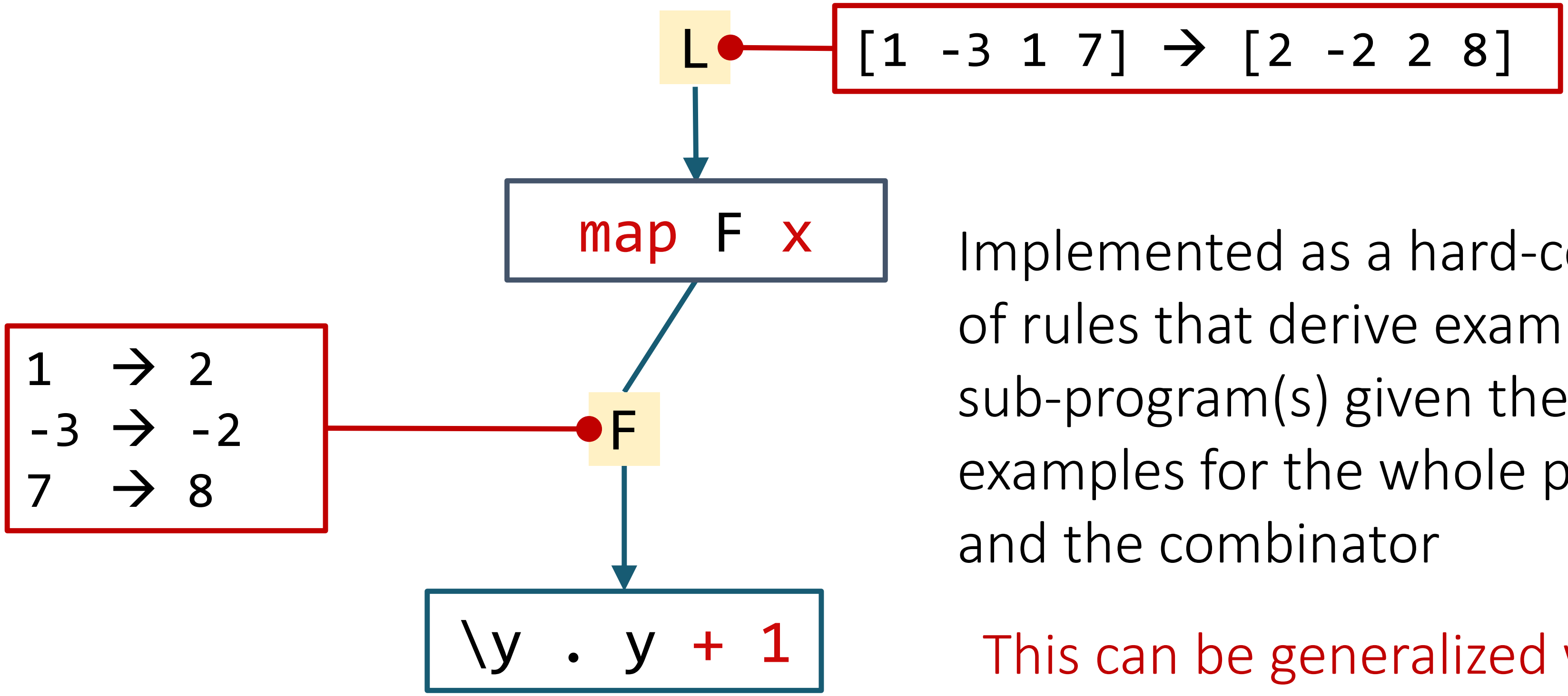
[Feser, Chaudhuri, Dillig '15]

```
map f x            map (\y . y + 1) [1, -3, 1, 7] → [2, -2, 2, 8]

filter f x         filter (\y . y > 0) [1, -3, 1, 7] → [1, 1, 7]

fold f acc x       fold (\y z . y + z) 0 [1, -3, 1, 7] → 6
```



```
                   fold (\y z . y + z) 0 [] → 0
```

λ²: TDP for list combinators



Implemented as a hard-coded set of rules that derive examples for sub-program(s) given the examples for the whole program and the combinator

This can be generalized with deductive reasoning!

