Mathematics 3 by Tessa Bold
Script*

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1 Differential equations

In many areas of economics, we are interested in how economic variables change over time. In contemporary macroeconomic models, first order optimality conditions yield an equation (or even systems of equations) which states how a specific variable advances through time. Time here can be defined in discrete or continuous terms. If an equation describes the continuous change of a variable we are dealing with differential equations (first part of the course) and if time is discrete we work with difference equations (second part). So what is a differential equation?

$$\ddot{x} + 3\dot{x} + 5x = 0$$

The dot over a variables denotes the time derivative of a variable, $\dot{x} = \frac{dx}{dt}$. Hence, a differential equation expresses a relation between a function and its derivative. The difference to previous equations you might have seen is that here we are not looking to solve for a number (e.g. $x = 5$), but rather for a function. The order of a differential equation is determined by the order of derivatives in the equation (the example above is a second order differential equation).

*This script is by no means meant to represent the complete literature for Tessa's part. It should rather guide you through the lectures and get you started with differential and difference equations. Further recommended readings can be found in the course syllabus. Please do not distribute without permission.

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1.1 First order differential equations

We will start with first order differential equations i.e. a relation between a function and its first derivative: $\dot{x} = F(x, t)$. Let me introduce you to the most important differential equation:

$$\dot{y} = ay. \tag{1}$$

To interpret this equation let us rearrange it first

$$\frac{\dot{y}}{y} = a.$$

The left hand side (LHS) is the absolute change in y divided by its absolute value and hence the percentage change in y. The RHS is equal to a constant. Equation (1) therefore gives us a differential equation where the percentage change in the variable at every instant in time is constant (i.e. y displays a constant growth rate). Before solving equation (1) let us look at a simpler differential equation: $\dot{y} = y$. Which function fulfills the property that the time derivative is the function itself? The exponential function e^t! Obviously the solution to equation (1) then is

$$y(t) = e^{at}.$$

How about uniqueness of the solution we found? In fact there are infinitely many equations that solve equation (1) namely any equation of the form $y(t) = Ce^{at}$, where C is any constant. Quick calculations show that this is indeed a solution:

LHS:

$$\dot{y} = aCe^{at}$$

RHS:

$$ay = aCe^{at}$$

and hence $\dot{y} = ay$.

How does the differential equation with constant percentage growth arise? The example that we are going to use is the one of the savings account model: let us assume that the balance on my bank account in $t = 0$ is y_0 and I receive an annual interest of a. The balance in the following years will be $y_1 = (1 + a)y_0$, $y_2 = (1 + a^2)y_0$ and so on. What happens if the bank does not compound annually but let us say once a month?

$$y_2 = \left(1 + \frac{a}{12}\right)^{24} y_0 = \left(1 + \frac{2a}{24}\right)^{24} y_0$$

Or even daily?

$$y_2 = \left(1 + \frac{a}{365}\right)^{730} y_0 = \left(1 + \frac{2a}{730}\right)^{730} y_0$$

Since $lim_{n\to\infty}\left(1 + \frac{at}{n}\right)^n = e^{at}$ we know what happens if the bank compounds continuously: $y(t) = e^{at}y_0$, which is exactly the solution we found above for equation (1).

Note that we found the solution by guessing and verifying. Along the way we will go through more analytic algorithms to solve for the solution. As the next step we will study how to solve for a solution of a differential equation if it is "separable".

What does separability mean? In general the first order differential equation (FODE) can be written $\dot{y} = F(y,t)$. The differential equation is called separable if the RHS is the product of two functions, one depending only on t and another depending only on y: $\dot{y} = f(t)g(y)$. We can then solve this separable differential equation with the following steps:

1. Start from $\dot{y} = \frac{dy}{dt} = f(t)g(y)$
2. Isolate the y variable on one side, t variable on the other: $\frac{dy}{g(y)} = f(t)dt$
3. Integrate each side: $\int \frac{dy}{g(y)} = \int f(t)dt$
4. Evaluate the two integrals and obtain a solution
5. Solve for y.

In addition to the solution found above, recover the constant solution i.e. $\dot{y} = 0$, namely every root of $g(y)$ (the solution to $g(y) = 0$). The constant solution is lost in the step 2 where we must assume $g(y) \neq 0$.

Example. *Let us walk through an example and apply the steps for* $\dot{y} = 2t(1-y)^2$ *which is a separable FODE.*

1. $\frac{dy}{dt} = 2t(1-y)^2$
2. $\frac{dy}{(1-y)^2} = 2tdt$
3. $\int \frac{dy}{(1-y)^2} = \int 2tdt$
4. $\frac{1}{(1-y)} = t^2 + C$
5. $y(t) = 1 - \frac{1}{t^2+C}$

Two comments are in order. First, we still have the unspecified constant C in the solution. We can nail C down if we are given a value of $y(t)$ at one point in time e.g. a starting value $y(0) = 0$. Until we have specified C it remains an unknown. Note that C does affect the shape of $y(t)$. Second, let us not forget about the constant solution $\frac{dy}{dt} = 0$. We obtain the constant solution by solving $2t(1-y)^2 = 0$, hence $y = 1$. We say that we have solved a differential equation when the derivative does not appear anymore.

Let us return to our most important FODE and solve it without guessing the solution:

$$
\begin{aligned}
\dot{y} &= ay \\
\Leftrightarrow \frac{dy}{dt} &= ay \\
\Leftrightarrow \frac{dy}{y} &= adt \\
\Leftrightarrow \int \frac{dy}{y} &= \int adt \\
\Leftrightarrow \ln(y) &= at + C_1 \\
\Leftrightarrow y(t) &= e^{at+C_1} = Ce^{at}
\end{aligned}
$$

where $C = e^{C_1}$. We see that we obtain the same solution as through guessing.

After seeing the first examples of differential equations let us talk about terminology. The **order of the equation** is the order of the largest derivative appearing in it (first order differential equations have only first derivatives, second has at most the second order derivative and so on). With a second order differential equation the change of the slope of y also comes into play: $m\ddot{y} + b\dot{y} + ky = q(t)$. A differential equation is called **ordinary** if it contains the relationship between a function and its derivative of one independent variable only. **Partial differential equations** may contain more than one independent variable. The differential equation $\dot{y} + a(t)y = q(t)$ is **linear** if the equation is a linear function of y and $\dot{y}$. If we put $q(t) = 0$ we obtain the **homogenous differential equation**. With $q(t) \neq 0$ the differential equation is called **inhomogeneous**. Lastly, $a(t) =$ constant, gives us a **constant coefficient differential equation**. If, $a(t) \neq$ constant, it is called **variable coefficients differential equation**. **Non-linear differential equations** are generally written $\frac{dy}{dt} = f(y,t)$. When there is no dependence on t i.e. $f(y)$ only it is called **autonomous**, otherwise **non-autonomous**.

Let us now return to our bank account example, but in more general terms. Assume that at each instant of time we receive a variable interest,

$a(t)y$, and add further cash to our deposit $q(t)$. Hence, the change of our bank account balance at each instant of time is $\dot{y} = a(t)y + q(t)$, which we can rewrite as $\dot{y} - a(t)y = q(t)$. The LHS is called the **system**, the RHS the **outside influence force** or **input signal**. The solution $y(t)$ is the **output signal** or the **system response**. In order to solve this linear differential equation, we apply the superposition principle. Note that the principle holds for general linear differential equations and not only in this example.

> Superposition principle: linear combinations of solutions are themselves solutions. In other words, if $q_1 \rightsquigarrow y_1$ ("input q_1 leads to system response y_1") and $q_2 \rightsquigarrow y_2$ then for any constants c_1 and c_2: $c_1q_1 + c_2q_2 \rightsquigarrow c_1y_1 + c_2y_2$.

Why is this theorem so useful? Because it implies that the solution to input signal $q(t)$ for any linear differential equation is the solution to the homogeneous equation $q(t) = 0$ and a particular solution (e.g. the constant solution where $\dot{y} = 0$) for $q(t)$. In many cases finding the solution to the homogenous equation and a particular solution to the inhomogenous equation simplifies the problem. We will make extensive use of this principle when we deal with second order differential equations.

Example. *Suppose you are given the following differential equations and their solutions.*

- $\dot{y} + 2y = 1$ *with the solution* $y(t) = \frac{1}{2}$
- $\dot{y} + 2y = e^{-2t}$ *with the solution* $y(t) = te^{-2t}$
- $\dot{y} + 2y = 0$ *with the solution* $y(t) = e^{-2t}$

Examples:

1. *Use the superposition principle to find a solution to* $\dot{y} + 2y = 1 + e^{-2t}$. *We see that the input is a combination of 1. and 2. with coefficients* $c = 1$. *Hence, the solution is a combination of the system response of 1. and 2. with* $c = 1$*:* $y(t) = \frac{1}{2} + te^{-2t}$.

2. *Find the solution to* $\dot{y}+2y = 2+3e^{-2t}$. *Same as in the previous example, only this time* $c_1 = 2, c_2 = 3$. *Hence,* $y(t) = 2 * \frac{1}{2} + 3 * te^{-2t}$.

3. *Find lots of solutions to* $\dot{y} + 2y = 1$. *Note that we can write the RHS as* $1 = 1 + c * 0$. *Hence, the solution is the sum of the system responses for* $\dot{y} + 2y = 1$ *and the homogeneous differential equation* $\dot{y} + 2y = 0$. *This is exactly why the superposition principle is so useful! We therefore obtain the solution* $y(t) = \frac{1}{2} + c * e^{-2t}$.

To solve the examples with the superposition principle only, we needed the solutions to 1. 2. and 3. as given. So how do we solve for the system response ourselves? 3. is easy to solve since it is a separable differential equation. So applying the previous steps yields the system response. Note, however, that 1. and 2. are not separable. To solve these we will make use of an integrating factor. Using an integrating factor we can obtain the complete solution (same as using the superposition principle).

