Recent changes in firm dynamics and the nature of economic growth*

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Abstract

Over the last decades, productivity growth has slowed, sales concentration has increased, and firm entry has fallen in many advanced economies. I document a new trend at the firm level in Swedish administrative data: firm life cycle growth of sales and employment accelerated from 1997 to 2017. Using a model of firm dynamics that features firm entry and incumbent markup growth through R&D, I show that a rise in the entry costs and a decline in the research productivity by incumbent firms account for the changes in life cycle growth. Sales and employment growth of active firms jointly accelerate due to rising entry costs. In contrast, falling research productivity accounts for the acceleration in employment relative to sales growth. At the aggregate level, these changes result in a decline in the long-run growth rate of 0.6pp, a drop in the firm entry rate by 8pp, a rise in sales concentration, and a reallocation of sales shares to low labor share firms. Changes in the innovation rates of the average firm and the reallocation of sales shares across firms that innovate at different rates contribute almost equally to changes in economic growth. However, the slowdown in economic growth is due to the fall in firm entry.

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1 Introduction

The U.S. economy has experienced several macroeconomic trends over the last decades: productivity growth has declined, sales concentration within industries has risen, and firm entry has fallen.[1] The U.S. economy is not an outlier; similar trends have been documented for many advanced economies worldwide.[2] I provide new insights about these macroeconomic trends based on a novel observation at the firm level in Swedish administrative data: for the universe of firms, life cycle growth of sales and employment accelerated from 1997 to 2017. Motivated by this result, I address the following four questions: What are the driving forces behind the acceleration in firm sales and employment growth? How do these forces affect the composition of incumbent firms? Are the changes in firm growth linked to the recent macroeconomic trends? What do the changes in firm growth imply about the source behind the slowdown in productivity growth?

The first contribution of this paper is empirical. I document a new stylized fact about firm growth using high-quality administrative data from Swedish tax records: life cycle growth of firm sales and employment accelerated. Measured over the first eight years of the firm, sales growth increased by 11.5 percentage points (pp) comparing cohorts of the late 1990s to the cohorts of the early 2010s. Employment growth increased by 17.8pp, respectively. Are these changes economically relevant? With the help of a structural model, I quantify the implications of these firm-level changes for the macroeconomy.

The model includes the following three elements to address the questions raised in the first paragraph. First, the model features a link between firm dynamics and economic growth in the spirit of Schumpeterian growth models (Aghion and Howitt, 1992; Grossman and Helpman, 1991; Klette and Kortum, 2004): firms capture market shares by replacing the previous incumbent firm through innovation (creative destruction).[3] Second, in the Klette and Kortum (2004) model, firm employment is proportional to sales. I include a second type of innovation in the model as in Peters (2020) that permits differential sales and employment growth (and changes therein) in line with the data.[4] Firms raise their markups by improving their own products through innovation (internal R&D). Markup growth drives a wedge between firm sales and employment growth. Third, I add firm-type heterogeneity to the model to quantify how the drivers behind the changes in firm life cycle growth affect the composition of firms. I follow Aghion, Bergeaud, Boppart, Klenow and Li (2023) and model type heterogeneity through persistent differences in productivity across firms. This

[1] Autor, Dorn, Katz, Patterson and Van Reenen (2017), Grullon, Larkin and Michaely (2019) and Akcigit and Ates (2021) document rising sales concentration in the U.S. The decline in firm entry is documented in Decker, Haltiwanger, Jarmin and Miranda (2016); Gourio, Messer and Siemer (2014); Karahan, Pugsley and Şahin (2022).

[2] See Andrews, Criscuolo and Gal (2016); Gopinath, Kalemli-Özcan, Karabarbounis and Villegas-Sanchez (2017); Autor, Dorn, Katz, Patterson and Van Reenen (2020); Karabarbounis and Neiman (2014).

[3] These models are analytically tractable yet capture salient features of firm dynamics (Lentz and Mortensen, 2008; Akcigit and Kerr, 2018).

[4] Similarly, Akcigit and Kerr (2018) features a quality-ladder model with creative destruction and innovation within product markets.

sort of firm-type heterogeneity generates ex-ante differences in (sales and employment) life cycle trajectories as documented in Sterk, Sedláček and Pugsley (2021) and can address reallocation effects between high and low labor share firms, highlighted by Kehrig and Vincent (2021).[5]

I estimate the model on Swedish administrative data matching firm sales and employment growth of cohorts in the late 1990s and other macroeconomic moments. As a comparative statics experiment, I reestimate two parameters of the model to match the acceleration of firm sales and employment life cycle growth of the latest cohorts in the data. Finally, I answer the raised research questions by comparing the economy across the two Balanced Growth Paths (BGPs).

As the paper's second contribution, I show that a rise in the entry costs (22%) and a decline in internal R&D productivity by incumbents (51%) quantitatively explain the increase in sales and employment growth. Rising entry costs incentivize incumbent firms to grow faster, accelerating their sales *and* employment growth. In contrast, the fall in the internal R&D productivity slows down the markup growth of active firms, accelerating their employment growth *relative* to sales growth. This replicates the disproportionate increase in employment growth. Bloom, Jones, Van Reenen and Webb (2020) show that falling research productivity is a pervasive trend in the U.S. economy. I show that a fall in research productivity explains the observed changes in firm growth. At the same time, the fall in research productivity gives rise to macroeconomic trends in, e.g., TFP growth, firm entry, and industry concentration that align with the data. Davis (2017) and Gutiérrez and Philippon (2018) further highlight rising barriers to entry.

As a third contribution, I quantify the extent of reallocation of market shares across firms following the rise in entry costs and fall in research productivity. Despite the symmetric (to the firm type) nature of the R&D cost changes, more productive firms overtake market shares from the less productive firms. The sales share of high productivity type firms increases by 17pp. Their share in the cross-section of firms increases by 12pp. The reallocation of sales shares to high productivity firms that feature relatively low labor shares and high markups is consistent with Kehrig and Vincent (2021), De Loecker, Eeckhout and Unger (2020) and Baqaee and Farhi (2020). The reallocation of sales shares has further implications for changes in the aggregate growth rate, which I focus on later.

Fourth, I use the structural model to quantify the aggregate effects of the rise in firm entry costs and the fall in R&D productivity. Alongside the changes in firm sales and employment growth, the aggregate growth rate declines by 0.62pp, the firm entry rate drops by 8pp, and concentration rises. These changes align with trends in the Swedish macroeconomy over the last three decades (Engbom, 2023). That rising entry and R&D costs are consistent with observed macroeconomic trends is well known in the literature. However, the comparative statics estimation does not target economic aggregates: the increase in firm sales and em-

[5] De Loecker, Eeckhout and Unger (2020) and Baqaee and Farhi (2020) document similar reallocation effects across high and low markup firms.

ployment growth discipline the changes in entry and R&D costs. That the estimated changes in entry costs and research productivity replicate the changes in firm growth while simultaneously delivering aggregate patterns that align with the data suggests that the firm-level and aggregate-level trends are directly linked.

As the fifth and final contribution, I decompose the fall in economic growth following the rise in entry costs and fall in research productivity into its sources. Changes in economic growth are due to changes in the innovation rates by (the average) firm, reallocation of sales shares across firms that innovate at different rates, or changes in firm entry. I quantify the size of these three channels. Changes in the innovation rates of the average firm and the reallocation of sales shares contribute *positively* to economic growth. The fall in research productivity of incumbents indeed lowers innovation rates of the average firm. However, the rise in entry costs incentivizes incumbent firms to innovate at higher rates. On the net, this within-firm effect is positive. The reallocation effect contributes positively because more productive firms innovate at higher rates. The increase in sales shares of these firms raises the aggregate growth rate. The magnitudes of the within and between firm effects are similar: rising innovation rates of the average firm increase the aggregate growth rate by 0.22pp, whereas the reallocation effect increases it by 0.27pp. Therefore, the fall in firm entry more than accounts for the total fall in the aggregate growth rate (-1.1pp compared to the total fall of 0.62pp). Falling firm entry squares the acceleration in firm growth with the fall in economic growth. That a decrease in firm entry drives the fall in economic growth is robust to alternative specifications. For an alternative estimation, where a rise in the R&D costs of incumbents and an increase in the productivity dispersion explain the fall in economic growth and the rise in concentration as in Aghion, Bergeaud, Boppart, Klenow and Li (2023), falling firm entry still accounts for the lion's share of the fall in economic growth. In this alternative specification, innovation rates of the average firm decline slightly; however, this channel compares small to the contribution by falling firm entry. Rising entry costs and increasing productivity dispersion both highlight falling firm entry as the driving force behind the slowdown in economic growth.

In addition to the previously mentioned studies, the findings further relate to the following strands of literature. Karahan, Pugsley and Şahin (2022) and Hopenhayn, Neira and Singhania (2022) document that firm employment conditional on age has been relatively constant in the U.S. since the 1980s. Karahan, Pugsley and Şahin (2022) document this stability for firms up to age ten, noting that for firms older than ten, firm size (conditional on age) increases significantly over time when holding the industry composition constant. An increase in firm size conditional on age over time implies that more recent cohorts grow faster, as documented in this paper. I replicate these size-conditional-on-age plots in Swedish administrative data. These raw plots already display an increase in average firm size for older firms over time. The acceleration in firm growth becomes even more apparent when using firm-level regressions to measure firm growth, controlling for industry composition.[6]

[6] Van Vlokhoven (2021) further documents that profits and sales of firms in Compustat data have become more back-loaded. While I share the observation that the sales growth over the firm's life cycle accelerated,

Sterk, Sedláček and Pugsley (2021) document changes in life cycle growth for U.S. firms over time. For the cohorts 1979 to 1993, the authors show that employment growth over the firm's life cycle slowed. The results presented in this paper are complementary rather than contradictory to theirs as I document trends for the cohorts from 1997 to 2017, suggesting a reversal of the previous trends. The rise in industry concentration, markups, and the fall in firm entry were particularly pronounced during the turn of the millennium, as shown by Autor, Dorn, Katz, Patterson and Van Reenen (2020), De Loecker, Eeckhout and Unger (2020) and Akcigit and Ates (2021). Firm-level changes during this period help to understand these macroeconomic changes.

The findings further relate to a literature that emphasizes the effects of reallocation on economic growth. China and East Germany are examples where long-term sustained growth followed the reallocation of market shares from state-owned enterprises to privately held companies (Song, Storesletten and Zilibotti, 2011; Findeisen, Lee, Porzio and Dauth, 2021). This reallocation potentially affects GDP per capita in a static sense through two channels. First, more productive firms gain market shares, thereby raising average productivity, and second, by reducing the extent of misallocation of production factors in the spirit of Hsieh and Klenow (2009). However, the reallocation could also affect the economy's long-run growth rate if privately held firms innovate (or imitate) at higher rates than state-owned enterprises. Acemoglu, Akcigit, Alp, Bloom and Kerr (2018) show that reallocating market shares can substantially raise the economy's long-run growth rate. For an advanced economy like Sweden, I quantify the effects of reallocation on the aggregate growth rate from 1997 to 2017. The effects of reallocation on the aggregate growth rate are of a similar magnitude (if not larger) than the effect of changes in the innovation rate of the average firm. I further show that the positive reallocation effects are squared with a fall in the aggregate growth rate through an even larger decline in firm entry.

The comparative statics exercise relates to a strand of literature explaining recent macroeconomic trends in the U.S. Proposed drivers for these trends are ever-increasing costs of R&D (Bloom, Jones, Van Reenen and Webb, 2020), increasing barriers to entry (Davis, 2017; Gutiérrez and Philippon, 2018), or rising productivity dispersion (Aghion, Bergeaud, Boppart, Klenow and Li, 2023).[7] The approach in this paper differs as the comparative statics estimation is disciplined instead by the change in firm sales and employment growth. As an outcome of the estimation, a rise in the entry costs and a fall in research productivity, which the above literature has highlighted as forces behind the macroeconomic trends, explain the changes in firm growth. This suggests that the same mechanisms drive the firm-level and aggregate trends.

I find firm size at entry relatively constant over time as Karahan, Pugsley and Şahin (2022) and Hopenhayn, Neira and Singhania (2022).

[7]Further explanations include an increasing importance of intangible capital and information and communications technology (ICT) (Crouzet and Eberly, 2019; Chiavari and Goraya, 2020; De Ridder, 2024; Hsieh and Rossi-Hansberg, 2023; Weiss, 2019), declining interest rates (Chatterjee and Eyigungor, 2019; Liu, Mian and Sufi, 2022), changes in the quality of ideas (Olmstead-Rumsey, 2019) or declining imitation rates (Akcigit and Ates, 2019).

Peters and Walsh (2021) further highlight demographic forces behind the trends of the U.S. economy.[8] In Peters and Walsh (2021), a decline in the growth rate of the population explains the fall in productivity growth, the rise in product market concentration, and the fall in the entry rate. Sweden's population growth rate gradually increased for two decades despite rising concentration, falling firm entry, and declining long-run productivity growth. This suggests that, at least for the Swedish economy, demographic forces are not behind the macroeconomic trends.

The paper proceeds as follows. Section 2 documents the trends in firm life cycle growth, and section 3 lays out the model. In section 4, I apply the model to answer the research questions. Section 5 concludes the paper.

2 Changes in firm life cycle growth

Using Swedish registry data, I show in this section that firm sales and employment growth over the firm's life cycle accelerated from 1997 to 2017. I describe the data in a first step.

2.1 Data

All data is provided by Statistics Sweden (SCB), the official statistical agency in Sweden. The main data set is *Företagens Ekonomi* (FEK), which covers information from balance sheets and profit and loss statements for the universe of Swedish firms. The unit of observation is the legal unit at an annual frequency covering the period 1997-2017. FEK contains the main variables of interest: sales and employment (in full-time units). Before 1997, FEK was a sample covering large Swedish firms. To ensure full representativeness, I focus on the years 1997 forward. The data further contains information on the firm's legal type and industry at the five-digit level. I focus on firms in the private economy. Throughout the paper, nominal variables are deflated to 2017 Swedish Krona (SEK) using the GDP deflator. For a detailed description of the data, see Section A in the Appendix.

I define the birth year of the firm as the year it hires its first employee. I obtain this information from the auxiliary data set *Registerbaserad Arbetsmarknadsstatistik* (RAMS). RAMS contains the universe of employer-employee matches. I further restrict myself to firms that employ at least one worker according to RAMS.[9]

Table 1 reports distributional statistics of firm sales, value added, and production inputs for the pooled data (1997 to 2017). The median firm has sales of roughly 2.7 million SEK (approx. 0.27 million US dollars), value added of 1.1 million SEK, and employs two workers.

[8] Bornstein (2018), Engbom (2023), Hopenhayn, Neira and Singhania (2022), Karahan, Pugsley and Şahin (2022) further emphasize the role of demographic forces behind macroeconomic trends.

[9] The acceleration in employment growth is very similar when computing firm employment based on the RAMS data.

Table 1: Summary statistics (1997-2017)

	25th Pct.	50th Pct.	75th Pct.	Mean	SD	Obs.
*Sales**	1.2	2.7	7.8	27.8	568.2	4,918,996
*Value added**	0.5	1.1	2.9	7.6	142.3	4,918,996
Employment	1	2	5	9.9	131.1	4,918,996
*Wage bill**	0.2	0.6	1.6	3.7	53.0	4,918,996
*Capital stock**	0.04	0.2	1.1	9.3	277.0	4,918,996
*Intermediate Inputs**	0.4	0.9	2.6	10.8	270.0	4,918,996

Note: *: in million 2017-SEK (1 SEK $\approx$ 0.1 US dollars). The capital stock is defined as fixed assets minus depreciation.

The distribution of sales, value added, and all production inputs is highly right-skewed, as indicated by the mean and the 25th, 50th, and 75th percentiles. Average firm sales are 27.8 million SEK, and average employment is 9.9. In total, the data includes about 4.9 million firm-year observations. For the age-specific empirical analysis, I focus on firms I observe since birth, i.e., firms established in 1997 or later, which reduces the sample size to 2.2 million firm-year observations. For these firms, age is not truncated.

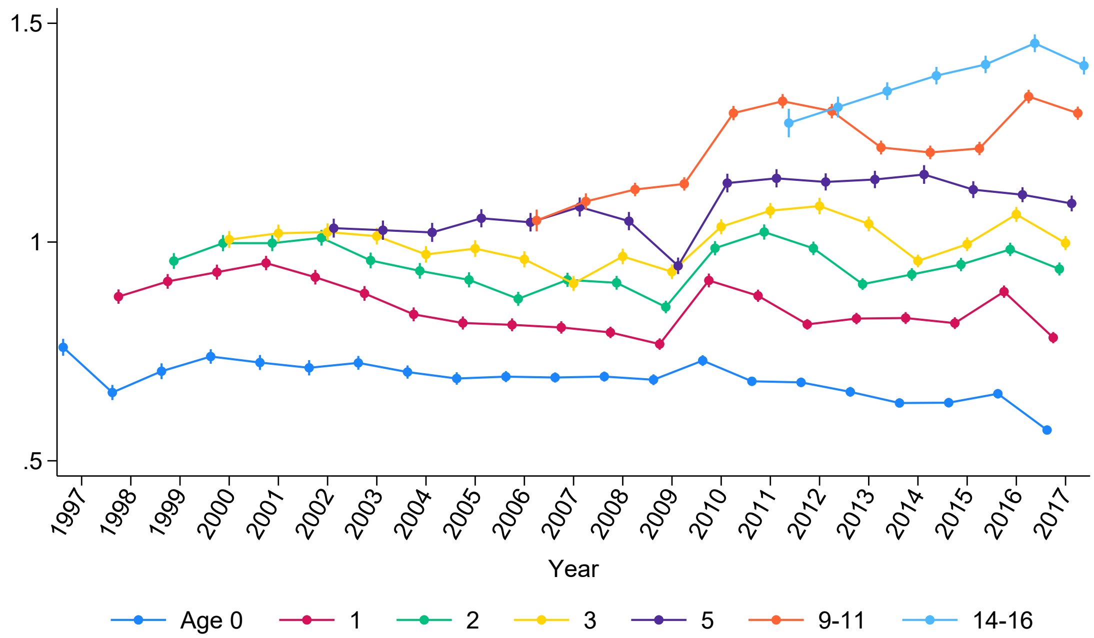
2.2 Changes in firm growth

I illustrate the change in firm growth in two different ways. First, I show patterns of average firm size conditional on age as Karahan, Pugsley and Şahin (2022) and Hopenhayn, Neira and Singhania (2022). An increase in average firm size conditional on age (while size at entry remains constant) implies an acceleration of firm growth. Such an increase in the average firm size of older firms over time is visible to the bare eye. These averages pool across all firms in the economy, so they do not consider industry composition. As a second step, I obtain firm life cycle growth (and report changes therein) using regression analysis controlling for detailed industry and cohort fixed effects.

Figure 1 displays the age-conditional average firm size patterns. 95% confidence intervals are included. For ages zero to three, firm size is relatively stable over time in line with Karahan, Pugsley and Şahin (2022) and Hopenhayn, Neira and Singhania (2022). Already for firms of age five, comparing firm size in 2002 and 2017 shows a slight increase. This increase is even more pronounced for older firms (ages 9-11 and 14-16). The average firm size displays an apparent positive trend growth for these ages. Karahan, Pugsley and Şahin (2022) note that, controlling for industry composition, firms older than age ten display a significant increase in the average firm size over time. This increase is even visible in the Swedish administrative data without controlling for the industry composition. The increase in average firm size for older firms is robust to alternative measures of firm size: Figure 3 in the Appendix shows the same trends for firm sales as a measure of size.

