1 Background

Survival analysis concerns the analysis of time to failure events. Let $\boldsymbol{T}$ denote a time to failure event with density $f(t)$. Further, denote the survivor function as $S(t) := 1 - F(t)$ and the hazard function as $h(t) = f(t)/S(t)$, where $F(t)$ is the cumulative density function.

1.1 Proportional hazards model

Let $\mathcal{D} = (t_i, \delta_i, x_i)_{i=1}^n$ denote our observed data for $t_i \in \mathbb{R}^+, \delta_i \in \{0, 1\}, x_i \in R^p$, where t_i is an observed time, δ_i is a censoring indicator with $\delta_i = 0$ denoting right censoring, and x_i are observed covariates. A popular approach for analysing failure data when there are covariates is through the proportional hazards models (PHM). In the PHM, it is assumed the hazard rate takes the form

$$h(t; x, \beta) = h_0(t) \exp(\beta^\top x) \tag{1}$$

where $\beta \in \mathbb{R}^p$. Under the PHM assumption, the density is given by

$$f(t; x, \beta, \delta) = h_0(t) \exp(\beta^\top x) \exp(-H(t)) \tag{2}$$

where $H(t) = \int_0^t h_(s)ds$, the cumulative hazard rate.

1.2 Bayesian survival analysis

Fitting PHMs in a Frequentist setting involves finding the coefficients that maximise the partial likelihood, given as

$$L_p(\mathcal{D}; \beta) = \prod_{\{i:\delta_i=1\}} \frac{\exp(\beta^\top x_i)}{\sum_{\{r \in \mathbb{R}_i\}} \exp(\beta^\top x_r)} \tag{3}$$

where $R_i := \{r : t_r \geq t_i\}$, known as the risk set. In a Bayesian setting we can approach posterior inference by the evaluating the posterior

$$\pi(\beta; \mathcal{D}) \propto L_p(\mathcal{D}; \beta)\pi(\beta) \tag{4}$$

where $\pi(\beta)$ is our prior distribution on β. Use of the partial likelihood in Equation (4) can be thought of as using a profile likelihood, where $H_0(t)$, a nuisance parameter, is given by the MLE of $H_0(t)$.

1.3 Spike and slab prior

Spike and slab priors are a class of prior that impose shrinkage. The general form of spike and slab priors is given by,

$$z_j \phi_{\text{slab}}(\beta_j) + (1 - z_j)\phi_{\text{spike}}(\beta_j) \tag{5}$$

where $z_j \in \{0, 1\}$, ϕ_{slab} is a density that spreads mass in the domain of β_j and ϕ_{spike} concentrates mass around some point (typically 0).

We let the slab be given by the Laplace density (centered at 0) and the spike by a Dirac mass on zero, formally,

$$\pi(\beta_j|z_j) \sim z_j \text{Laplace}(\lambda) + (1 - z_j)\delta(\beta_j) \tag{6}$$

$$\pi(z_j|w) \sim \text{Bernoulli}(w) \tag{7}$$

$$\pi(w) \sim \text{Beta}(a_0, b_0) \tag{8}$$

for $a_0, b_0, \lambda > 0$.

2 Posterior Inference

We can perform posterior inference by constructing the Gibbs sampler presented in Algorithm 1.

Algorithm 1 Gibbs sampler for Equation (4)

Initialise $z^{(0)}, \beta^{(0)}, w^{(0)}$
2: **for** $i = 1, \dots N$ **do**
  $w^{(i)} \sim p(w|\mathcal{D}, \beta^{(i-1)}, z^{(i-1)})$
4: **for** $j = 1, \dots p$ **do**
   $z_j^{(i)} \sim p(z_j|\mathcal{D}, \beta^{(i-1)}, z_{1:(j-1)}^{(i)}, z_{(j+1):p}^{(i-1)}, w^{(i)})$
6: **for** $j = 1, \dots p$ **do**
   $\beta_j^{(i)} \sim p(\beta_j|\mathcal{D}, \beta_{1:(j-1)}^{(i-1)}, \beta_{(j+1):p}^{(i-1)}, z^{(i)}, w^{(i)})$

where $x_{k:l} = (x_i)_{i=k}^{l}$, for $1 \leq l < k \leq p$, $x \in \mathbb{R}^p$ and N are the number of iterations we run our sampler for. Ignoring the superscript for clarity, the conditional distribution for $p(z_j|\mathcal{D}, \beta, z_{-j}, w)$ is given by

$$p(z_j|\mathcal{D}, \beta, z_{-j}, w) \propto L_p(\mathcal{D}; \beta, z, w)\pi(z_j|w) \tag{9}$$

we can evaluate the RHS for $z_j = 0$ and $z_j = 1$ to obtain the unnormalised probabilities. For $p(\beta_j|\mathcal{D}, \beta_{-j}, z, w)$, we turn to the Metropolis Hastings algorithm to sample from the conditional distribution, noting,

$$p(\beta_j|\mathcal{D}, \beta_{-j}, z, w) \propto L_p(\mathcal{D}; \beta, z, w)\pi(\beta_j). \tag{10}$$

