Recent Changes in Firm Dynamics and the Nature of Economic Growth*

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Abstract

This paper documents a novel observation on firm growth in high-quality administrative data: cumulative sales and employment growth over a firm's life cycle has systematically increased since the late 1990s. I rationalize these trends in a model of creative destruction, where incumbent firms and potential entrants replace competing firms in new product markets through successful innovations. Incumbent firms innovate at heterogeneous rates that arise endogenously. Through the lens of the model, the changes in firm growth provide three insights into recent trends in aggregate productivity growth. First, the slowdown in productivity growth is mainly due to changes in incumbents' and entrants' R&D costs. Second, innovation by entrants has declined since the 1990s, while average incumbent innovation has increased. Third, rising average incumbent innovation is due to increasing innovation rates within incumbents and a reallocation of sales shares to incumbents that innovate at systematically higher rates.

Keywords: Productivity growth, Firm dynamics, Reallocation, Administrative data

JEL codes: D22, E24, O31, O47, O50

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1 Introduction

The U.S. economy has experienced several macroeconomic trends over the last decades: productivity growth has declined, sales concentration within industries has risen, and firm entry has fallen.[1] The U.S. economy is not an outlier; similar trends have been documented for many advanced economies worldwide.[2] This paper provides new insights about these macroeconomic trends based on a novel observation in high-quality administrative data at the firm level: for the universe of Swedish firms, cumulative sales and employment growth over the firm's life cycle (henceforth, firm life cycle growth) has systematically increased since the late 1990s. Theories of firm dynamics that map from the micro- to the macroeconomy imply that changes at the firm level contain information about economic aggregates. To this end, the changes in firm dynamics inform about the observed macroeconomic trends, such as the forces behind the slowdown in productivity growth. Further, changes in firm dynamics provide insights into how these forces transmit from the micro- to the macroeconomy. Whether this occurs, for instance, through changes within the firm or between firms matters not only for policy but also for economic modeling as it guides which features models of firm dynamics should include.

The first contribution of this paper is empirical. I document a new stylized fact about firm growth using administrative data from tax records: life cycle growth of firm sales and employment accelerated. Over the first eight years of the firm, sales increased by 55.9 percent for firms established in the late 1990s compared to 67.4 percent for the cohorts of the early 2010s. For employment growth, these differences are even more significant. Firm employment increased by 28.8 percent over the first eight years for the cohorts of the late 1990s compared to 46.6 percent for the cohorts of the 2010s. What do the changes in firm growth imply about the slowdown in aggregate productivity growth? I view the firm-level changes through the lens of a structural model and analyze their implications for the macroeconomy.

The model includes the following three elements. First, the model features a link between firm dynamics and economic growth in the spirit of Schumpeterian growth models (Aghion and Howitt, 1992; Grossman and Helpman, 1991; Klette and Kortum, 2004): incumbent firms and potential entrants gain sales shares by replacing competing firms in new product markets through innovation (creative destruction).[3] Second, in standard models of creative destruction with constant markups, firm sales and employment growth are identical. In line with the data, I include a second type of product innovation that permits differential sales and employment growth (and changes therein). This type of innovation (internal R&D)

[1] Autor, Dorn, Katz, Patterson and Van Reenen (2017), Grullon, Larkin and Michaely (2019) and Akcigit and Ates (2021) document rising sales concentration in the U.S. The decline in firm entry is documented in Decker, Haltiwanger, Jarmin and Miranda (2016); Gourio, Messer and Siemer (2014); Karahan, Pugsley and Şahin (2022).

[2] See Andrews, Criscuolo and Gal (2016); Gopinath, Kalemli-Özcan, Karabarbounis and Villegas-Sanchez (2017); Autor, Dorn, Katz, Patterson and Van Reenen (2020); Karabarbounis and Neiman (2014); Engbom (2023).

[3] These models are analytically tractable yet capture salient features of firm dynamics (Lentz and Mortensen, 2008; Akcigit and Kerr, 2018).

allows incumbent firms to distance their competitors vertically in the product space and, in equilibrium, charge a higher markup as in Peters (2020).[4] Markup growth drives a wedge between firm sales and employment growth. Third, the model includes permanent differences in firm productivity as in Aghion, Bergeaud, Boppart, Klenow and Li (2023). As an equilibrium outcome, permanent differences in productivity generate heterogeneity in innovation rates and, hence, expected life cycle trajectories that differ across firms (Sterk, Sedláček and Pugsley, 2021).[5] This allows the model to explain the observed changes in firm life cycle growth through changes in (within) firm growth and firm composition. The composition of firms has aggregate implications: the reallocation of sales shares across firms that innovate at systematically different rates affects long-run growth (Acemoglu, Akcigit, Alp, Bloom and Kerr, 2018). Disciplined by the changes in firm life cycle growth, the model answers how much such reallocation across firms that innovate at different rates has contributed to changes in long-run economic growth since the 1990s. The richness of the Swedish data allows me to leverage information on the capital stock and intermediate input usage for the universe of firms to provide suggestive evidence that differences in firm life cycle growth arise from permanent productivity heterogeneity.

I estimate the model on a balanced growth path (BGP) using Swedish administrative data matching firm sales and employment growth of cohorts in the late 1990s and other macroeconomic moments. As a comparative statics experiment, I re-estimate two parameters of the model to match the acceleration of firm sales and employment life cycle growth of the latest cohorts in the data. The estimation highlights a rise in the cost of entry (+22%) and an increase in incumbents' internal R&D costs (+51%) as the cause behind the acceleration of firm life cycle growth. Rising firm entry costs incentivize incumbent firms to expand into new product lines, accelerating their sales *and* employment growth. In contrast, the fall in the internal R&D productivity slows firm markup growth, accelerating firm employment *relative* to sales growth. Whereas the rise in entry costs accounts for the joint acceleration in firm sales and employment growth, the fall in the internal R&D productivity explains the relative acceleration of employment growth. The rise in entry costs is consistent with Davis (2017) and Gutiérrez and Philippon (2018), who argue that the increasing complexity of regulatory requirements and lobbying expenditures disadvantage entrants. The rise in internal R&D costs further relates to Bloom, Jones, Van Reenen and Webb (2020), who document that falling R&D productivity is a pervasive feature of the U.S. economy.[6]

The structural model quantifies the aggregate effects of the rise in the firm entry and incumbent R&D costs, both over the transition period and in the long run. Despite the symmetric nature (to the firm-productivity type) of the R&D and entry cost changes, more productive

[4]Similarly, Akcigit and Kerr (2018) features a quality-ladder model with creative destruction and innovation within product markets.

[5]Other examples of creative destruction frameworks with heterogeneous innovation rates across firms include Lentz and Mortensen (2008), Acemoglu, Akcigit, Alp, Bloom and Kerr (2018) and De Ridder (2024). In these models, the heterogeneity arises from systematic differences in the step size of quality improvements or the firms' cost structure.

[6]Olmstead-Rumsey (2019) also documents evidence of declining innovativeness.

firms steal product markets from less productive ones, resulting in a reallocation of sales shares. The reallocation to more productive firms that feature relatively low labor shares and high markups is consistent with Kehrig and Vincent (2021), De Loecker, Eeckhout and Unger (2020), and Baqaee and Farhi (2020). Alongside rising concentration, the firm entry rate drops by eight percentage points (pp), and the aggregate growth rate declines by 0.62pp in the long run. Economic growth declines gradually over the transition, resulting in a welfare loss. The decline in growth and firm entry account for roughly 60% of the measured decline in TFP growth and 80% of the decline in the firm entry rate in Sweden over the last three decades (Engbom, 2023).

The acceleration of sales and employment life cycle growth points towards changes in the cost of firm entry and incumbent R&D as the driver behind recent macroeconomic trends. These changes rationalize the acceleration of firm life cycle growth and, at the aggregate level, generate a slowdown in economic growth, falling firm entry, and rising concentration in line with the data.