I use a regression framework to quantify the changes in firm life cycle growth over time. More specifically, I run the following regression

Figure 1: Average firm size (log employment) conditional on age



Notes: the figure shows avg. firm size (log employment) conditional on firm age over time. 95% confidence intervals are shown.

$$\ln \text{Size}_{j,t} = \gamma_0 + \sum_{a_f=1}^{20} \gamma_{a_f} \mathbb{1}_{\text{Age}_{j,t}=a_f} + \theta_c + \theta_k + \epsilon_{j,t}, \tag{1}$$

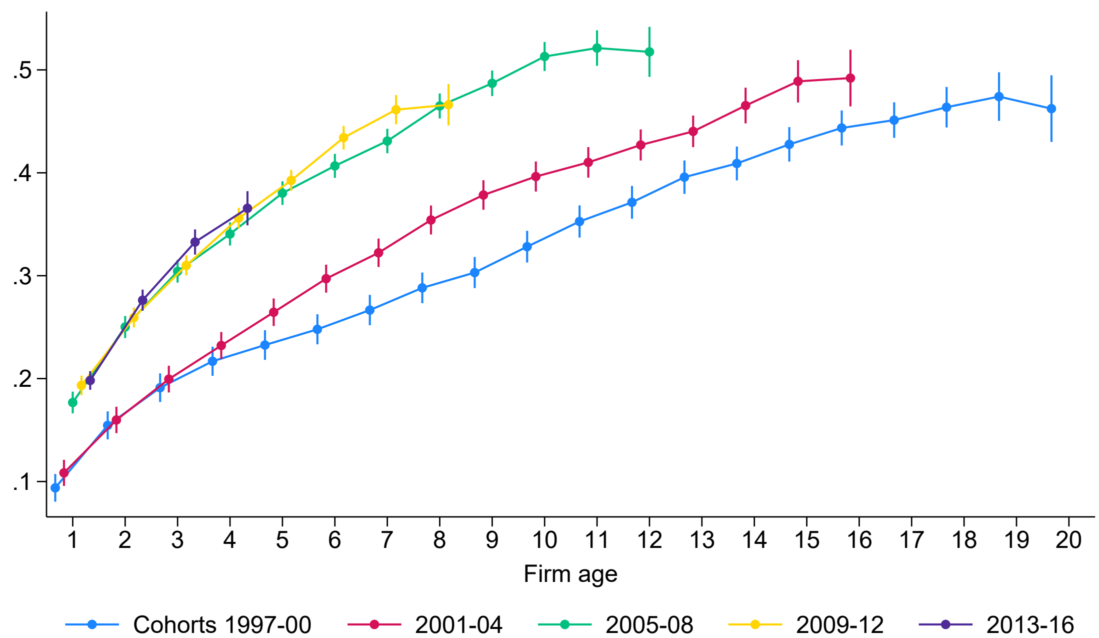
where $\mathbb{1}_{\text{Age}_{j,t}=a_f}$ is an indicator function for firms of age a_f. θ_k is a 5-digit industry fixed effect and as Sterk, Sedláček and Pugsley (2021), I control for cohort fixed effects, θ_c.[10] γ_0 captures the average log firm size at entry (age zero) and γ_{a_f} captures the log difference in average firm size between age a_f and age zero, i.e., γ_1 to γ_{20} provide the non-parametric estimates of life cycle growth. I use both employment and sales as a measure of firm size.

I run the regression for consecutive cohort groups (each group includes four cohorts) to capture changes in the growth profile over time. Figure 2 plots the age coefficients, γ_{a_f}, for employment as the size measure. A clear, gradual upward shift of the life cycle profiles is visible for the later cohorts, and the shift is even monotonic. Every cohort group displays faster employment life cycle growth than the previous cohorts. When measured over the first eight years of the firm, employment growth increased from about 29% (cohorts 1997 to 2000) to about 47% (cohorts 2009 to 2012). Figure 2 further shows that the gap between the early and the more recent cohorts opens up with firm age. This is consistent with the observation in Figure 1 that the average firm size of older firms increases significantly,

[10] The cohort and industry dependence of the other variables is suppressed for clarity.

whereas for younger firms, it is relatively stable.

Figure 2: Employment life cycle growth (by cohort)



Notes: the figure shows cumulative employment growth over the firm's life cycle, measured as the difference between average log employment at age a_f and age zero according to eq. (1). Cohorts are pooled as indicated in the legend. Firm employment is filtered at its 1% tails. The figure includes 95% confidence intervals.

A similar observation holds for sales as the measure of firm size. Figure 4 in the Appendix shows the same patterns for sales growth over the life cycle. Over the first eight years of the firm, sales increased by about 56% for the cohorts 1997 to 2000, whereas sales increased by about 67% for the cohorts 2009 to 2012. The acceleration in sales life cycle growth is smaller than for employment, but a clear upward shift of the life cycle profiles over time is apparent.

3 Model

This section outlines the theory I will use to address the raised research questions.

3.1 Preferences and aggregate economy

The model is formulated in continuous time. The economy consists of a representative household that chooses the path of consumption C_t and wealth A_t to maximize lifetime utility

$$U = \int_0^\infty \exp(-\rho t) \ln C_t dt,$$

subject to the budget constraint $\dot{A}_t = r_t A_t + w_t L_t - C_t$ and a standard no-Ponzi game condition. ρ denotes the discount factor, r_t the interest rate and w_t the real wage. The household supplies one unit of labor inelastically such that $L_t = 1$. The optimality condition (Euler equation) for the household problem reads

$$\frac{\dot{C}_t}{C_t} = r_t - \rho.$$

Aggregate output is produced competitively using a Cobb-Douglas technology over a continuum of different products indexed by i (time subscripts suppressed)

$$Y = \exp\left(\int_0^1 \ln\left[q_i y_i\right] di\right),$$

where y_i and q_i denote the quantity and quality of product i. Output is consumed entirely such that $Y = C$. Expenditure minimization leads to the standard demand function for product y_i

$$y_i = \frac{YP}{p_i}.$$

Here P is defined as the aggregate price index

$$P \equiv \exp\left(\int_0^1 \ln\left[p_i / q_i\right] di\right),$$

which is normalized to 1.

3.2 Production

Firms can produce in every product market i with the following technology

$$y_{ij} = \varphi_j l_{ij},$$

where y_{ij} is the amount of product i produced by firm j, l_{ij} is the amount of labor hired, and φ_j denotes the physical productivity of firm j producing product i. A firm active in multiple markets produces the products using the same productivity, i.e., φ_j varies with j, but not with i. As in Aghion, Bergeaud, Boppart, Klenow and Li (2023), the firm's productivity is fixed over time, which captures the notion that some firms are persistently more efficient at producing than others, e.g., due to a better business plan. For simplicity, firms are either of the high or low productivity type, i.e., $\varphi_j \in \{\varphi^h, \varphi^l\}$ with $\varphi^h/\varphi^l > 1$, which I refer to as high and low-type firms.

Permanent differences in productivity across firms have implications for firm dynamics. The productivity type of the firm affects the equilibrium markup that the firm sets in a product line, as explained below. Therefore, more productive firms that are able to set larger markups have a higher incentive to expand into new product markets and grow faster. Without permanent productivity differences, all firms choose the same creative destruction (firm expansion) rates. In this case, firm size growth is entirely determined by ex-post shocks. Sterk, Sedláček and Pugsley (2021) emphasize that ex-ante heterogeneity rather than ex-post shocks explain differences in firm growth. In addition, differences in firm productivity types introduce reallocation effects on economic aggregates, e.g., the aggregate growth rate, which will be highlighted later. Aghion, Bergeaud, Boppart, Klenow and Li (2023) further show that an increasing spread between high and low-productivity firms is consistent with a reallocation of market shares to low labor share firms as documented in Kehrig and Vincent (2021).

3.3 Static allocation

Taking the distribution of product qualities and the number of firms as exogenous in this section, I characterize static allocations at the product, firm and aggregate levels.

3.3.1 Product level

Firms within a product market i compete in prices (Bertrand competition). In equilibrium, only the firm with the highest quality-adjusted productivity $q_{ij}\varphi_j$ produces product i.

Under Bertrand competition, the leader (the firm with the highest quality-adjusted productivity) engages in limit pricing and sets its price equal to the quality-adjusted marginal costs of the follower (the firm with the second highest quality-adjusted productivity). The leader's price in equilibrium is hence given by

$$p_{ij} = \frac{q_{ij}}{q_{ij'}} \frac{w}{\varphi_{j'}}, \tag{2}$$

where j' indexes the follower in a market. According to eq. (2), the price that the leader sets is increasing in the quality gap between the leader and the follower.

The equilibrium price-cost markup in market i for producer j is defined as the output price over marginal costs, hence

$$\mu_{ij} \equiv \frac{p_{ij}}{w/\varphi_j} = \frac{q_{ij}}{q_{ij'}} \frac{\varphi_j}{\varphi_{j'}}. \tag{3}$$

The leader's markup for product i is increasing in the quality and productivity gap between the leader and the follower.

The price setting of the leader gives rise to the following equilibrium profits for product

i

$$\pi_{ij} = p_{ij}y_{ij} - wl_{ij} = Y\left(1 - \frac{1}{\mu_{ij}}\right),$$

with labor demand for product i given by

$$l_{ij} = \frac{Y}{w}\mu_{ij}^{-1}.$$

Employment in product line i is decreasing in the markup.

3.3.2 Firm level

Summing employment per product over the set of products that firm j is producing, N_j, gives employment at the firm level:

$$l_j = \sum_{i \in N_j} l_{ij} = \frac{Y}{w}\left(\sum_{i \in N_j} \mu_{ij}^{-1}\right).$$

Employment at the firm level is increasing in the number of products the firm produces. The firm's sales are $n_j Y$, which follows from the fact that revenue per line is equalized.

3.3.3 Aggregate level

Summing firm employment over all firms yields the total workforce in production:

$$L_P = \int_j l_j dj = \frac{Y}{w}\int_0^1 \mu_{ij}^{-1} di. \tag{4}$$

An expression for the wage can be found from the markup equation (3). After taking logs and integrating one obtains

$$w = \exp\left(\int_0^1 \ln q_{ij} di\right) \times \exp\left(\int_0^1 \ln \varphi_{j(i)} di\right) \times \exp\left(\int_0^1 \ln \mu_{ij}^{-1} di\right). \tag{5}$$

To find an expression for aggregate output, insert eq. (5) into eq. (4) to obtain

$$Y = Q\Phi\mathcal{M}L_P,$$

where

$$Q = \exp\left(\int_0^1 \ln q_{ij} di\right), \quad \Phi = \exp\left(\int_0^1 \ln \varphi_{j(i)} di\right), \quad \mathcal{M} = \frac{\exp\left(\int_0^1 \ln \mu_{ij}^{-1} di\right)}{\int_0^1 \mu_{ij}^{-1} di}.$$

Aggregate output Y depends on geometric averages of quality Q and productivity Φ across all product lines as well as on the dispersion of markups $\mathcal{M}$ and the total labor force L_P. Aggregate TFP is captured by $Q\Phi\mathcal{M}$. $\mathcal{M}$, the measure of misallocation, is less (or equal) than unity as a geometric mean (numerator) is weakly lower than an arithmetic mean (denominator). Misallocation of production factors that increase the dispersion of markups reduces aggregate TFP as in Peters (2020).

Using again equation (4), monopoly power affects factor prices by reducing labor demand. The aggregate labor income share is given by

$$\Lambda \equiv \frac{wL_P}{Y} = \int_0^1 \mu_{ij}^{-1} di.$$

Aggregate TFP depends on the dispersion of markups. The aggregate labor income share depends on the level of markups.

3.4 Dynamic firm problem

Firms continuously improve the quality of products, q_i, in the economy through two different types of R&D.[11] Internal R&D raises the quality of an item that the firm produces (within-product market R&D), whereas, through creative destruction, the firm improves the quality of a competitor's product. As highlighted below, internal R&D is a source of markup growth that allows for differential firm sales and employment growth. Item quality is improved step-wise such that every time quality is improved (either through creative destruction or through own innovation), quality increases by a factor of λ. As Aghion, Bergeaud, Boppart, Klenow and Li (2023) I assume $\lambda > \varphi^h/\varphi^l$. This assumption guarantees that the firm with the highest quality version in a product line is always the active producer.[12] Denote by λ^{Δ_i} the ratio of product qualities between the active producer and the second best firm (firm with the second highest value of $q_{ij}\varphi_j$) in product line i such that

$$\lambda^{\Delta_i} = \frac{q_{ij}}{q_{ij'}}.$$

Markups determine firm profits in each product line, which depend on the productivity and quality gap to the follower. To infer the firm's current profits, the markups per line are sufficient; however, for the dynamic problem of the firm, one needs to keep track of the firm's productivity type separately. The (expected) productivity gap in a new line depends on the productivity type of the firm. To save on notation, denote by $[\mu_i]$ the set of markups in the

[11] Alternatively, the model could be set up with firms improving the product-line specific productivity and firm types representing systematic differences in firm quality. This alternative formulation is isomorphic to the one presented.

[12] Relaxing this assumption would give room for a race for incumbency between low-productivity entrants facing a high-productivity incumbent from which I abstract.

product lines where the firm is the incumbent producer. Firm profits are given by

$$\pi_t(n, [\mu_i]) = \sum_{k=1}^{n} Y_t \left(1 - \frac{1}{\mu_{kjt}}\right) = \sum_{k=1}^{n} Y_t \left(1 - \frac{1}{\lambda^{\Delta_{kt}}} \frac{1}{\frac{\varphi_{kj}}{\varphi_{kj'}}}\right) \equiv \sum_{k=1}^{n} \pi(\mu_{kt}).$$

Whereas $\pi_t(n, [\mu_i])$ denotes total firm profits, $\pi(\mu_{kt})$ denotes product line specific profits.

Incumbent firms choose the rate of internal R&D, I_i, and the rate of expansion R&D, x_i, for each of their product lines, i. When choosing optimal internal R&D and expansion R&D rates, firms take aggregate output Y_t, the real wage w_t, the share of lines operated by high productivity producers S_t, the interest rate r_t and the rate of creative destruction τ_t as given. Denoting the time derivative by $\dot{V}_t^h()$, the value function of a high type firm indexed by h satisfies the following HJB equation:

$$\begin{aligned}
&r_t V_t^h(n, [\mu_i]) - \dot{V}_t^h(n, [\mu_i]) = \\
&\quad \sum_{k=1}^{n} \underbrace{\pi(\mu_k)}_{\text{Flow profits}} + \sum_{k=1}^{n} \tau_t \underbrace{\Big[V_t^h\left(n-1, [\mu_i]_{i \neq k}\right) - V_t^h(n, [\mu_i])\Big]}_{\text{Creative destruction}} \\
&\quad + \max_{[x_k, I_k]} \Bigg\{ \sum_{k=1}^{n} I_k \underbrace{\Big[V_t^h\left(n, [[\mu_i]_{i \neq k}, \mu_k \times \lambda]\right) - V_t^h(n, [\mu_i])\Big]}_{\text{Internal R\&D}} \\
&\quad + \sum_{k=1}^{n} x_k \underbrace{\Big[S_t V_t^h\left(n+1, [[\mu_i], \lambda]\right) + (1 - S_t) V_t^h\left(n+1, \left[[\mu_i], \lambda \times \varphi^h/\varphi^l\right]\right) - V_t^h(n, [\mu_i])\Big]}_{\text{Expansion R\&D}} \\
&\quad - \underbrace{w_t \Gamma^h\left([x_i, I_i]; n, [\mu_i]\right)}_{\text{R\&D costs}} \Bigg\}.
\end{aligned}$$

As in Peters (2020), the value of a firm consists of flow profits, research costs and three parts related to internal R&D, expansion R&D and creative destruction. At rate τ_t (determined in equilibrium), the firm loses one of its n products, in which case $n-1$ of its products remain. The firm chooses internal R&D rates I_k and expansion R&D rates x_k for each product. If R&D turns out successful, the firm charges a λ times higher markup (internal R&D), or acquires a new product (expansion R&D). When acquiring a new product, the probability of facing a high-type second-best firm is S, in which case the high-type entrant charges a markup of λ. With probability $1 - S$, the second-best firm is of the low type and the high-type entrant charges a markup of $\lambda \times \varphi^h/\varphi^l$. The HJB equation of a low productivity firm is listed in Appendix, section C.1. It follows the same structure. However, the term related to expansion R&D differs since the low productivity firms set a different markup in expectation when entering a new product line.

Type heterogeneity introduces new elements to the value function compared to Peters (2020).

First, the value function (and the resulting firm policies) are specific to the productivity type of the firm. Second, the value function depends on the distribution of firm productivity types across product lines S. Firms build expectations about the likelihood of facing a high or low-type firm when expanding into a new product line. Firms take S_t as given when making optimal decisions; however, they affect it through their expansion R&D efforts in equilibrium.

$\Gamma([x_i, I_i]; n, [\mu_i])$ denote the cost of internal and expansion R&D. For their R&D activities, firms pay a cost of

$$\Gamma^h([x_i, I_i]; n, [\mu_i]) = \sum_{k=1}^{n} c(x_k, I_k; \mu_k) = \sum_{k=1}^{n} \left[\mu_k^{-1} \frac{1}{\psi_I} (I_k)^\zeta + \frac{1}{\psi_x} (x_k)^\zeta \right],$$

with $\zeta > 1$. R&D costs are additively separable to ensure a closed-form solution for the value function along the balanced growth path. Profits within a product line are concave in the markup. Therefore, the incentives for internal R&D decrease with the quality gap that the firm has accumulated. I scale the internal R&D costs by the inverse markup, which renders the product line-specific own-innovation rate independent of the product markup in equilibrium as in Peters (2020). The R&D cost parameters are estimated later by matching firm sales and employment growth.

Firm entry is determined as follows: using a linear production technology, potential entrants produce a flow of marketable ideas ψ_z per unit of labor that improves the quality of a randomly selected product line. Entrants start with a one-step quality gap. I assume that after entering, firms get assigned the high productivity type with probability p^h. Denoting by z_t the equilibrium flow rate of entry, the free entry condition requires that the expected value of firm entry equals the entry costs

$$p^h E[V_t^h(1, \mu_i)] + (1 - p^h) E[V_t^l(1, \mu_i)] = \frac{1}{\psi_z} w_t, \tag{6}$$

where

$$\begin{aligned} E[V_t^h(1, \mu_i)] &= S_t V_t^h(1, \lambda) + (1 - S_t) V_t^h(1, \lambda \times \varphi^h / \varphi^l) \\ E[V_t^l(1, \mu_i)] &= S_t V_t^l(1, \lambda \times \varphi^l / \varphi^h) + (1 - S_t) V_t^l(1, \lambda) \end{aligned}$$

denote the expected value of entering as a high-type or low-type firm. With firms growing at systematically different rates, firm entry is necessary, i.e., $z > 0$ to generate a stable distribution of firm types across product lines.

Labor market clearing requires that production labor L_{Pt} and research labor L_{Rt} add up to one, the aggregate labor endowment:

$$L_{Pt} + L_{Rt} = 1, \tag{7}$$

where research labor is devoted to internal R&D, expansion R&D and entry according to the R&D cost functions.