> The complete solution $y(t)$ is the sum of a particular solution to the differential equation $y_p(t)$ and the general solution to the homogeneous equation $y_h(t)$ (the null solution):
>
> $$y(t) = y_h(t) + y_p(t).$$

So let us solve the general differential equation

$$\dot{y} + a(t)y = q(t). \tag{2}$$

Note first that the solution to the homogeneous equation is easy to find as it is separable:

$$\begin{aligned} \frac{dy}{y} &= -a(t)dt \\ \Leftrightarrow \ln(y) &= \int -a(t)dt + c_1 \\ \Leftrightarrow y &= e^{c_1} e^{\int -a(t)dt} \\ \Leftrightarrow y &= Ce^{\int -a(t)dt} \end{aligned}$$

plus the constant solution $y(t) = 0$.

Let us proceed to find the solution to (2). Here we will use the integrating factor $e^{A(t)}$. Multiply both sides by the factor to obtain:

$$\dot{y}e^{A(t)} + a(t)ye^{A(t)} = q(t)e^{A(t)} \tag{3}$$

where we chose $A(t)$ to satisfy $A(t) = \int a(t)dt$. Note that the LHS of (3) is the derivative of $ye^{A(t)}$ w.r.t. t using the product rule. Hence we can write

(3) as

$$\begin{aligned}
\frac{d(ye^{A(t)})}{dt} &= q(t)e^{A(t)} \\
\Leftrightarrow \int \frac{d(ye^{A(t)})}{dt}dt &= \int q(t)e^{A(t)}dt \\
\Leftrightarrow ye^{A(t)} &= \int q(t)e^{A(t)}dt + C \\
\Leftrightarrow y &= Ce^{-A(t)} + e^{-A(t)} \int q(t)e^{A(t)}dt
\end{aligned}$$

which is the complete solution to (2). The formula found here is generally valid, also in the case where $a(t) = a$, the constant coefficient case.

Example. *Let us use the formula to find the complete solution to* $\dot{y} + \frac{1}{t}y = t^2$*. To make it consistent with the formula we must have* $a(t) = \frac{1}{t}$ *and* $q(t) = t^2$*, and hence* $A(t) = \ln(t)$*. Insert this into the formula to get*

$$\begin{aligned}
y &= Ce^{-A(t)} + e^{-A(t)} \int q(t)e^{A(t)}dt \\
&= Ce^{-\ln(t)} + e^{-\ln(t)} \int t^2 e^{\ln(t)}dt \\
&= \frac{C}{t} + \frac{1}{t} \int t^3 dt \\
&= \frac{C}{t} + \frac{1}{4}t^3.
\end{aligned}$$

The solution is the sum of the homogenous solution and a particular solution $(y(t) = y_h(t) + y_p(t))$*. Notice that the homogeneous solution is one over the integrating factor, which generally holds true.*

Let us do another example with constant coefficients: $\dot{y} + ay = q$*. The solution to the homogeneous equation is* $y_h = Ce^{-at}$*. How about the particular solution* $e^{-A(t)} \int q(t)e^{A(t)}dt$*? In this case* $A(t) = at$ *so that the particular solution becomes* $e^{-at} \int qe^{at}dt \Leftrightarrow e^{-at}\frac{1}{a}qe^{at} \Leftrightarrow \frac{q}{a}$*. So the general solution is:* $y(t) = Ce^{-at} + \frac{q}{a}$*. If we are given* $y(0)$ *we can determine* C*:* $y(0) = C + \frac{q}{a} \Leftrightarrow C = (y(0) - \frac{q}{a})$*. The general solution then is* $y(t) = (y(0) - \frac{q}{a})e^{-at} + \frac{q}{a}$*. Notice that the particular solution is the constant solution. Instead of using the integrating factor we could have found the solution much quicker by applying the superposition principle and solving for the homogeneous solution plus the solution to the differential equation where* $\dot{y} = 0$*.*

Let us check that both the homogeneous and the particular solution solve the differential equation. The homogeneous solution $(y(0) - \frac{q}{a})e^{-at}$ should solve $\dot{y} + ay = 0$. Inserting our solution into the differential equation we obtain $-a(y(0) - \frac{q}{a})e^{-at} + a(y(0) - \frac{q}{a})e^{-at} = 0$. The equation holds. Let us look at the particular solution $\frac{q}{a}$ which should solve $\dot{y} + ay = q$. Again, inserting the derivative of the particular solution and the solution multiplied by a yields $0 + q = q$. Hence, the equation holds which confirms that we have indeed found the solution to the differential equation.

Notice that if $a > 0$ in the general solution $y(t) = (y(0) - \frac{q}{a})e^{-at} + \frac{q}{a}$ the homogeneous solution vanishes as $t \to \infty$. What remains is $\frac{q}{a}$ the particular solution to $\dot{y} = 0$. With $\dot{y} = 0$ the system does not change over time, it is in its steady state: deposits exactly balance depreciations. If $a < 0$ the exponent of the exponential function is positive and hence the solution explodes over time; we move away from the steady state.

We have seen that by applying the superposition principle we can find the solution to any differential equation as the sum of the solution to the homogeneous equation and a particular solution. To find the solution to the homogeneous equation turned out to be a simple task, because of its separability. In case of constant input to the system we found the particular solution by setting $\dot{y} = 0$ (the steady state solution). However, a steady state does not necessarily have to exist. In that case we can use the method of matching coefficients. Here we guess that the particular solution is of a general form, plug the solution into the differential equation and adjust the coefficients in a way to make the equation hold.

Example. *Let us look at an example: the exponential input. Suppose the differential equation is $\dot{y} + ay = e^{ct}$. Here, the input has the same functional form as what is inside the system. With exponential input we can model growth if $c > 0$, decay if $c < 0$, and oscillations for $c = i\omega$. Let us specify our differential equation to be $\dot{y} + 5y = 3e^{4t}$. The solution to the homogeneous equation is quickly found to be $y_h = Ce^{-5t}$. If you guess a particular solution it is always good to start with a general form of the input to the system, here $y_p = be^{4t}$. We can determine b by inserting our guess into the differential equation:*

$$\begin{aligned} 4be^{4t} + 5be^{4t} &= 3e^{4t} \\ \Leftrightarrow 9b &= 3 \\ \Leftrightarrow b &= \frac{1}{3} \end{aligned}$$

The complete solution becomes:

$$\begin{aligned} y(t) &= y_h(t) + y_p(t) \\ \Leftrightarrow y(t) &= Ce^{-5t} + \frac{1}{3}e^{4t} \end{aligned}$$

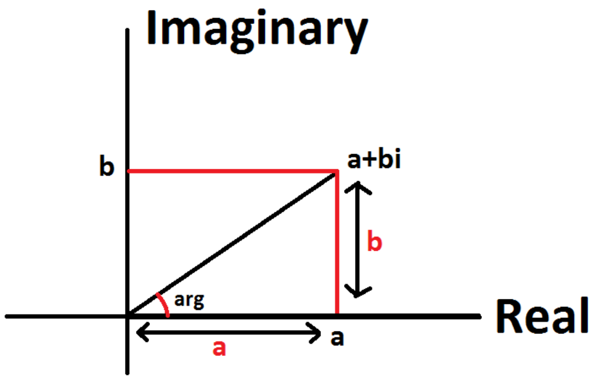
Suppose we are given $y(0) = 2$. With a value for $y(t)$ given we can determine

C.

$$y(0) = C + \frac{1}{3} = 2 \Leftrightarrow C = \frac{5}{3}$$
$$\implies y(t) = \frac{5}{3}e^{-5t} + \frac{1}{3}e^{4t}$$

We have seen that with the exponential input we can model growth and decay. If we wanted to model business cycles we would need an oscillating input. Functions which display oscillating patters are the trigonometric functions sines, cosines or combinations of them. It turns out that complex exponentials can model oscillations in the same way as cosines and sines, but are much easier to work with. For that we need to get used to working with complex numbers[1]. They arise naturally when we look for the square root of a negative number: $i^2 = -1 \Leftrightarrow i = \sqrt{-1}$. We can write each number as a linear combination of its real (a) and imaginary part (ib): $z = a + ib$. Notice that we can write any real number as a complex number with zero imaginary part. We usually plot complex numbers in an Argand diagram which has the real numbers on its x-axis and the imaginary part on its y-axis. With complex numbers we

Figure 1: Argand diagram



can do all operations as usual i.e. $i * i = -1$. Define the complex conjugate of $z = a + ib$ as $\bar{z} = a - ib$. Multiplying a complex number by its conjugate yields $z\bar{z} = a^2 + b^2$. The length of z is defined in the Argand diagram as $|z| = \sqrt{a^2 + b^2}$. To see how the complex numbers and trigonometric functions are connected let us look at the sinusoidal function first. A sinusoidal function is one that can be written in the form $q(t) = R\cos(\omega t - \phi)$. R is the amplitude; it determines the height of the oscillations. ϕ is called the phase lag: the

[1] For an introduction to complex numbers consult Mark Voorneveld's Maths 2 script.

value of ωt for which the graph reaches its maximum. The sinusoidal function and complex numbers are linked by the sinusoidal identity:

$$a\cos(\omega t) + b\sin(\omega t) = A\cos(\omega t - \phi) \tag{4}$$

where $A = \sqrt{a^2+b^2}, \tan\phi = \frac{b}{a}, a + bi = Ae^{i\phi}$. The sum of two sinusoidal functions of the same frequency is another sinusoidal function with that frequency. The proof of (4) is the following:

$$\begin{aligned} A\cos(\omega t - \phi) =& Re(Ae^{i(\omega t - \phi)}) \\ =& Re(e^{i\omega t}Ae^{-i\phi}) \\ =& Re((\cos(\omega t) + i\sin(\omega t)) * (a - ib)) \\ =& Re(a\cos(\omega t) + b\sin(\omega t) + i(a\sin(\omega t) - b\cos(\omega t))) \\ =& a\cos(\omega t) + b\sin(\omega t) \end{aligned}$$

where we have made use of Euler's formula in the third line: $e^{it} = \cos(t) + i\sin(t)$.

