Sparsity Patterns

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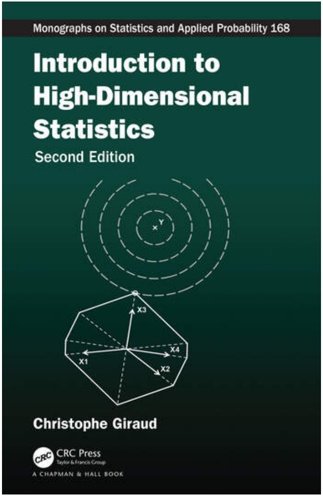
Outline

About this presentation

About this presentation

Based on the 2nd Chapter of “An introduction to High-dimensional Statistics” by Giraud (2021).

Can be found online for free



Motivation: Why should we care?

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Making use of the data requires tools that take into account the patterns within it.

The aim of this presentation is to introduce some of these patterns and the corresponding tools to analyze them

Statistical Setting

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Statistical setting

Formally, we'll be working in a regression setting where,

$$y_i = f(x_i) + \epsilon_i \tag{1}$$

which links our response $y \in \mathbb{R}$, to p variables stored in a p-dimensional real valued vector $x_i \in \mathbb{R}^p$.

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This is because many forms for $f(x)$ can be re-written in this way, e.g. piecewise constant regression, additive models etc.

Sparsity Patterns

Types of Sparsity

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- Coordinate sparsity (which we've all come across)
- Group-wise sparsity
- Sparse group-wise sparsity

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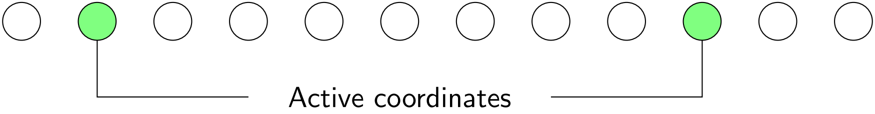

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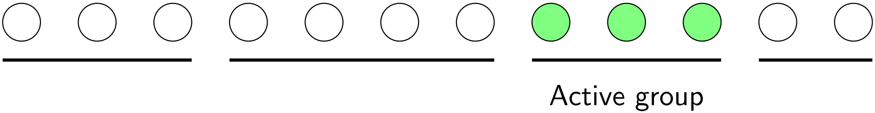

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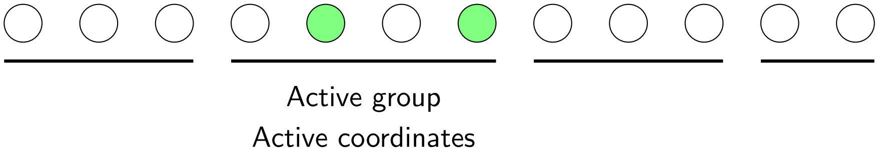

Examples in Practice

These examples are taken from

"A Sparse-group LASSO" by Simon et al. Sec 5.

A SPARSE-GROUP LASSO

NOAH SIMON, JEROME FRIEDMAN, TREVOR HASTIE, AND ROB TIBSHIRANI

Abstract. For high dimensional supervised learning problems, often using problem specific assumptions can lead to greater accuracy. For problems with grouped covariates, which are believed to have sparse effects both on a group and within group level, we introduce a regularized model for linear regression with ℓ_1 and ℓ_2 penalties. We discuss the sparsity and other regularization properties of the optimal fit for this model, and show that it has the desired effect of group-wise and within group sparsity. We propose an algorithm to fit the model via accelerated generalized gradient descent, and extend this model and algorithm to convex loss functions. We also demonstrate the efficacy of our model and the efficiency of our algorithm on simulated data.

Keywords: penalize, regularize, regression, model, nesterov

Example: Breast cancer data

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Dataset

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Dataset

- Gene expression values of $n = 60$ patients with estrogen positive breasts
- Patients were treated with tamoxifen for 5 years
- Classified according to whether cancer recurred
- After pre-processing $p \approx 12,000$ genes
- Genes are groups by cytogenetic position data (GSEA C1 data)
- 30 patients chosen at random used in the training set.