3.5 Distribution over quality and productivity gaps

In this section, I characterize the two-dimensional distribution of incumbents' quality and productivity gaps across all product lines, $\nu_t\left(\Delta, \frac{\varphi_j}{\varphi_{j'}}\right)$. This two-dimensional distribution characterizes the share of product lines operated by each productivity type, S, and labor demand for production and R&D. S and the labor demand, in return, enter the firms' optimization problem and the labor market clearing condition. From the firm's maximization problem, it will turn out that along the balanced growth path, the internal and expansion R&D rates are time-invariant and do not depend on the quality gap in a product line. Further, as shown below, the optimal internal R&D rates do not depend on the productivity type $I^h = I^\ell = I$. The productivity type-specific expansion R&D rates are denoted by x^h and x^ℓ.

The distribution of quality and productivity gaps is characterized by a set of infinitely many differential equations. These differential equations capture the change in the mass of product lines that are of a specific quality gap, λ^Δ, and productivity gap, $\frac{\varphi_j}{\varphi_{j'}}$ as follows

$$\dot{\nu}_t\left(\Delta, \frac{\varphi_j}{\varphi_{j'}}\right) = I\nu_t\left(\Delta - 1, \frac{\varphi_j}{\varphi_{j'}}\right) - \nu_t\left(\Delta, \frac{\varphi_j}{\varphi_{j'}}\right)(I + \tau_t) \quad \text{for} \quad \Delta \geq 2. \tag{8}$$

For product lines with a unitary quality gap, $\Delta = 1$, the differential equations read:

$$\begin{aligned}
\dot{\nu}_t\left(1, \frac{\varphi^l}{\varphi^h}\right) &= (1 - S_t)x^l S_t + z_t(1 - p^h)S_t - \nu_t\left(1, \frac{\varphi^l}{\varphi^h}\right)(I + \tau_t) \\
\dot{\nu}_t\left(1, \frac{\varphi^l}{\varphi^l}\right) &= (1 - S_t)x^l(1 - S_t) + z_t(1 - p^h)(1 - S_t) - \nu_t\left(1, \frac{\varphi^l}{\varphi^l}\right)(I + \tau_t) \\
\dot{\nu}_t\left(1, \frac{\varphi^h}{\varphi^h}\right) &= S_t x^h S_t + z_t p^h S_t - \nu_t\left(1, \frac{\varphi^h}{\varphi^h}\right)(I + \tau_t) \\
\dot{\nu}_t\left(1, \frac{\varphi^h}{\varphi^l}\right) &= S_t x^h(1 - S_t) + z_t p^h(1 - S_t) - \nu_t\left(1, \frac{\varphi^h}{\varphi^l}\right)(I + \tau_t).
\end{aligned} \tag{9}$$

The law of motion for the mass of product lines with quality gap Δ and a given productivity gap is characterized by the inflow minus the outflow of product mass. Outflows are due to successful internal R&D (quality gap increases from Δ to $\Delta + 1$) and creative destruction (quality gap gets reset from Δ to unity). For $\Delta \geq 2$, inflows into state Δ are due to successful internal R&D in product lines that previously had a quality gap of $\Delta - 1$. For $\Delta = 1$, inflows result from product lines where creative destruction resets previously accumulated quality gaps back to unity.

Summing the measure $\nu_t\left(\Delta, \frac{\varphi_j}{\varphi_{j'}}\right)$ over product lines where the incumbent firm is of the high type defines the market share of high type firms S

$$S_t = \sum_{i=1}^{\infty}\left[\nu_t\left(i, \frac{\varphi^h}{\varphi^h}\right) + \nu_t\left(i, \frac{\varphi^h}{\varphi^l}\right)\right]. \tag{10}$$

From the differential equations in eqs. (8) and (9) it follows that

$$\dot{S}_t = S_t x^h(1 - S_t) - (1 - S_t)x^l S_t + z_t\left(p^h(1 - S_t) - (1 - p^h)S_t\right). \tag{11}$$

Changes in S_t are due to high-productivity firms expanding into markets with low-productivity incumbents (first term), low-productivity firms expanding into markets with high-productivity incumbents (second term), high-productivity entrants replacing low-productivity incumbents and low-productivity entrants replacing high-productivity incumbents (final term).

The rate of creative destruction is defined by

$$\tau_t = S_t x^h + (1 - S_t)x^l + z_t. \tag{12}$$

Note that τ_t is a function of S_t. The rate of creative destruction depends on the distribution of productivity types across product lines.

3.6 Balanced growth path (BGP) characterization

I define a BGP of the economy as follows.

Definition 1. *A balanced growth path (BGP) is a set of allocations* $[x_{it}, I_{it}, \ell_{it}, z_t, S_t, y_{it}, C_t]_{it}$ *and prices* $[r_t, w_t, p_{it}]_{it}$ *such that firms choose* $[x_{it}, I_{it}, p_{it}]$ *optimally, the representative household maximizes utility choosing* $[C_t, y_{it}]_{it}$*, the growth rate of aggregate variables is constant, the free-entry condition holds, all markets clear and* S_t *is consistent with the stationary distribution of quality and productivity gaps.*

Along the BGP, the value function, the distribution of productivity types across product lines, the aggregate labor share, markup, misallocation measure, and growth rate can be characterized in closed form.

Proposition 1. *In the above setup, along a balanced growth path:*

1. *The value function for a firm of productivity type* $d \in \{h, l\}$ *is given by*

$$\begin{aligned} V_t^d(n, [\mu_i]) =& V_{t,P}^d(n) + \sum_{k=1}^{n} V_{t,M}(\mu_k) \\ =& n \frac{1}{(\rho+\tau)} \frac{\zeta - 1}{\psi_x} (x^d)^\zeta w_t + \sum_{k=1}^{n} \frac{\pi(\mu_k) + \frac{\zeta-1}{\psi_I} I^\zeta w_t \mu_k^{-1}}{\rho + \tau}, \end{aligned} \tag{13}$$

where $I \equiv I^h = I^l$ *and* $x^h > x^l$*.*

2. S_{φ^k, φ^p}*, the constant share of product lines where the incumbent firm is of productivity type* k *and the second-best firm of type* p *is*

$$\begin{aligned} S_{\varphi^l,\varphi^h} \equiv \sum_{i=1}^{\infty} \nu\left(i, \frac{\varphi^l}{\varphi^h}\right) =& \frac{(1-S)x^l S + z(1-p^h)S}{\tau} \\ S_{\varphi^l,\varphi^l} \equiv \sum_{i=1}^{\infty} \nu\left(i, \frac{\varphi^l}{\varphi^l}\right) =& \frac{(1-S)x^l(1-S) + z(1-p^h)(1-S)}{\tau} \\ S_{\varphi^h,\varphi^h} \equiv \sum_{i=1}^{\infty} \nu\left(i, \frac{\varphi^h}{\varphi^h}\right) =& \frac{Sx^h S + zp^h S}{\tau} \\ S_{\varphi^h,\varphi^l} \equiv \sum_{i=1}^{\infty} \nu\left(i, \frac{\varphi^h}{\varphi^l}\right) =& \frac{Sx^h(1-S) + zp^h(1-S)}{\tau}, \end{aligned}$$

which implicitly defines the share of product lines operated by the high-productivity type

$$S = S_{\varphi^h,\varphi^h} + S_{\varphi^h,\varphi^l} = \frac{Sx^h + zp^h}{\tau}. \tag{14}$$

3. *The growth rate of aggregate variables is given by*

$$g = \frac{\dot{Q}_t}{Q_t} = \Bigg(\underbrace{I}_{\textit{Incumbent internal R\&D}} + \underbrace{Sx^h + (1-S)x^l}_{\textit{Incumbent expansion R\&D}} + \underbrace{z}_{\textit{Entry}} \Bigg) \times \ln(\lambda). \tag{15}$$

Proof. The Appendix, sections C.1, C.3 and C.5, contains the proofs. □

The value function is additive across products. The first part of the value function that represents the option value of expanding into new product markets scales linearly in the number of products and is productivity-type specific. The second part consists of flow profits and the option value to increase markups further. Both terms are scaled by the sum of the discount factor and the rate of creative destruction, the rate at which products get replaced. I show in Appendix C.1 that type heterogeneity in firm productivity introduces type heterogeneity in the optimal expansion R&D rate. For each productivity type, the expected value of adding a new product line, $E[V_t^h(1, \mu_i)]$ or $E[V_t^l(1, \mu_i)]$, equals the marginal cost of expansion R&D at the optimum. The ability to set higher markups increases the expected value of a new product line and incentivizes the more productive firms to expand into new product lines at a higher rate, i.e., $x^h > x^l$.

The share of products where the incumbent firm is of type k and the second-best firm of type p is constant. This share equals the fraction of creatively destroyed products that start in a product line where the incumbent is of type k and the second best firm of type p at each instant in time.

Economic growth in this model results from R&D at the product level. This occurs through successful internal R&D, expansion R&D, or firm entry. The growth rate is equal to the aggregate arrival rate of innovation times the step size of innovation, $\ln(\lambda)$. Since firms with different productivities innovate at different rates, the aggregate growth rate depends on the distribution of productivity types across product lines, S.

Proposition 2. *Let I and τ denote the rates of internal R&D and creative destruction and* $\theta = \frac{\ln(1+\tau/I)}{\ln(\lambda)}$.

1. *The aggregate labor income share $\Lambda = \frac{wL_P}{Y}$ is given by*

$$\Lambda = \frac{\theta}{\theta+1} \sum_{k\in\{h,l\}} \sum_{n\in\{h,l\}} \frac{1}{\varphi_k/\varphi_n} S_{\varphi_k,\varphi_n}.$$

2. *The misallocation measure $\mathcal{M}$ is given by*

$$\mathcal{M} = \frac{e^{\sum_{k\in\{h,l\}} \sum_{n\in\{h,l\}} \left[S_{\varphi_k,\varphi_n} \left(\ln\left(\frac{1}{\frac{\varphi_k}{\varphi_n}} \right) - \frac{1}{\theta} \right) \right]}}{\Lambda}.$$

3. *The aggregate markup $E[\mu] = \int_0^1 \mu_i di$ is given by*

$$E[\mu] = \frac{\theta}{\theta-1} \sum_{k\in\{h,l\}} \sum_{n\in\{h,l\}} \frac{\varphi_k}{\varphi_n} \times S_{\varphi_k,\varphi_n}.$$

Proof. The Appendix, sections C.3 and C.4 contains the proofs. □

The stationary two-dimensional distribution of productivity and quality gaps characterizes (1) the aggregate labor income share Λ, (2) the TFP misallocation measure $\mathcal{M}$ that captures the static loss in output that arises from markup dispersion, and (3) the average markup in the economy. All three have in common that they depend on the speed of creative destruction relative to internal R&D, θ, the size of productivity gaps, φ_k/φ_n, and the distribution of productivity gaps across product lines, S_{φ_k,φ_n}. Without productivity differences across firms, $\varphi_k/\varphi_n = 1$, these measures boil down to the aggregate labor income share, misallocation measure, and aggregate markup in Peters (2020).

To find the BGP solution of the model, I reduce the optimality conditions of the firm (derived in Appendix C.1), the free entry condition, eq. (6), the labor market clearing condition, eq. (7), and the system of differential equations characterizing the distribution of productivity and quality gaps to seven equations in seven unknowns, which is solved using a root-finder. Appendix C.2 contains the details.

3.6.1 Discussion of the stationary firm-type distribution

In equilibrium, high-productivity firms expand into new product markets faster than low-productivity firms. Firm entry prevents high-productivity firms from capturing all product lines. To see this note that in steady state $\dot{S} = 0$ such that eq. (11) turns into

$$z\left(S - p^h\right) = S(1 - S)(x^h - x^l). \tag{16}$$

It is worthwhile to discuss eq. (16) since it provides intuition on the relationship between expansion rates and firm entry. Suppose high-productivity incumbents expand at higher rates than low-productivity firms ($x^h > x^l$). In that case, for the share of high-productivity incumbents to be constant along the BGP, S needs to be greater than p^h, the share of entrants of the high-productivity type. In other words, the share of high-productivity firms among entrants must be lower than the share of product lines operated by high-productivity firms in the economy. In this case, "sufficient" entry by low-productivity firms balances the relatively higher expansion rate by existing high-productivity incumbents, and the share of lines operated by high-productivity firms remains constant. Eq. (16) highlights the role of firm entry. Without entry ($z = 0$), higher expansion rates by high-productivity incumbents would result in those firms eventually overtaking all product lines. Given $x^h > x^l$ and $0 < S < 1$, for eq. (16) to hold, firm entry, z, needs to be positive.

In the special case where all entrants are of the low productivity type ($p^h = 0$), eq. (16) can be written as

$$Sx^h(1 - S) - (1 - S)x^l S = zS.$$

Entry by low-productivity firms that replace high-productivity incumbents (zS) makes up precisely for the lost market share of incumbent low-productivity firms, $Sx^h(1 - S) - (1 - S)x^l S$, such that the aggregate share of high-productivity firms remains constant.

3.7 Firm dynamics

In this section, I derive how firm markups, sales and survival evolve with age and characterize the firm size distribution. The results of this section will again be used when estimating the model.

3.7.1 Markup dynamics

Firm markups are defined as $\mu_f = \frac{py_f}{wl_f} = \left(\frac{1}{n_f}\sum_{i\in N_f}\mu_{if}^{-1}\right)^{-1}$. The firm markup is the harmonic mean of its product markups. In Appendix C.6, I show that for a high-productivity type firm, the expected log markup conditional on firm age a_f is

$$E\left[\ln\mu_f|\text{firm age}=a_f,\varphi^h\right] = \underbrace{\ln\lambda\times\left(1+I\times E[a_P^h|a_f]\right)}_{\text{Quality improvements}} + \underbrace{(1-S)\times\ln\left(\frac{\varphi^h}{\varphi^l}\right)}_{\text{Productivity advantage}}, \tag{17}$$

where $E[a_P^h|a_f]$ denotes the average product age of a high process efficiency firm conditional on firm age, which is given by

$$\begin{aligned}
E[a_P^h|a_f] &= \frac{1}{x^h}\left(\frac{\frac{1}{\tau}\left(1-e^{-\tau a_f}\right)}{\frac{1}{x^h+\tau}\left(1-e^{-(x^h+\tau)a_f}\right)}-1\right)\left(1-\phi^h(a_f)\right)+a_f\phi^h(a_f)\\
\phi^h(a) &= e^{-x^h a}\frac{1}{\gamma^h(a)}\ln\left(\frac{1}{1-\gamma^h(a)}\right)\\
\gamma^h(a) &= \frac{x^h\left(1-e^{-(\tau-x^h)a}\right)}{\tau-x^h e^{-(\tau-x^h)a}},
\end{aligned}$$

Expected markups conditional on age consist of two terms. The first term in eq. (17) reflects how average quality gaps across the firm's products evolve with firm age. It captures the effects that as the firm ages, it improves the quality of its continuing items, the firm loses product lines for which it has accumulated quality gaps and acquires new products with initially low-quality gaps. This term is akin to the markup expression in Peters (2020). In Peters (2020), this term holds for all firms, whereas in this model, this term is specific to the productivity type of the firm as internal R&D and expansion R&D rates vary by firm type. Permanent differences in the productivity type across firms affect not only expected markup growth (captured by the first term) but also introduce a level effect captured by the second term in eq. (17). The intuition behind the second term is that if a high process efficiency firm faces a low process efficiency second best producer in a given line, it can charge a φ^h/φ^l higher markup, which occurs in expectation in $1-S$ of the firm's product lines.

The expected markup conditional on firm age for a low process efficiency firm is

$$E\left[\ln\mu_f|\text{firm age}=a_f,\varphi^l\right] = \underbrace{\ln\lambda\times\left(1+I\times E[a_P^l|a_f]\right)}_{\text{Quality improvements}} + \underbrace{S\times\ln\left(\frac{\varphi^l}{\varphi^h}\right)}_{\text{Productivity disadvantage}}. \tag{18}$$

The first term capturing the quality advantage is equivalent to the first term in eq. (17) for the high type. $E[a_P^l|a_f]$ follows the same expression as $E[a_P^h|a_f]$ with h replaced by l. The second term in eq. (18) differs from eq. (17) as low productivity firms face a productivity disadvantage in a share S of their product lines, in which they face a high productivity second best producer. Since $\varphi^l < \varphi^h$, this term is negative.

3.7.2 Sales dynamics

Firms lose products according to the same stochastic process as in Klette and Kortum (2004) that, in their model, results in a skewed sales distribution, a decreasing variance of sales growth in size, a declining exit probability in age and size and firm size growth being independent of size. In this model, the rate at which firms add products is heterogeneous across firms. Firms of the high productivity type add new products at a faster rate than low-type firms, which in turn affects firm sales and survival. Therefore, the properties related to firm size and survival in Klette and Kortum (2004) hold for each firm type. In particular, conditional on the type, firm size and (expected) growth are unrelated, i.e., Gibrat's law holds conditionally as in Lentz and Mortensen (2008).

Firm sales are proportional to the number of products a firm produces. Since more productive firms add new products faster, sales growth is specific to the productivity type. Expected log sales growth for a firm with process efficiency $\varphi^j, j \in \{h, l\}$ between age zero and age a_f is $E\left[\ln nY|a_f, \varphi^j\right] - E\left[\ln nY|0, \varphi^j\right]$, where n is the number of products a firm is producing. Firm sales growth stems from aggregate growth and from the firm gaining and losing products as it ages

$$E\left[\ln nY|a_f, \varphi^j\right] - E\left[\ln nY|0, \varphi^j\right] = \underbrace{g \times a_f}_{\text{Aggregate growth}} + \underbrace{E\left[\ln n|a_f, \varphi^j\right]}_{\text{Firm's product growth}} .$$

To derive $E\left[\ln n|a_f, \varphi^j\right]$ note that the probability of a high process efficiency firm producing n products at age a conditional on survival is $(1-\gamma^j(a))\left(\gamma^j(a)\right)^{n-1}$, where $\gamma^j(a) = x^j \frac{1-e^{-(\tau - x^j)a}}{\tau - x^j e^{-(\tau - x^j)a}}$. Therefore sales growth is given by

$$E\left[\ln nY|a_f, \varphi^j\right] - E\left[\ln nY|0, \varphi^j\right] = \underbrace{g \times a_f}_{\text{Aggregate growth}} + \underbrace{\left(1-\gamma^j(a_f)\right)\sum_{n=1}^{\infty} \ln n \times \left(\gamma^j(a_f)\right)^{n-1}}_{\text{Firm's product growth}} . \tag{19}$$

Relative sales growth of the firm is equal to the firm's product growth.

3.7.3 Employment dynamics

Employment of the firm can be decomposed into the number of products that the firm produces and its markup as in Peters (2020). In my model, product and markup dynamics depend on the productivity type of the firm such that

$$E[\ln l_f|\text{firm age} = a_f, \varphi^j] =$$
$$E\left[\ln\left(\frac{nY}{w\mu_f}\right)|a_f, \varphi^j\right] = \ln\left(\frac{Y}{w}\right) + E\left[\ln n|a_f, \varphi^j\right] - E\left[\ln \mu_f|a_f, \varphi^j\right],$$

where $\varphi^j \in \{\varphi^h, \varphi^l\}$. Employment growth then simply is

$$E[\ln l_f|a_f, \varphi^j] - E[\ln l_f|0, \varphi^j] = \underbrace{E\left[\ln n|a_f, \varphi^j\right]}_{\text{Firm's product growth}} - \underbrace{\left(E\left[\ln \mu_f|a_f, \varphi^j\right] - E\left[\ln \mu_f|0, \varphi^j\right]\right)}_{\text{Firm's markup growth}}. \tag{20}$$

$E\left[\ln \mu_f|a_f, \varphi^j\right] - E\left[\ln \mu_f|0, \varphi^j\right]$ and $E\left[\ln n|a_f, \varphi^j\right]$ are derived in eq. (17), (18) and (19).