Example. *Let us look at an example with oscillating input. Suppose the differential equation is* $\dot{x} + 2x = 2\cos(2t)$. *Our goal is to put this into a complex exponential so we should optimally have a* sin *part as well. The idea now is to define a second differential equation* $\dot{y} + 2y = 2\sin(2t)$ *and to generate the variable* $z = x + iy$. *The trick is to solve the differential equation for* z *and to back out its real part to obtain the solution for* x. *The differential equation for* z *becomes:* $\dot{z} + 2z = 2(\cos(2t) + i\sin(2t))$. *Notice that the RHS is equal to* $2e^{2it}$ *by Euler's formula. Based on the input we guess (and match coefficients) that the solution is of the form* $z_p = Ae^{2it}$. *Inserting the guess in the differential equation gives us:* $\dot{z} + 2z = 2iAe^{2it} + 2Ae^{2it} = (2+2i)Ae^{2it} = 2e^{2it}$. *Hence we must have* $A = \frac{1}{1+i}$. *As a last step we transform* $1+i$ *into polar coordinates* $Re^{i\phi}$. *With* $1 + i = 1 + 1 * i$ *we obtain* $R = \sqrt{1+1} = \sqrt{2}$. *To get* ϕ *we solve* $\tan(\phi) = \frac{1}{1} \Leftrightarrow \phi = \frac{\pi}{4}$. *This gives us* $1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$. *The particular solution therefore is* $z_p(t) = \frac{e^{2it}}{1+i} = \frac{e^{2it}}{\sqrt{2}e^{i\frac{\pi}{4}}} = \frac{e^{i(2t-\frac{\pi}{4})}}{\sqrt{2}} = \frac{1}{\sqrt{2}}(\cos(2t - \frac{\pi}{4}) - i\sin(2t - \frac{\pi}{4}))$. *The particular solution for* $x(t)$ *is the real part of* $z_p(t)$. *Again, the complete solution is the particular solution plus the solution to the homogeneous equation.*

1.2 Systems of first order differential equations

Suppose we are not given a single differential equation, but a whole system of equations. Systems of differential equations occur e.g. in macroeconomic modelling. If your model includes two or more choice variables, your first order conditions constitute a system of at least two differential equations (under the assumption that time is continuous). The system might look like the following:

$$\begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 & \dots & a_{1n}x_n \\ \vdots & & \vdots \\ a_{n1}x_1 & \dots & a_{nn}x_n \end{pmatrix} \Leftrightarrow \mathbf{\dot{x}} = \mathbf{Ax}$$

If $\mathbf{A}$ were a diagonal matrix we could simply solve each $\dot{x}_i = a_{ii}x_i$. If some off-diagonal elements of $\mathbf{A}$ are non-zero we will use a trick to diagonalize the system first and then solve for it.

Suppose that $\mathbf{A}$ has n distinct real eigenvalues $r_1, \dots, r_n$ with corresponding eigenvectors $\mathbf{v_1} \dots \mathbf{v_n}$:

$$\mathbf{Av_i} = r_i \mathbf{v_i}. \tag{5}$$

Define the matrix $\mathbf{P}$ whose columns are the n eigenvectors:

$$\mathbf{P} = \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} & \dots & \mathbf{v_n} \end{bmatrix}$$

Then, (5) can be rewritten as $\mathbf{AP} = \mathbf{PD}$, where $\mathbf{D}$ is the diagonal matrix containing the eigenvalues as its elements. As the eigenvectors for distinct eigenvalues are linearly independent we can invert $\mathbf{P}$ to obtain $\mathbf{D} = \mathbf{P^{-1}AP}$. The trick to diagonalize the system is to introduce the change of variables $\mathbf{y} = \mathbf{P^{-1}x}$ with its inverse $\mathbf{x} = \mathbf{Py}$:

$$\begin{aligned} \mathbf{\dot{y}} &= \mathbf{P^{-1}\dot{x}} && \text{differentiate the transformation w.r.t. } t \\ &= \mathbf{P^{-1}Ax} && \text{definition of our old system} \\ &= \mathbf{P^{-1}APy} && \text{since } \mathbf{x} = \mathbf{Py} \\ &= \mathbf{Dy} && \text{using the definition of } \mathbf{D} \end{aligned}$$

Let us stop here for a second and think about why this transformation is so useful. Why do we transform the system of differential equations in $\mathbf{x}$ into a system of differential equations in $\mathbf{y}$? Remember that $\mathbf{A}$ was not a diagonal matrix so $\dot{x_1}$ depends not only on x_1, but also on other $x_i, i \neq 1$. It is therefore no ordinary differential equation any longer. This is what distinguishes the system $\mathbf{\dot{y}} = \mathbf{Dy}$ from the system $\mathbf{\dot{x}} = \mathbf{Ax}$. Since $\mathbf{D}$ is a diagonal matrix, we can write the system as $\dot{y_1} = r_1 y_1, \dots, \dot{y_n} = r_n y_n$ with the solution:

$$\begin{pmatrix} \dot{y}_1(t) \\ \vdots \\ \dot{y}_n(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{r_1 t} \\ \vdots \\ c_n e^{r_n t} \end{pmatrix}.$$

Now that we have found the solution for $\mathbf{y}$ we can back out the solution for $\mathbf{x}$ from the original transformation $\mathbf{x} = \mathbf{P}\mathbf{y}$

$$\begin{aligned}
\mathbf{x}(\mathbf{t}) &= \mathbf{P}\mathbf{y}(\mathbf{t}) \\
&= [\mathbf{v_1} \quad \mathbf{v_2} \quad \ldots \quad \mathbf{v_n}] \begin{pmatrix} c_1 e^{r_1 t} \\ \vdots \\ c_n e^{r_n t} \end{pmatrix} \\
&= c_1 e^{r_1 t}\mathbf{v_1} + c_2 e^{r_2 t}\mathbf{v_2} + \cdots + c_n e^{r_n t}\mathbf{v_n},
\end{aligned}$$

which is the solution to our original system in $\mathbf{x}$.

A note on stability of systems of differential equations. The expression in the final line goes to zero for $t \to \infty$ if the eigenvalues r_i are negative. In this case the system converges to the same state independently of the starting point and is hence called **globally asymptotically stable**. This is the case if and only if trace($\mathbf{A}$) < 0 and det($\mathbf{A}$) > 0 or equivalently all eigenvalues of $\mathbf{A}$ have negative real part. The system explodes for $t \to \infty$ if all eigenvalues of $\mathbf{A}$ have positive real part. In the case where the eigenvalues are of opposite sign (if and only if det($\mathbf{A}$) < 0), the solution trajectory is called saddle path stable. Given initial conditions there exists a unique path to the steady state of the system. Only solutions on this path converge to the equilibrium point as $t \to \infty$.

The previous discussion about systems of differential equation is not only restricted to real eigenvalues. Complex eigenvalues work equally well in the solution of systems of differential equations.

1.3 Second order differential equations

Second order differential equations (SODEs) are the straight forward extension of first order equations including the second time derivative of the function in its differential equation:

$$m\ddot{y} + b\dot{y} + ky = 0 \tag{6}$$

a linear, constant coefficient, homogeneous second order differential equation. We have seen in the first order case that the solution consists of an exponential function. Let us do the same here and guess that a solution for $y(t)$ is of the form e^{rt}. Substitute the guess into (6) to obtain

$$(mr^2 + br + k)e^{rt} = 0 \tag{7}$$

$$\Leftrightarrow (mr^2 + br + k) = 0 \tag{8}$$

The LHS of (8) is called the characteristic polynomial. All we have to do to find the solution of (6) is to solve the quadratic equation (8). The roots of the quadratic equation can be real, repeated or complex.

Suppose that both $u_1(t)$ and $u_2(t)$ satisfy (6). The general solution of the homogeneous differential equation is then $y(t) = Au_1(t) + Bu_2(t)$ where $u_1(t)$ and $u_2(t)$ are any two solutions (obtained from the characteristic equation) that are not proportional, and A and B are arbitrary constants. To see this notice that since $\dot{y} = A\dot{u}_1 + B\dot{u}_2$ and $\ddot{y} = A\ddot{u}_1 + B\ddot{u}_2$, we have (with $m = 1$)

$$\begin{aligned}\ddot{y} + b\dot{y} + ky &= A\ddot{u}_1 + B\ddot{u}_2 + b(A\dot{u}_1 + B\dot{u}_2) + k(Au_1(t) + Bu_2(t)) \\ &= A[\ddot{u}_1 + b\dot{u}_1 + ku_1] + B[\ddot{u}_2 + b\dot{u}_2 + ku_2].\end{aligned}$$

If both u_1 and u_2 satisfy (6), both terms in the square brackets are zero and hence $\dot{y} = A\dot{u}_1 + B\dot{u}_2$ satisfies (6) as well.

Example. *Solve the differential equation $\ddot{y} + 8\dot{y} + 7y = 0$. Set up the characteristic equation: $r^2 + 8r + 7 = 0$ or simply $(r + 1)(r + 7) = 0$ with the two roots $r_1 = -1$ and $r_2 = -7$. Given the two solutions the complete solution is hence $y(t) = Ae^{-t} + Be^{-7t}$.*

Now that we know how to solve homogeneous SODEs we turn our attention to nonhomogeneous equations:

$$\ddot{y} + b\dot{y} + ky = f(t) \tag{9}$$

where $f(t)$ could take the form of any polynomial. Ultimately we are interested in the general solution $y(t)$ of (9). Suppose for a second that we know $y(t)$ and have found a particular solution $u^* = u^*(t)$ of (9). Then the difference $v = y(t) - u^*(t)$ is a solution of the homogeneous equation (6):

$$\begin{aligned}\ddot{v} + b\dot{v} + kv &= \ddot{y} - \ddot{u} + b(\dot{y} - \dot{u}) + k(y - u) \\ &= [\ddot{y} + b\dot{y} + ky] - [\ddot{u} + b\dot{u} + ku] \\ &= f(t) - f(t) = 0.\end{aligned}$$

The general solution minus a particular solution to the nonhomogeneous equations yields the solution to the homogeneous equation. In mathematical terms: $y(t) - u^*(t) = y_h(t)$ or equivalently $y(t) = y_h(t) + u^*(t)$.

This implies that the general solution is the solution to the homogeneous equation plus a particular solution to the nonhomogeneous equation. This is the superposition principle all over again! Since we can find $y_h(t)$ by solving (8) all we have to do is to find a particular solution $u^*(t)$ to obtain the general solution $y(t)$. This gives us the general solution to any linear SODE.