To sample β_j we propose a kernel Q and sample $\beta_j^{(i+1)} \sim Q$, letting

$$A := \min\left(1, \frac{L_p(\mathcal{D}; \beta_{-j}, \beta_j^{(i+1)}, z, w)\pi(\beta_j^{(i+1)})}{L_p(\mathcal{D}; \beta_{-j}, \beta_j^{(i)}, z, w)\pi(\beta_j^{(i)})} \frac{Q(\beta_j^{(i)}|\beta_j^{(i+1)})}{Q(\beta_j^{(i+1)}|\beta_j^{(i)})}\right) \tag{11}$$

we accept $\beta_j^{(i+1)}$ with probability A, and we reject the proposal with probability $1 - A$; letting $\beta_j^{(i+1)} \leftarrow \beta_j^{(i)}$ if we reject the proposal.

3 Example

`R` code implementing Algorithm 1 is available at `https://github.com/mkomod/mcmc_ss_surv`. Within our implementation we have taken $Q = \mathcal{N}(\beta_j; \beta_j^{(i)}; 10^{(1-z_j^{(i)})}\sigma^2)$ where $\sigma = 0.3$. We simulated $n = 250$ observations with a censoring rate of 0.4. We let $p = 50$ and $\beta = (2, 2, 0, \dots, 0)$. Further, we chose a value of $\lambda = 1$ and ran our sampler for $N = 10,000$ (using a burn-in 1,000). We present the densities for $\beta_1, \dots, \beta_4$ in Figure 1.