Example: Breast cancer data

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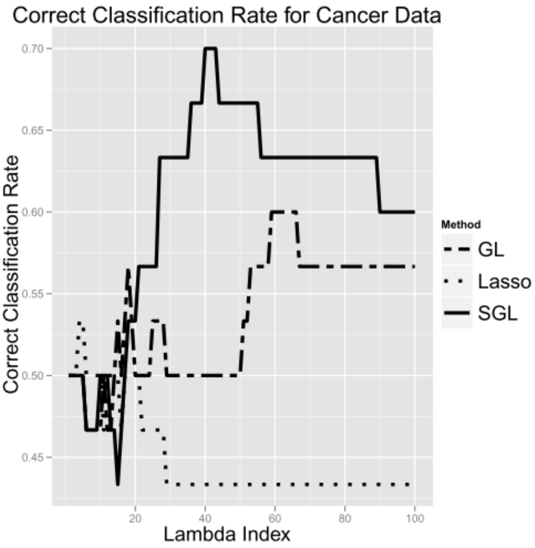

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Examining the peak classification accuracy for each method

Example: Breast cancer data

Examining the peak classification accuracy for each method

Method	Classification Accuracy	Num. Features
Sparse group LASSO	70%	54 (11 groups)
Group LASSO	60%	74 (14 groups)
LASSO	53%	3

Example: Colitis data

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- 127 patients
- 85 with colitis, 42 controls
- $p \approx 8,300$ after pre-processing
- grouped into gene-sets using cytogenetic information giving 277 disjoint groups.
- 50 observations used to fit models, 77 used to test.

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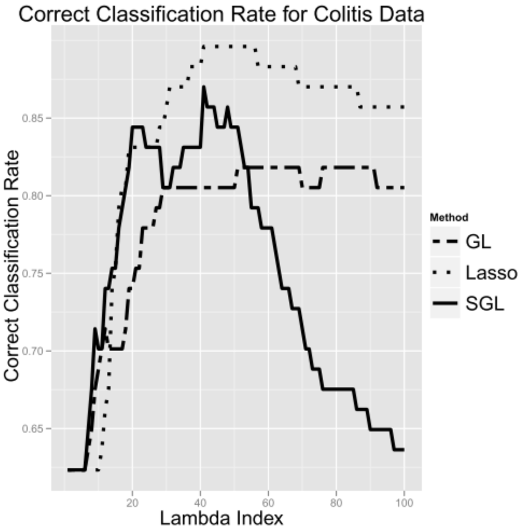

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In the second example the sparse group LASSO with the chosen gene sets did not perform as well as the LASSO - specialist information about the gene set may improve the results.

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When to use which? A rule of thumb

Method	Number of Groups	Size of groups
Coordinate sparse	Small	Large
Group sparse	Large	Small
Sparse group sparse	Large	Large

Questions so far?

We're about to jump into some theory

so now's the time to delay that happening!

Model selection

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General idea

Compare different statistical models corresponding to different possible structures and select the model best suited to estimation.

Model selection: Framework

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Framework

We're assuming

$$y_i = f_i + \epsilon_i, \quad \epsilon_i \overset{iid}{\sim} N(0, \sigma^2) \tag{3}$$

where $f_i = \langle x_i, \beta \rangle$, with $\beta = (\beta_1, \ldots, \beta_p)^\top$ and $x_i = (x_{i,1}, \ldots, x_{i,p}) \in \mathbb{R}^p$ for $i = 1, \ldots, n$.

We will assume a true and unknown $\beta^* \in \mathbb{R}^p$, used to generate the data.

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To keep things simple we're going to be considering the model selection process for co-ordinate sparsity.

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2. Estimating the model $\widehat{f}_m$ for each m using maximum likelihood.
3. Estimating f by selecting the **best** model $\widehat{f}_m$ from the collection $m \in \mathcal{M}$

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As we don't know f^* a natural idea is to estimate r_m using an unbiased estimator, and since,

$$r_m = \mathbb{E}\left[\|Y - f_m\|^2\right] + (2d_m - n)\sigma^2 \tag{5}$$

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$$\widehat{r}_m = \|Y - f_m\|^2 + (2d_m - n)\sigma^2 \tag{6}$$

AIC

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In turn we can estimate m using

$$\widehat{m} \in \underset{m \in \mathcal{M}}{\operatorname{argmin}} \left\{ \| Y - f_m \|^2 \; + 2 d_m \sigma^2 \right\} \tag{7}$$

Where we have dropped the $-n\sigma^2$ term (which does not change $\widehat{m}$). This estimator for $\widehat{m}$ is the popular Akaike information criterion.