The particular solution can be found by the method of matching coefficients, similar to the FODE case. We guess a solution, substitute it into the SODE and adjust the unknown coefficients to solve the differential equation. Even though finding the right guess can turn out to be a pain, a good start is to guess a solution equivalent to $f(t)$: if $f(t) = c$ guess that the particular solution is a constant $u^*(t) = 0$, if $f(t) = 4t^2 + 7$ guess that the particular solution is a quadratic polynomial of the general form $u^*(t) = at^2 + bt + c$ etc. For the latter case it is important to include the bt term in the guess even though it does not appear in $f(t)$. This holds true for any $f(t)$. Guess the polynomial in its most general form.

Let us discuss complex solutions to SODEs in more detail since those are usually the ones which take the most time to solve for. When solutions to the characteristic equation are complex, we find the real solutions via Euler's equation and the real solution theorem.

> The real solution theorem says that if $y(t)$ is a complex-valued solution to
>
> $$\ddot{y} + a\dot{y} + by = 0 \tag{10}$$
>
> then the real and imaginary parts of $y(t)$ are also solutions to (10).

Suppose we know that $y(t) = u(t) + iv(t)$ solves (10). Then

$$\begin{aligned}\ddot{y} + a\dot{y} + by &= \ddot{u} + i\ddot{v} + a(\dot{u} + i\dot{v}) + b(u + iv)\\ &= [\ddot{u} + a\dot{u} + bu] + i[\ddot{v} + a\dot{v} + bv]\\ &= 0.\end{aligned}$$

For the last equation to hold we must have that both terms in the square brackets are zero and hence both $u(t)$ (the real part) and $v(t)$ (the imaginary part) solve (10).

Example. *Solve* $\ddot{y}+4\dot{y}+5y = 0$*. The characteristic equation* $r^2+4r+5 = 0$ *is solved by* $r_{1/2} = -2 \pm \sqrt{-1} = -1 \pm i$*. This gives us two roots. Let us start with* $-2 + i$*:*

$$y_1(t) = e^{(-2+i)t} = e^{-2t}e^{it} = e^{-2t}(\cos(t) + i\sin(t)).$$

So the real part is $e^{-2t}\cos(t)$ *and the imaginary part is* $e^{-2t}\sin(t)$*. Applying the real solution theorem we notice that both the real and the imaginary part must solve the differential equation and hence the complete solution becomes*

$$y(t) = Ae^{-2t}\cos(t) + Be^{-2t}\sin(t).$$

Note that we could have also started with the second root $-2 - i$. This would have given us the same linear combination.

1.4 Nonlinear differential equations

In most cases we cannot solve nonlinear differential equations explicitly. One exception is the logistic model, a model of growth slowed down by competition:

$$\frac{dy}{dt} = f(y) = ay - by^2 \qquad \text{where} \quad a, b > 0.$$

This is an autonomous equation. The change in y does not depend on t. Interesting questions are whether this function has a steady state and if it converges for values close to it.

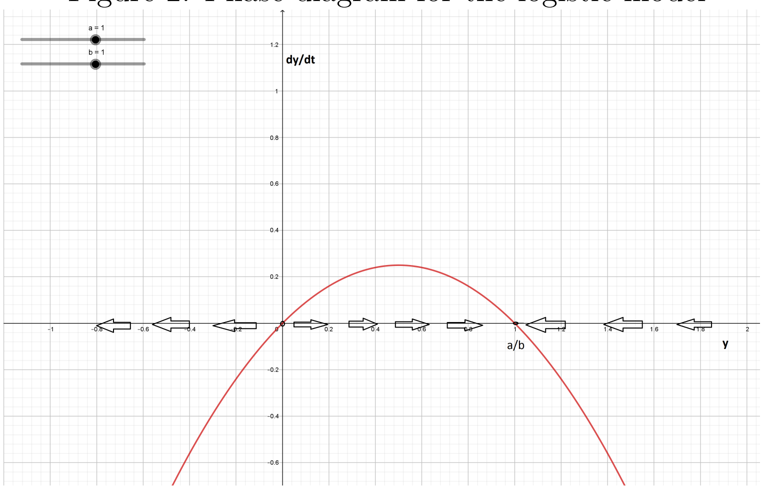
$$\begin{aligned} \frac{dy}{dt} &= 0 = ay - by^2 \\ &= y(a - by) \qquad \Longrightarrow\; y = 0 \quad \text{or} \quad y = \frac{a}{b} \end{aligned}$$

If we arrive at one of the y's, the system will stay there forever. Before we solve the differential equation explicitly let us examine its stability properties. To think about questions such as stability it is useful to draw what is called a phase diagram, a diagram displaying the relationship between y and its time derivative $f(y)$. The phase diagram for $\frac{dy}{dt} = f(y) = ay - by^2$ is displayed in Figure 2. In the figure I picked $a = b = 1$ so the steady states of the system are at $y \in \{0, 1\}$. For $y \in (0, \frac{a}{b})$ the change in y is positive. Hence, if we start off inside this interval, we move towards the steady state at $y = \frac{a}{b}$. For values greater than $\frac{a}{b}$ the change in y is negative and we move back to $y = \frac{a}{b}$. Hence, the steady state at $y = \frac{a}{b}$ is locally stable. The opposite is the case for the steady state at $y = 0$. If we slightly depart from zero, we move further away from the steady state. The system diverges to $-\infty$ for $y < 0$ and converges to the steady state at $\frac{a}{b}$ for $y > 0$. The steady state at zero is hence not stable.

So what is the criterion determining whether a steady state, y^*, is at least locally stable or not? To converge to y^* from both sides, the change of the system must increase for values smaller than the steady state and decrease for values greater than the steady state. In mathematical terms: $f'(y^*) < 0$.

To complete this exercise let us solve the differential equation. There are two ways to do so. One way is to solve it through a transformation of variables, the other through partial fractions. Let us start with the transformation of variables. Without loss of generality we will look at the case where $a = b = 1$ so $\dot{y} = y - y^2$. Introduce the transformation $z = \frac{1}{y}$ so that $\frac{dz}{dt} = -\frac{1}{y^2}\frac{dy}{dt} \Leftrightarrow \frac{dy}{dt} = -\frac{dz}{dt}y^2$. Substitute this expression into our original differential equation to obtain $\frac{dz}{dt} = -\frac{1}{y^2}(y - y^2) = -\frac{1}{y} + 1 = -z(t) + 1$. Hence, we arrive at the transformed differential equation $\dot{z} = -z + 1$ which we know how to solve. By the superposition principle, the complete solution is the

Figure 2: Phase diagram for the logistic model

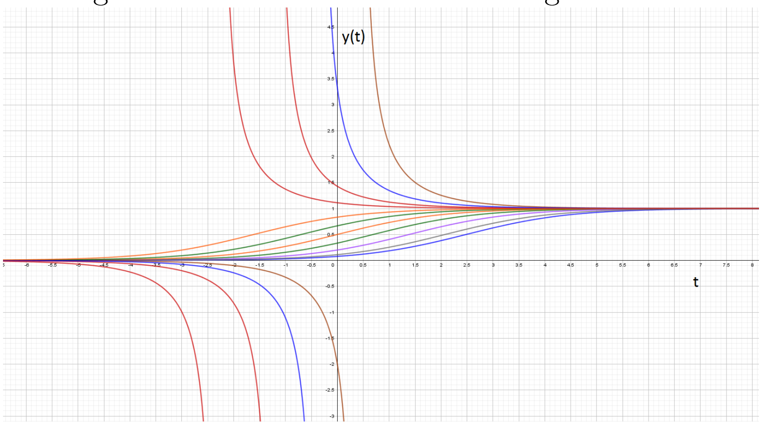


solution to the homogeneous equation ($\dot{z} + z = 0$) plus a particular solution to $\dot{z} + z = 1$[2]. To find the solution to the homogeneous equation we solve $\frac{dz}{dt} = -z$. Applying the steps from the first chapter we find $z(t) = Ce^{-t}$. To find the particular solution to the nonhomogeneous equation notice that the input to the system is a constant. We guess that there exists a steady state where $\dot{z} = 0$, hence $z(t) = 1$. The complete solution is $z(t) = Ce^{-t} + 1$. Having solved for $z(t)$ we can now back out our variable of interest $y(t)$ via the original transformation $y = \frac{1}{z}$. Hence, $y(t) = \frac{1}{Ce^{-t}+1}$, the logistic function. Some interesting properties: $\lim_{t\to\infty} y(t) = 1$ and $\lim_{t\to-\infty} y(t) = 0$. To get a more intuitive picture of our solution we can plot the relationship between t and $y(t)$. The solution curves for different values of C are shown in Figure 3. We could pin down C if we were given a value of y at one point in time e.g. $y(0)$. Notice that the solution curves go hand in hand with our prior observation on stability. If we start off with an initial value for y greater than $\frac{a}{b} = 1$ we converge towards the steady state at one from above. For values inside the unit interval, $y(t)$ converges to the steady state at one from below. For negative values the system diverges to negative infinity. This is why phase diagrams can provide useful information before having solved a differential equation.

The second way to solve for the logistic equation is via partial fractions.

[2] Very important note: the superposition principle holds true for *linear* differential equations. We cannot apply it to the nonlinear differential equation $\frac{dy}{dt} = f(y) = ay - by^2$ directly. Our transformed differential equation in z, however, is linear so we can make use of the superposition principle here!