3.7.4 Firm survival and pooled life cycle growth

Firm size dynamics determine firm survival since firms that lose their final product become inactive. Firms that grow faster in size have a higher probability of survival. Hence, firm survival is productivity-type specific. The share of high and low productivity type firms surviving until age a is given by

$$\chi^h(a_f) = 1 - \tau\frac{1 - e^{-(\tau - x^h)a_f}}{\tau - x^h e^{-(\tau - x^h)a_f}} \tag{21}$$
$$\chi^l(a_f) = 1 - \tau\frac{1 - e^{-(\tau - x^l)a_f}}{\tau - x^l e^{-(\tau - x^l)a_f}}. \tag{22}$$

The firm survival function can be used to compute firm markup, sales, and employment growth for the pooled sample of firms. In particular, the share of high-productivity firms among firms at age a_f is given by

$$s^h(a_f) = \frac{p^h\chi^h(a_f)}{p^h\chi^h(a_f) + (1 - p^h)\chi^l(a_f)}$$

The share corresponds to the mass of high-type firms at entry multiplied by the probability of surviving until age a_f relative to the total mass of firms surviving until age a_f. For the example of employment growth, pooled life cycle growth at age a_f is then given by

$$s^h(a_f)\left(E[\ln l_f|a_f, \varphi^h] - E[\ln l_f|0, \varphi^h]\right) + \left(1 - s^h(a_f)\right)\left(E[\ln l_f|a_f, \varphi^l] - E[\ln l_f|0, \varphi^l]\right) \tag{23}$$

The productivity specific employment growth is defined in eq. (20). When estimating the model, I match observed employment growth in the data by pooling employment growth in the model over both productivity types as in eq. (23). Pooled sales growth is computed similarly.

3.7.5 Firm size distribution

The mass of high and low process efficiency firms with $n \geq 2$ products follows the differential equations

$$\begin{aligned}\dot{M}_t^h(n) &= (n-1)x^h M_t^h(n-1) + (n+1)\tau M_t^h(n+1) - n(x^h+\tau)M_t^h(n) \\ \dot{M}_t^l(n) &= (n-1)x^l M_t^l(n-1) + (n+1)\tau M_t^l(n+1) - n(x^l+\tau)M_t^l(n), \end{aligned} \tag{24}$$

whereas the mass of firms with one product evolves according to

$$\begin{aligned}\dot{M}_t^h(1) &= zp^h + 2\tau M_t^h(2) - (x^h+\tau)M_t^h(1) \\ \dot{M}_t^l(1) &= z(1-p^h) + 2\tau M_t^l(2) - (x^l+\tau)M_t^l(1). \end{aligned} \tag{25}$$

The mass of firms with n products increases through firms with $n-1$ products expanding to size n at rate x^h or x^l per product or through firms with $n+1$ products losing one product at the rate of aggregate creative destruction. The mass of firms with n products decreases through firms with n products either gaining or losing a product through expansion or creative destruction. The mass of firms with one product additionally increases through firm entry. The change in the total mass of firms at any point in time is $z - \tau(M_t^h(1) + M_t^l(1))$.

Proposition 3. *The stationary firm size distribution along the BGP is characterized as follows.*

1. *The mass of high and low productivity firms with n products is*

$$\begin{aligned} M^h(n) &= \frac{(x^h)^{n-1} zp^h}{n\tau^n} = \frac{zp^h}{x^h}\frac{1}{n}\left(\frac{x^h}{\tau}\right)^n \\ M^l(n) &= \frac{(x^l)^{n-1} z(1-p^h)}{n\tau^n} = \frac{z(1-p^h)}{x^l}\frac{1}{n}\left(\frac{x^l}{\tau}\right)^n. \end{aligned}$$

2. *The total mass of firms with n products is*

$$M(n) = M^h(n) + M^l(n) = \frac{(x^h)^{n-1} zp^h + (x^l)^{n-1} z(1-p^h)}{n\tau^n}.$$

3. The mass of firms of each productivity type is

$$M^h = \sum_{n=1}^{\infty} M^h(n) = \frac{zp^h}{x^h} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{x^h}{\tau}\right)^n = \frac{zp^h}{x^h} \ln\left(\frac{\tau}{\tau - x^h}\right)$$
$$M^l = \sum_{n=1}^{\infty} M^l(n) = \frac{z(1-p^h)}{x^l} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{x^l}{\tau}\right)^n = \frac{z(1-p^h)}{x^l} \ln\left(\frac{\tau}{\tau - x^l}\right)$$

4. The total mass of firms is

$$M = M^h + M^l.$$

Proof. These results follow from setting the time derivatives in equations (24) and (25) equal to zero and solving the system of equations. □

For each firm type, the share of firms with n products, $M^h(n)/M^h$ and $M^l(n)/M^l$, follows the pdf of a logarithmic distribution with parameter x^h/τ and x^l/τ as in Lentz and Mortensen (2008). The firm size distribution is highly skewed to the right.

Since there is a continuum of mass one of products and each product is mapped to one firm $\sum_{i=1}^{\infty} M(n) \times n = 1$. Further, the mass of high process efficiency firms producing n products is related to the previously defined share of product lines operated by high process efficiency firms, S, as follows

$$S = \sum_{n=1}^{\infty} M^h(n) \times n = \frac{zp^h}{\tau - x^h}.$$

From the firm size distribution, I obtain the share of high process efficiency firms

$$S_{M^h} = \frac{M^h}{M},$$

and the entry rate

$$\text{Entry rate} = \frac{z}{M}. \tag{26}$$

4 Comparative statics across balanced growth paths

In this section, I use the model to study the drivers behind the changes in sales and employment life cycle growth, measure compositional changes, quantify the implications for economic aggregates, and decompose the fall in economic growth. To this extent, I estimate model parameters twice. The initial BGP captures firm life cycle growth and aggregate economic conditions during the end of the 1990s. I then re-estimate model parameters to explain the changes in firm life cycle growth vis-a-vis the latest cohorts in the data.

4.1 Initial BGP

There are, in total, eight parameters in the model. The internal R&D efficiency ψ_I, the expansion efficiency ψ_x, the innovation cost curvature ζ, the entry efficiency ψ_z, the step size of innovation λ, the productivity differential φ^h/φ^ℓ, the probability of being assigned the high productivity type conditional on entry p^h, and the discount rate ρ. Two parameters are set exogenously, and the remaining parameters are estimated. I follow Acemoglu, Akcigit, Alp, Bloom and Kerr (2018) and Peters (2020) that set ζ equal to two based on evidence from the microeconometric innovation literature (Blundell, Griffith and Windmeijer, 2002; Hall and Ziedonis, 2001). The discount rate ρ is set to 0.02, resulting in an annual discount factor of roughly 0.97%.

The remaining six parameters are estimated, targeting moments of firm life cycle growth as well as cross-sectional firm heterogeneity and economic aggregates. In particular, I target firms' sales and employment growth, cross-sectional dispersion in inverse labor shares across firms at age zero, and the firm entry rate, TFP growth, and markups at the aggregate level. Despite all parameters being identified jointly in the estimation, there is a tight mapping between parameters and targets.

Sales and employment growth disciplines the firms' R&D cost parameters. Successful expansion R&D translates into sales growth in the model. Therefore, matching sales growth disciplines the expansion R&D cost ψ_x of firms. The internal R&D cost parameter ψ_I governs the markup growth of firms. Since markup growth drives a wedge between sales and employment growth in the model, targeting employment and sales growth jointly disciplines markup growth and, hence, the internal R&D cost. The advantage of targeting employment instead of markup growth is that the firm's employment is directly observable in the data. I target sales and employment growth over the first eight years of the firm. This period is long enough to average out transitory shocks during the firm's early years and still allows for estimating separate balanced growth paths (one for the early cohorts and one for the latest cohorts) over the data coverage period of 1997 to 2017. The model matches firm growth at any age well, so the specific age target is not restrictive. In the model, sales and employment growth are specific to the productivity type of the firm. In the data, the productivity type is unobserved. I match observed sales and employment growth by pooling firms in the model over all productivity types as in eq. (23). The alternative is to identify the productivity type of a firm in the data and to measure type-specific sales and employment growth separately. I match pooled firm growth to avoid classifying firms incorrectly into productivity types that would affect the parameter estimates and the firm composition.[13] Therefore, the composition of productivity types in the model (an endogenous outcome of the estimation) is such that pooled life cycle growth matches observed growth in the data. For the initial BGP, I target sales and employment growth of the cohorts 1997 to 2000. For these cohorts,

[13]Matching type-specific growth further implies that the productivity cutoff is set exogenously.

sales grow by 55.9% and employment by 28.8% over the first eight years of the firm.

I target the firm entry rate to parametrize the entry efficiency of firms ψ_z. Eq. (26) defines the entry rate in the model. I compute the entry rate in the data as the share of firms equal to or less than one year of age. This results in an average entry rate of 14.3% over the period 1997-2005, in line with Engbom (2023).

Aggregate productivity (TFP) growth in the data further disciplines the step-size improvements of innovation λ: the aggregate growth rate in the model in eq. (15) directly depends on λ. I obtain aggregate productivity growth for the Swedish economy from Federal Reserve Economic Data (FRED) in labor augmenting terms.[14] After suffering a financial crisis in the early 90s, Sweden's economy featured strong growth towards the end of the century. During 1997–2005, aggregate productivity grew by 3.02% per year.

To help pin down the productivity differential φ^h/φ^ℓ, I target the aggregate markup. The aggregate markup is a weighted average of product markups that, in return, depend on φ^h/φ^ℓ. Sandström (2020) and De Loecker and Eeckhout (2018) report sales-weighted markups for the Swedish economy. Sandström (2020) computes the markup in Swedish registry data focusing on firms with at least ten employees, whereas De Loecker and Eeckhout (2018) focus mainly on publicly listed firms. I target the average of both reported aggregate markups, resulting in a conservative estimate of 7.5%. Lastly, I target the standard deviation of log (inverse) labor shares across entering firms. Given φ^h/φ^ℓ, the dispersion of log labor shares at entry depends on the share of product lines operated by high-type firms (an endogenous equilibrium object) and the share of high-type firms among entrants (the parameter p^h). The dispersion of inverse labor shares across firms, hence, disciplines p^h. The standard deviation of log (inverse) labor shares of entering firms averaged over 1997-2005 equals 0.053.[15]

The estimation follows a two-step approach. In the first (global) step, the algorithm computes the sum of squared percentage deviations from the targeted moments for a large Sobol sequence of parameter values (vectors). In the second (local) step, I take the best candidates from the first step and perform a local search using the Nelder-Mead algorithm, minimizing the distance from the targets again.[16] All targets receive equal weights. The best parameter vectors from the second estimation step converge to the same parameter values.

Table 2 shows the estimation results. The model replicates all targeted moments well. The estimated parameters can be interpreted as follows: successful innovation increases product quality by 13.6%. High and low-type firms' productivity differs by 9.1%, and 68.3% of firms

[14] FRED series RTFPNASEA632NRUG. The labor share is obtained from FRED, series LABSHPSEA156NRUG, averaged over 1997–2005

[15] For firms with a low wage bill, inverse labor shares explode. Therefore, I focus on firms with a sales-to-wage bill ratio between one and three (model implied markups between 0% and 200%). Further, sales relative to the wage bill in the data may vary for reasons outside the model. I bin firms into equally sized groups based on their capital and intermediate inputs and compute the dispersion of log inverse labor shares across firms within these groups.

[16] In the global step, I evaluate 100,000 parameter vectors from the Sobol sequence and run the local search for the 20 best candidates from the global step.

Table 2: Initial BGP. Moments and parameters

	Data	Model
Moments		
Sales growth by age 8 (cohorts 1997–2000)	55.9%	55.8%
Employment growth by age 8 (cohorts 1997–2000)	28.8%	28.8%
Cross-sectional SD of log labor shares across entrants (1997–2005)	0.053	0.053
Agg. productivity growth g (1997–2005; FRED)	3.02%	3.02%
Entry rate (1997–2005)	14.3%	14.3%
Agg. markup μ (Sandström, 2020; De Loecker and Eeckhout, 2018)	7.5%	7.5%
Parameters		
ψ_I *Internal R&D efficiency*		0.144
ψ_x *Expansion R&D efficiency*		0.282
ψ_z *Entry R&D efficiency*		1.483
λ *Step size of innovation*		1.136
φ^h/φ^ℓ *Productivity gap*		1.091
p^h *Share of high type among entrants*		0.683
Set exogenously		
ρ *Discount rate*		0.02
ζ *R&D cost curvature*		2

Notes: except for g and μ, the moments are computed using Swedish registry data. Aggregate productivity (TFP) growth is obtained from Federal Reserve Economic Data (FRED), series RTFPNASEA632NRUG, in labor augmenting terms (the labor share is obtained from FRED, series LABSHPSEA156NRUG, averaged over the same period 1997–2005).

enter the economy as high-type firms.

Along the BGP, high-productivity type firms make up 74% of the stationary, cross-sectional firm type distribution. This number is larger than their share at entry ($p^h = 0.683$) due to high-type firms expanding faster into new product markets after entry than low-type firms, i.e., $x^h - x^\ell = 0.075$. This is reflected in their life cycle growth: over the first eight years of the firm, sales grow by 63% for high-type firms relative to 37% for low-type firms. Weighted by their survival probabilities as in eq. (23), this results in pooled sales growth of 55.8%. Sterk, Sedláček and Pugsley (2021) show that ex-ante heterogeneity rather than ex-post realized shocks explain differences in firm growth rates. Type heterogeneity in firms' growth rates, generated through permanent productivity differences across firms as in this model, is consistent with their findings. I find suggestive evidence that firm productivity relates positively to size growth in the data, see section 4.4.2.

4.2 Ending BGP

I replicate the observed acceleration in firm sales and employment growth in the data vis-a-vis the initial BGP in the model. To match the changes in the two moments, I re-estimate two parameters, particularly the internal R&D efficiency ψ_I and the entry R&D efficiency ψ_z. Those two parameters successfully replicate the changes in firm sales and employment growth because rising entry costs allows incumbent firms to expand into new product markets, thereby simultaneously increasing firm sales and employment. In contrast, the rising internal R&D costs increase firm employment growth *relative* to sales growth. Internal R&D costs govern firms' markup growth, which drives a wedge between sales and employment growth. Slower markup growth ceteris paribus increases the employment growth of firms.[17]

Table 3: Ending BGP. Moments and parameters

	Data	Model	Change
Moments			
Sales growth by age 8 (cohorts 2009–2012)	67.4%	67.4%	+11.5pp
Employment growth by age 8 (cohorts 2009–2012)	46.6%	46.6%	+17.8pp
Parameters			
ψ_I *Internal R&D efficiency*		0.07	-51%
ψ_z *Entry R&D efficiency*		1.157	-22%

Notes: the column Change compares moments and parameters of the ending and initial BGP.

Table 3 shows the changes in the targeted moments and the estimated parameters. For the cohorts 2009 to 2012, sales grow by 67.4% over the first eight years (an increase of 11.5pp relative to the cohorts 1997 to 2000) and employment grows by 46.6% (an increase of 17.8pp). The model matches these changes by lowering the internal R&D efficiency by 51% and the entry R&D efficiency by 22%. The estimated fall in the R&D efficiency relates to Bloom, Jones, Van Reenen and Webb (2020), who find that research productivity has declined over time. The story of ideas getting harder to find is consistent with falling R&D efficiency at the product level, as highlighted in the above estimation.[18] The estimated fall in the entry R&D efficiency is further in line with Davis (2017) and Gutiérrez and Philippon (2018), who emphasize rising barriers to entry behind recent macroeconomic trends.

How do the estimated rise in internal R&D and entry R&D costs affect firm size growth? Table 4 shows the effect of each parameter change on firm sales and employment growth in isolation relative to the initial BGP.[19] The acceleration in firm size growth is mainly due to rising entry costs, which increase sales and employment growth by age eight by 12.94pp

[17] The sign of the parameter change is not restricted in the estimation.

[18] A decline in the expansion R&D efficiency would also result in declining research productivity. However, in the estimated model, declining expansion R&D efficiency counterfactually leads to falling concentration, rising entry, and a slowdown in both sales and employment life cycle growth. In contrast, declining internal R&D efficiency results in rising concentration, falling entry, and an acceleration in employment relative to sales life cycle growth, as I document in Swedish data. Declining internal R&D efficiency is further consistent with increasing within-firm labor shares that Autor, Dorn, Katz, Patterson and Van Reenen (2020) find for most U.S. sectors.

[19] I report the effect relative to the initial BGP for all exercises to ease the comparison with section 4.4.1, where I estimate an alternative ending BGP.

Table 4: Contributions to changes in firm size growth

	Initial BGP	$\psi_I \downarrow$	$\psi_z \downarrow$	$\psi_I \downarrow, \psi_z \downarrow$
Sales growth by age 8	55.8%	-1.78pp	+12.94pp	+11.5pp
Employment growth by age 8	28.8%	+2.23pp	+14.80pp	+17.8pp

Notes: the table shows the change in firm sales and employment growth relative to the initial BGP in percentage points. $\psi_I \downarrow$ denotes the 51% fall in the internal R&D efficiency and $\psi_z \downarrow$ the 22% fall in the entry efficiency.

and 14.8pp, respectively. The acceleration in employment growth relative to sales growth is mainly due to rising internal R&D costs that lower sales growth by 1.78pp and increase employment growth by 2.23pp. The rise in internal R&D costs slows markup growth, thereby raising employment relative to sales growth.

Are the changes in pooled life cycle growth at a given firm age driven by changes in the growth of the average firm or by selection of firm types? The answer depends on the age in question: in both BGPs, at age zero, a (constant) share p^h of firms are of the high type, and for a high enough age, the share of high-type firms equals unity because $x^h > x^l$. Therefore, firm-type selection plays a small role in explaining changes in pooled life cycle growth at low or high ages. Decomposing changes in pooled employment growth in eq. (23) using standard shift-share techniques shows that, at age eight, 17pp of the total 17.8pp increase is driven by changes in employment growth of the average firm, holding the composition of firm types constant. However, the selection of firm types, unconditional on age, is substantial. The share of high-type firms in the cross-section of firms increases by 12pp across the BGPs.

What are the implications of the R&D cost changes for the aggregate economy? Changes in the aggregate economy have not been targeted for the estimation of the ending BGP. Changes in firm sales and employment growth discipline the change in the R&D and entry costs. Nevertheless, the implied changes for economic aggregates align with recent macroeconomic trends: the aggregate growth rate declines by 0.6pp, the firm entry rate drops by 8pp, and concentration rises. Aggregate TFP growth, measured from 2010 to 2015, declined by about 1pp relative to 1997–2005 in the data. Further, Engbom (2023) documents a fall in the entry rate by about 10pp from the early 1990s to the mid 2010s in the Swedish economy. In the Appendix, section B.2, I further show that sales concentration within industries increased over the same period. In the model, high-type firms gain market shares at the expense of low-type firms. The share of product lines (or total sales) operated by high-type firms, S, increases by 17pp. The estimated parameter change is consistent with a reallocation of sales shares to firms with relatively low labor shares, as documented by Kehrig and Vincent (2021) for U.S. manufacturing firms. Similarly, De Loecker, Eeckhout and Unger (2020) and Baqaee and Farhi (2020) document a reallocation of sales shares to firms with a high sales to cost-of-goods-sold ratio in Compustat data.