I decompose the implied fall in economic growth into the different contributions by incumbent firms and entrants. Changes in long-run growth are due to (i) changes in incumbents' innovation rates (holding sales shares constant), (ii) reallocation of sales shares across incumbents that innovate at different rates, and (iii) changes in firm entry. Changes in incumbent R&D (the first two channels) and firm entry work in opposite directions. First, incumbent innovation rates increase (i). The rise in entry costs boosts incumbents' innovation more than rising internal R&D costs lower it. Second, the reallocation of sales shares to more productive firms contributes positively to long-run growth (ii). This is because more productive firms innovate at systematically higher rates in equilibrium. Channel (ii) exceeds (i), suggesting that standard models of creative destruction in which firms innovate at identical rates miss the main channel through which incumbents have affected long-run economic growth since the 1990s. As incumbent R&D contributes positively, the fall in firm entry (iii) more than accounts for the 0.62pp fall in the aggregate growth rate. The fact that a decrease in firm entry drives the fall in economic growth holds over the transition period, where falling entry further dominates rising average productivity. These results are robust to an alternative estimation, where an increase in the productivity dispersion explains the magnitude of the decline in U.S. TFP growth as in Aghion, Bergeaud, Boppart, Klenow and Li (2023).

Related Literature. The comparative statics exercise is linked to recent studies explaining trends in the U.S. economy. Proposed drivers for these trends are increasing costs of R&D (Bloom, Jones, Van Reenen and Webb, 2020), increasing barriers to entry (Davis, 2017; Gutiérrez and Philippon, 2018), or rising productivity dispersion (Aghion, Bergeaud, Boppart, Klenow and Li, 2023).[7] The approach in this paper differs as the comparative statics

[7] Further explanations include the increasing importance of intangible capital and information and communications technology (ICT) (Crouzet and Eberly, 2019; Chiavari and Goraya, 2020; De Ridder, 2024; Hsieh and Rossi-Hansberg, 2023; Weiss, 2019), declining interest rates (Chatterjee and Eyigungor, 2019; Liu, Mian and Sufi, 2022), changes in the quality of ideas (Olmstead-Rumsey, 2019) or declining imitation

estimation is informed by changes in firm life cycle growth. It turns out that the rise in the entry costs and fall in research productivity, estimated to match the changes in firm life cycle growth, is consistent with recent trends in the macroeconomy.

Peters and Walsh (2021) further highlight demographic forces behind recent trends in the U.S. economy.[8] In Peters and Walsh (2021), a decline in population growth explains the fall in productivity growth, the rise in product market concentration, and the fall in the entry rate. Population growth in Sweden gradually increased over the period of study for two decades despite rising concentration, falling firm entry, and declining long-run productivity growth. This suggests that, at least for the Swedish economy, falling population growth is not the driving force behind the macroeconomic (or firm-level) trends. Nevertheless, an increase in firm life cycle growth is also implied in their theory.

The findings further relate to a literature that emphasizes the effects of reallocation on economic growth. China and East Germany are examples where long-term sustained growth followed the reallocation of market shares from state-owned enterprises to privately held companies (Song, Storesletten and Zilibotti, 2011; Findeisen, Lee, Porzio and Dauth, 2021). This reallocation potentially affects GDP per capita in a static sense through two channels. First, more productive firms gain market shares, thereby raising average productivity, and second, by reducing the extent of misallocation of production factors in the spirit of Hsieh and Klenow (2009). However, the reallocation could also affect the economy's long-run growth rate if privately held firms innovate (or imitate) at higher rates than state-owned enterprises (Acemoglu, Akcigit, Alp, Bloom and Kerr, 2018). I account for the effect of reallocation on economic growth through all three channels: over the transition to the new balanced growth path, the reallocation of sales shares across firms affects aggregate output through changes in average productivity, misallocation, and quality growth. The growth decomposition shows that incumbent firms have mainly contributed to changes in long-run growth since the 1990s through reallocation effects.

Akcigit and Kerr (2018), Garcia-Macia, Hsieh and Klenow (2019), and Peters (2020) decompose economic growth into the contributions by entrants and incumbent firms. These studies conclude that economic growth is mainly due to incumbent firms rather than entrants. While this is also the case in the parametrized model in this paper, I show that entrants rather than incumbents account for the *fall* in economic growth since the late 1990s. This finding is consistent with the observation in Garcia-Macia, Hsieh and Klenow (2019) that the share of economic growth accounted for by entrants has declined in the U.S. from the 1990s to the 2010s.[9]

rates (Akcigit and Ates, 2019).

[8] Bornstein (2018), Engbom (2023), Hopenhayn, Neira and Singhania (2022), Karahan, Pugsley and Şahin (2022) further emphasize the role of demographic forces behind macroeconomic trends.

[9] Bartelsman and Doms (2000), Haltiwanger, Foster and Krizan (2001), Lentz and Mortensen (2008) and Acemoglu, Akcigit, Alp, Bloom and Kerr (2018) decompose productivity growth further into within- and between firm effects. This paper studies which of these channels explains the *changes* in productivity growth since the 1990s.

The empirical results further relate to Karahan, Pugsley and Şahin (2022) and Hopenhayn, Neira and Singhania (2022), who document that firm employment conditional on age has been relatively constant in the U.S. since the 1980s. Karahan, Pugsley and Şahin (2022) report this stability for firms up to age ten, noting that for firms older than ten, firm size (conditional on age) increases significantly over time when holding the industry composition constant. An increase in firm size conditional on age over time implies that more recently established firms grow faster, as documented in this paper. I report the size-conditional-on-age patterns in Swedish administrative data. These raw plots already display an increase in average firm size (conditional on age) over time. The acceleration in firm life cycle growth becomes even more apparent when using firm-level regressions to measure life cycle growth within industries.[10]

Sterk, Sedláček and Pugsley (2021) document changes in life cycle growth for U.S. firms over time. For the cohorts 1979 to 1993, the authors show that employment growth over the firm's life cycle slowed. The results presented in this paper are complementary rather than contradictory to theirs as I document trends for the cohorts from 1997 to 2017, suggesting a reversal of the previous trends. The rise in industry concentration, and the fall in firm entry accelerated strongly during the turn of the millennium, as shown by Autor, Dorn, Katz, Patterson and Van Reenen (2020), and Akcigit and Ates (2021). Firm-level changes during this period are particularly useful to understand the forces behind these macroeconomic trends.

The paper proceeds as follows. Section 2 documents the acceleration in firm life cycle growth, and Section 3 lays out the model. In Section 4, I apply the model to study the aggregate implications of the changes in firm growth. This includes first a balanced growth path analysis followed by an investigation of the transitional dynamics in Section 5. Section 6 provides robustness, and Section 7 concludes.

2 Changes in firm life cycle growth

This section documents systematic changes in sales and employment growth over the firms' life cycle. I describe the data in a first step.

2.1 Data

All data is provided by Statistics Sweden (SCB), the official statistical agency in Sweden. The main data set is *Företagens Ekonomi* (FEK), which covers information from balance sheets and profit and loss statements for the universe of Swedish firms. The unit of observation is the legal unit at an annual frequency covering the period 1997-2017. FEK contains the main

[10] Van Vlokhoven (2021) further documents that profits and sales of firms in Compustat data have become more back-loaded. While I share the observation that the sales growth over the firm's life cycle accelerated, I find firm size at entry relatively constant over time as Karahan, Pugsley and Şahin (2022) and Hopenhayn, Neira and Singhania (2022).

variables of interest: sales and employment (in full-time units). Before 1997, FEK was a sample covering large Swedish firms. To ensure full representativeness, I focus on the years 1997 forward. The data further contains information on the firm's legal type and industry at the five-digit level. I focus on firms in the private economy. Throughout the paper, nominal variables are deflated to 2017 Swedish Krona (SEK) using the GDP deflator. For a detailed description of the data, see Section A in the Appendix.

Table 1: Summary statistics (1997-2017)

	25th Pct.	50th Pct.	75th Pct.	Mean	SD	Obs.
*Sales**	1.2	2.7	7.8	27.8	568.2	4,918,996
*Value added**	0.5	1.1	2.9	7.6	142.3	4,918,996
Employment	1	2	5	9.9	131.1	4,918,996
*Wage bill**	0.2	0.6	1.6	3.7	53.0	4,918,996
*Capital stock**	0.04	0.2	1.1	9.3	277.0	4,918,996
*Intermediate Inputs**	0.4	0.9	2.6	10.8	270.0	4,918,996

Note: variables marked with * are in units of million 2017-SEK (1 SEK $\approx$ 0.1 US dollars). The capital stock is defined as fixed assets minus depreciation.