Figure 3: Solution curves for the logistic model



Let us know solve the differential equation in its general form:

$$
\begin{aligned}
\frac{dy}{dt} &= ay - by^2 && (11)\\
&= a(y - \frac{b}{a}y^2) && (12)\\
\Leftrightarrow \frac{dy}{y - \frac{b}{a}y^2} &= adt && (13)
\end{aligned}
$$

We can always apply the approach of partial fractions if we have a polynomial in the denominator. The idea is to split the denominator up into two parts with both parts being linear in the denominator. We want to find numbers A, B s.t.

$$
\begin{aligned}
\frac{A}{y} + \frac{B}{1 - \frac{b}{a}y} &= \frac{1}{y(1 - \frac{b}{a}y)}\\
\Leftrightarrow \frac{A(1 - \frac{b}{a}y) + By}{y(1 - \frac{b}{a}y)} &= \frac{1}{y(1 - \frac{b}{ay})}
\end{aligned}
$$

The equation holds for $A = 1$ and $B = \frac{b}{a}$. Using this in (13) we obtain:

$$
\begin{aligned}
\int \frac{1}{y}dy + \int \frac{\frac{b}{a}}{1 - \frac{b}{a}y}dy &= \int adt\\
\Leftrightarrow \ln(y) - ln(1 - \frac{b}{a}y) &= \int adt\\
\Leftrightarrow \ln\left(\frac{y}{1 - \frac{b}{a}y}\right) &= at + B\\
\Leftrightarrow y(t) &= \frac{1}{Ce^{-at} + \frac{b}{a}}
\end{aligned}
$$

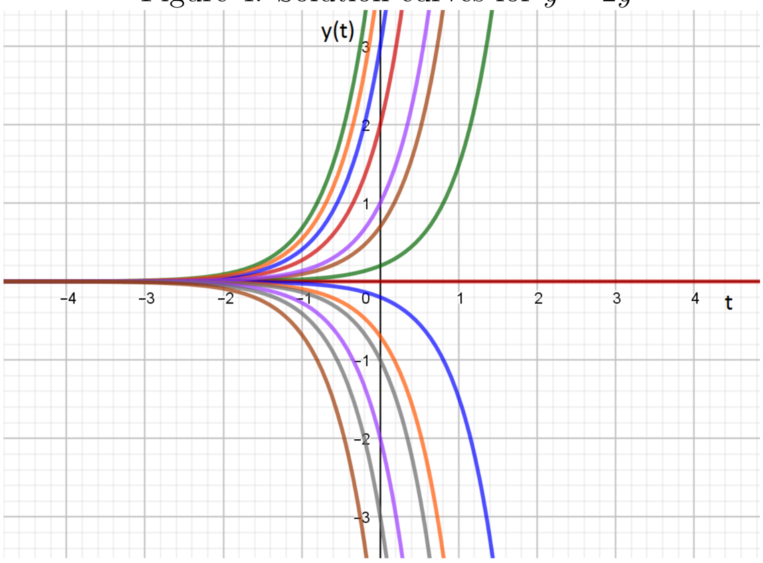
which coincides with the previously found solution for $a = b = 1$.

1.5 Qualitative/quantitative analysis of differential equations

In the previous section we already started displaying properties of differential equations graphically. Let us go into more detail here. The graph of $y(t)$, t on the x-axis, is called a solution curve. Unless we can nail down the constant C (which pops up when integrating) by some constraint on $y(t)$, the solution curve varies with the constant. This gives us an infinite set of solutions.

Example. *Consider the differential equation* $\dot{y} = 2y$ *with its solution* $y(t) = Ce^{2t}$. *For every* C *this gives a solution curve from* $t = -\infty$ *to* $t = \infty$ *(Figure 4). The whole* $t - y$ *plane is filled with solution curves. Every point*

Figure 4: Solution curves for $\dot{y} = 2y$

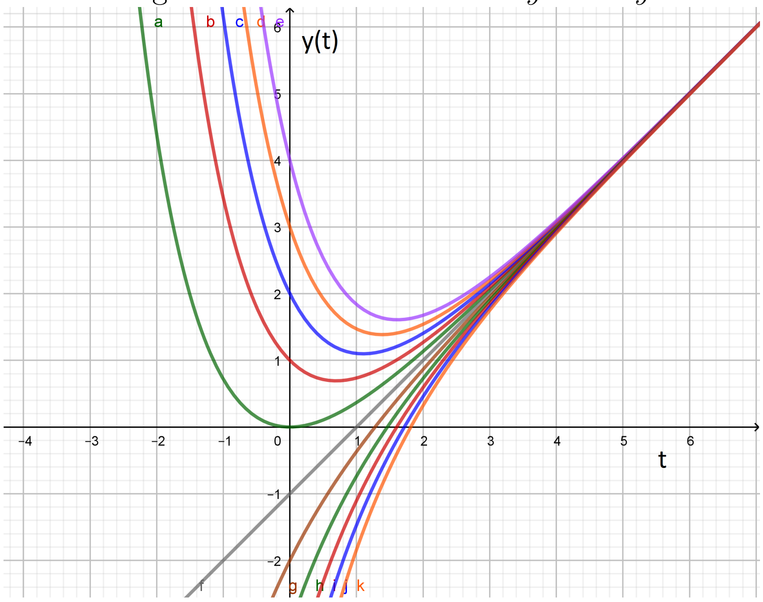


$(t, y(t))$ *in the plane has one solution curve going through it for the appropriate* C.

Unfortunately we cannot always display solution curves for differential equations since for some nonlinear differential equations there do not exist explicit solutions. What we can do in this case is to work with the expression for the differential equation itself: $\dot{y} = f(y, t)$. We can simply plot the slope of the differential equation given by $f(y, t)$ in the $t - y$ plane. The slope then indicates in which direction the solution curve is heading.

Let us go through the steps for the differential equation $\dot{y} = t - y$. Notice that this differential equation is indeed linear and we can solve it. To find the solution to the homogeneous equation set $t = 0$. To find the particular solution to the inhomogeneous equation guess that the solution takes the form $y_p(t) = at + b$, substitute the guess into the differential equation and match coefficients[3]. Verify yourself that the complete solution is $y(t) = Ce^{-t} + t - 1$. The solution curves are displayed in Figure 5. Now imagine that we could not

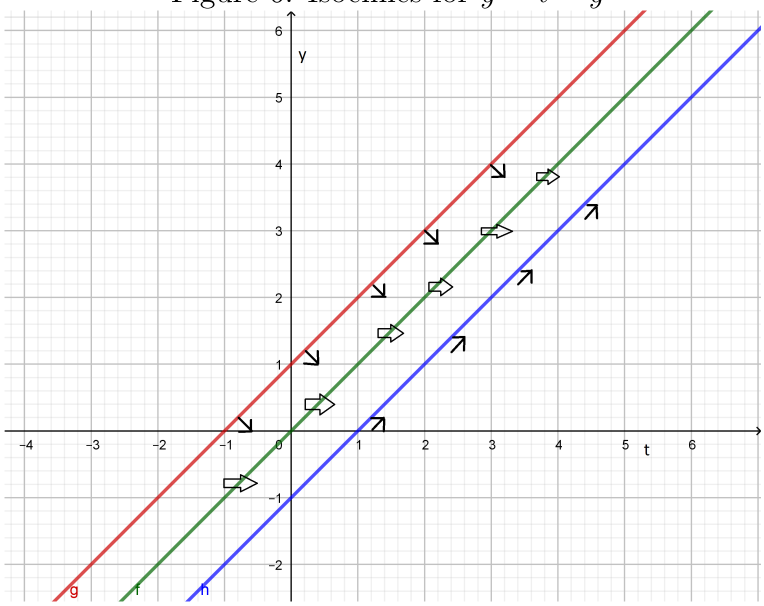
Figure 5: Solution curves for $\dot{y} = t - y$



solve the differential equation explicitly and we were to work with $\dot{y} = t - y$ only. Let us plot the slopes in the $t - y$ space. We could do this by picking different values for t and y and calculate $\dot{y}$ at each point. This is very cumbersome so here is an alternative approach: find the isoclines; lines where the slope of different solution curves to the differential equation takes on the same value. Set $\dot{y}$ equal to a constant and then solve $\dot{y} = t - y$. This gives you all the points in the $t - y$ plane where the solution curve takes on that specific slope. For this example we find that along $y(t) = t$ the slope is zero (solve $\dot{y} = 0 = t - y$), along $y(t) = t - 1$ the slope equals one and along $y(t) = t + 1$ the slope is minus one etc. If you plot the isoclines in $t - y$ space you obtain what is called the direction field or phase diagram of a differential equation (Figure 6). Compare the isoclines to the solution curves in Figure

[3] Common mistake: note that the input to the differential equation is a first order polynomial. Hence we also guess that the solution is a first order polynomial (and do not forget the constant b). Guessing that $\dot{y} = 0$ leads you down the wrong path. You are implicitly assuming that there exists a solution where $y(t)$ is constant. This is never the case for this differential equation as the solution curves in Figure 5 display.

Figure 6: Isoclines for $\dot{y} = t - y$



5 to see that the solution curves actually do take on the slopes drawn on the isoclines. If you keep adding isoclines for different slopes to Figure 6 you can obtain a good intuition about the shape of the solution curves. Use a computer program of your choice to finish this task for you. In MATLAB the "quiver" command produces more professional and cleaner looking direction fields (than Figure 6 which I drew in Microsoft Paint...) for you (see Figure 7).

Now that we have seen how to analyze a differential equation graphically, let us talk about theoretical properties of solution curves. Two question arise:

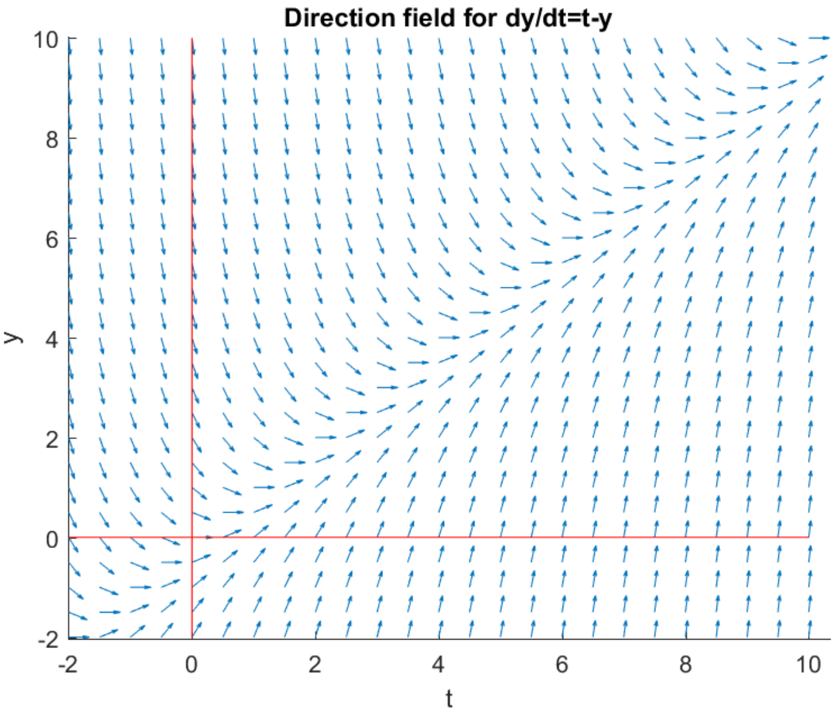
1. Starting from $y(0)$ at $t = 0$, does $\frac{dy}{dt} = f(t, y)$ always have a solution?