4.3 Incumbent innovation, reallocation, entry and growth

What do the changes in firm growth imply about the fall in economic growth? In other words, what can we learn from the changes in firm growth about the contribution to the fall in economic growth by incumbent firms and entrants? The aggregate growth rate of the economy along the BGP naturally lends itself to a decomposition of growth into its sources. The aggregate growth rate defined in eq. (15), in a slightly rewritten formulation, reads

$$g = Sg^h + (1-S)g^\ell + g^z,$$

where $g^h \equiv (I + x^h)\ln(\lambda)$, $g^\ell \equiv (I + x^\ell)\ln(\lambda)$ and $g^z \equiv z\ln(\lambda)$ denote the contributions to economic growth by high type incumbents, low type incumbents and entrants. Note that for the total contribution by incumbents, their innovation rates and the composition of types matter. Using a shift-share decomposition, I decompose changes in the growth rate across BGPs, $\Delta g \equiv g_{new} - g_{old}$, as follows

$$\Delta g = \underbrace{S_{old}\Delta g^h + (1-S_{old})\Delta g^\ell}_{\Delta\text{Within}} + \underbrace{g^h_{old}\Delta S - g^\ell_{old}\Delta S}_{\Delta\text{Between}} + \underbrace{\Delta g^h \Delta S - \Delta g^\ell \Delta S}_{\Delta\text{Cross}} + \underbrace{\Delta g^z}_{\Delta\text{Entry}}, \tag{27}$$

where *old* and *new* index BGP variables before and after the parameter change. Changes in the aggregate growth rate are due to changes in innovation rates of incumbent firms holding the composition of firm types constant (ΔWithin), due to changes in the composition of incumbent firm types holding innovation rates of firms constant (ΔBetween), due to changes in both innovation rates and firm composition (ΔCross) as well as due to changes in firm entry (Δg^z). The ΔWithin, ΔBetween and ΔCross terms capture changes in the contribution by incumbents, whereas Δg^z captures by definition changes in the contribution by firm entry (ΔEntry). Because the ΔCross term is absent without firm type heterogeneity, I group the ΔBetween and ΔCross-term into a common ΔReallocation term.[20]

What drives the decline in the aggregate growth rate? Is it falling innovation rates among incumbent firms, reallocating market shares across firms, or falling firm entry? Table 5 quantifies the different contributions to the fall in the aggregate growth rate.

Changes in the innovation behavior by incumbent firms and entrants work in opposite directions. First, the ΔWithin term is positive at 0.22pp, indicating that innovation rates of the average incumbent firm increased. Second, the reallocation of market shares to firms with higher innovation rates contributes positively to economic growth. The ΔReallocation term is positive at 0.27pp. Therefore, changes in incumbent innovation *raise* the aggregate growth rate by a total of 0.49pp with almost equal contributions by ΔWithin and ΔReallocation.

[20] The ΔCross-term compares small to the ΔBetween term.

Table 5: Decomposing the fall in economic growth

	$\psi_I \downarrow, \psi_z \downarrow$	$\psi_I \downarrow$	$\psi_z \downarrow$
ΔWithin	0.22pp	-0.23pp	0.47pp
ΔReallocation	0.27pp	0.01pp	0.20pp
ΔEntry	-1.10pp	-0.11pp	-0.93pp
Δg	-0.62pp	-0.33pp	-0.26pp

Notes: the table shows the contributions to the change in the aggregate growth rate according to the decomposition in eq. (27) in percentage points. ΔReallocation is the sum of the ΔBetween and ΔCross terms. The growth rate in the initial BGP stands at 3.02%. $\psi_I \downarrow$ denotes the 51% fall in the internal R&D efficiency and $\psi_z \downarrow$ the 22% fall in the entry efficiency.

Lastly, falling firm entry lowers the aggregate growth rate substantially by 1.1pp. The fall in firm entry dominates the positive contribution by incumbents, resulting in a total decline in the growth rate of 0.62pp.

That the ΔWithin term is positive may be surprising given that R&D costs of incumbents have increased. Columns 3 and 4 of Table 5 repeat the decomposition for each parameter change in isolation. The ΔWithin effect of a rise in the R&D cost for incumbents is negative (-0.23pp), i.e., innovation rates of the average incumbent fall. At the same time, the rise in the entry R&D costs incentivizes incumbent firms to innovate faster, resulting in a positive ΔWithin effect. Overall, the positive ΔWithin effect following the rise in the entry R&D costs outweighs the negative ΔWithin effect of the rising R&D costs for incumbents. Note also that the rise in the entry costs drives the positive ΔReallocation effect. A fall in firm entry reallocates market shares among the incumbents towards the relatively faster-growing high-type firms. Overall, the fall in the internal R&D efficiency and the entry R&D efficiency almost contribute equally to the total fall in the aggregate growth rate (-0.33pp and -0.26pp, respectively).

In sum, the decomposition suggests that the fall in the aggregate growth rate is driven by a fall in firm entry that dominates both rising innovation rates by incumbents and the reallocation of market shares towards more innovative firms.

4.4 Robustness and further model validation

In this section, I show that an alternative explanation for the recent macroeconomic trends, namely a rise in the productivity dispersion, also leads to lower economic growth through a fall in firm entry and test other model predictions.

4.4.1 Entry and economic growth

The previous decomposition highlights a fall in firm entry as the driver behind the decline in the aggregate growth rate. One might argue that the relative contributions of the ΔWithin, ΔReallocation, and ΔEntry terms to the decline in economic growth are specific to the estimated parameter changes. In this section, I show that alternative explanations that align with the macroeconomic trends also highlight a fall in firm entry as the key driver. However, these alternative explanations cannot fully explain the changes in firm growth.

Aghion, Bergeaud, Boppart, Klenow and Li (2023) explain the fall in economic growth and the rise in concentration in the US economy through changes in the R&D efficiency of incumbent firms and rising productivity dispersion across firms. In line with their story, I estimate an alternative ending BGP where the parameters subject to change are the internal R&D efficiency ψ_I (as in the previous case) and the productivity gap φ^h/φ^ℓ (instead of the entry efficiency).

Table 6: Alternative ending BGP. Moments and parameters

	Data	Model
Moments		
Sales growth by age 8 (cohorts 2009–2012)	+11.5pp	+2.1pp
Employment growth by age 8 (cohorts 2009–2012)	+17.8pp	+7.4pp
Parameters		
ψ_I *Internal R&D efficiency*		-54%
φ^h/φ^ℓ *Productivity gap*		+6%

Notes: the table shows changes in moments and parameters with respect to the initial BGP.

Table 6 shows the results of the estimation. The internal R&D efficiency falls by 54% (compared to 51% in the previous estimation) and the productivity gap increases by 6%.[21] The implied changes in firm sales and employment growth are qualitatively in line with the data, yet fall short in explaining them quantitatively.[22] Therefore, changes in the productivity gap are not able to fully account for the changes in firm growth. Nevertheless, changes in economic aggregates match recent trends in the macroeconomy: the long-run aggregate growth rate falls by 0.49pp, concentration rises and the firm entry rate declines by 3pp. Therefore, the estimated fall in the R&D efficiency and increase in the productivity gap give rise to a similar fall in the aggregate growth rate as the one targeted in Aghion, Bergeaud, Boppart, Klenow and Li (2023) (-0.42pp).

I decompose the fall in the aggregate growth rate for the alternative ending BGP according to eq. (27.) as before. In line with the previous estimation, the fall in firm entry more than explains the fall in the aggregate growth rate: -0.53pp compared to -0.49pp, see Table 7. Changes in incumbent innovation rates, ΔWithin, lower the growth rate slightly (-0.13pp),

[21] For this estimation, I assume that entrants always replace incumbents as the estimated productivity gap exceeds the step size of innovation λ. Otherwise, low type entrants facing high type incumbents engage in a race for incumbency. Estimating the parameters with the constraint $\varphi^h/\varphi^\ell < \lambda$ results in the constraint binding at $\varphi^h/\varphi^\ell = 1.136$, which is the value of λ in the initial BGP.

[22] For a large enough productivity disadvantage, low-type firms stop expanding into new markets and remain one-product firms, which reduces the degrees of freedom in the model to match the increase in sales and employment growth for the pooled sample of firms.

Table 7: Decomposing the fall in economic growth revisited

	$\psi_I \downarrow, \varphi^h/\varphi^\ell \uparrow$	$\psi_I \downarrow$	$\varphi^h/\varphi^\ell \uparrow$
ΔWithin	-0.13pp	-0.24pp	0.11pp
ΔReallocation	0.18pp	0.01pp	0.13pp
ΔEntry	-0.53pp	-0.12pp	-0.35pp
Δg	-0.49pp	-0.35pp	-0.11pp

Notes: the table shows the contributions to the change in the aggregate growth rate according to the decomposition in eq. (27) in percentage points. ΔReallocation is the sum of the ΔBetween and ΔCross terms. The growth rate in the initial BGP stands at 3.02%. $\psi_I \downarrow$ denotes the 54% fall in the internal R&D efficiency and $\varphi^h/\varphi^\ell \uparrow$ the 6% rise in the productivity gap.

whereas the reallocation of market shares, ΔReallocation, towards firms with higher innovation rates generates a positive growth effect (+0.18pp). Therefore, the conclusion that the decline in the aggregate growth rate is driven by a fall in firm entry even holds for alternative mechanisms that keep the entry technology constant. The reason why the decomposition yields similar results to before is that the rising productivity gap works similarly as rising entry costs on growth: both generate positive ΔWithin and ΔReallocation effects that are dominated by a negative ΔEntry effect (last column of Table 5 and 7). The ΔWithin, ΔReallocation and ΔEntry contributions resulting from a fall in the internal R&D efficiency (column 3) are quantitatively almost identical to the previous estimation.

In Aghion, Bergeaud, Boppart, Klenow and Li (2023), all firms innovate at the same rate, and there is no firm entry such that falling innovation rates of the average firm, ΔWithin, fully explain the decline in the aggregate growth rate. Table 7 suggests that reallocation effects and firm entry matter for changes in the aggregate growth rate. The ΔReallocation effect outweighs the ΔWithin effect, and ΔEntry dominates both.

Would the role of entry change when relaxing the assumption of a unitary demand elasticity? With a demand elasticity greater than one, firms gain market shares through successful internal R&D. This suggests that, ceteris paribus, a larger fall in firm entry would be required to offset the negative size-growth effect from rising internal R&D costs when matching the increase in firm size growth.

4.4.2 Firm productivity and size growth

The model generates heterogeneity in ex-ante growth profiles in line with Sterk, Sedláček and Pugsley (2021) through firm-type heterogeneity. I document suggestive evidence in the data that supports firm productivity as one such type of heterogeneity that drives differences in (realized) firm growth.

Firm productivity is generally unobserved in the data. I use a model-based approach to infer the firms' productivity. As firms enter the model economy with one product, eq. (3) captures firm markups upon entry. Their productivity advantage allows more productive firms to charge higher markups in expectation. Therefore, I proxy firm productivity by its markup (sales relative to wage bill) at age zero and test whether the proxy of firm productivity correlates with subsequent size growth in the data. I control for cohort and 5-digit industry fixed effects. More specifically, I run the following regression

$$\ln \text{Size}_{\text{Age}_{j,t}=a_f} - \ln \text{Size}_{\text{Age}_{j,t}=0} = \beta_0 + \beta_1 \log \left(\frac{py}{wl}\right)_{\text{Age}_{j,t}=0} + \theta_c + \theta_k + \epsilon_{j,t}. \tag{28}$$

py/wl denotes inverse labor shares, otherwise the notation follows eq. (1). In line with the model estimation, I focus on firm size growth over the first eight years, i.e., $a_f = 8$. As the measure of firm size, I use employment to avoid sales at age zero on both sides of eq. (28).

Table 8: Firm productivity and size growth

	$\Delta \ln \text{Size}_{\text{Age}=8}$	$\Delta \ln \text{Size}_{\text{Age}=8}$	$\Delta \ln \text{Size}_{\text{Age}=8}$	$\Delta \ln \text{Size}_{\text{Age}=8}$
$\log\left(\frac{py}{wl}\right)_{\text{Age}=0}$	0.130	0.198	0.222	0.237
	(0.006)	(0.005)	(0.005)	(0.006)
$\log K_{\text{Age}=0}$			-0.041	0.003
			(0.003)	(0.003)
$\log M_{\text{Age}=0}$				-0.107
				(0.004)
Cohort fixed effects	✓	✓	✓	✓
Industry fixed effects	✓	✓	✓	✓
$\log\left(\frac{py}{wl}\right)_{\text{Age}=0} > 0$		✓	✓	✓
N	66,817	65,875	60,950	60,832
R^2	0.06	0.08	0.08	0.10

Notes: the table reports the regression coefficient β_1 of eq. (28). Firm size growth over the first eight years, $\Delta \ln \text{Size}_{\text{Age}_{j,t}=8} \equiv \ln \text{Size}_{\text{Age}_{j,t}=8} - \ln \text{Size}_{\text{Age}_{j,t}=0}$, is measured using firm employment. $\log (py/wl)_{\text{Age}_{j,t}=0}$ denotes the log inverse labor share at age zero, the proxy of firm productivity, as explained in the main text. $\log K$ and $\log M$ denote the firm's capital stock and intermediate inputs, respectively. Robust standard errors are in parentheses.

Table 8 shows the regression results. The regression coefficient of interest, β_1, stands at 0.13, i.e., within the same 5-digit industry and cohort, firms with 1% higher inverse labor shares are associated with approximately 0.13% faster employment growth over the first eight years. For the model-relevant sub-sample of firms with positive markups (firms with inverse labor shares larger than one), the regression coefficient increases to 0.198 (column two). Higher inverse labor shares at entry are positively related to subsequent size growth, even when controlling for capital and intermediate inputs. Including capital or intermediate inputs at age zero in the regression increases β_1 to 0.222 and 0.237, respectively (third and fourth column).[23] Across all specifications, the coefficient of interest remains highly significant,

[23] I obtain similar results when using $TFPR$ at age zero instead of labor productivity as a proxy for firm productivity, where $TFPR \equiv \frac{py}{K^\alpha (wl)^{1-\alpha}}$ with α estimated at the industry level using cost shares.

with an almost constant (robust) standard error of 0.005. The data confirms that firms with high inverse labor shares at entry, perhaps due to high firm productivity as suggested by the model, display faster subsequent size growth.

5 Conclusion

I document a novel observation about firm growth in this paper. Sales and employment growth over the firm's life cycle accelerated from 1997 to 2017. Comparing cohorts from the late 1900s to the early 2010s, sales growth over the firm's first eight years increased by 11.5pp. Employment growth increased by 17.8pp. Motivated by this observation, I study the drivers behind the changes in firm growth, as well as their implications for the composition of firms, economic aggregates, and the source behind the fall in economic growth.

I use a model of firm dynamics with firm markup growth and productivity-type heterogeneity for the quantitative analysis. Markup growth (and changes therein) permits differential trends in firm sales and employment growth. In contrast, the productivity type heterogeneity allows compositional forces to affect economic aggregates. I estimate the model on two BGPs, one representing economic conditions at the macro level and firm growth during the late 1990s and one reflecting firm growth in the early 2010s.

Comparing the BGPs highlights a rise in the entry costs and a fall in the R&D efficiency for incumbent firms behind the acceleration in firm sales and employment growth. Sales and employment growth jointly accelerate in response to the increase in entry costs. At the same time, the rise in the markup-generating R&D cost increases employment growth relative to sales growth as markup growth falls. At the aggregate level, the implications of the cost changes align with macroeconomic trends. The aggregate growth rate declines by 0.6pp, the firm entry rate falls by 8pp, and concentration rises. More productive firms take over market shares from less productive ones, increasing their sales share by 17pp. The share of high productivity type firms in the cross-section of firms increases by 12pp. Lastly, I decompose the fall in the aggregate growth rate into the contributions by changes in innovation rates by the average firm, reallocation of sales shares across firms with different innovation rates, and changes in firm entry. The reallocation of sales shares to more productive firms that innovate faster entails positive growth effects that outweigh the effect of changes in innovation rates by the average firm. However, the fall in firm entry dominates the positive reallocation effects, causing the decline in economic growth.

How does the measured effect of the sales share reallocation on the aggregate growth rate compare to other countries that have experienced similar or even more severe episodes of reallocation? Over the last decades, many Western economies have privatized their education, health care, transportation, or communication industries. It would be interesting to decompose changes in the growth rate of these industries following the privatization into changes in the innovation rates of the average firm, reallocation across firms with different innovation rates, and firm entry, as in this paper. If privately held firms innovate at higher rates than state-owned enterprises, one might expect positive reallocation effects on the long-run growth rate. Alternatively, the timing of the reallocation could correlate with changes in the average firm's innovation rates or in firm entry such that these forces, instead of the reallocation, explain changes in industry-productivity growth. A growth decomposition, disciplined by changes in firm dynamics, could separate these forces.

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Appendices

A Data

The main data set, *Företagens Ekonomi* (FEK), covers information from balance sheets and profit and loss statements for the universe of Swedish firms. From this data, I obtain the main variables of interest, namely sales (*"Nettoomsättning"*, variable name: *Nettoomsattning*) and employment (*"Antal anställda"*, variable name: *MedelantalAnstallda*). In the FEK codebook by Statistics Sweden, these variables are defined as follows.[24] Sales refer to income from the companies' main business for goods sold and provided services. Employment refers to the average number of employees in full-time units in accordance with the company's annual report. As described in the main text, I focus on firms in the private sector. These firms have a legal type (variable name: *JurForm*) less than "50" or equal to "96".

The 5-digit industry classification (SNI codes) changed twice between 1997 and 2017, once in 2002 and once in 2007. I ensure a consistent industry classification using the following steps. During the year of the change, I observe both the old and the new industry classifications. For the firms present in the data this year, extending the new industry classification further back in time before the change in the classification is straightforward. This way, the industry codes of almost all firms are updated. A firm might be in the data before and after the cutoff year but not at the cutoff year. For these firms, the above method does not work. If the firm appears in the data one year after the classification change, I use the observed classification after the change and update the classification before the change accordingly. For firms that are absent for several years around the year of change, I use industry mappings provided by Statistics Sweden. These mappings do not always provide a 1:1 mapping between industries before and after the classification change, so I use the most common transitions for the m:m mappings.

One concern is that changes in the firm structure, e.g., when firms merge with other firms, change the firm ID. To address this concern, I impute changes in firm IDs using worker flows between firms. The auxiliary data set *Registerbaserad Arbetsmarknadsstatistik* (RAMS) contains the universe of employer-employee matches. I impute changes in the firm ID of firms with at least five employees as follows: if more than 50% of the workforce of firm A in year t makes up for more than 50% of the workforce of firm B in year $t+1$, I substitute firm B's firm ID by firm A's firm ID following $t+1$. The results remain virtually unchanged when excluding firms for whom the imputed firm ID differs from the observed firm ID.