Morpheus: TDP with deduction

[Feng et al‘17]

Input-output examples

ID	Name	Age
1	Alice	8
2	Bob	18
3	Tom	12

→

ID	Name	Age
2	Bob	18
3	Tom	12

Synthesizer

straight-line program

Components

with partial specifications!

```
select : Table → [Col] → Table
```

```
out.rows = in.rows
&& out.cols < in.cols
```

```
filter : Table → (Row → Bool) → Table
```

```
our.rows < in.rows
&& out.cols = in.cols
```

Morpheus: TDP with deduction

[Feng et al‘17]

x

ID	Name	Age
1	Alice	8
2	Bob	18
3	Tom	12

→

y

ID	Name	Age
2	Bob	18
3	Tom	12

??

~~select~~ ?? x

filter ?? x

$$\exists x\ y{:}\ y.\mathrm{rows} = x.\mathrm{rows} \wedge y.\mathrm{cols} < x.\mathrm{cols} \\ \wedge\, x.\mathrm{rows} = 3 \wedge x.\mathrm{cols} = 3 \\ \wedge\, y.\mathrm{rows} = 2 \wedge y.\mathrm{cols} = 3$$

$\xrightarrow{\text{SMT}}$ UNSAT

$$\exists x\ y{:}\ y.\mathrm{rows} < x.\mathrm{rows} \wedge y.\mathrm{cols} = x.\mathrm{cols} \\ \wedge\, x.\mathrm{rows} = 3 \wedge x.\mathrm{cols} = 3 \\ \wedge\, y.\mathrm{rows} = 2 \wedge y.\mathrm{cols} = 3$$

$\xrightarrow{\text{SMT}}$ SAT

```
select : Table → [Col] → Table
filter : Table → (Row → Bool) → Table
```

```
out.rows = in.rows && out.cols < in.cols
our.rows < in.rows && out.cols = in.cols
```

Synthesis-friendly verification

Good deductive system for synthesis?

1. good at rejecting incomplete programs
2. general
3. expressive

Type checkers can do 1 and 2!

- and type checkers for *expressive type systems* can do 3 as well

Type Systems

What is a type system?

Formalization of a typing discipline of a language

- independently of a particular type checking algorithm (more or less)
- if a type checking algorithm exists, type system is *decidable*

Deductive system for proving facts about programs and types

- defined using *inference rules* over *judgments*

environment / context
(declares free variables of $\mathfrak{I}$)

$$\Gamma \vdash \mathfrak{I}$$

assertion
for example:

typically:

$$x_1 : T_1, \ldots, x_n : T_n$$

$e :: T$	"e has type T"
T	"T is well-formed"
$T' <: T$	"T' is a subtype of T"

Simple type system

$$e \ ::= \text{true} \mid \text{false} \mid n \mid e + e$$

Syntax of terms (programs)

$$T \ ::= \text{Bool} \mid \text{Int}$$

Syntax of types

Inference Rules

$$\text{T-true} \ \frac{}{\Gamma \vdash \text{true} :: \text{Bool}} \qquad \text{T-false} \ \frac{}{\Gamma \vdash \text{false} :: \text{Bool}} \qquad \text{T-num} \ \frac{(n = 0, 1, \dots)}{\Gamma \vdash n :: \text{Int}}$$

label →

$$\text{T-plus} \ \frac{\Gamma \vdash e_1 :: \text{Int} \qquad \Gamma \vdash e_2 :: \text{Int}}{\Gamma \vdash e_1 + e_2 :: \text{Int}}$$

← premises

← conclusion

Type derivations

$\emptyset \vdash 1 + 2 :: \mathrm{Int}$ is a valid judgment, because....

$$\dfrac{\dfrac{}{\emptyset \vdash 1 :: \mathrm{Int}}\,\text{T-num} \qquad \dfrac{}{\emptyset \vdash 2 :: \mathrm{Int}}\,\text{T-num}}{\emptyset \vdash 1 + 2 :: \mathrm{Int}}\,\text{T-plus}$$

We say that $1 + 2$ is *well-typed* (and has type Int)