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To correct this, we can introduce a penalization term taking into account the number of models per dimension.

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where $|\cdot|$ is the cardinality of the set m and $\lambda = (1 + \sqrt{2 \log p})^2 \sigma^2$.

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Theory is covered at the end of Chapter 2 and start of Chapter 5 for Giraud (2021).

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$$\widehat{m} \in \underset{m \in \mathcal{M}}{\operatorname{argmin}} \min_{\beta : \mathrm{supp}(\beta) = m} \left\{ \|Y - f_m\|^2 + \lambda |\beta|_0 \right\} \tag{9}$$

where $|\beta|_0 = \#\{\beta_j \neq 0 : j = 1, \ldots, p\}$

Penalization: LASSO

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Solving

$$\widehat{\beta}_{\lambda} \in \underset{\beta:\mathrm{supp}(\beta)=m}{\operatorname{argmin}} \left\{ \|Y - f_m\|^2 + \lambda|\beta|_0 \right\} \tag{10}$$

is non-convex due to $|\beta|_0$.

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is non-convex due to $|\beta|_0$.

Relaxing $|\beta|_0$ to be $|\beta|_1$ is convex and gives us the popular LASSO estimator

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How we got here:

- We wanted to do model selection
- But the estimator $\widehat{m}$ when p is large tends to not work well.
- Penalizing the estimator has better (statistical) proprieties but is computationally intractable
- Relaxing our penalization scheme is computationally tractable and has decent statistical proprieties

A Bayesian perspective

Some history

Some history

Model selection priors started gaining attention around the 1990s

Bayesian Variable Selection in Linear Regression

T. J. MITCHELL and J. J. BEAUCHAMP*

This article is concerned with the selection of subsets of predictor variables in a linear regression model for the prediction of a dependent variable. It is based on a Bayesian approach, intended to be as objective as possible. A probability distribution is first assigned to the dependent variable through the specification of a family of prior distributions for the unknown parameters in the regression model. The method is not fully Bayesian, however, because the ultimate choice of prior distribution from

Recent work has been aimed at scaling these prior to suit high-dimensional problems.

Variable Selection Via Gibbs Sampling

EDWARD I. GEORGE and ROBERT E. McCULLOCH*

A crucial problem in building a multiple regression model is the selection of predictors to include. The main thrust of this article is to propose and develop a procedure that uses probabilistic considerations for selecting promising subsets. This procedure entails embedding the regression setup in a hierarchical normal mixture model where latent variables are used to identify subset choices. In this framework the promising subsets of predictors can be identified as those with higher posterior probability. The computational burden is then alleviated by using the Gibbs sampler to indirectly sample from this multinomial posterior distribution on the set of possible subset choices. Those subsets with higher probability—the promising ones—can then be identified by their more frequent appearance in the Gibbs sample.

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Construct a posterior that updates our prior belief about model m using the data.

This is a (conceptually) rich tool, as it assigns a posterior probability to each model m and therefore gives us a nice way to do model and variable selection.

Bayesian model selection: Prior

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Co-ordinate sparse priors

Bayesian model selection: Prior

Co-ordinate sparse priors

$$\begin{aligned} \beta_j | z_j &\overset{ind}{\sim} z_j \Phi(\beta_j) + (1 - z_j)\delta_0 \\ z_j &\overset{iid}{\sim} \text{Bernoulli}(p) \end{aligned} \tag{11}$$

for $j = 1, \ldots, p$ and where Φ is a continuous distribution for β and δ_0 is a Dirac mass at 0.

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where $G_k \subset \{1, \ldots, p\}$ for $k = 1, \ldots, m$ are disjoint sets such that $\cup_{k=1}^{m} G_k = \{1, \ldots, p\}$ and δ_0 is a multivariate Dirac mass at 0.

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Group-wise and Sparse-Group sparsity

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Sparse-group sparsity can be achieved by ensuring Φ induces sparsity e.g. using another spike and slab prior.

Computational problems

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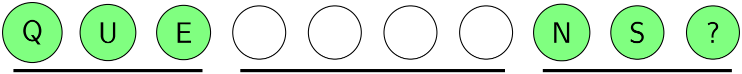
There are some work-arounds, involving integrating our z (these are known as continuous shrinkage priors), but we no longer explore the model space.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