I define the birth year of the firm as the year it hires its first employee. I obtain this information from the auxiliary data set *Registerbaserad Arbetsmarknadsstatistik* (RAMS), containing the universe of employer-employee matches. I further restrict myself to firms that employ at least one worker, according to RAMS.[11]

Table 1 reports distributional statistics of firm sales, value added, and production inputs for the pooled data (1997 to 2017). The median firm has sales of roughly 2.7 million SEK (approx. 0.27 million US dollars), value added of 1.1 million SEK, and employs two workers. The distribution of sales, value added, and all production inputs is highly right-skewed, as indicated by the mean and the 25th, 50th, and 75th percentiles. Average firm sales are 27.8 million SEK, and average employment is 9.9. In total, the data includes about 4.9 million firm-year observations. For the age-specific empirical analysis, I focus on firms established in 1997 or later, which reduces the sample size to 2.2 million firm-year observations. For these firms, age is not truncated.

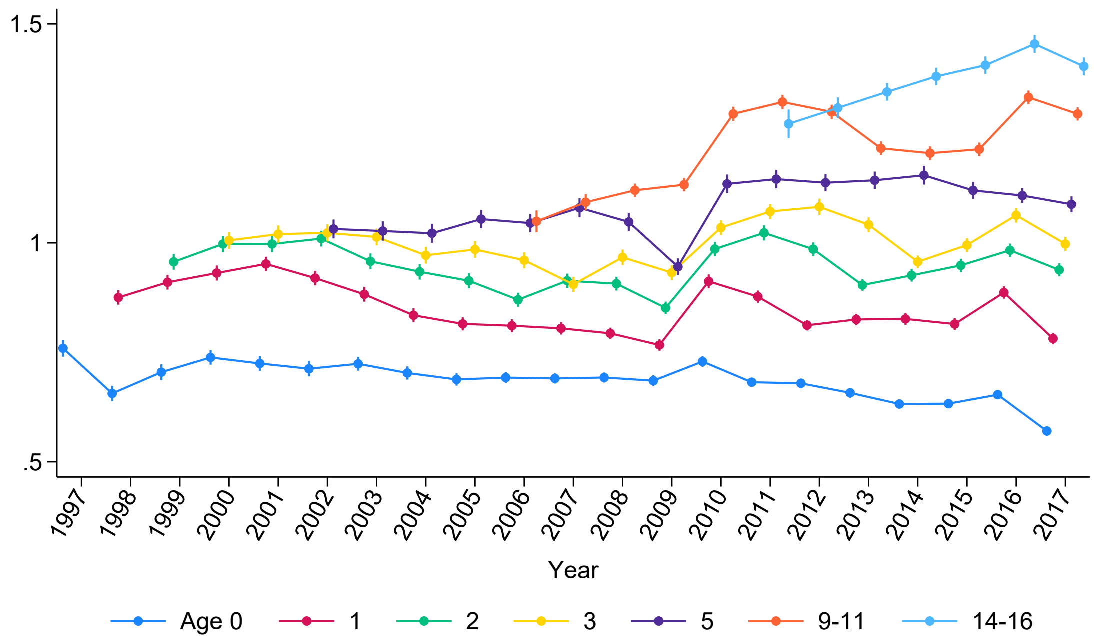
2.2 Changes in firm growth

I illustrate the change in firm growth in two different ways. Karahan, Pugsley and Şahin (2022) and Hopenhayn, Neira and Singhania (2022) show patterns of average firm size conditional on age. An increase in average firm size conditional on age over time (while size at entry remains constant) implies an acceleration of cumulative size growth over the firm's life cycle. I show these size-conditional-on-age patterns in a first step. These averages pool across all firms in the economy, so they do not consider, e.g., industry composition. As a

[11]The acceleration in employment growth is very similar when measuring firm employment using RAMS.

second step, I obtain firm life cycle growth (and changes therein) using regression analysis controlling for detailed industry and cohort fixed effects.

Figure 1: Average firm size (log employment) conditional on age



Notes: the figure shows avg. firm size (log employment) conditional on firm age over time. 95% confidence intervals are shown.

Figure 1 displays the average firm size patterns conditional on age. 95% confidence intervals are included. For ages zero to three, firm size is relatively stable over time in line with Karahan, Pugsley and Şahin (2022) and Hopenhayn, Neira and Singhania (2022). Already for firms of age five, comparing firm size in 2002 and 2017 shows a slight increase. This increase is even more pronounced for older firms (ages 9-11 and 14-16). The average firm size displays an apparent positive trend for these ages. Karahan, Pugsley and Şahin (2022) note that, controlling for industry composition, firms older than age ten display a significant increase in the average firm size over time. Such an increase is visible in the Swedish administrative data even without controlling for the industry composition, as Figure 1 shows. The increase in average firm size for older firms is robust to alternative measures of firm size: Figure 5 in the Appendix shows the same trends for firm sales.

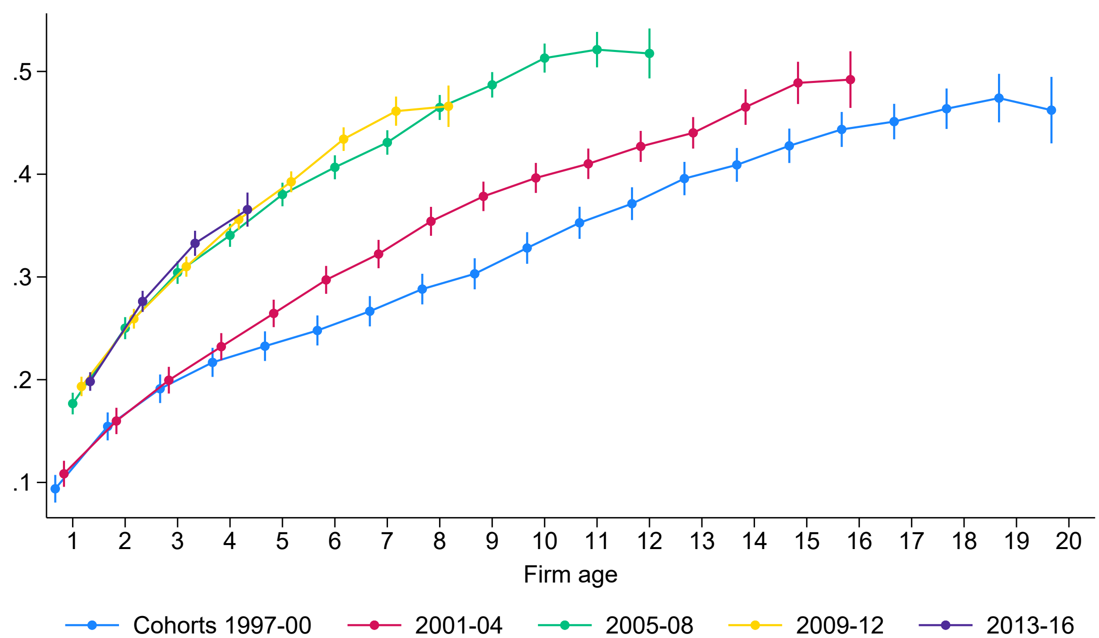
I use a regression framework to quantify the changes in firm life cycle growth over time. More specifically, I run the following regression

$$\ln \mathrm{Size}_{j,t} = \gamma_0 + \sum_{a_f=1}^{20} \gamma_{a_f} \mathbb{1}_{\mathrm{Age}_{j,t}=a_f} + \theta_c + \theta_k + \epsilon_{j,t}, \tag{1}$$

where $\mathbb{1}_{\text{Age}_{j,t}=a_f}$ is an indicator function for firms of age a_f. θ_k is a 5-digit industry fixed effect and as Sterk, Sedláček and Pugsley (2021), I control for cohort fixed effects, θ_c.[12] γ_0 captures the average log firm size at entry (age zero) and γ_{a_f} captures the log difference in average firm size between age a_f and zero, i.e., γ_1 to γ_{20} provide the non-parametric estimates of cumulative life cycle growth. I use employment and sales as a measure of firm size.

I run the regression for consecutive cohort groups (each group includes four cohorts) to capture changes in the growth profile over time. Figure 2 plots the age coefficients, γ_{a_f}, for employment as the size measure. Cumulative employment growth over the firm's life cycle has systematically increased. When measured over the first eight years of the firm, employment growth increased from about 29% (cohorts 1997 to 2000) to about 47% (cohorts 2009 to 2012). Figure 2 further shows that the gap opens up with firm age. This is consistent with the observation in Figure 1 that the average firm size of older firms increases significantly, whereas for younger firms, it is relatively stable.

Figure 2: Log employment relative to age zero (by cohort)



Notes: the figure shows cumulative employment growth over the firm's life cycle, measured as the difference between log employment at age a_f and age zero according to eq. (1). Cohorts are pooled as indicated in the legend. Firm employment is filtered at its 1% tails. The figure includes 95% confidence intervals.