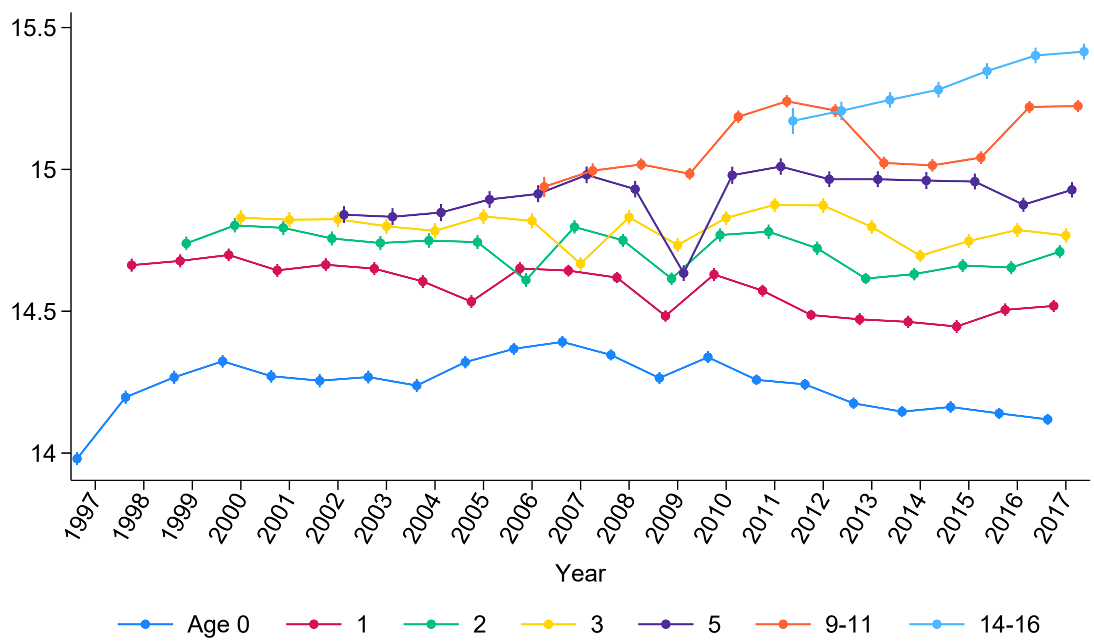
[24] https://www.scb.se/contentassets/9dd20ce462644cc19f6f04eb2edbbe28/nv0109_kd_2017_bv_190508_v2.pdf, accessed 07.02.2024.

B Trends in the Swedish economy

B.1 Changes in firm growth

Figure 3 shows the age-conditional average firm size patterns for sales as the measure of firm size. Similarly to the patterns for employment, average firm sales are relatively stable for young firms, whereas a positive trend is apparent for older firms.

Figure 3: Average firm size (log sales) conditional on age



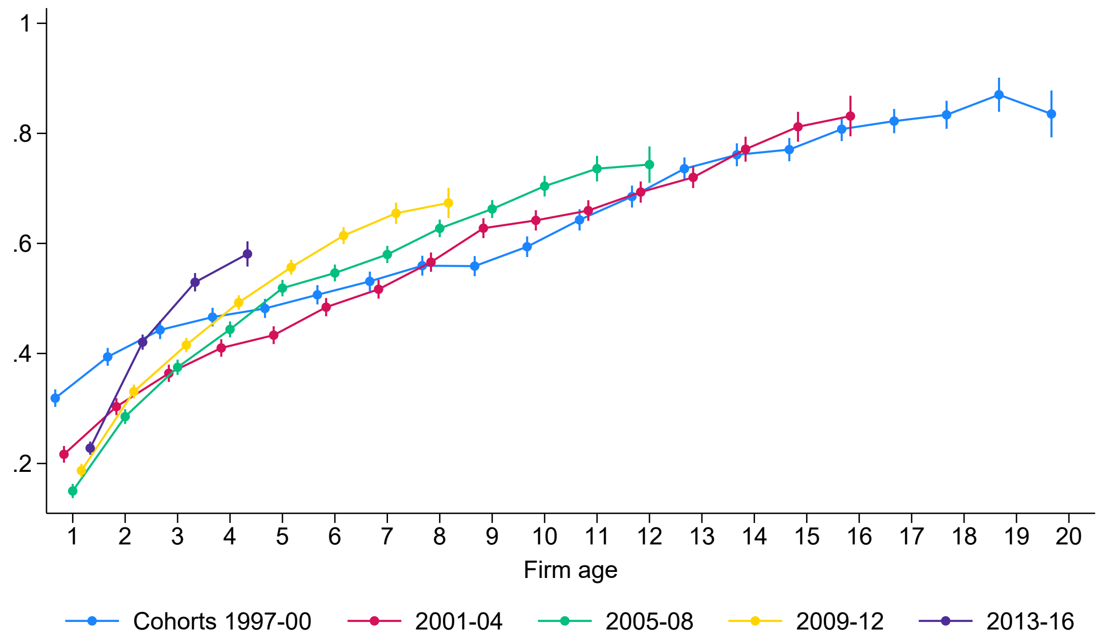
Notes: the figure shows avg. firm size (log sales) conditional on firm age over time. Sales are deflated to 2017 Swedish Krona (SEK) using the GDP deflator. 95% confidence intervals are shown.

I run regression (1) using sales as the firm size measure to quantify the changes in firm life cycle growth. Figure 4 plots the age coefficients for the different cohorts. As for employment, sales life cycle growth accelerates over time.

B.2 Changes in industry concentration

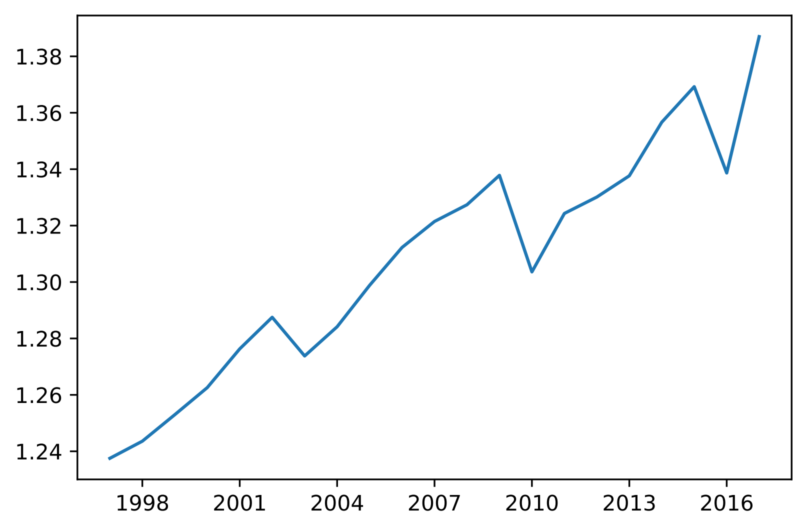
I compute the standard deviation of log sales within industries to measure industry concentration. Note that this measure coincides with the standard deviation of log sales shares. The more dispersed sales (or sales shares), the more concentrated the industry. I filter firm sales at the 1% tails for each year and compute the standard deviation for industries with at least 50 firms to avoid changes in industry size affecting the concentration measure. Figure 5 shows the standard deviation, averaged across all industries. Concentration displays positive trend growth. Only the crisis episodes of the early 2000s and the financial crisis temporarily put rising concentration on hold.

Figure 4: Sales life cycle growth (by cohort)



Notes: the figure shows cumulative sales growth over the firm's life cycle, measured as the difference between average log sales at age a_f and age zero according to eq. (1). Cohorts are pooled as indicated in the legend. Firm sales are filtered at their 1% tails. The figure includes 95% confidence intervals.

Figure 5: Within-industry sales concentration



Notes: the figure shows the within-industry standard deviation of log sales, averaged over all industries with at least 50 firms.

C Model

C.1 Solving the dynamic firm problem

The HJB for a high productivity-type firm h reads[25]

$$
\begin{aligned}
& r_t V_t^h(n,[\mu_i]) - \dot{V}_t^h(n,[\mu_i]) = \\
& \quad \sum_{k=1}^{n} \pi(\mu_k) + \sum_{k=1}^{n} \tau_t \Big[V_t^h\left(n-1,[\mu_i]_{i\neq k}\right) - V_t^h(n,[\mu_i]) \Big] \\
& \quad + \max_{[x_k, I_k]} \Bigg\{ \sum_{k=1}^{n} I_k \Big[V_t^h\left(n,[[\mu_i]_{i\neq k}, \mu_k \times \lambda]\right) - V_t^h(n,[\mu_i]) \Big] \\
& \quad + \sum_{k=1}^{n} x_k \Big[S_t V_t^h\left(n+1,[[\mu_i],\lambda]\right) + (1-S_t) V_t^h\left(n+1, \left[[\mu_i], \lambda \times \varphi^h/\varphi^l\right]\right) - V_t^h(n,[\mu_i]) \Big] \\
& \quad - w_t \left[\mu_k^{-1} \frac{1}{\psi_I} (I_k)^\zeta + \frac{1}{\psi_x} (x_k)^\zeta \right] \Bigg\}
\end{aligned}
$$

The HBJ for a low productivity-type firm l reads

$$
\begin{aligned}
& r_t V_t^l(n,[\mu_i]) - \dot{V}_t^l(n,[\mu_i]) = \\
& \quad \sum_{k=1}^{n} \pi(\mu_k) + \sum_{k=1}^{n} \tau_t \Big[V_t^l\left(n-1,[\mu_i]_{i\neq k}\right) - V_t^l(n,[\mu_i]) \Big] \\
& \quad + \max_{[x_k, I_k]} \Bigg\{ \sum_{k=1}^{n} I_k \Big[V_t^l\left(n,[[\mu_i]_{i\neq k}, \mu_k \times \lambda]\right) - V_t^l(n,[\mu_i]) \Big] \\
& \quad + \sum_{k=1}^{n} x_k \Big[S_t V_t^l\left(n+1, \left[[\mu_i], \lambda \times \varphi^l/\varphi^h\right]\right) + (1-S_t) V_t^l\left(n+1,[[\mu_i],\lambda]\right) - V_t^l(n,[\mu_i]) \Big] \\
& \quad - w_t \left[\mu_k^{-1} \frac{1}{\psi_I} (I_k)^\zeta + \frac{1}{\psi_x} (x_k)^\zeta \right] \Bigg\}.
\end{aligned}
$$

I solve for the value function of a high-type firm, however the steps for the low-type firm are equivalent. Guess that the value function of the firm consists of a component that is common to all lines and a line-specific component

$$
V_t^h(n,[\mu_i]) = V_{t,P}^h(n) + \sum_{k=1}^{n} V_{t,M}^h(\mu_k)
$$

[25]The notation follows Peters (2020) where possible.

so that

$$\begin{aligned} \dot{V}_t^h(n,[\mu_i]) &= \dot{V}_{t,P}^h(n) + \sum_{k=1}^n \dot{V}_{t,M}^h(\mu_k) \\ V_t^h\left(n-1,[\mu_i]_{i\neq k}\right) - V_t^h(n,[\mu_i]) &= V_{t,P}^h(n-1) - V_{t,P}^h(n) - V_{t,M}^h(\mu_k) \\ V_t^h\left(n,[[\mu_i]_{i\neq k},\mu_k\times\lambda]\right) - V_t^h(n,[\mu_i]) &= V_{t,M}^h(\mu_k\times\lambda) - V_{t,M}^h(\mu_k) \end{aligned}$$

and

$$\begin{aligned} &S_t V_t^h\left(n+1,[[\mu_i],\lambda]\right) + (1-S_t)V_t^h\left(n+1,\left[[\mu_i],\lambda\times\varphi^h/\varphi^l\right]\right) - V_t^h(n,[\mu_i] = \\ &\quad S_t\left(V_{t,P}^h(n+1) + \sum_{k=1}^n V_{t,M}^h(\mu_k) + V_{t,M}^h(\lambda)\right) + (1-S_t)\left(V_{t,P}^h(n+1) + \sum_{k=1}^n V_{t,M}^h(\mu_k) + V_{t,M}^h(\lambda\times\varphi^h/\varphi^l)\right) \\ &\quad - V_{t,P}^h(n) - \sum_{k=1}^n V_{t,M}^h(\mu_k) = \\ &\quad V_{t,P}^h(n+1) - V_{t,P}^h(n) + S_t V_{t,M}^h(\lambda) + (1-S_t)V_{t,M}^h(\lambda\times\varphi^h/\varphi^l). \end{aligned}$$

Substituting the guess into the HJB

$$\begin{aligned} r_t\Big[V_{t,P}^h(n) + \sum_{k=1}^n V_{t,M}^h(\mu_k)\Big] - \dot{V}_{t,P}^h(n) - \sum_{k=1}^n \dot{V}_{t,M}^h(\mu_k) = \\ &\sum_{k=1}^n \pi(\mu_k) + \sum_{k=1}^n \tau_t\Big[V_{t,P}^h(n-1) - V_{t,P}^h(n) - V_{t,M}^h(\mu_k)\Big] \\ &+ \max_{[x_k,I_k]}\Bigg\{\sum_{k=1}^n I_k\Big[V_{t,M}^h(\mu_k\times\lambda) - V_{t,M}^h(\mu_k)\Big] \\ &+ \sum_{k=1}^n x_k\Big[V_{t,P}^h(n+1) - V_{t,P}^h(n) + S_t V_{t,M}^h(\lambda) + (1-S_t)V_{t,M}^h(\lambda\times\varphi^h/\varphi^l)\Big] \\ &- w_t\left[\sum_{k=1}^n \mu_k^{-1}\frac{1}{\psi_I}(I_k)^\zeta + \frac{1}{\psi_x}(x_k)^\zeta\right]\Bigg\} \end{aligned}$$

and rearranging terms yields

$$\begin{aligned} &r_t V_{t,P}^h(n) - \dot{V}_{t,P}^h(n) + \sum_{k=1}^n\Big[r_t V_{t,M}^h(\mu_k) - \dot{V}_{t,M}^h(\mu_k)\Big] = \\ &\qquad \sum_{k=1}^n\left\{\pi(\mu_k) - \tau_t V_{t,M}^h(\mu_k) + \max_{I_k}\left\{I_k\Big[V_{t,M}^h(\mu_k\times\lambda) - V_{t,M}^h(\mu_k)\Big] - w_t\mu_k^{-1}\frac{1}{\psi_I}(I_k)^\zeta\right\}\right\} \\ &\qquad + \sum_{k=1}^n \tau_t\Big[V_{t,P}^h(n-1) - V_{t,P}^h(n)\Big] \\ &\qquad + \max_{[x_k]}\left\{\sum_{k=1}^n x_k\Big[V_{t,P}^h(n+1) - V_{t,P}^h(n) + S_t V_{t,M}^h(\lambda) + (1-S_t)V_{t,M}^h(\lambda\times\varphi^h/\varphi^l)\Big] - w_t\frac{1}{\psi_x}(x_k)^\zeta\right\}. \end{aligned}$$

First solve

$$r_t V^h_{t,M}(\mu_i) - \dot{V}^h_{t,M}(\mu_i) = \pi(\mu_i) - \tau_t V^h_{t,M}(\mu_i) \\ + \max_{I_i} \left\{ I_i \left[V^h_{t,M}(\mu_i \times \lambda) - V^h_{t,M}(\mu_i) \right] - w_t \mu_i^{-1} \frac{1}{\psi_I} (I_i)^\zeta \right\}. \tag{29}$$

Once we know $V^h_{t,M}$, we can solve for $V^h_{t,P}$ in

$$r_t V^h_{t,P}(n) - \dot{V}^h_{t,P}(n) = \sum_{k=1}^{n} \tau_t \left[V^h_{t,P}(n-1) - V^h_{t,P}(n) \right] \\ + \max_{[x_k]} \left\{ \sum_{k=1}^{n} x_k \left[V^h_{t,P}(n+1) - V^h_{t,P}(n) + S_t V^h_{t,M}(\lambda) + (1 - S_t) V^h_{t,M}(\lambda \times \varphi^h / \varphi^l) \right] - w_t \frac{1}{\psi_x} (x_k)^\zeta \right\} \tag{30}$$

Assume (and verified below) that in steady-state $V^h_{t,P}$ and $V^h_{t,M}$ grow at the constant rate g such that

$$\dot{V}^h_{t,P}(n) = g V^h_{t,P}(n)$$
$$\dot{V}^h_{t,M}(\mu_i) = g V^h_{t,M}(\mu_i)$$

In steady-state we can then write eq. (29) as

$$(r - g + \tau) V^h_{t,M}(\mu_i) = \pi(\mu_i) + \max_{I_i} \left\{ I_i \left[V^h_{t,M}(\mu_i \times \lambda) - V^h_{t,M}(\mu_i) \right] - w_t \mu_i^{-1} \frac{1}{\psi_I} (I_i)^\zeta \right\}. \tag{31}$$

Guess that[26]

$$V^h_{t,M}(\mu_i) = \kappa_t - \alpha_t \mu_i^{-1}$$

so that

$$V^h_{t,M}(\mu_i \times \lambda) - V^h_{t,M}(\mu_i) = \alpha_t \mu_i^{-1} \left(1 - \frac{1}{\lambda} \right).$$

The FOC for I_i then reads

$$\alpha_t \mu_i^{-1} \left(1 - \frac{1}{\lambda} \right) = w_t \mu^{-1} \frac{1}{\psi_I} \zeta \left(I_i \right)^{\zeta - 1}.$$

[26]It will turn out from eq. (33) that $\alpha_t > 0$, otherwise I_i would not be positive, such that $V^h_{t,M}(\mu_i)$ is increasing in μ_i.

Rearranging yields

$$\left(\frac{\alpha_t}{w_t}\left(1-\frac{1}{\lambda}\right)\frac{\psi_I}{\zeta}\right)^{\frac{1}{\zeta-1}} = I_i. \tag{32}$$

It will turn out that α_t/w_t is constant such that I_i is time independent. Using the guess for the value function, the FOC, and the Euler equation $\rho = r - g$ in eq. (31) we get

$$\begin{aligned}(\rho+\tau)\left[\kappa_t - \alpha_t\mu_i^{-1}\right] =& Y_t\left(1-\frac{1}{\mu_i}\right) + w_t\mu^{-1}\frac{1}{\psi_I}\zeta\left(I_i\right)^\zeta - w_t\mu_i^{-1}\frac{1}{\psi_I}(I_i)^\zeta \\ =& Y_t\left(1-\frac{1}{\mu_i}\right) + \frac{\zeta-1}{\psi_I}w_t\mu_i^{-1}(I_i)^\zeta.\end{aligned}$$

Matching coefficients we obtain

$$\begin{aligned}(\rho+\tau)\kappa_t =& Y_t \\ \Leftrightarrow \kappa_t =& \frac{Y_t}{\rho+\tau}\end{aligned}$$

and

$$\begin{aligned}-\alpha_t\mu_i^{-1} =& \frac{-Y_t\mu_i^{-1} + \frac{\zeta-1}{\psi_I}w_t\mu_i^{-1}(I_i)^\zeta}{\rho+\tau} \\ \Leftrightarrow \alpha_t =& \frac{Y_t - \frac{\zeta-1}{\psi_I}(I_i)^\zeta w_t}{\rho+\tau}.\end{aligned}$$

This confirms that α_t/w_t is indeed time independent. The value function is

$$\begin{aligned}V^h_{t,M}(\mu_i) =& \kappa_t - \alpha_t\mu_i^{-1} \\ =& \frac{Y_t}{\rho+\tau} - \frac{Y_t\mu_i^{-1} - \frac{\zeta-1}{\psi_I}(I_i)^\zeta w_t\mu_i^{-1}}{\rho+\tau} \\ =& \frac{\pi(\mu_i) + \frac{\zeta-1}{\psi_I}(I_i)^\zeta w_t\mu_i^{-1}}{\rho+\tau}.\end{aligned}$$

Inserting α into the optimality condition, I_i solves

$$I_i = \left(\left(\frac{Y_t}{w_t} - \frac{\zeta-1}{\psi_I}(I_i)^\zeta\right)\left(1-\frac{1}{\lambda}\right)\frac{\psi_I}{\zeta(\rho+\tau)}\right)^{\frac{1}{\zeta-1}}. \tag{33}$$

Internal innovation rates I_i are time invariant, and independent of the product line and the productivity type of the firm as the optimality condition shows: $I_i = I$.