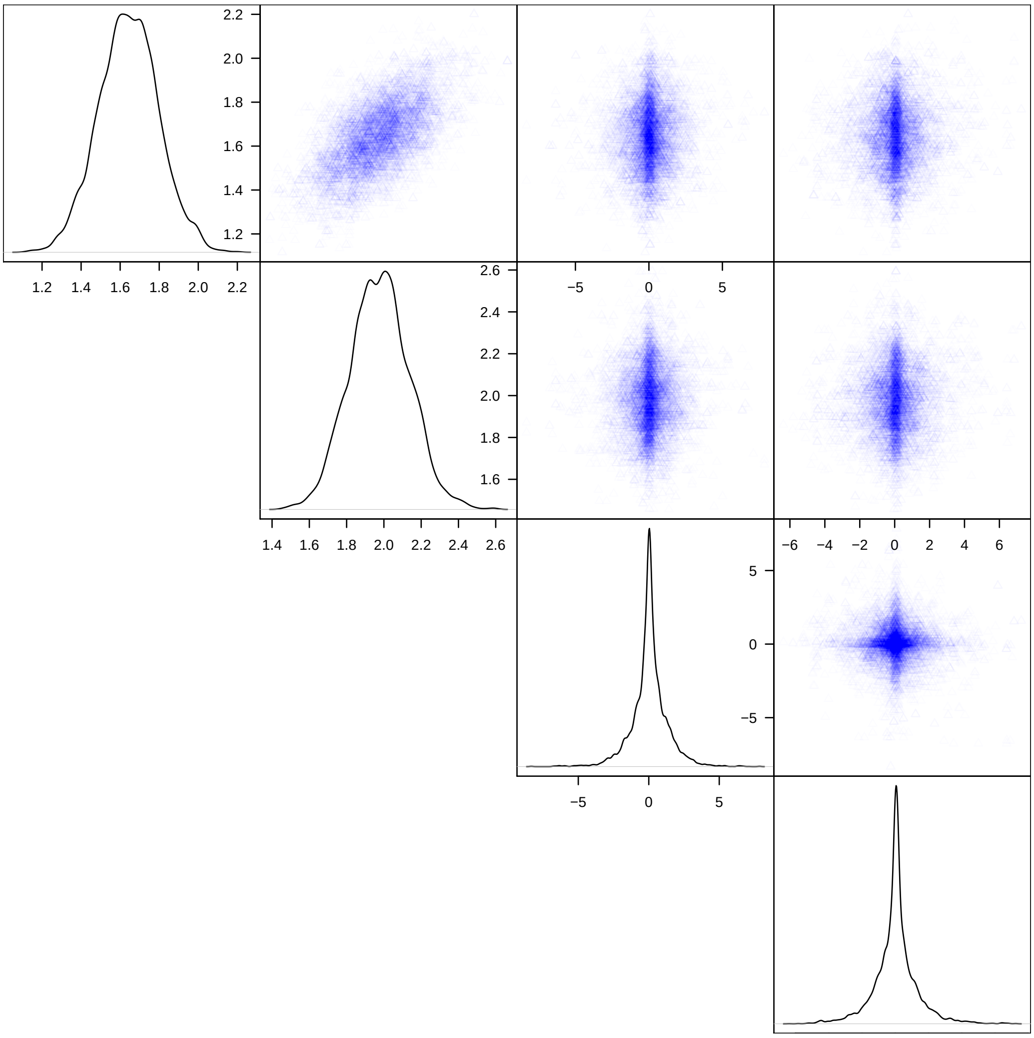

Figure 1: Mixing and marginal densities for $\beta_1, \dots \beta_4$

The posterior means for each element of β are presented in Table 1

1.644	1.977	0.054	0.035	0.013	-0.035	0.103	0.020	0.049	-0.040
0.033	0.086	-0.035	-0.035	-0.036	-0.066	-0.038	0.040	0.020	-0.001
-0.004	-0.003	-0.066	-0.058	0.012	-0.053	-0.105	-0.029	-0.037	0.037
-0.056	0.034	-0.090	-0.013	0.020	-0.047	0.028	0.066	0.143	0.036
-0.056	-0.020	-0.031	-0.009	0.035	0.015	0.008	-0.021	0.122	0.026

Table 1: Posterior means for β

The chains for $\beta_1, \ldots, \beta_{16}$ are present in Figure 2.

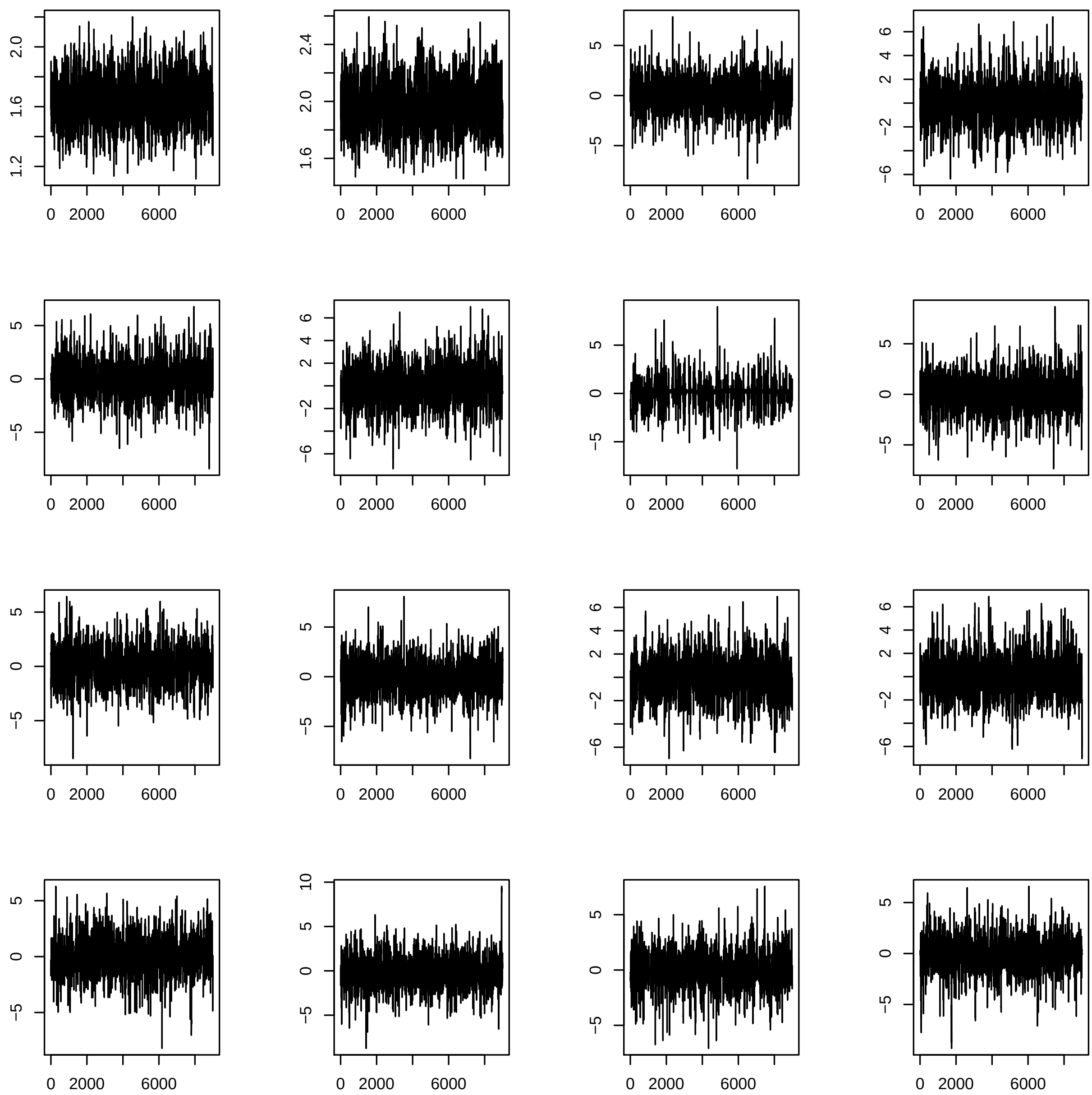

Figure 2: Mixing for $\beta_1, \ldots \beta_{16}$

References

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