Type derivations

$\emptyset \vdash 1 + \text{true} :: \text{Int}$ is not a valid judgment, because....

$$\dfrac{\dfrac{}{\emptyset \vdash 1 :: \text{Int}}\,\text{T-num} \qquad \emptyset \vdash \text{true} :: \text{Int}}{\emptyset \vdash 1 + \text{true} :: \text{Int}}\,\text{T-plus}$$

We say that $1 + \text{true}$ is *ill-typed* (or *not typable*)

Type checking vs inference

The problem of discovering the derivation of $\Gamma \vdash e :: T$ is called *type reconstruction* or *type checking*

The problem of discovering the type T such that there exists a derivation of $\Gamma \vdash e :: T$ is called *type inference*

If we have a mechanism for inference, we can also do checking

- How?

The goal of inference is to free the programmer from writing *type annotations*

Function types

$$e \ ::= \text{true} \mid \text{false} \mid n \mid e + e$$

Syntax of terms (programs)

$$\mid x \mid e\ e \mid \lambda x{:}T.\,e$$

(variable, application, lambda abstraction)

$$T \ ::= \text{Bool} \mid \text{Int}$$

(basic types)

$$\mid T_1 \rightarrow T_2$$

(function types)

Syntax of types

T-var $$\frac{(x{:}T \in \Gamma)}{\Gamma \vdash x :: T}$$

T-abs $$\frac{\Gamma; x{:}T \vdash e :: T'}{\Gamma \vdash \lambda x{:}T.\,e :: T \rightarrow T'}$$

T-app $$\frac{\Gamma \vdash e_1 :: T \rightarrow T' \qquad \Gamma \vdash e_2 :: T}{\Gamma \vdash e_1\ e_2 :: T'}$$

Exercise 1

Infer the type of $\lambda x\!:\mathrm{Int}.\, x + x$ in $\emptyset$ using the rules

T-num $$\frac{(n = 0, 1, \dots)}{\Gamma \vdash n :: \mathrm{Int}}$$

T-plus $$\frac{\Gamma \vdash e_1 :: \mathrm{Int} \qquad \Gamma \vdash e_2 :: \mathrm{Int}}{\Gamma \vdash e_1 + e_2 :: \mathrm{Int}}$$

T-var $$\frac{(x\!: T \in \Gamma)}{\Gamma \vdash x :: T}$$

T-abs $$\frac{\Gamma; x\!: T \vdash e :: T'}{\Gamma \vdash \lambda x\!: T.\, e :: T \to T'}$$

T-app $$\frac{\Gamma \vdash e_1 :: T \to T' \qquad \Gamma \vdash e_2 :: T}{\Gamma \vdash e_1\ e_2 :: T'}$$

Exercise 1a

Infer the type of $(\lambda x{:}\,\mathrm{Int}.\, x + x)\ 5$ in $\emptyset$ using the rules

T-num $$\frac{(n = 0, 1, \dots)}{\Gamma \vdash n :: \mathrm{Int}}$$

T-plus $$\frac{\Gamma \vdash e_1 :: \mathrm{Int} \qquad \Gamma \vdash e_2 :: \mathrm{Int}}{\Gamma \vdash e_1 + e_2 :: \mathrm{Int}}$$

T-var $$\frac{(x{:}\,T \in \Gamma)}{\Gamma \vdash x :: T}$$

T-abs $$\frac{\Gamma; x{:}\,T \vdash e :: T'}{\Gamma \vdash \lambda x{:}\,T.\, e :: T'}$$

T-app $$\frac{\Gamma \vdash e_1 :: T \to T' \qquad \Gamma \vdash e_2 :: T}{\Gamma \vdash e_1\ e_2 :: T'}$$

Type checking vs inference

In type inference, we interpret rules left-to-top-to-right:

$$\dfrac{\dfrac{\cdots}{x : Int \vdash x ::} \qquad \dfrac{\cdots}{x : Int \vdash x ::}}{\dfrac{x : Int \vdash x + x ::}{\emptyset \vdash \lambda x{:}\,\mathrm{Int}.\, x + x ::}\;\text{T-abs}}\;\text{T-plus}$$

Type information flows leaves-to-root ("bottom-up")

That's why we need type annotations on lambda arguments!