A similar observation holds for sales as the measure of firm size. Figure 6 in the Appendix shows the same patterns for sales growth over the life cycle. Over the first eight years of the firm, sales increased by about 56% for the cohorts 1997 to 2000, whereas sales increased by about 67% for the cohorts 2009 to 2012. The acceleration in sales life cycle growth is smaller than for employment, but a clear upward shift of the life cycle profiles is apparent.

[12]The cohort and industry dependence of the other variables is suppressed for clarity.

3 Model

This section outlines an endogenous growth model with firm dynamics. The model allows for inference about the slowdown in aggregate productivity growth from the documented changes in firm life cycle growth, which will be the subject of later sections.

3.1 Preferences and aggregate economy

Time is continuous. The economy consists of a representative household that chooses the path of consumption C_t and wealth A_t to maximize lifetime utility

$$U = \int_0^\infty \exp(-\rho t) \ln C_t dt,$$

subject to the budget constraint $\dot{A}_t = r_t A_t + w_t L_t - C_t$ and a standard no-Ponzi game condition. ρ denotes the discount factor, r_t the interest rate and w_t the real wage. The household supplies one unit of labor inelastically such that $L_t = 1$. The optimality condition (Euler equation) for the household problem reads

$$\frac{\dot{C}_t}{C_t} = r_t - \rho.$$

Aggregate output is produced competitively using a Cobb-Douglas technology over a continuum of different products indexed by i (time subscripts suppressed)

$$Y = \exp\left(\int_0^1 \ln\left[q_i y_i\right] di\right),$$

where y_i and q_i denote the quantity and quality of product i. Output is consumed entirely such that $Y = C$. Expenditure minimization leads to the standard demand function

$$y_i = \frac{YP}{p_i}.$$

P is defined as the aggregate price index, which I normalize to 1.

3.2 Production

Firms potentially produce in a product market i with the following technology

$$y_{ij} = \varphi_j l_{ij},$$

where y_{ij} is the amount of product i produced by firm j, l_{ij} is the amount of labor hired, and φ_j denotes the productivity of firm j producing product i. Firm j produces different products with the same productivity, i.e., φ_j varies with j, but not with i. As in Aghion, Bergeaud, Boppart, Klenow and Li (2023), the firm's productivity is fixed over time, which captures the notion that some firms are persistently more efficient at producing than others, e.g., due to a better business plan. For simplicity, firms are of a high or low productivity type, i.e., $\varphi_j \in \{\varphi^h, \varphi^l\}$ with $\varphi^h/\varphi^l > 1$, which I refer to as high- and low-type firms.

3.3 Static allocation

Taking the joint distribution of product qualities and firm productivity as exogenous in this section, I characterize the static allocations at the product, firm and aggregate levels.

3.3.1 Product level

Firms within a product market i compete in prices (Bertrand competition). In equilibrium, only the firm with the highest quality-adjusted productivity $q_{ij}\varphi_j$ produces product i (henceforth, incumbent). Under Bertrand competition, the incumbent firm engages in limit pricing and sets its price equal to the quality-adjusted marginal costs of the follower (the firm with the second highest quality-adjusted productivity)

$$p_{ij} = \frac{q_{ij}}{q_{ij'}} \frac{w}{\varphi_{j'}}, \tag{2}$$

where j' indexes the follower in product market i. According to eq. (2), the price that the incumbent sets is increasing in the quality gap between the incumbent and the follower. The equilibrium price-cost markup in market i for producer j is defined as the output price over marginal costs, hence

$$\mu_{ij} \equiv \frac{p_{ij}}{w/\varphi_j} = \frac{q_{ij}}{q_{ij'}} \frac{\varphi_j}{\varphi_{j'}}. \tag{3}$$

The incumbent's markup for product i is increasing in the quality and productivity gap with respect to the follower. The price setting of the incumbent gives rise to the following profits for product i

$$\pi_{ij} = p_{ij} y_{ij} - w l_{ij} = Y \left(1 - \frac{1}{\mu_{ij}}\right),$$

with labor demand for product i

$$l_{ij} = \frac{Y}{w} \mu_{ij}^{-1}.$$

Employment in product line i is decreasing in the markup.

3.3.2 Firm level

Summing employment over the set of product lines where firm j is the incumbent, N_j,

$$l_j = \sum_{i \in N_j} l_{ij} = \frac{Y}{w} \left(\sum_{i \in N_j} \mu_{ij}^{-1} \right).$$

Firm employment decreases in product markups but increases in the number of product lines. Firm sales are given by $\left|N_j\right| Y \equiv n_j Y$, as revenues are equalized across product lines.

3.3.3 Aggregate level

Integrating employment across firms or products yields the total workforce in production:

$$L_P = \int_j l_j dj = \frac{Y}{w} \int_0^1 \mu_{ij}^{-1} di. \tag{4}$$

Taking logs and integrating eq. (3), one obtains an expression for the wage

$$w = \exp\left(\int_0^1 \ln q_{ij} di \right) \times \exp\left(\int_0^1 \ln \varphi_{j(i)} di \right) \times \exp\left(\int_0^1 \ln \mu_{ij}^{-1} di \right). \tag{5}$$

To find an expression for aggregate output, insert eq. (5) into eq. (4) to obtain

$$Y = Q \Phi \mathcal{M} L_P, \tag{6}$$

where

$$Q = \exp\left(\int_0^1 \ln q_{ij} di \right), \quad \Phi = \exp\left(\int_0^1 \ln \varphi_{j(i)} di \right), \quad \mathcal{M} = \frac{\exp\left(\int_0^1 \ln \mu_{ij}^{-1} di \right)}{\int_0^1 \mu_{ij}^{-1} di}.$$

Aggregate output Y depends on geometric averages of quality Q and productivity Φ as well as on the misallocation statistic $\mathcal{M}$ and production labor L_P. Aggregate TFP is captured by the term $Q\Phi\mathcal{M}$. As in Peters (2020), misallocation arises from markup dispersion that lowers aggregate TFP through $\mathcal{M}$. In this model, markup dispersion is due to both quality and productivity differences between incumbent and second-best firms.

Using again equation (4), monopoly power affects factor prices by reducing labor demand. The aggregate labor income share is given by

$$\Lambda \equiv \frac{wL_P}{Y} = \int_0^1 \mu_{ij}^{-1} di.$$

TFP depends on the dispersion of markups, whereas markup levels affect production labor and the aggregate labor income share.

3.4 Dynamic firm problem

Incumbents continuously improve the quality of products, q_i, in the economy through different types of R&D. Internal R&D increases the quality of one of the incumbent's product lines, whereas, through expansion R&D, the firm improves the quality of a random product line of a competing incumbent. Item quality is improved step-wise such that every innovation (either internal or expansion R&D) increases q_i by a factor of λ. As Aghion, Bergeaud, Boppart, Klenow and Li (2023), I assume that the step size of quality improvements exceeds the productivity differential, $\lambda > \varphi^h/\varphi^l$. This assumption guarantees that the firm with the highest quality version in a product line is always the incumbent producer.[13] Denote by λ^{Δ_i} the relative qualities of incumbent and second-best firms within a product line, i.e.,

$$\lambda^{\Delta_i} = \frac{q_{ij}}{q_{ij'}}.$$

Denote by $[\mu_i]$ the set of markups, where firm j is the incumbent. Firm profits are then

$$\pi_{jt}(n, [\mu_i]) = \sum_{k=1}^{n} Y_t \left(1 - \frac{1}{\mu_k}\right) = \sum_{k=1}^{n} Y_t \left(1 - \frac{1}{\lambda^{\Delta_k}} \frac{1}{\frac{\varphi_{jk}}{\varphi_{j'k}}}\right) \equiv \sum_{k=1}^{n} \pi(\mu_k),$$

where $\pi(\mu_i)$ denotes profits in product line i. Incumbent firms choose the rate of internal R&D, I_i, and the rate of expansion R&D, x_i, for each of their product lines, i. When choosing optimal internal and expansion R&D rates, firms take aggregate output Y_t, the real wage w_t, the share of lines operated by high-productivity firms S_t, the interest rate r_t and the rate of creative destruction τ_t as given. Denoting the time derivative by $\dot{V}_t^h()$, the value function of a high-productivity type firm (indexed by h) satisfies the following HJB equation:

$$\begin{aligned}
&r_t V_t^h(n, [\mu_i], S_t) - \dot{V}_t^h(n, [\mu_i], S_t) = \\
&\quad \sum_{k=1}^{n} \underbrace{\pi(\mu_k)}_{\text{Flow profits}} + \sum_{k=1}^{n} \underbrace{\tau_t \Big[V_t^h\left(n-1, [\mu_i]_{i \neq k}, S_t\right) - V_t^h(n, [\mu_i], S_t) \Big]}_{\text{Creative destruction}} \\
&\quad + \max_{[x_k, I_k]} \Bigg\{ \sum_{k=1}^{n} \underbrace{I_k \Big[V_t^h\left(n, [[\mu_i]_{i \neq k}, \mu_k \cdot \lambda], S_t\right) - V_t^h(n, [\mu_i], S_t) \Big]}_{\text{Internal R\&D}} \\
&\quad + \sum_{k=1}^{n} \underbrace{x_k \left[S_t V_t^h\left(n+1, [[\mu_i], \lambda], S_t\right) + (1 - S_t) V_t^h \left(n+1, \left[[\mu_i], \lambda \cdot \frac{\varphi^h}{\varphi^l} \right], S_t \right) - V_t^h(n, [\mu_i], S_t) \right]}_{\text{Expansion R\&D}} \\
&\quad - \underbrace{w_t \Gamma^h\left([x_i, I_i]; n, [\mu_i]\right)}_{\text{R\&D costs}} \Bigg\}.
\end{aligned}$$

[13] Relaxing this assumption would give room for a race for incumbency between low-productivity entrants facing a high-productivity incumbent from which I abstract.

As in Peters (2020), the value of a firm consists of flow profits, research costs, and three parts related to internal R&D, expansion R&D, and creative destruction. At the rate of creative destruction τ_t (determined in equilibrium), the firm loses one of its n products, in which case, it remains with $n-1$ products. At the optimally chosen rate I_k, internal R&D turns out successful (third row), and the firm charges a λ times higher markup on its product according to eq. (3). Alternatively, at the optimally chosen rate x_k, expansion R&D is successful (fourth row), and the firm acquires a new product (n increases by one).

Firm-type heterogeneity introduces new elements to the value function compared to Peters (2020). First, the value function is specific to the productivity type of the firm. Second, the share of product lines operated by each productivity type is a state variable (with two types, it is sufficient to keep track of S_t). When taking over a new product line through expansion R&D (fourth row), the probability of replacing a high-type incumbent is S_t, in which case the high-type entrant charges a markup of λ. With probability $1-S_t$, the replaced incumbent is of the low type, and the high-type entrant charges a markup of $\lambda \cdot \varphi^h/\varphi^l$. Firms take S_t as given; however, they affect it through their expansion R&D x_k in equilibrium. The HJB equation for a low-productivity firm follows the same structure and is listed in the Appendix, Section C.1. The term related to expansion R&D (fourth row) varies since low-productivity firms build different markup expectations when entering a new product line.

$\Gamma\left([x_i, I_i]; n, [\mu_i]\right)$ denote the R&D costs. For their R&D activities, firms pay a cost of

$$\Gamma^h\left([x_i, I_i]; n, [\mu_i]\right) = \sum_{k=1}^{n} c\left(x_k, I_k; \mu_k\right) = \sum_{k=1}^{n}\left[\mu_k^{-1}\frac{1}{\psi_I}(I_k)^{\zeta} + \frac{1}{\psi_x}(x_k)^{\zeta}\right]$$

in terms of labor. $\zeta > 1$ ensures convexity of the cost function. R&D costs are additively separable to render a closed-form solution for the value function along the balanced growth path.[14] ψ_I and ψ_x scale the internal and expansion R&D costs and capture the R&D efficiency.

Firm entry is determined as follows: using a linear production technology, potential entrants produce a flow of marketable ideas ψ_z per unit of labor that improves the quality of a randomly selected product line. Entrants get assigned the high productivity type with probability p^h and start with a one-step quality gap. Denoting by z_t the equilibrium flow rate of entry, the free entry condition requires that the expected value of firm entry equals the entry costs

$$p^h E[V_t^h(1,\mu_i)] + (1-p^h)E[V_t^l(1,\mu_i)] = \frac{1}{\psi_z} w_t, \tag{7}$$

[14] The incentives for internal R&D decrease with the quality gap that the firm has accumulated as profits within a product line are concave in the markup. I scale the internal R&D costs by the inverse markup to keep internal R&D incentives constant as in Peters (2020).

where the expected value of entering as a high- or low-type firm is

$$\begin{aligned}E[V_t^h(1,\mu_i)] =& S_t V_t^h(1,\lambda) + (1-S_t)V_t^h(1,\lambda\times\varphi^h/\varphi^l)\\ E[V_t^l(1,\mu_i)] =& S_t V_t^l(1,\lambda\times\varphi^l/\varphi^h) + (1-S_t)V_t^l(1,\lambda).\end{aligned}$$

Labor market clearing requires that production labor L_{Pt} and research labor L_{Rt} add up to one, the aggregate labor endowment

$$1 = L_{Pt} + L_{Rt} = \int_0^1 \frac{Y_t}{w_t}\mu_{it}^{-1}di + \int_0^1 \left(\mu_{it}^{-1}\frac{I_{it}^{\zeta}}{\psi_I} + \frac{x_{it}^{\zeta}}{\psi_x}\right) di + \frac{z_t}{\psi_z}. \tag{8}$$

3.5 Cross-sectional distribution of quality and productivity gaps

The joint (cross-sectional) distribution of quality and productivity gaps is the key equilibrium object that characterizes aggregates in the model. On the one hand, quality and productivity gaps characterize the markup distribution that determines labor demand. On the other hand, the distribution keeps track of the share of product lines operated by each productivity type, which is a state variable in the firm's optimization problem. This section characterizes the joint distribution of quality and productivity gaps as a function of firm policies, which allows the equilibrium distribution to be solved jointly with the policies.

The distribution of quality and productivity gaps ν is characterized by a set of infinitely many differential equations. For product lines where the incumbent is at least two quality steps ahead of the follower ($\Delta \geq 2$), the measure ν follows

$$\dot{\nu}_t\left(\Delta, \frac{\varphi_j}{\varphi_{j'}}\right) = I_t\nu_t\left(\Delta - 1, \frac{\varphi_j}{\varphi_{j'}}\right) - \nu_t\left(\Delta, \frac{\varphi_j}{\varphi_{j'}}\right)(I_t + \tau_t). \tag{9}$$

For product lines where the incumbent is one step ahead ($\Delta = 1$), the measure follows

$$\begin{aligned}\dot{\nu}_t\left(1, \frac{\varphi^l}{\varphi^h}\right) &= (1-S_t)x_t^l S_t + z_t(1-p^h)S_t - \nu_t\left(1, \frac{\varphi^l}{\varphi^h}\right)(I_t+\tau_t)\\ \dot{\nu}_t\left(1, \frac{\varphi^l}{\varphi^l}\right) &= (1-S_t)x_t^l(1-S_t) + z_t(1-p^h)(1-S_t) - \nu_t\left(1, \frac{\varphi^l}{\varphi^l}\right)(I_t+\tau_t)\\ \dot{\nu}_t\left(1, \frac{\varphi^h}{\varphi^h}\right) &= S_t x_t^h S_t + z_t p^h S_t - \nu_t\left(1, \frac{\varphi^h}{\varphi^h}\right)(I_t+\tau_t)\\ \dot{\nu}_t\left(1, \frac{\varphi^h}{\varphi^l}\right) &= S_t x_t^h(1-S_t) + z_t p^h(1-S_t) - \nu_t\left(1, \frac{\varphi^h}{\varphi^l}\right)(I_t+\tau_t),\end{aligned} \tag{10}$$

The distribution is characterized for firm-type specific expansion R&D rates, x_t^h and x_t^ℓ, and uniform internal R&D rates, I_t, as proven shortly in Proposition 1. Changes in the measure $\dot{\nu}$ are due to inflows and outflows. Outflows arise from successful internal R&D (quality gap increases from Δ to $\Delta + 1$) and creative destruction (quality gap gets reset to unity). For $\Delta \geq 2$, inflows into state Δ are due to successful internal R&D in product lines with a quality gap of $\Delta - 1$. For $\Delta = 1$, inflows result from creative destruction. As one example, the measure of products with a low-type incumbent and high-type second best firm $\nu_t\left(1, \frac{\varphi^l}{\varphi^h}\right)$ increases due to low-type incumbents and entrants replacing high-type incumbents, captured by $(1 - S_t)x_t^l S_t + z_t(1 - p^h)S_t$.