With this at hand, we can turn back to the differential equation for $V_{t,P}^h(n)$ in eq. (30).

$$
\begin{aligned}
r_t V_{t,P}^h(n) - \dot{V}_{t,P}^h(n) = & \sum_{k=1}^{n} \tau_t \Big[V_{t,P}^h(n-1) - V_{t,P}^h(n) \Big] \\
& + \max_{[x_k]} \Big\{ \sum_{k=1}^{n} x_k \Big[V_{t,P}^h(n+1) - V_{t,P}^h(n) + S_t V_{t,M}^h(\lambda) + (1-S_t) V_{t,M}^h(\lambda \times \varphi^h/\varphi^l) \Big] - w_t \frac{1}{\psi_x}(x_k)^\zeta \Big\}
\end{aligned}
$$

In addition to the guess that $V_{t,P}^h(n)$ grows at rate g, conjecture that $V_{t,P}^h(n) = n \times v_t^h$. Combined with the Euler we get

$$
(\rho+\tau) n v_t^h = \max_{[x_k]} \Big\{ \sum_{k=1}^{n} x_k \Big[v_t^h + S_t V_{t,M}^h(\lambda) + (1-S_t) V_{t,M}^h(\lambda \times \varphi^h/\varphi^l) \Big] - w_t \frac{1}{\psi_x}(x_k)^\zeta \Big\}. \quad (34)
$$

The optimality condition for x_k is given by

$$
v_t^h + S_t V_{t,M}^h(\lambda) + (1-S_t) V_{t,M}^h(\lambda \times \varphi^h/\varphi^l) = w_t \frac{\zeta}{\psi_x}(x_k)^{\zeta-1}. \quad (35)
$$

Several observations are noteworthy. First, the FOC shows that optimal expansion rates are independent of quality and productivity gaps in line k. We can hence drop the item indexation: $x_k = x^d$, where $d \in \{h, \ell\}$. Second, $v_t, V_{t,M}^h, w_t$ all grow at the same rate g, which implies that expansion rates are constant over time. We can hence write eq. (34) as

$$
(\rho+\tau) n v_t^h = n w_t \frac{\zeta}{\psi_x}(x^h)^\zeta - n w_t \frac{1}{\psi_x}(x^h)^\zeta.
$$

or

$$
v_t^h = \frac{1}{(\rho+\tau)} \frac{\zeta-1}{\psi_x}(x^h)^\zeta w_t.
$$

Gathering all terms, the value function is given by

$$
\begin{aligned}
V_t^h(n, [\mu_i]) = & V_{t,P}^h(n) + \sum_{k=1}^{n} V_{t,M}(\mu_k) \\
= & n v_t^h + \sum_{k=1}^{n} V_{t,M}(\mu_k) \\
= & n \frac{1}{(\rho+\tau)} \frac{\zeta-1}{\psi_x}(x^h)^\zeta w_t + \sum_{k=1}^{n} \frac{\pi(\mu_k) + \frac{\zeta-1}{\psi_I} I^\zeta w_t \mu_k^{-1}}{\rho+\tau}. \quad (36)
\end{aligned}
$$

To see that high-type firms expand at different rates than low-type firms, assume that $x^h = x^\ell$. In this case, $v_t^h = v_t^\ell$, however $E[V_t^h(1, \mu_i)] > E[V_t^\ell(1, \mu_i)]$, because the value function is increasing in the markup. This is true because $Y - \frac{\zeta-1}{\psi_I} I^\zeta w > 0$, otherwise the optimal internal R&D rate defined in eq. (33) would be negative (or zero). The optimality condition for expansion R&D in eq. (35) relates the expected value of expanding into a new product market to the marginal cost of expanding. Given $E[V_t^h(1,\mu_i)] > E[V_t^\ell(1,\mu_i)]$, the cost of expansion R&D (the right hand side of eq. (35)) must be larger for high-type than for low-type firms, which implies $x^h > x^\ell$; contradicting the initial assumption. As in Lentz and Mortensen (2008), the fact that the marginal value of a product line increases in profits per line implies that firms' expansion rates increase with profitability (productivity).

Using the expression for v_t^h in the optimality condition for the expansion rate, write eq. (35) as

$$\frac{1}{(\rho+\tau)}\frac{\zeta-1}{\psi_x}(x^h)^\zeta w_t + S_t \frac{\pi(\lambda) + \frac{\zeta-1}{\psi_I} I^\zeta w_t \lambda^{-1}}{\rho+\tau} + (1-S_t)\frac{\pi(\lambda\times\varphi^h/\varphi^l) + \frac{\zeta-1}{\psi_I} I^\zeta w_t \lambda^{-1}\frac{\varphi^l}{\varphi^h}}{\rho+\tau} = w_t \frac{\zeta}{\psi_x}(x^h)^{\zeta-1}.$$

Simplifying gives

$$\frac{\zeta-1}{\psi_x}(x^h)^\zeta + S_t\left(\pi(\lambda)/w_t + \frac{\zeta-1}{\psi_I} I^\zeta \lambda^{-1}\right) + (1-S_t)\left(\pi(\lambda\times\varphi^h/\varphi^l)/w_t + \frac{\zeta-1}{\psi_I} I^\zeta \lambda^{-1}\frac{\varphi^l}{\varphi^h}\right) = (\rho+\tau)\frac{\zeta}{\psi_x}(x^h)^{\zeta-1}.$$

Inserting the profit function

$$\frac{\zeta-1}{\psi_x}(x^h)^\zeta + S_t\left(\frac{Y_t}{w_t}\left(1-\frac{1}{\lambda}\right) + \frac{\zeta-1}{\psi_I} I^\zeta \lambda^{-1}\right) + (1-S_t)\left(\frac{Y_t}{w_t}\left(1-\frac{\varphi^l}{\varphi^h}\frac{1}{\lambda}\right) + \frac{\zeta-1}{\psi_I} I^\zeta \lambda^{-1}\frac{\varphi^l}{\varphi^h}\right) = (\rho+\tau)\frac{\zeta}{\psi_x}(x^h)^{\zeta-1}.$$

The optimality condition for the expansion rate of the low productivity type reads

$$\frac{\zeta-1}{\psi_x}(x^l)^\zeta + S_t\left(\frac{Y_t}{w_t}\left(1-\frac{1}{\lambda}\frac{\varphi^h}{\varphi^l}\right) + \frac{\zeta-1}{\psi_I} I^\zeta \lambda^{-1}\frac{\varphi^h}{\varphi^l}\right) + (1-S_t)\left(\frac{Y_t}{w_t}\left(1-\frac{1}{\lambda}\right) + \frac{\zeta-1}{\psi_I} I^\zeta \lambda^{-1}\right) = (\rho+\tau)\frac{\zeta}{\psi_x}(x^l)^{\zeta-1}.$$

C.2 Solving for the steady state equilibrium

In the model there are the seven unknown variables $x^h, x^l, I, z, \tau, \frac{Y_t}{w_t}, S$ and the markup distribution $\nu()$ in seven equations plus the system of differential equations characterizing $\nu()$.

Optimality condition for the internal innovation rate

$$I = \left(\left(\frac{Y_t}{w_t} - \frac{\zeta - 1}{\psi_I} I^\zeta \right) \left(1 - \frac{1}{\lambda} \right) \frac{\psi_I}{\zeta(\rho + \tau)} \right)^{\frac{1}{\zeta - 1}}$$

Optimality condition for high productivity expansion rate

$$\frac{\zeta - 1}{\psi_x}(x^h)^\zeta + S\left(\frac{Y_t}{w_t}\left(1 - \frac{1}{\lambda}\right) + \frac{\zeta - 1}{\psi_I} I^\zeta \lambda^{-1} \right) + (1 - S)\left(\frac{Y_t}{w_t}\left(1 - \frac{\varphi^l}{\varphi^h}\frac{1}{\lambda}\right) + \frac{\zeta - 1}{\psi_I} I^\zeta \lambda^{-1} \frac{\varphi^l}{\varphi^h} \right)$$
$$= (\rho + \tau)\frac{\zeta}{\psi_x}(x^h)^{\zeta - 1}$$

Optimality condition for low productivity expansion rate

$$\frac{\zeta - 1}{\psi_x}(x^l)^\zeta + S\left(\frac{Y_t}{w_t}\left(1 - \frac{1}{\lambda}\frac{\varphi^h}{\varphi^l}\right) + \frac{\zeta - 1}{\psi_I l} I^\zeta \lambda^{-1} \frac{\varphi^h}{\varphi^l} \right) + (1 - S)\left(\frac{Y_t}{w_t}\left(1 - \frac{1}{\lambda}\right) + \frac{\zeta - 1}{\psi_I l} I^\zeta \lambda^{-1} \right)$$
$$= (\rho + \tau)\frac{\zeta}{\psi_x}(x^l)^{\zeta - 1}$$

Free entry condition

$$p^h \Big(S V_t^h(1, \lambda) + (1 - S) V_t^h(1, \lambda \times \varphi^h/\varphi^l) \Big) + (1 - p^h)\Big(S V_t^l(1, \lambda \times \varphi^l/\varphi^h) + (1 - S) V_t^l(1, \lambda) \Big) = \frac{1}{\psi_z} w_t,$$

where

$$V_t^d(1, \mu) = \frac{1}{(\rho + \tau)} \frac{\zeta - 1}{\psi_x}(x^d)^\zeta w_t + \frac{Y_t\left(1 - \frac{1}{\mu}\right) + \frac{\zeta - 1}{\psi_I} I^\zeta w_t \mu^{-1}}{\rho + \tau}$$

Labor market clearing condition

$$1 = \frac{Y_t}{w_t} \sum_{\frac{\varphi_j}{\varphi_{j'}}} \sum_i \frac{1}{\lambda^i \frac{\varphi_j}{\varphi_{j'}}} \nu\left(i, \frac{\varphi_j}{\varphi_{j'}}\right) + \frac{1}{\psi_I} I^\zeta \sum_{\frac{\varphi_j}{\varphi_{j'}}} \sum_i \frac{1}{\lambda^i \frac{\varphi_j}{\varphi_{j'}}} \nu\left(i, \frac{\varphi_j}{\varphi_{j'}}\right) + S\frac{1}{\psi_x}(x^h)^\zeta + (1 - S)\frac{1}{\psi_x}(x^l)^\zeta + \frac{z}{\psi_z}$$

Creative destruction

$$\tau = z + Sx^h + (1 - S)x^l$$

Share of high productivity type

$$S = \sum_{i=1}^{\infty} \left[\nu\left(i, \frac{\varphi^h}{\varphi^h}\right) + \nu\left(i, \frac{\varphi^h}{\varphi^l}\right) \right],$$

where ν, the stationary distribution of quality and productivity gaps, is characterized by

$$0 = \dot{\nu}\left(\Delta, \frac{\varphi_j}{\varphi_{j'}}\right) = I\nu\left(\Delta - 1, \frac{\varphi_j}{\varphi_{j'}}\right) - \nu\left(\Delta, \frac{\varphi_j}{\varphi_{j'}}\right)(I+\tau) \quad \text{for} \quad \Delta \geq 2$$

and for the case of a unitary quality gap

$$\begin{aligned}
0 = \dot{\nu}\left(1, \frac{\varphi^l}{\varphi^h}\right) &= (1-S)x^l S + z_t(1-p^h)S - \nu\left(1, \frac{\varphi^l}{\varphi^h}\right)(I+\tau) \\
0 = \dot{\nu}\left(1, \frac{\varphi^l}{\varphi^l}\right) &= (1-S)x^l(1-S) + z_t(1-p^h)(1-S) - \nu\left(1, \frac{\varphi^l}{\varphi^l}\right)(I+\tau) \\
0 = \dot{\nu}\left(1, \frac{\varphi^h}{\varphi^h}\right) &= Sx^h S + z_t p^h S - \nu\left(1, \frac{\varphi^h}{\varphi^h}\right)(I+\tau) \\
0 = \dot{\nu}\left(1, \frac{\varphi^h}{\varphi^l}\right) &= Sx^h(1-S) + z_t p^h(1-S) - \nu\left(1, \frac{\varphi^h}{\varphi^l}\right)(I+\tau)
\end{aligned}$$

To simplify the system of equations, first rewrite the creative destruction equation

$$z = (\tau - Sx^h - (1-S)x^l)$$

such that z can be substituted out from the remaining equations. Second, as derived in the main text, from the differential equations characterizing the distribution of quality and productivity gaps in BGP, we obtain the share of high-productivity incumbents in the economy

$$\begin{aligned}
S =& S_{\varphi^h,\varphi^h} + S_{\varphi^h,\varphi^l} \\
=& \frac{Sx^h + zp^h}{\tau}.
\end{aligned}$$

Third, the optimality conditions for expansion rates (multiplied by p^h and $(1-p^h)$) and the free entry condition together imply

$$\frac{1}{\psi_x}p^h(x^h)^{\zeta-1} + \frac{1}{\psi_x}(1-p^h)(x^l)^{\zeta-1} = \frac{1}{\psi_z\zeta}.$$

The system of equilibrium conditions can hence be reduced to:

Optimality condition for the internal innovation rate

$$I = \left(\left(\frac{Y_t}{w_t}\psi_I - (\zeta-1)I^\zeta\right)\frac{\left(1-\frac{1}{\lambda}\right)}{\zeta(\rho+\tau)}\right)^{\frac{1}{\zeta-1}}$$

Optimality condition for high productivity expansion rate

$$\frac{\zeta-1}{\psi_x}(x^h)^\zeta + S\left(\frac{Y_t}{w_t}\left(1-\frac{1}{\lambda}\right)+(\zeta-1)I^\zeta\lambda^{-1}\frac{1}{\psi_I}\right)+(1-S)\left(\frac{Y_t}{w_t}\left(1-\frac{\varphi^l}{\varphi^h}\frac{1}{\lambda}\right)+(\zeta-1)I^\zeta\lambda^{-1}\frac{\varphi^l}{\psi_I\varphi^h}\right)$$
$$=(\rho+\tau)\frac{\zeta}{\psi_x}(x^h)^{\zeta-1}$$

Optimality condition for low productivity expansion rate

$$\frac{\zeta-1}{\psi_x}(x^l)^\zeta + S\left(\frac{Y_t}{w_t}\left(1-\frac{1}{\lambda}\frac{\varphi^h}{\varphi^l}\right)+(\zeta-1)I^\zeta\lambda^{-1}\frac{\varphi^h}{\psi_I l\varphi^l}\right)+(1-S)\left(\frac{Y_t}{w_t}\left(1-\frac{1}{\lambda}\right)+(\zeta-1)I^\zeta\lambda^{-1}\frac{1}{\psi_I l}\right)$$
$$=(\rho+\tau)\frac{\zeta}{\psi_x}(x^l)^{\zeta-1}$$

Free entry

$$p^h\frac{(x^h)^{\zeta-1}}{\psi_x}+(1-p^h)\frac{(x^l)^{\zeta-1}}{\psi_x}=\frac{1}{\psi_z\zeta}$$

Labor market clearing condition

$$1=\frac{Y_t}{w_t}\Lambda+\Lambda_I+S\frac{1}{\psi_x}(x^h)^\zeta+(1-S)\frac{1}{\psi_x}(x^l)^\zeta+\frac{\tau-Sx^h-(1-S)x^l}{\psi_z},$$

where[27]

$$\Lambda=\frac{\theta}{\theta+1}\sum_{k\in\{h,l\}}\sum_{n\in\{h,l\}}\frac{1}{\varphi_k/\varphi_n}S_{\varphi_k,\varphi_n}$$
$$\Lambda_I=\frac{1}{\psi_I}I^\zeta\frac{\theta}{\theta+1}\sum_{k\in\{h,l\}}\sum_{n\in\{h,l\}}\frac{1}{\varphi_k/\varphi_n}S_{\varphi_k,\varphi_n}$$
$$\theta=\frac{\ln(I+\tau)-\ln(I)}{\ln(\lambda)}$$

Share of high productivity type

$$S=\frac{Sx^h+(\tau-Sx^h-(1-S)x^l)p^h}{\tau}$$

This constitutes a system of seven equations in seven unknowns $(x^h, x^l, I, \tau, \frac{Y_t}{w_t}, S)$, which I solve using a root finder.