Type annotations

$$\text{T-abs'}\ \frac{\Gamma; x : ? \vdash e ::}{\Gamma \vdash \lambda x. e :: ? \to ?}$$

Without the annotation, we don't know what type to give x while analyzing e

If we were doing checking (not inference), this is not a problem:

$$\text{T-abs''}\ \frac{\Gamma; x : T_1 \vdash e :: T_2}{\Gamma \vdash \lambda x. e :: T_1 \to T_2}$$

Bidirectional type-system

Rules differentiate between type inference and checking

$$\Gamma \vdash e \uparrow T \qquad\qquad \Gamma \vdash e \downarrow T$$

"e generates T in Γ" "e checks against T in Γ"

$$\text{I-var}\ \frac{(x : T \in \Gamma)}{\Gamma \vdash x \uparrow T} \qquad \text{C-abs}\ \frac{\Gamma; x : T_1 \vdash e \downarrow T_2}{\Gamma \vdash \lambda x. e \downarrow T_1 \rightarrow T_2} \qquad \text{C-I}\ \frac{\Gamma \vdash e \uparrow T' \quad \Gamma \vdash T = T'}{\Gamma \vdash e \downarrow T}$$

Exercise 2

Check $\lambda x.\, x + x$ against $\text{Int} \to \text{Int}$ in $\emptyset$ using the rules

T-num $$\frac{(n = 0, 1, \ldots)}{\Gamma \vdash n \uparrow \text{Int}}$$

T-plus $$\frac{\Gamma \vdash e_1 \uparrow \text{Int} \qquad \Gamma \vdash e_2 \uparrow \text{Int}}{\Gamma \vdash e_1 + e_2 \uparrow \text{Int}}$$

I-var $$\frac{(x : T \in \Gamma)}{\Gamma \vdash x \uparrow T}$$

C-abs $$\frac{\Gamma; x : T_1 \vdash e \downarrow T_2}{\Gamma \vdash \lambda x.\, e \downarrow T_1 \to T_2}$$

C-I $$\frac{\Gamma \vdash e \uparrow T' \qquad \Gamma \vdash T = T'}{\Gamma \vdash e \downarrow T}$$

Can we infer the type of $\lambda x.\, x + x$ using these rules?

Polymorphism (aka "generics")

$$e ::= \text{true} \mid \text{false} \mid n \mid e + e \mid x \mid e\ e \mid \lambda x{:}T.\, e$$

Terms

$$T ::= \text{Bool} \mid \text{Int}$$ (basic types)

$$\mid T_1 \rightarrow T_2$$ (function types)

$$\mid \alpha$$ (type variables)

Types

$$S ::= T \mid \forall \alpha.\, S$$

Type schemas

T-gen $$\frac{\Gamma; \alpha \vdash e :: S}{\Gamma \vdash e :: \forall \alpha.\, S}$$

T-inst $$\frac{\Gamma \vdash e :: \forall \alpha.\, S \qquad \Gamma \vdash T}{\Gamma \vdash e :: S[\alpha \mapsto T]}$$

Exercise 3

Let's infer the type of $id\ 5$ in Γ
where $\Gamma = [\mathrm{id} : \forall\alpha.\alpha \to \alpha]$

using the following rules:

T-num $$\frac{(n = 0, 1, \dots)}{\Gamma \vdash n :: \mathrm{Int}}$$

T-var $$\frac{(x{:}\,T \in \Gamma)}{\Gamma \vdash x :: T}$$

T-app $$\frac{\Gamma \vdash e_1 :: T \to T' \quad \Gamma \vdash e_2 :: T}{\Gamma \vdash e_1\ e_2 :: T'}$$

T-gen $$\frac{\Gamma; \alpha \vdash e :: S}{\Gamma \vdash e :: \forall\alpha.S}$$

T-inst $$\frac{\Gamma \vdash e :: \forall\alpha.S \qquad \Gamma \vdash T}{\Gamma \vdash e :: S[\alpha \mapsto T]}$$