The measure ν defines the share of product lines operated by high-productivity type firms

$$S_t = \sum_{i=1}^{\infty} \left[\nu_t \left(i, \frac{\varphi^h}{\varphi^h} \right) + \nu_t \left(i, \frac{\varphi^h}{\varphi^l} \right) \right]. \tag{11}$$

Expansion R&D, firm entry, and S characterize the rate of creative destruction

$$\tau_t = S_t x_t^h + (1 - S_t)x_t^l + z_t. \tag{12}$$

3.6 Balanced growth path characterization

I define a balanced growth path of the economy as follows.

Definition 1. *A balanced growth path (BGP) is a set of allocations* $[x_{it}, I_{it}, \ell_{it}, z_t, S_t, y_{it}, C_t]_{it}$ *and prices* $[r_t, w_t, p_{it}]_{it}$ *such that firms choose* $[x_{it}, I_{it}, p_{it}]$ *optimally, the representative household maximizes utility choosing* $[C_t, y_{it}]_{it}$*, the growth rate of aggregate variables is constant, the free-entry condition holds, all markets clear and the distribution of quality and productivity gaps is stationary.*

Along the balanced growth path, the economy can be characterized in closed form.

Proposition 1. *In the above setup, along a balanced growth path:*

1. *The value function for a firm of productivity type* $d \in \{h, l\}$ *is given by*

$$\begin{aligned} V_t^d(n, [\mu_i]) =& V_{t,P}^d(n) + \sum_{k=1}^{n} V_{t,M}(\mu_k) \\ =& n \frac{1}{(\rho + \tau)} \frac{\zeta - 1}{\psi_x} (x^d)^\zeta w_t + \sum_{k=1}^{n} \frac{\pi(\mu_k) + \frac{\zeta - 1}{\psi_I} I^\zeta w_t \mu_k^{-1}}{\rho + \tau}, \end{aligned} \tag{13}$$

   where $I \equiv I^h = I^l$ *and* $x^h > x^l$.

2. S_{φ^k, φ^p}*, the constant share of product lines where the incumbent firm is of productivity*

type k and the second-best firm of type p is

$$S_{\varphi^l,\varphi^h} \equiv \sum_{i=1}^{\infty} \nu\left(i, \frac{\varphi^l}{\varphi^h}\right) = \frac{(1-S)x^l S + z(1-p^h)S}{\tau}$$

$$S_{\varphi^l,\varphi^l} \equiv \sum_{i=1}^{\infty} \nu\left(i, \frac{\varphi^l}{\varphi^l}\right) = \frac{(1-S)x^l(1-S) + z(1-p^h)(1-S)}{\tau}$$

$$S_{\varphi^h,\varphi^h} \equiv \sum_{i=1}^{\infty} \nu\left(i, \frac{\varphi^h}{\varphi^h}\right) = \frac{Sx^h S + zp^h S}{\tau}$$

$$S_{\varphi^h,\varphi^l} \equiv \sum_{i=1}^{\infty} \nu\left(i, \frac{\varphi^h}{\varphi^l}\right) = \frac{Sx^h(1-S) + zp^h(1-S)}{\tau},$$

which implicitly defines the share of product lines operated by the high-productivity type

$$S = S_{\varphi^h,\varphi^h} + S_{\varphi^h,\varphi^l} = \frac{Sx^h + zp^h}{\tau}. \tag{14}$$

3. *The growth rate of aggregate variables is given by*

$$g = \frac{\dot{Q}_t}{Q_t} = \Bigg(\underbrace{I}_{\text{Incumbent internal R\&D}} + \underbrace{Sx^h + (1-S)x^l}_{\text{Incumbent expansion R\&D}} + \underbrace{z}_{\text{Entry}} \Bigg) \times \ln(\lambda). \tag{15}$$

Proof. The Appendix, Sections C.1, C.2 and C.3, contains the proofs. □

The value function is additive across products. The first part of the value function represents the option value of expanding into new product markets and scales linearly in the number of products. This term is productivity-type specific, as different productivity types choose heterogeneous expansion R&D rates. Intuitively, the (expected) value of a product line equals the marginal cost of expansion R&D for the optimal expansion R&D rate. The value of a product line increases in firm productivity as profits rise in the markup such that in equilibrium, more productive firms choose higher expansion R&D rates, i.e., $x^h > x^l$. The second part of the value function consists of flow profits and the option value to increase markups in the future. The value function is scaled by the sum of the rate of creative destruction (the rate at which incumbents get replaced) and the discount factor.

Proposition 1 further shows that the share of products where the incumbent firm is of type k and the second-best firm of type p is constant along a balanced growth path. This share equals the fraction of creatively destroyed products that start in a product line where the incumbent is of type k and the second best firm of type p at each instant in time.

Long-run growth results from R&D at the product level. This occurs through successful internal R&D, expansion R&D, or firm entry. The growth rate is equal to the aggregate arrival rate of innovation times the step size of innovation, $\ln(\lambda)$. Since expansion R&D

rates are heterogeneous, changes in the share of product lines operated by each productivity type, S and $1-S$, affect the aggregate growth rate as Proposition 1 shows.

The stationary distribution of productivity and quality gaps further characterizes (1) the aggregate labor income share, (2) the TFP misallocation measure that captures the static loss in output that arises from markup dispersion, and (3) the aggregate markup.

Proposition 2. *Let I and τ denote the rates of internal R&D and creative destruction and* $\theta = \frac{\ln(1+\tau/I)}{\ln(\lambda)}$.

1. *The aggregate labor income share $\Lambda = \frac{wL_P}{Y}$ is given by*

$$\Lambda = \frac{\theta}{\theta+1} \sum_{k\in\{h,l\}} \sum_{n\in\{h,l\}} \frac{1}{\varphi_k/\varphi_n} S_{\varphi_k,\varphi_n}.$$

2. *The misallocation measure $\mathcal{M}$ is given by*

$$\mathcal{M} = \frac{e^{\sum_{k\in\{h,l\}} \sum_{n\in\{h,l\}} \left[S_{\varphi_k,\varphi_n} \left(\ln\left(\frac{1}{\frac{\varphi_k}{\varphi_n}} \right) - \frac{1}{\theta} \right) \right]}}{\Lambda}.$$

3. *The aggregate markup $E[\mu] = \int_0^1 \mu_i di$ is given by*

$$E[\mu] = \frac{\theta}{\theta-1} \sum_{k\in\{h,l\}} \sum_{n\in\{h,l\}} \frac{\varphi_k}{\varphi_n} \times S_{\varphi_k,\varphi_n}.$$

Proof. The Appendix, Section C.2, contains the proofs. □

As in Peters (2020), the rate of creative destruction relative to internal R&D, captured by θ, affects $\Lambda, \mathcal{M}$ and $E[\mu]$. In this model, they further depend on the size of the productivity gap, φ^h/φ^l, and the share of product lines with a given productivity gap, S_{φ_k,φ_n}. For example, a rise in the productivity gap or a reallocation of sales shares towards high-productivity firms lowers the aggregate labor income share and raises the markup, ceteris paribus.

To find the balanced growth path, I solve the optimality conditions of the firm (derived in Appendix C.1), the free entry condition, eq. (7), the labor market clearing condition, eq. (8), and the system of differential equations characterizing the distribution of productivity and quality gaps jointly. Appendix C.4 contains the details.

3.6.1 Discussion of the stationary firm-type distribution

Given the systematic heterogeneity in expansion R&D rates, why is the stationary distribution of firm types along the balanced growth path non-degenerate? Taking the time derivative of (11) and inserting (9) and (10), one obtains the differential equation for S_t

$$\dot{S}_t = S_t x_t^h (1 - S_t) - (1 - S_t) x_t^l S_t + z_t \left(p^h (1 - S_t) - (1 - p^h) S_t \right). \tag{16}$$

Changes in the sales share of high-productivity type incumbents are due to high-type incumbents stealing product lines from low-type ones (first term), low-type incumbents stealing product lines from high-type ones (second term), high-type entrants replacing low-type incumbents and low-type entrants replacing high-type incumbents (final term). Along the balanced growth path, $\dot{S}_t = 0$ such that eq. (16) turns into

$$z \left(S - p^h \right) = S(1 - S)(x^h - x^l). \tag{17}$$

With $x^h > x^l$, for eq. (17) to hold, S needs to be greater than p^h, the share of entrants of the high-productivity type. In other words, the share of high-type firms among entrants must be lower than the share of product lines operated by high-type incumbents. In this case, sufficient entry by low-type firms exactly balances high-type incumbents' relatively higher expansion rate, and S_t remains constant. Eq. (17) highlights the role of firm entry. Without entry ($z = 0$), higher expansion rates by high-type incumbents would result in those firms eventually overtaking all product lines. In the special case where all entrants are of the low productivity type ($p^h = 0$), eq. (17) can be written as

$$Sx^h(1 - S) - (1 - S)x^l S = zS.$$

Along the balanced growth path where S_t is constant, entry by low-type firms that replace high-type incumbents (zS) makes up precisely for the lost sales share of low-type incumbents, $Sx^h(1 - S) - (1 - S)x^l S$.