2. Is the solution unique i.e. could there be two or more solutions that start from the same $y(0)$?

Answering the first question: if $f(t, y)$ is a continuous function for t close to zero and y near $y(0)$, then a solution exists. To the second question: if $\frac{\partial f}{\partial y}$ is also continuous, there cannot be two solutions with the same $y(0)$. Note that existence and uniqueness of a solution curve imply that the curves cannot cross: every point in the $t - y$ space lies on exactly one solution curve.

An important point to stress is that even if f and $\frac{\partial f}{\partial y}$ are continuous at all points, solutions do not necessarily have to reach $t = \infty$. We know that there exists a solution starting at $y(0)$ and that the solution will be unique, but the solution curve could explode for some finite t. Consider the nonlinear

Figure 7: Phase diagrams in MATLAB



differential equation $\frac{\partial f}{\partial y} = y^2$ with the solution $y(t) = \frac{1}{1-t}$. To obtain the solution introduce the variable $z = y^2$ and solve the transformed differential equation in z. Go through the steps to solve the differential equation as an exercise. $f = y^2$ and its derivative $\frac{\partial f}{\partial y} = 2$ are continuous so, but the solution curve explodes at $t = 1$. To ensure that the solution does not blow up at a finite t we need to impose an upper bound L on the continuous function $\frac{\partial f}{\partial y}$: if $|\frac{\partial f}{\partial y}| \leq L \quad \forall t, y$, there exists a unique solution through $y(0)$ reaching all t.

A final note on the special case of autonomous equations. If $\dot{y}$ is independent of t, the time dimension is irrelevant for the change of the system. If we draw a phase diagram for an autonomous equation, arrows on the same horizontal line point towards the same direction since they are independent of time. This implies that all solution curves are simply horizontal transformations of each other. The solution curves for the logistic model in Figure 3 hint that this is indeed the case.

To get a clear picture of a differential equation it is always helpful to plot phase diagrams and solution curves (if possible). In the case of autonomous equations we can be more systematic:

1. Find critical points: $f(y) = 0$.

2. Plot $f(y)$ against y and determine where $\dot{y} > 0, \dot{y} < 0$.
3. Draw what is called a phase/stability line: a one dimensional space (just a straight line for y values; t dimension not needed), indicate critical points and add arrows pointing in the direction where the system is heading for each interval in between critical points.
4. Sketch solution curves

Note that information from all four steps are included in a phase diagram for an autonomous differential equations so choose yourself which diagrams you are most comfortable with.

2 Difference equations

Difference equations are the discrete time counterpart of differential equations. As we go along you will notice many similarities between difference and differential equations. A difference equation still describes how a system is changing over time. The contrast is that time is now defined in discrete steps, so $t \in \mathbb{Z}$.

2.1 First order difference equations

Given a known function f, a difference equation relates a future value of x_t to its current value (or a current value to its past):

$$\begin{aligned} x(t+1) &= f(t, x(t)) \\ \text{or} \quad x_{t+1} &= f(t, x_t) \end{aligned} \tag{14}$$

A **particular** solution of (14) is a function or sequence $(x(t))_{t=0,1,2,..}$ such that $x(t+1) = f(t, x(t))$. The **general** solution of (14) is the class of all functions or sequences $(x(t))_{t=0,1,2,..}$ that solve (14).

So how do we find a solution $(x(t))_{t=0,1,2,..}$? First of all we need some $x_0 \in \mathbb{R}$ to initialize the sequence. Given a starting point we can repeatedly apply f to obtain the full sequence. This approach is called the insertion method.

$$\begin{aligned} x_1 &= f(0, x_0) \\ x_2 &= f(1, x_1) = f(1, f(0, x_0)) \\ x_3 &= f(2, x_2) = f(2, f(1, f(0, x_0))) \\ &\vdots \end{aligned}$$

Hence, given any x_0 we can always find a solution to (14). Despite its simplicity, this approach suffers from two drawbacks. It is prone to errors. Once we have an error at t, the whole sequence is off for the ensuing time periods. Second, this approach calculates values, we do not obtain an explicit function. If we want to know the value of x_{1000} we have to calculate 999 values before. Hence, other approaches might be faster.

In some cases we are able to derive explicit functional forms as a solution to our difference equation. One of such cases is a linear equation with constant coefficients:

$$x_{t+1} = ax_t + b \tag{15}$$

where a, b are constants.

We can repeatedly use the difference equation and apply the formula for finite geometric series to see that the following solution to (15) holds $\forall t$:

$$x_t = a^t \left(x_0 - \frac{b}{1-a} \right) + \frac{b}{1-a} \tag{16}$$

with x_0 given. (16) is not defined for $a = 1$ (dividing by zero). Substituting $a = 1$ into (15) gives us $x_t = x_0 + bt$.

How do we find the steady state of the difference equation (15)? Recall that in the previous chapter on differential equations we were solving for a particular constant solution by setting $\dot{y} = 0$. In the discrete case the idea is the same. Across time periods, x_t should display no change: $x_{t+1} = x_t = x^*$. Hence, we can easily solve (15) for the unique steady state:

$$\begin{aligned} x^* =& ax^* + b \\ x^* =& \frac{b}{1-a} \end{aligned}$$

What can we say about stability of the steady state? Notice that if $|a| < 1$ in (16), for $t \to \infty$ the first term vanishes. What remains is $\frac{b}{1-a}$, the steady state. No matter where we start, sooner or later we will reach the steady state. x^* is globally asymptotically stable. If $|a| > 1$ our difference equation explodes (unless $x_t = x^*$) and hence x^* is unstable.

In the case where the coefficients a and b vary over time

$$x_{t+1} = a_t x_t + b_t \tag{17}$$

we can show by induction that the following equation provides a solution to (17):

$$x_t = \left(\prod_{s=0}^{t-1} a_s\right) x_0 + \sum_{k=0}^{t-1} b_k \left(\prod_{s=k+1}^{t-1} a_s\right) \quad \forall t \in \mathbb{N}_0 \tag{18}$$

with $\prod_{s=0}^{t-1} a_s = 1$ and $\sum_{k=0}^{t-1} b_k(\circ) = 0$ if $t-1 < s, k$. The proof can be completed in two steps: first show that (18) holds for $t = 0$. Second, assuming that (18) holds for t, substitute the solution for x_t into (17). Multiply by a_t and add b_t to see that the obtained expression equals the formula from (18) for $t+1$. Hence, we established that the solution holds $\forall t \in \mathbb{N}_0$.

One more note on the difference/differential equation comparison. Consider the differential equation $\dot{x} = ax(t) + b$. Assume now that $\dot{x} \approx x_{t+1} - x_t$ holds approximately. We can then rewrite the differential equation as a difference equation:

$$\begin{aligned} x_{t+1} - x_t &= ax_t + b \\ x_{t+1} &= (1+a)x_t + b \\ x_t &= (1+a)^t(x_0 + \frac{b}{a}) - \frac{b}{a}. \end{aligned}$$

Both the difference and the differential equation have an equilibrium state at $x^* = -\frac{b}{a}$. The distinction now is that for the steady state in the difference equation to be globally asymptotically stable we need $|1+a| < 1$. For stability in the differential equation we need $a < 0$.

The previous argument was on linear first order difference equations. What about the nonlinear case? Assume you are given the general difference equation

$$x_{t+1} = f(x_t). \tag{19}$$

A steady state for (19) is a constant x^* such that the equation $x^* = f(x^*)$ holds. An equilibrium state X^* is called (suppose $x_t = f^t(x_0)$)

- **locally asymptotically stable** if

$$\exists \epsilon > 0 : \forall x_0 \in (x^* - \epsilon, x^* + \epsilon) : \lim_{t \to \infty} x_t = x^*$$

  In words: there exists an epsilon-environment around the steady state such that we always converge to the steady state if we start within that environment.

- **globally asymptotically stable** if

$$\forall x_0 : \lim_{t\to\infty} x_t = x^*$$

  In words: no matter where we start we converge to the steady state.

- **unstable** if

$$\exists \epsilon > 0 : \forall \delta > 0 : \exists x_0, t : |x_0 - x^*| < \delta, |x_t - x^*| \geq \epsilon$$

  In words: no matter how close we start from the steady state, at one point we will be arbitrarily far off from the steady state.

One way to analyze stability properties of a difference equation is to solve it. If the difference equation is nonlinear it can turn out impossible to find an explicit solution. However, we can still assess whether a nonlinear difference equation is stable (suppose f is first-order continuously differentiable):

- if $|f'(x^*)| < 1$, then x^* is locally asymptotically stable
- if $|f'(x^*)| > 1$, then x^* is unstable.

If $|f'(x^*)| < 1$ over the whole domain of f, then x^* is **globally asymptotically stable**. To convince yourself that the proposition is intuitive graph yourself some difference equations and see what happens if the difference equation intersects the 45^o line (at the steady state) with a slope bigger or smaller than one. Think also about what happens if the slope is exactly one.

2.2 Second order difference equations

Second order difference equations take on the form

$$x_{t+2} = f(t, x_t, x_{t+1}).$$

We will start with linear equations: $x_{t+2}+a_t x_{t+1}+b_t x_t = c_t$ with $(a_t)_t, (b_t)_t, (c_t)_t$ given. Keeping the same terminology as for the differential equations, the following is an example of a **homogeneous, constant coefficients** difference equation:

$$x_{t+2} + ax_{t+1} + bx_t = 0. \tag{20}$$

We find the solution to (20) with a similar approach as we did with the homogeneous differential equation. Guess that the solution is of the form $x_t = m^t, x_{t+1} = m^{t+1} = m \times m^t, m^{t+2} = m^2 \times m^t$. (20) simplifies to:

$$\begin{aligned} m^t(m^2 + am + b) &= 0 \\ m^2 + am + b &= 0 \end{aligned}$$

which is the characteristic equation of the difference equation. It has two roots (not necessarily distinct): $m_1 = -\frac{1}{2}a + \frac{1}{2}\sqrt{a^2-4b}, m_2 = -\frac{1}{2}a - \frac{1}{2}\sqrt{a^2-4b}$. The general solution of (20) with $b \neq 0$ is:

1. If we obtain two distinct real roots to the homogeneous equation ($a^2 - 4b > 0$):
   $x_t = Am_1^t + Bm_2^t$, with $m_{1,2} = -\frac{1}{2}a \pm \frac{1}{2}\sqrt{a^2-4b}$
2. If the characteristic equation has one real double root ($a^2-4b = 0$):
   $x_t = (A+Bt)m^t$, with $m = -\frac{1}{2}a$.
3. If the root is a complex number ($a^2 - 4b < 0$):
   $x_t = r^t(A\cos(\theta t) + B\sin(\theta t))$, with $r = \sqrt{b}, \cos(\theta) = -\frac{a}{2\sqrt{b}}, \theta \in [0,\pi]$.