C.3 Joint distribution of quality and productivity gaps

I characterize the two-dimensional distribution of quality and productivity gaps along the BGP as a function of firm policies. This allows for optimal policies and the distribution

[27] For the derivation of Λ I assume a continuous distribution of quality gaps.

to be solved jointly. I solve for the steady state distribution over quality and productivity gaps by setting the differential equations characterizing the law-of-motion in eq. (8) and (9) equal to zero. The stationary mass of product lines with quality gap λ^{Δ} and productivity gap φ^i/φ^j is given by

$$\nu\left(\Delta, \frac{\varphi^l}{\varphi^h}\right) = \left(\frac{I}{I+\tau}\right)^{\Delta} \frac{(1-S)x^l S + z(1-p^h)S}{I}$$
$$\nu\left(\Delta, \frac{\varphi^l}{\varphi^l}\right) = \left(\frac{I}{I+\tau}\right)^{\Delta} \frac{(1-S)x^l(1-S) + z(1-p^h)(1-S)}{I}$$
$$\nu\left(\Delta, \frac{\varphi^h}{\varphi^h}\right) = \left(\frac{I}{I+\tau}\right)^{\Delta} \frac{Sx^h S + zp^h S}{I}$$
$$\nu\left(\Delta, \frac{\varphi^h}{\varphi^l}\right) = \left(\frac{I}{I+\tau}\right)^{\Delta} \frac{Sx^h(1-S) + zp^h(1-S)}{I}$$

It follows that

$$\Pr\left(\Delta \leq d, \frac{\varphi^l}{\varphi^h}\right) = \sum_{i=1}^{d} \nu\left(i, \frac{\varphi^l}{\varphi^h}\right) = S_{\varphi^l,\varphi^h}\left(1 - \left(\frac{I}{I+\tau}\right)^d\right)$$
$$\Pr\left(\Delta \leq d, \frac{\varphi^l}{\varphi^l}\right) = \sum_{i=1}^{d} \nu\left(i, \frac{\varphi^l}{\varphi^l}\right) = S_{\varphi^l,\varphi^l}\left(1 - \left(\frac{I}{I+\tau}\right)^d\right)$$
$$\Pr\left(\Delta \leq d, \frac{\varphi^h}{\varphi^h}\right) = \sum_{i=1}^{d} \nu\left(i, \frac{\varphi^h}{\varphi^h}\right) = S_{\varphi^h,\varphi^h}\left(1 - \left(\frac{I}{I+\tau}\right)^d\right)$$
$$\Pr\left(\Delta \leq d, \frac{\varphi^h}{\varphi^l}\right) = \sum_{i=1}^{d} \nu\left(i, \frac{\varphi^h}{\varphi^l}\right) = S_{\varphi^h,\varphi^l}\left(1 - \left(\frac{I}{I+\tau}\right)^d\right).$$

Focusing on product lines where a low-productivity incumbent faces a high-productivity second-best firm:

$$\begin{aligned} P\left(\frac{\varphi^l}{\varphi^h}, \Delta \leq d\right) =& S_{\varphi^l,\varphi^h}\left(1 - \left(\frac{I}{I+\tau}\right)^d\right) \\ =& S_{\varphi^l,\varphi^h}\left(1 - e^{\ln\left(\left(\frac{I}{I+\tau}\right)^d\right)}\right) \\ =& S_{\varphi^l,\varphi^h}\left(1 - e^{-d[\ln(I+\tau) - \ln I]}\right) \end{aligned}$$

and

$$
\begin{aligned}
P\left(\frac{\varphi^l}{\varphi^h}, \ln\left(\lambda^{\Delta}\right) \leq d\right) =& P\left(\frac{\varphi^l}{\varphi^h}, \Delta \ln\left(\lambda\right) \leq d\right) \\
=& P\left(\frac{\varphi^l}{\varphi^h}, \Delta \leq \frac{d}{\ln\left(\lambda\right)}\right) \\
=& S_{\varphi^l, \varphi^h}\left(1 - e^{-\frac{\ln(I+\tau)-\ln I}{\ln(\lambda)} d}\right)
\end{aligned}
$$

Conditional on the productivity gap, $\ln\left(\lambda^{\Delta}\right)$ is exponentially distributed with parameter $\frac{\ln(I+\tau)-\ln I}{\ln(\lambda)}$. Further

$$
\begin{aligned}
P\left(\frac{\varphi^l}{\varphi^h}, \lambda^{\Delta} \leq d\right) =& P\left(\frac{\varphi^l}{\varphi^h}, \Delta \leq \frac{\ln(d)}{\ln(\lambda)}\right) \\
=& S_{\varphi^l, \varphi^h}\left(1 - e^{-\frac{\ln(I+\tau)-\ln I}{\ln(\lambda)} \ln(d)}\right) \\
=& S_{\varphi^l, \varphi^h}\left(1 - d^{-\frac{\ln(I+\tau)-\ln I}{\ln(\lambda)}}\right)
\end{aligned}
$$

Conditional on the productivity gap, quality gaps follow a Pareto distribution with parameter $\frac{\ln(I+\tau)-\ln I}{\ln(\lambda)}$. Denote $\theta = \frac{\ln(I+\tau)-\ln I}{\ln(\lambda)}$. We then have

$$
P\left(\frac{\varphi^l}{\varphi^h}, \lambda^{\Delta} \leq m\right) = S_{\varphi^l, \varphi^h}\left(1 - m^{-\theta}\right).
$$

Conditional on the productivity gap, quality gaps follow a Pareto distribution with parameter θ. The Pareto shape parameter is affected by the rate of own-innovation I. The more own-innovation, the more mass is in the tail of the quality gap distribution. In Peters (2020), markups follow a Pareto distribution. I introduce ex-ante differences in firm productivities, which leads to the result that markups *conditional* on the productivity gap between incumbents and second-best firms are Pareto distributed.

Doing the same steps for lines with different productivity gaps, the aggregate labor share can be computed as

$$\begin{aligned}
\Lambda &= \sum_{k\in\{h,l\}}\sum_{n\in\{h,l\}}\int_1^\infty \frac{1}{\varphi_k/\varphi_n}\frac{1}{m}S_{\varphi_k,\varphi_n}\theta m^{-(\theta+1)}dm \\
&= \sum_{k\in\{h,l\}}\sum_{n\in\{h,l\}}\frac{1}{\varphi_k/\varphi_n}S_{\varphi_k,\varphi_n}\theta\int_1^\infty m^{-(\theta+2)}dm \\
&= \sum_{k\in\{h,l\}}\sum_{n\in\{h,l\}}\frac{1}{\varphi_k/\varphi_n}S_{\varphi_k,\varphi_n}\theta\left[-\frac{1}{\theta+1}m^{-(\theta+1)}\right]_1^\infty \\
&=\frac{\theta}{\theta+1}\sum_{k\in\{h,l\}}\sum_{n\in\{h,l\}}\frac{1}{\varphi_k/\varphi_n}S_{\varphi_k,\varphi_n}.
\end{aligned}$$

The TFP misallocation statistic $\mathcal{M}$ is then

$$\begin{aligned}
\mathcal{M} &=\frac{e^{\sum_{k\in\{h,l\}}\sum_{n\in\{h,l\}}\int\left[\ln\left(\frac{1}{\frac{\varphi_k}{\varphi_n}}\frac{1}{m}\right)S_{\varphi_k,\varphi_n}\theta m^{-(\theta+1)}\right]dm}}{\Lambda} \\
&=\frac{e^{\sum_{k\in\{h,l\}}\sum_{n\in\{h,l\}}\int\left[\ln\left(\frac{1}{\frac{\varphi_k}{\varphi_n}}\right)S_{\varphi_k,\varphi_n}\theta m^{-(\theta+1)}+\ln\left(\frac{1}{m}\right)S_{\varphi_k,\varphi_n}\theta m^{-(\theta+1)}\right]dm}}{\Lambda} \\
&=\frac{e^{\sum_{k\in\{h,l\}}\sum_{n\in\{h,l\}}\left[\ln\left(\frac{1}{\frac{\varphi_k}{\varphi_n}}\right)S_{\varphi_k,\varphi_n}+\int\ln\left(\frac{1}{m}\right)S_{\varphi_k,\varphi_n}\theta m^{-(\theta+1)}dm\right]}}{\Lambda} \\
&=\frac{e^{\sum_{k\in\{h,l\}}\sum_{n\in\{h,l\}}\left[S_{\varphi_k,\varphi_n}\ln\left(\frac{1}{\frac{\varphi_k}{\varphi_n}}\right)-S_{\varphi_k,\varphi_n}\frac{1}{\theta}\right]}}{\Lambda} \\
&=\frac{e^{\sum_{k\in\{h,l\}}\sum_{n\in\{h,l\}}\left[S_{\varphi_k,\varphi_n}\left(\ln\left(\frac{1}{\frac{\varphi_k}{\varphi_n}}\right)-\frac{1}{\theta}\right)\right]}}{\Lambda}
\end{aligned}$$

where I have made use of

$$\int_1^\infty \ln\left(\frac{1}{m}\right)S_{\varphi_k,\varphi_n}\theta m^{-(\theta+1)}dm = \left[\frac{\theta\ln(m)+1}{\theta m^\theta}+C\right]_1^\infty = -\frac{1}{\theta}.$$

Alternatively note that this expression is equal to $-S_{\varphi_k,\varphi_n}E[\ln(\lambda^\Delta)|\varphi_k,\varphi_n]$. I have shown above that $\ln(\lambda^\Delta)$ conditional on the productivity gap is exponentially distributed with parameter θ. From the characteristics of an exponential distribution, its expected value is $1/\theta$.

C.4 Moments of the markup distribution

Mean of markups (unweighted or sales weighted with Cobb-Douglas aggregator)

$$
\begin{aligned}
E[\mu] =& \sum_{k\in\{h,l\}} \sum_{n\in\{h,l\}} \int_1^{\infty} \frac{\varphi_k}{\varphi_n} m S_{\varphi_k,\varphi_n} \theta m^{-(\theta+1)} dm \\
=& \sum_{k\in\{h,l\}} \sum_{n\in\{h,l\}} \frac{\varphi_k}{\varphi_n} S_{\varphi_k,\varphi_n} \theta \int_1^{\infty} m^{-\theta} dm \\
=& \sum_{k\in\{h,l\}} \sum_{n\in\{h,l\}} \frac{\varphi_k}{\varphi_n} S_{\varphi_k,\varphi_n} \theta \left[\frac{1}{1-\theta} m^{1-\theta}\right]_1^{\infty} \\
=& \frac{\theta}{\theta-1} \sum_{k\in\{h,l\}} \sum_{n\in\{h,l\}} \frac{\varphi_k}{\varphi_n} S_{\varphi_k,\varphi_n},
\end{aligned}
$$

where in the last equation it is assumed that $\theta > 1$, which is true if $\frac{\tau}{I} > \lambda - 1$. Otherwise, the mean is ∞. Note that this is simply the mean of a Pareto distribution (once $\frac{\varphi_k}{\varphi_l} S_{\varphi_k,\varphi_l}$ is taken out of the integral). The geometric mean is computed from previously derived expressions:

$$
E[\mu^{geo}] = e^{-\ln(\mathcal{M}\times\Lambda)}.
$$

2nd moment of markups

$$
\begin{aligned}
E[\mu^2] =& \sum_{k\in\{h,l\}} \sum_{n\in\{h,l\}} \int_1^{\infty} \left(\frac{\varphi_k}{\varphi_n} m\right)^2 S_{\varphi_k,\varphi_n} \theta m^{-(\theta+1)} dm \\
=& \sum_{k\in\{h,l\}} \sum_{n\in\{h,l\}} \left(\frac{\varphi_k}{\varphi_n}\right)^2 S_{\varphi_k,\varphi_n} \theta \int_1^{\infty} m^{1-\theta} dm \\
=& \sum_{k\in\{h,l\}} \sum_{n\in\{h,l\}} \left(\frac{\varphi_k}{\varphi_n}\right)^2 S_{\varphi_k,\varphi_n} \theta \left[\frac{1}{2-\theta} m^{2-\theta}\right]_1^{\infty} \\
=& \frac{\theta}{\theta-2} \sum_{k\in\{h,l\}} \sum_{n\in\{h,l\}} \left(\frac{\varphi_k}{\varphi_n}\right)^2 S_{\varphi_k,\varphi_n}.
\end{aligned}
$$

Variance of markups

$$
\begin{aligned}
Var(\mu) = E[\mu^2] - E[\mu]^2 =& \frac{\theta}{\theta-2} \sum_{k\in\{h,l\}} \sum_{n\in\{h,l\}} \left(\frac{\varphi_k}{\varphi_n}\right)^2 S_{\varphi_k,\varphi_n} \\
& - \left(\frac{\theta}{\theta-1} \sum_{k\in\{h,l\}} \sum_{n\in\{h,l\}} \frac{\varphi_k}{\varphi_n} S_{\varphi_k,\varphi_n}\right)^2
\end{aligned}
$$

Without any differences in productivity ($\varphi_h = \varphi_l$), the variance collapses to $\frac{\theta}{\theta-2} - \left(\frac{\theta}{\theta-1}\right)^2 = \frac{\theta}{(\theta-2)(\theta-1)^2}$, which is just the variance of the Pareto distribution μ collapses to.

Mean of log markups

$$
\begin{aligned}
E[\ln \mu] &= \sum_{k\in\{h,l\}} \sum_{n\in\{h,l\}} \int_1^{\infty} \ln\left(\frac{\varphi_k}{\varphi_n} m\right) S_{\varphi_k,\varphi_n} \theta m^{-(\theta+1)} dm \\
&= \sum_{k\in\{h,l\}} \sum_{n\in\{h,l\}} \int_1^{\infty} \left(\ln\left(\frac{\varphi_k}{\varphi_n}\right) + \ln m\right) S_{\varphi_k,\varphi_n} \theta m^{-(\theta+1)} dm \\
&= \sum_{k\in\{h,l\}} \sum_{n\in\{h,l\}} \left(\ln\left(\frac{\varphi_k}{\varphi_n}\right) S_{\varphi_k,\varphi_n} + S_{\varphi_k,\varphi_n} \theta \int_1^{\infty} \ln(m) m^{-(\theta+1)} dm\right) \\
&= \sum_{k\in\{h,l\}} \sum_{n\in\{h,l\}} \left(S_{\varphi_k,\varphi_n} \left(\ln\left(\frac{\varphi_k}{\varphi_n}\right) + \frac{1}{\theta}\right)\right)
\end{aligned}
$$

2nd moment of log markups

$$
\begin{aligned}
E[(\ln \mu)^2] &= \sum_{k\in\{h,l\}} \sum_{n\in\{h,l\}} \int_1^{\infty} \left[\ln\left(\frac{\varphi_k}{\varphi_n} m\right)\right]^2 S_{\varphi_k,\varphi_n} \theta m^{-(\theta+1)} dm \\
&= \sum_{k\in\{h,l\}} \sum_{n\in\{h,l\}} S_{\varphi_k,\varphi_n} \frac{\theta \ln\left(\frac{\varphi_k}{\varphi_n}\right)\left(\theta \ln\left(\frac{\varphi_k}{\varphi_n}\right) + 2\right) + 2}{\theta^2}
\end{aligned}
$$

The variance of log markups is then

$$
Var(\ln \mu) = E[(\ln \mu)^2] - E[(\ln \mu)]^2,
$$

which is computed using the above expressions. Without any differences in productivity ($\varphi_h = \varphi_l$) and own-innovation rates ($\theta_h = \theta_l$), the variance collapses to $1/\theta^2$, which is just the variance of the exponential distribution that $\ln \mu$ collapses to.

C.5 Deriving the steady-state growth rate of aggregate variables

The growth rate of Q_t determines the growth rate of aggregate variables.

$$
g = \frac{\dot{Q}_t}{Q_t} = \frac{\partial \ln(Q_t)}{\partial t}
$$

Quality of a product in a given product line increases through own-innovation, firm expansion or firm entry. For the growth rate of Q_t we have

$$\begin{aligned}
\ln(Q_{t+\Delta}) &= \int_0^1 \ln(q_{t+\Delta,i})di \\
&= \int_0^1 \Big[(\Delta S I + \Delta(1-S)I + \Delta S x^h + \Delta(1-S)x^l + \Delta z)\ln(q_{t,i}\lambda) \\
&\quad + (1 - \Delta S I - \Delta(1-S)I + \Delta S x^h + \Delta(1-S)x^l + \Delta z)\ln(q_{t,i})\Big]di \\
&= \int_0^1 \Big[(\Delta S I + \Delta(1-S)I + \Delta S x^h + \Delta(1-S)x^l + \Delta z)\ln(\lambda) + \ln(q_{t,i})\Big]\, di
\end{aligned}$$

so that

$$\begin{aligned}
\frac{\ln(Q_{t+\Delta}) - \ln(Q_t)}{\Delta} &= \Big(I + Sx^h + (1-S)x^l + z\Big)\ln(\lambda) \\
&= (I + \tau)\ln(\lambda).
\end{aligned}$$

For $\Delta \to 0$, $g = (I+\tau)\ln(\lambda)$.

C.6 Markup dynamics

Firm markups are defined by $\mu_f = \frac{py_f}{wl_f} = \left(\frac{1}{n}\sum_{k=1}^{n}\mu_{kf}^{-1}\right)^{-1}$. Therefore

$$\ln\mu_f = -\ln\left(\frac{1}{n}\sum_{k=1}^{n}\mu_k^{-1}\right).$$

Rewrite the term in brackets (for a high-productivity firm) as

$$\frac{1}{n}\sum_{k=1}^{n}\mu_k^{-1} = \frac{1}{n}\sum_{k=1}^{n}e^{-\ln\mu_k} = \frac{1}{n}\left(\sum_{i=1}^{n_i}e^{-\ln\frac{\varphi^h}{\varphi^l}-\Delta_i\ln\lambda} + \sum_{j=1}^{n_j}e^{-\Delta_j\ln\lambda}\right), \tag{37}$$

where i indexes the product lines where the high productivity firm faces a low productivity second best producer, j the lines where it faces a high productivity second best producer and $n_i + n_j = n$. A two-dimensional linear Taylor expansion around $\ln\lambda = 0$ and $\ln\frac{\varphi^h}{\varphi^l} = 0$ gives

$$\frac{1}{n}\left(\sum_{i=1}^{n_i}e^{-\ln\frac{\varphi^h}{\varphi^l}-\Delta_i\ln\lambda} + \sum_{j=1}^{n_j}e^{-\Delta_j\ln\lambda}\right) \approx 1 - \left(\frac{1}{n}\sum_{k=1}^{n}\Delta_k\right)\ln\lambda - \frac{n_i}{n}\ln\left(\frac{\varphi^h}{\varphi^l}\right)$$

such that

$$E\left[\ln \mu_f|\text{firm age} = a_f, \varphi^h\right] \approx E\left[\frac{1}{n}\sum_{k=1}^{n}\Delta_k|\text{firm age} = a_f, \varphi^h\right]\ln\lambda + (1-S)\ln\left(\frac{\varphi^h}{\varphi^l}\right),$$

where I have used the fact that the expected share of the firm's product lines with a low productivity second best producer is equal to the aggregate share of product lines where the active producer is of the low productivity type. From Peters (2020) I know that

$$E\left[\frac{1}{n}\sum_{k=1}^{n}\Delta_k|\text{firm age} = a_f, \varphi^h\right]\ln\lambda = \left(1 + I \times E[a_P^h|a_f]\right)\ln\lambda,$$

where $E[a_P^h|a_f]$ denotes the average product age of a high process efficiency firm conditional on firm age a_f and

$$\begin{aligned}
E[a_P^h|a_f] &= \frac{1}{x^h}\left(\frac{\frac{1}{\tau}\left(1-e^{-\tau a_f}\right)}{\frac{1}{x^h+\tau}\left(1-e^{-(x^h+\tau)a_f}\right)} - 1\right)\left(1-\phi^h(a_f)\right) + a_f\phi^h(a_f) \\
\phi^h(a) &= e^{-x^h a}\frac{1}{\gamma^h(a)}\ln\left(\frac{1}{1-\gamma^h(a)}\right) \\
\gamma^h(a) &= \frac{x^h\left(1-e^{-(\tau-x^h)a}\right)}{\tau - x^h e^{-(\tau-x^h)a}},
\end{aligned}$$

which gives the expression in the main text.

For a firm of the low process efficiency type, the last term in eq. (37) reads

$$\frac{1}{n}\left(\sum_{i=1}^{n_i} e^{-\ln\Delta_i \ln\lambda} + \sum_{j=1}^{n_j} e^{-\ln\frac{\varphi^l}{\varphi^h} - \ln\Delta_j\ln\lambda}\right),$$

where i indexes the product lines where the low productivity producer faces a low productivity second best producer, j the lines where it faces a high productivity second best producer and $n_i + n_j = n$. Following the same steps as for a high productivity firm, this time linearizing around $\ln\frac{\varphi^l}{\varphi^h} = 0$ (and $\ln\lambda = 0$) gives

$$E\left[\ln\mu_f|\text{firm age} = a_f, \varphi^l\right] \approx \left(1 + I \times E[a_P^l|a_f]\right)\ln\lambda + S\ln\left(\frac{\varphi^l}{\varphi^h}\right),$$

where again I have made use of the fact that the share of the firm's product lines with a high productivity second best producer is equal to the aggregate share of product lines where the active producer is of the high productivity type. $E[a_P^l|a_f]$ is exactly defined as $E[a_P^h|a_f]$ with x^h replaced by x^l in the above expressions.