3.7 Firm dynamics

Firms lose products according to the same stochastic process as in Klette and Kortum (2004). However, in this model, firms add products at systematically different rates as optimally chosen expansion R&D rates vary with the firm's productivity type.[15] This generates ex-ante heterogeneity in firm life cycle trajectories as highlighted by Sterk, Sedláček and Pugsley (2021). I derive firm-type specific sales and employment life cycle growth that will be mapped to the data to explain the documented trends in firm growth.

[15] Therefore, the properties related to firm size growth and survival in Klette and Kortum (2004) hold conditional on the firm type. In particular, conditional on the type, firm size, and (expected) growth are unrelated, i.e., Gibrat's law holds conditionally as in Lentz and Mortensen (2008). For the unconditional firm size and growth correlation, two forces are at play. On the one hand, young (small) firms tend to grow quicker due to survival bias. On the other hand, more productive firms (with faster growth rates) are more likely to end up large. In the estimated (initial) balanced growth path, 74% of the firms are of the high productivity type. Hence, size is unrelated to growth (in expectation) for the vast majority of firms.

3.7.1 Firm sales growth

Firm sales are proportional to the number of products a firm produces, such that successful expansion R&D increases firm sales. Since optimal expansion R&D rates depend on the firm's productivity type, expected sales growth is type-specific. Expected log sales growth for a firm with productivity $\varphi^j, j \in \{h, l\}$ between age zero and age a_f is $E[\ln n_f Y|a_f, \varphi^j] - E[\ln n_f Y|0, \varphi^j]$, where n_f is the number of products the firm is producing. Hence,

$$E\left[\ln n_f Y|a_f, \varphi^j\right] - E\left[\ln n_f Y|0, \varphi^j\right] = \underbrace{g \times a_f}_{\text{Aggregate growth}} + \underbrace{E\left[\ln n_f|a_f, \varphi^j\right]}_{\text{Firm's product growth}}.$$

To derive $E[\ln n_f|a_f, \varphi^j]$ note that the probability of a high-productivity type firm producing n products at age a conditional on survival is $(1 - \gamma^j(a))(\gamma^j(a))^{n-1}$, where $\gamma^j(a) = x^j \frac{1-e^{-(\tau - x^j)a}}{\tau - x^j e^{-(\tau - x^j)a}}$. Therefore expected sales growth is given by

$$E\left[\ln n_f Y|a_f, \varphi^j\right] - E\left[\ln n_f Y|0, \varphi^j\right] = \underbrace{g \times a_f}_{\text{Aggregate growth}} + \underbrace{\left(1 - \gamma^j(a_f)\right) \sum_{n=1}^{\infty} \ln n \times \left(\gamma^j(a_f)\right)^{n-1}}_{\text{Firm's product growth}}. \tag{18}$$

Relative sales growth of the firm is equal to the firm's product growth.

3.7.2 Firm markup growth

Firm markups are defined as $\mu_f = \frac{py_f}{wl_f}$. The Appendix, Section C.5 shows that for a high-productivity type firm, the expected log markup conditional on firm age a_f is

$$E\left[\ln \mu_f|a_f, \varphi^h\right] = \underbrace{\ln \lambda \times \left(1 + I \times E[a_P^h|a_f]\right)}_{\text{Quality improvements}} + \underbrace{(1 - S) \times \ln\left(\frac{\varphi^h}{\varphi^l}\right)}_{\text{Productivity advantage}}, \tag{19}$$

where $E[a_P^h|a_f]$, the average product age of a high-type firm conditional on firm age, is

$$\begin{aligned}
E[a_P^h|a_f] &= \frac{1}{x^h}\left(\frac{\frac{1}{\tau}\left(1 - e^{-\tau a_f}\right)}{\frac{1}{x^h+\tau}\left(1 - e^{-(x^h+\tau)a_f}\right)} - 1\right)\left(1 - \phi^h(a_f)\right) + a_f \phi^h(a_f) \\
\phi^h(a) &= e^{-x^h a} \frac{1}{\gamma^h(a)} \ln\left(\frac{1}{1 - \gamma^h(a)}\right) \\
\gamma^h(a) &= \frac{x^h\left(1 - e^{-(\tau - x^h)a}\right)}{\tau - x^h e^{-(\tau - x^h)a}}.
\end{aligned}$$

The expected firm markup conditional on age consists of two terms. The first term in eq. (19) is akin to Peters (2020) and, as marvelously illustrated in the paper, reflects that internal R&D translates quality improvements within the firm's product line into markup growth at the firm level as the firm ages. In Peters (2020), this term holds for all firms, whereas in this model, this term is specific to the productivity type of the firm, as the average product age varies by firm type. The second term in eq. (19) captures a level effect that differences in productivity introduce. The intuition is that if a high-productivity type incumbent faces a low-type second-best firm in a given line, it can charge a φ^h/φ^l higher markup, which occurs in expectation in $1 - S$ of the incumbent's product lines.

The expected markup conditional on firm age for a low-productivity type firm follows

$$E\left[\ln \mu_f | a_f, \varphi^l\right] = \underbrace{\ln \lambda \times \left(1 + I \times E[a_P^l | a_f]\right)}_{\text{Quality improvements}} + \underbrace{S \times \ln\left(\frac{\varphi^l}{\varphi^h}\right)}_{\text{Productivity disadvantage}}. \tag{20}$$

The first term captures quality improvements through internal R&D, equivalently to eq. (19). $E[a_P^l|a_f]$ follows the same expression as $E[a_P^h|a_f]$ with h replaced by l. The second term in eq. (20) differs from eq. (19). Low-productivity incumbents face a high-productivity second-best firm in a share S of their product lines. Since $\varphi^l < \varphi^h$, this term is negative.

3.7.3 Firm employment growth

Firm employment conditional on age and firm type is equal to

$$E[\ln l_f| = a_f, \varphi^j] = \ln\left(\frac{Y}{w}\right) + E\left[\ln n_f | a_f, \varphi^j\right] - E\left[\ln \mu_f | a_f, \varphi^j\right].$$

Since $\frac{Y}{w}$ is constant along the balanced growth path, employment growth is given by

$$E[\ln l_f | a_f, \varphi^j] - E[\ln l_f | 0, \varphi^j] = \underbrace{E\left[\ln n_f | a_f, \varphi^j\right]}_{\text{Firm's product growth}} - \underbrace{\left(E\left[\ln \mu_f | a_f, \varphi^j\right] - E\left[\ln \mu_f | 0, \varphi^j\right]\right)}_{\text{Firm's markup growth}}, \tag{21}$$

where $E\left[\ln n_f | a_f, \varphi^j\right]$ and $E\left[\ln \mu_f | a_f, \varphi^j\right] - E\left[\ln \mu_f | 0, \varphi^j\right]$ are defined in eqs. (18)-(20). Employment growth equals product growth minus markup growth as in Peters (2020). In this model, expected employment growth is further productivity type specific.

3.7.4 Firm survival and unconditional life cycle growth

Firm size dynamics determine firm survival. Since firm size growth is type-dependent, so is firm survival. The survival function in Klette and Kortum (2004) holds conditional on the firm type, i.e., the share of high and low type firms surviving until age a_f is

$$\chi^h(a_f) = 1 - \tau \frac{1 - e^{-(\tau - x^h)a_f}}{\tau - x^h e^{-(\tau - x^h)a_f}} \tag{22}$$

$$\chi^l(a_f) = 1 - \tau \frac{1 - e^{-(\tau - x^l)a_f}}{\tau - x^l e^{-(\tau - x^l)a_f}}. \tag{23}$$

The firm survival function can be used to compute firm sales, and employment growth unconditionally of the firm type. The share of high-type firms among firms at age a_f is

$$s^h(a_f) = \frac{p^h \chi^h(a_f)}{p^h \chi^h(a_f) + (1 - p^h)\chi^l(a_f)}.$$

The share corresponds to the mass of high-type survivors relative to the total mass of survivors. Unconditional employment growth between age zero and a_f is then given by

$$s^h(a_f)\left(E[\ln l_f|a_f, \varphi^h] - E[\ln l_f|0, \varphi^h]\right) + \left(1 - s^h(a_f)\right)\left(E[\ln l_f|a_f, \varphi^l] - E[\ln l_f|0, \varphi^l]\right). \tag{24}$$

Unconditional sales growth is defined similarly. When estimating the model, I match observed employment growth in the data using eq. (24).