A and B are arbitrary coefficients. With x_0 and x_1 given the coefficients are uniquely determined. For the full proof please consult "Further Mathematics for Economic Analysis (FMEA)", page 405.

Let us generalize the previous result for homogeneous difference equations with variable coefficients: the solution for any difference equation of the type

$$x_{t+2} + a_t x_{t+1} + b_t x_t = 0 \tag{21}$$

is $x_t = Au_t + Bv_t$, where u_t, v_t are arbitrary solutions to (21) that are linearly independent ($\nexists\alpha : \alpha v_t - u_t = 0 \forall t$) and A and B are arbitrary constants.

The general solution of the nonhomogeneous equation

$$x_{t+2} + a_t x_{t+1} + b_t x_t = c_t \tag{22}$$

is $x_t = Au_t + Bv_t + w_t^*$ where (w_t^*) is an arbitrary particular solution of (22) and $(Au_t + Bv_t)$ is the general solution of the associated homogeneous equation (21).

The proof is analogous to the one for differential equations. First, we want to show that if $(u_t), (v_t)$ each solve the homogeneous equation, then $(Au_t + Bv_t)$ solves it as well.

$$\begin{aligned} x_{t+2} + a_t x_{t+1} + b_t x_t =& (Au_{t+2} + Bv_{t+2}) + a_t(Au_{t+1} + Bv_{t+1}) + b_t(Au_t + Bv_t) \\ =& A(u_{t+2} + a_t u_{t+1} + b_t u_t) + B(v_{t+2} + a_t v_{t+1} + b_t v_t) \\ =& A \cdot 0 + B \cdot 0 = 0 \end{aligned}$$

Second, let (x_t) be the general- and (w_t^*) a particular solution to (22). Then $(x_t - w_t^*)$ solves the homogeneous equation (21). Hence, it must hold that $(x_t - w_t^*) = Au_t + Bv_t \Leftrightarrow x_t = Au_t + Bv_t + w_t^*$.

Do you see the resemblance to the differential equations? The solution to the nonhomogeneous equation is the general solution to the homogeneous equation plus a particular solution to the inhomogeneous equation: it is the superposition principle over again! This makes our life easy since to obtain the solution to a linear second order equation we first find two linearly independent solutions of the homogeneous equation and second we look for a particular solution of the nonhomogeneous equation to obtain the general solution. The two solutions to the homogeneous equation must be linearly independent. This is why we cannot use the same root twice if the characteristic equation has only one real root. To check for independence the following result can be of help: $u_t^{(1)}$ and $u_t^{(2)}$ are linearly independent if and only if $\det(M) \neq 0$, where M is defined by

$$M = \begin{bmatrix} u_0^{(1)} & u_0^{(2)} \\ u_1^{(1)} & u_1^{(2)} \end{bmatrix}.$$

Example. *Let us do three examples: solve the following homogeneous difference equations*

1. $x_{t+2} - 5x_{t+1} + 6x_t = 0$.
Set up the characteristic equation $m^2 - 5m + 6 = 0$ *which has the two distinct and real roots* $m_1 = 2, m_2 = 3$. *Hence, the complete solution to the difference equation is* $x_t = A2^t + B3^t$.

2. $x_{t+2} - 6x_{t+1} + 9x_t = 0$.
The characteristic equation becomes $m^2 - 6m + 9$. *The double root is* $m = 3$. *To obtain two independent solutions to the homogeneous equation we multiply the solution we found,* 3^t*, by* t *(which we know will also satisfy the homogeneous equation) to find the second independent solution. The complete solution becomes* $x_t = (A + Bt)3^t$.

3. $x_{t+2} - x_{t+1} + x_t = 0$.
Intuitive (long) way: the characteristic equation $m^2 - m + 1 = 0$ *has the complex roots* $m_{1,2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}\sqrt{-1} = 1 \cdot (\cos(\frac{\pi}{3}) \pm \sin(\frac{\pi}{3}))$. *By de Moivre's formula*[4]: $m_1^t = \cos(\frac{\pi}{3}t) + \sin(\frac{\pi}{3}t)$ *and* $m_2^t = \cos(\frac{\pi}{3}t) - \sin(\frac{\pi}{3}t)$. *The complex functions* m_1^t *and* m_2^t *both satisfy the difference equation, and so does every complex linear combination of them. Hence, the complete solution is* $x_t = A\cos(\frac{\pi}{3}t) + B\sin(\frac{\pi}{3}t)$.
Short way: just plug in the numbers into the given solution from above

[4] $(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta), \quad n \in \mathbb{N}$

by $x_t = r^t(A\cos(\theta t) + B\sin(\theta t))$*, with* $r = \sqrt{b}, \cos(\theta) = -\frac{a}{2\sqrt{b}}, \theta \in [0, \pi]$ *where* a *and* b *are the coefficient on* x_{t+1} *and* x_t *respectively in the difference equation. The solution you obtain is obviously the same.*

Once we know how to solve homogeneous difference equations, the extension to nonhomogeneous difference equations with constant coefficients is straightforward. First, find the general solution of the associated homogeneous equation. Second, find a particular solution to the nonhomogeneous equation by the guess and verify method (aka method of undetermined coefficients), which you are already familiar with from differential equations. Suppose the difference equation is of the form

$$x_{t+2} + a_t x_{t+1} + b_t x_t = c_t, \tag{23}$$

where c_t is a polynomial of degree n in t. Again, a good start is to guess that the particular solution, w_t, is a polynomial of degree n as well.

$$w(t) = \gamma_0 + \gamma_1 t + \gamma_2 t^2 + \cdots + \gamma_n t^n$$

Even if some monomials are missing in c_t, do not forget to include them in w_t. Once we have set our guess, substitute it into the difference equation

$$w(t+2) + aw(t+1) + bw(t) = c_t \tag{24}$$

and solve for the coefficients $\gamma_0, \gamma_1, \ldots, \gamma_n$ such that (24) holds. The method of undetermined coefficients works as long as c_t is a sum, difference, or product of polynomials, exponential functions or trigonometric functions.

Example. *Find the general solution of* $x_{t+2} - 5x_{t+1} + 6x_t = t^2 + 3$*. We have already solved for the general solution of the associated homogeneous difference equation, namely* $x_t = A2^t + B3^t$*. Hence, all we still need to do to find the general solution of the nonhomogeneous equation is to determine a particular solution to the nonhomogeneous equation. For that guess that the particular solution is of the form* $at^2 + bt + c$ *and plug it into the difference equation to obtain*

$$a(t+2)^2 + b(t+2) + c - 5(a(t+1)^2 + b(t+1) + c) + 6(at^2 + bt + c) = t^2 + 3.$$

Multiplying out the terms and summing them together gives

$$2at^2 + (2b - 6a)t - a - 3b + 2c = t^2 + 3.$$

Comparing both LHS and RHS we obtain three equations to be fulfilled such that our guess constitutes a solution:

$$\begin{aligned} 2a =& 1 \quad &\Leftrightarrow a = \frac{1}{2} \\ 2b - 6a =& 0 \quad &\Leftrightarrow b = \frac{3}{2} \\ -a - 3b + 2c =& 3 \quad &\Leftrightarrow c = 4. \end{aligned}$$

The complete solution is $x_t = A2^t + B3^t + \frac{1}{2}t^2 + \frac{3}{2}t + 4$.

Similarly to the previous case of differential equations one can ask the question: when is a solution to a difference equation stable? Consider the general solution $x_t = Au_t + Bv_t + w_t$. The solution is globally asymptotically stable if $\forall A, B \in \mathbb{R}$:

$$\lim_{t\to\infty} Au_t + Bv_t = 0.$$

In words, no matter what A and B are, the solution of the homogeneous equation converges to zero. Notice that given two conditions on the solution, the constants are uniquely determined. Since global asymptotic stability implies that the solution converges to zero independently of the constants, it does not matter what the two conditions are. If the two conditions are e.g. starting points, then independently of where we start, the system converges to the same point. To determine whether a difference equation is stable the following theorem might be helpful.

Given a constant coefficients linear difference equation $x_{t+2} + ax_{t+1} + bx_t = c_t$, the solution is globally asymptotically stable if and only if either

- the roots of the characteristic equation $m^2 + am + b = 0$ have moduli strictly less than one
- or $|a| < 1 + b$ and $b < 1$.

The modulus is defined as the absolute value of a real number or as $\sqrt{x^2 + y^2}$ if the number is complex: $x + iy$.

2.3 Systems of first order difference equations

Suppose you are given the following system of difference equations:

$$\begin{aligned} x_1(t+1) =& a_{11}(t)x_1(t) + \cdots + a_{1n}(t)x_n(t) + b_1(t) \\ &\vdots \\ x_n(t+1) =& a_{n1}(t)x_1(t) + \cdots + a_{nn}(t)x_n(t) + b_n(t). \end{aligned}$$

To rewrite the system in matrix form define
$$x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \dots & a_{nn}(t) \end{pmatrix}, \quad b(t) = \begin{pmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{pmatrix}.$$ The system can then be written as

$$x(t+1) = A(t)x(t) + b(t) \quad \forall t.$$

Keep iterating to obtain the expression (and assume that $A(t) = A$: the system has constant coefficients)

$$x(t) = A^t x(0) + \sum_{k=1}^{t} A^{t-k} b(k-1) \tag{25}$$

as long as you analogously define A^0 to be the identity matrix, I_n. There are two ways to calculate $x(t)$. The first one is the "brute force" method by calculating A^t. This computation is simplified if A is diagonalizable. Remember from the previous Maths class that an $n \times n$ matrix is diagonalizable if it has n linearly independent eigenvectors. Stack the eigenvectors as a matrix containing in each column a different eigenvector and denote this matrix by P. It then follows that

$$A = P \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} P^{-1},$$

where the λ_i correspond to the n distinct eigenvalues of A. We can then calculate the power of A in an easy fashion:

$$A^2 = P \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \underbrace{P^{-1}P}_{I_n} \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} P^{-1} = P \begin{pmatrix} \lambda_1^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_n^2 \end{pmatrix} P^{-1}.$$

Generalize this to

$$A^t = P \begin{pmatrix} \lambda_1^t & & 0 \\ & \ddots & \\ 0 & & \lambda_n^t \end{pmatrix} P^{-1}. \tag{26}$$

The system of difference equations is called globally asymptotically stable if $\forall x(0) \in \mathbb{R}^n : \lim_{t\to\infty} A^t x(0) = 0$. We see from (26) that this is the case if and only if $\lambda_1, \dots, \lambda_n$ have moduli strictly less than one.

Now suppose the difference equation is stable so that $\lim_{t\to\infty} A^t x(0) = 0$ and the coefficient vector $b(t) = b$ is also constant. In the limit (25) turns into

$$\begin{aligned} \lim_{t\to\infty} x(t) =& b + Ab + A^2 b + A^3 b + A^4 b + \dots \\ =& \underbrace{(1 + A + A^2 + A^3 + A^4 + \dots)}_{V} b. \end{aligned}$$

Let us determine what the series V is. First note that since we assume that the system is stable we know that all eigenvalues of A have moduli strictly less than one. Hence, the powers of A will not explode and the series V is well defined. To find V we write it recursively. This is a trick which you can apply to find the limit of any converging series. See that the following equation holds:

$$\begin{aligned} V =& 1 + AV \\ \Leftrightarrow (I_n - A)V =& 1 \\ \Leftrightarrow V =& (I_n - A)^{-1} \end{aligned}$$

and therefore $\lim_{t\to\infty} x(t) = (I_n - A)^{-1} b$, which is the stationary solution. Hence, with constant coefficients, if the system is globally asymptotically stable ($\Leftrightarrow$ moduli of eigenvalues < 1), it converges to the stationary solution $x = (I_n - A)^{-1} b$. So when are the moduli <1? You can use the following sufficient condition: if $\forall i = 1, \dots, n : \sum_{j=1}^n |a_{ij}| < 1$ where a_{ij} denotes the entry in matrix A in line i and column j, then all eigenvalues of A have moduli <1.

You may wonder whether there is a more elegant way of solving systems of difference equations than calculating the powers of a matrix by brute force. Recall that we solved systems of differential equations using a change of variables approach using the matrix of eigenvectors to decouple the system. Is there a similar solution approach to systems of difference equations? Yes there is!

Suppose the system of difference equations is given by

$$\mathbf{z_{t+1}} = \mathbf{A z_t},$$

where entries of $\mathbf{A}$ not lying on the diagonal are nonzero so that we are in fact dealing with a system. "Of course" you come up with the right guess $\mathbf{z} = \mathbf{PZ} \Leftrightarrow \mathbf{Z} = \mathbf{P^{-1}z}$ as a change of variables. We then rewrite $\mathbf{Z_{t+1}}$ as

$$\begin{aligned}
\mathbf{Z_{t+1}} =& \mathbf{P^{-1}z_{t+1}} && \text{using our guess}\\
=& \mathbf{P^{-1}Az_t} && \text{using the original difference equation}\\
=& \mathbf{P^{-1}APZ_t} && \text{using our guess.}
\end{aligned}$$

Using the change of variables we transformed $\mathbf{z_{t+1}} = \mathbf{Az_t}$ into $\mathbf{Z_{t+1}} = \mathbf{P^{-1}APZ_t}$. At first glance the transformation looks more sophisticated than the previous formulation, so when does the transformation turn out useful? Exactly in the case where $\mathbf{P^{-1}AP}$ is a diagonal matrix so that the transformed system is a collection of decoupled difference equations. If we choose $\mathbf{P}$ as the matrix containing the eigenvectors, $\mathbf{v_i}$, of $\mathbf{A}$ in each column, then $\mathbf{P^{-1}AP} = \mathbf{D}$, where $\mathbf{D}$ is the diagonal matrix containing the eigenvalues, r_i, of $\mathbf{A}$ as entries. The uncoupled system then has the solution

$$\begin{aligned}
Z_t^1 =& c_1 r_1^t\\
&\vdots\\
Z_t^k =& c_k r_k^t.
\end{aligned}$$

We have now solved for the transformed variables $\mathbf{Z}$. As a last step use the original transformation to back out the variables we wanted to solve for in the first place.

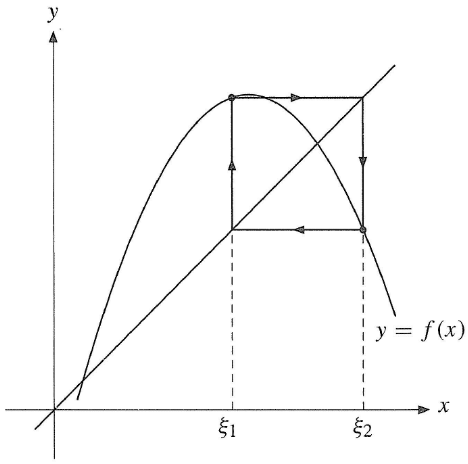
$$\mathbf{z_t} = \begin{pmatrix} z_t^1 \\ \vdots \\ z_t^k \end{pmatrix} = \mathbf{P} \begin{pmatrix} Z_t^1 \\ \vdots \\ Z_t^k \end{pmatrix} = [\mathbf{v_1} \dots \mathbf{v_k}] \begin{pmatrix} c_1 r_1^t \\ \vdots \\ c_k r_k^t \end{pmatrix} = \mathbf{v_1} c_1 r_1^t + \cdots + \mathbf{v_k} c_k r_k^t$$

The change of variables approach shows that $\mathbf{z_t}$ can be written as a combination of eigenvectors and eigenvalues of $\mathbf{A}$. Notice the resemblance to systems of differential equations we went through in the previous chapter.

3 Cycles

A cycle of $x_{t+1} = f(x_t)$ with period two is a solution such that $f(f(x_t)) = x_t$. Hence, $x_{t+2} = x_t$ with $x_{t+1} \neq x_t$. The difference equation admits a cycle of period two iff $f(x_t) = x_{t+1}$ and $f(x_{t+1}) = x_{t+2} = x_t$ with $x_{t+2} \neq x_{t+1}$. See Figure 8 for a graphical representation. The cycle is said to be locally asymptotically stable if every solution that comes close to the cycle points

Figure 8: Cycle of period two



converges to the cycle. For stability we need that the difference equation $F(x_t) = f(f(x_t))$ is stable around the cycle points. Using the proposition on stability for nonlinear difference equations, the cycle is locally asymptotically stable if $F'(x_t) = f'(f(x_t))f'(x_t) = |f'(x_{t+1})f'(x_t)| < 1$.

Suppose we work with a quadratic difference equation:

$$f(x_t) = x_{t+1} = ax_t^2 + bx_t + c. \tag{27}$$

If there are steady states, they are given by

$$x_{1,2}^* = \frac{1 - b \pm \sqrt{(b-1)^2 - 4ac}}{2a}.$$

Insert x^* into (27) and apply the *abc* formula (*pq* formula) to find the roots. (27) admits a cycle of period two if we find an x_t and x_{t+1} such that $f(x_t) = x_{t+1}$ and $f(x_{t+1}) = x_t$. These numbers will solve $F(x_t) = f(f(x_t)) = x_t$. Remember that $f(x_t)$ is a polynomial of degree two so solving $F(x_t) = x_t$ implies finding the root of a fourth order polynomial. Luckily we know that any point that solves $x_t = f(x_t)$ also solves $x_t = f(f(x_t)) \Leftrightarrow x_t - f(f(x_t)) = 0$. We can therefore factor out the polynomial[5] $x_t - f(x_t)$ to rewrite

$$x_t - f(f(x_t)) = (x_t - f(x_t))g(x_t),$$

[5]If you have fun doing it apply the difference equation twice and factor out to see that the equation indeed holds.

where $g(x_t) = a^2x_t^2 + a(b+1)x_t + ac + b + 1$. Now apply the *abc* formula to $g(x_t) = 0$ to find your cycle points:

$$x^*_{t,t+1} = \frac{-(b+1) \pm \sqrt{(b-1)^2 - 4ac - 4}}{2a}.$$

Following the discussion on complex numbers, the cycle points exist and are distinct iff $(b-1)^2 > 4ac + 4$. To assess stability of the cycle points note that

$$\begin{aligned}
|f'(x_{t+1})f'(x_t)| =& |(2ax_{t+1} + b)(2ax_t + b)| \\
=& |4a^2 x_{t+1x_t} + 2ax_{t+1}b + 2abx_t + b^2| \\
=& |4(ac+1) - b^2 - 2b|
\end{aligned}$$

so for local stability we require $|4(ac+1) - b^2 - 2b| < 1$.

4 Literature

Please consult the syllabus for a complete list of the course literature. In this script I made use of

- Sydsæter, K., Hammond, P., Seierstad, A., & Strom, A. (2008). *Further mathematics for economic analysis.* Pearson education.
- Simon, C. P., & Blume, L. (1994). *Mathematics for economists (Vol. 7).* New York: Norton.

Other scripts that I have not made use of, but highly recommend to read through at some point are John Hassler's script (on Maths 3/formerly Maths 2) and Mark Voorneveld's Maths 2 script (the chapter on complex numbers).

Let me end this script with some personal advice. When I took the course it turned out to be very helpful to just google concepts/theorems and read through lecture-material that you find online. The same holds for the books mentioned in the syllabus. Try to read as many as possible of them. Often authors stress different aspects of the theory. Reading someone else's approach to differential equations (difference equations, solutions to systems etc.) will make you think of concepts in a different way. Also, only going through the material over and over again will make you understand the theory in detail. Even reading the course material again as a preparation for TA'ing this course helped me to connect some concepts from Maths 3 and Macro 1 in a way I had previously not thought of...