3.7.5 Firm size distribution

The model also makes precise predictions about the firm size distribution. The mass of high- and low-productivity type firms with $n \geq 2$ products follows the differential equations

$$\dot{M}_t^h(n) = (n-1)x_t^h M_t^h(n-1) + (n+1)\tau_t M_t^h(n+1) - n(x_t^h + \tau_t)M_t^h(n)$$

$$\dot{M}_t^l(n) = (n-1)x_t^l M_t^l(n-1) + (n+1)\tau_t M_t^l(n+1) - n(x_t^l + \tau_t)M_t^l(n), \tag{25}$$

whereas the mass of firms with one product evolves according to

$$\dot{M}_t^h(1) = z_t p^h + 2\tau_t M_t^h(2) - (x_t^h + \tau_t)M_t^h(1)$$

$$\dot{M}_t^l(1) = z_t(1 - p^h) + 2\tau_t M_t^l(2) - (x_t^l + \tau_t)M_t^l(1). \tag{26}$$

The mass of firms with n products increases through firms with $n-1$ products expanding to size n at rate x_t^h or x_t^l per product or through firms with $n+1$ products losing a product at the rate of aggregate creative destruction τ_t. The mass of firms with n products decreases through firms with n products either gaining or losing a product through expansion or creative destruction. The mass of firms with one product additionally increases through firm entry.

Proposition 3. *The stationary firm size distribution along the balanced growth path is characterized as follows.*

1. *The mass of high and low productivity firms with n products is*

$$M^h(n) = \frac{(x^h)^{n-1} z p^h}{n\tau^n} = \frac{zp^h}{x^h}\frac{1}{n}\left(\frac{x^h}{\tau}\right)^n$$
$$M^l(n) = \frac{(x^l)^{n-1} z(1-p^h)}{n\tau^n} = \frac{z(1-p^h)}{x^l}\frac{1}{n}\left(\frac{x^l}{\tau}\right)^n.$$

2. *The total mass of firms with n products is*

$$M(n) = M^h(n) + M^l(n) = \frac{(x^h)^{n-1} zp^h + (x^l)^{n-1} z(1-p^h)}{n\tau^n}.$$

3. *The mass of firms of each productivity type is*

$$M^h = \sum_{n=1}^{\infty} M^h(n) = \frac{zp^h}{x^h}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{x^h}{\tau}\right)^n = \frac{zp^h}{x^h}\ln\left(\frac{\tau}{\tau - x^h}\right)$$
$$M^l = \sum_{n=1}^{\infty} M^l(n) = \frac{z(1-p^h)}{x^l}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{x^l}{\tau}\right)^n = \frac{z(1-p^h)}{x^l}\ln\left(\frac{\tau}{\tau - x^l}\right)$$

4. *The total mass of firms is*

$$M = M^h + M^l.$$

Proof. These results follow from setting the time derivatives in equations (25) and (26) equal to zero and solving the system of equations. □

For each firm type, the share of firms with n products, $M^h(n)/M^h$ and $M^l(n)/M^l$, follows the pdf of a logarithmic distribution with parameter x^h/τ and x^l/τ as in Lentz and Mortensen (2008). The firm size distribution is highly skewed to the right.

From the firm size distribution, I obtain the share of high-productivity type firms

$$S_{M^h} = \frac{M^h}{M}, \tag{27}$$

and the firm entry rate

$$\text{Firm entry rate} = \frac{z}{M}. \tag{28}$$

4 Comparative statics across balanced growth paths

This section applies the model to study the implications of the changes in firm life cycle growth for aggregate economic growth. To this extent, I estimate the model along two balanced growth paths. The initial balanced growth path captures firm life cycle growth and aggregate economic conditions during the 1990s. I then re-estimate model parameters to explain the changes in firm life cycle growth of the latest cohorts in the data.

4.1 Initial balanced growth path

There are, in total, eight parameters in the model. The internal R&D efficiency ψ_I, the expansion R&D efficiency ψ_x, the innovation cost curvature ζ, the entry efficiency ψ_z, the step size of innovation λ, the productivity differential φ^h/φ^ℓ, the share of high-productivity type firms among entrants p^h, and the discount rate ρ. Two parameters are set exogenously, and the remaining parameters are estimated. I follow Acemoglu, Akcigit, Alp, Bloom and Kerr (2018) and Peters (2020) that set ζ equal to two based on evidence from the microeconometric innovation literature (Blundell, Griffith and Windmeijer, 2002; Hall and Ziedonis, 2001). The discount rate ρ is set to 0.02, resulting in an annual discount factor of roughly 0.97%.

The remaining six parameters are estimated, targeting moments of firm life cycle growth as well as cross-sectional firm heterogeneity and economic aggregates. In particular, I target firms' sales and employment growth, dispersion in inverse labor shares across entrants, the firm entry rate, TFP growth, and the aggregate markup. Despite all parameters being identified jointly, there is a tight mapping between parameters and targets.

Sales and employment growth disciplines the firms' R&D cost parameters. In the model, successful expansion R&D translates into sales growth, such that sales growth identifies the expansion R&D cost ψ_x. The internal R&D cost parameter ψ_I governs a firm's markup growth. Since markup growth drives a wedge between sales and employment growth, targeting employment and sales growth jointly disciplines markup growth and, hence, the internal R&D cost. The advantage of targeting employment instead of markup growth is that employment is directly observed in the data. I target sales and employment growth over the first eight years of the firm. This period is long enough to average out transitory shocks during the firm's early years and still allows for estimating separate balanced growth paths (one for the early cohorts and one for the latest cohorts) over the data coverage period from 1997 to 2017. The model matches firm growth at any age well, so the specific age target is not restrictive. In the model, sales and employment growth are specific to the productivity type of the firm. In the data, the productivity type is unobserved. I match observed sales and employment growth in the data with unconditional (of the productivity type) firm growth in the model, defined in eq. (24).[16] Therefore, the composition of firm types conditional on

[16] The alternative is to impute the productivity type of a firm in the data and to measure type-specific sales and employment growth. I match firm life cycle growth, unconditional of the productivity type, to avoid classifying firms incorrectly into types that would affect the parameter estimates and the firm composition. Matching type-specific growth would further require setting the productivity cutoff exogenously.

firm age (an endogenous outcome) is such that type-specific life cycle growth weighted by firm type composition matches observed growth in the data. For the initial balanced growth path, I target sales and employment growth of the cohorts 1997 to 2000. For these cohorts, sales grew by 55.9% and employment by 28.8% over the first eight years of the firm.

The entry rate helps identify the entry efficiency of firms ψ_z. I compute the entry rate in the data as the share of firms equal to or less than one year of age. This results in an average entry rate over the period 1997-2005 of 14.3%, in line with Engbom (2023). I match this number with the model-implied entry rate in eq. (28).

Aggregate productivity growth disciplines the step-size improvement of innovation λ: the aggregate growth rate in eq. (15) directly depends on λ. I obtain aggregate productivity growth for the Swedish economy from Federal Reserve Economic Data (FRED) in labor augmenting terms.[17] After suffering a financial crisis in the early 90s, Sweden's economy featured strong growth towards the end of the century. During 1997–2005, aggregate productivity grew by 3.02% per year.

To pin down the productivity differential φ^h/φ^ℓ, I target the aggregate markup. The aggregate markup is a weighted average of product markups that, in return, depend on φ^h/φ^ℓ. Sandström (2020) and De Loecker and Eeckhout (2018) report sales-weighted markups for the Swedish economy. Sandström (2020) computes the markup in Swedish registry data focusing on firms with at least ten employees, whereas De Loecker and Eeckhout (2018) focus mainly on publicly listed firms. I target the average of both reported aggregate markups, resulting in a conservative estimate of 7.5%. Lastly, I target the standard deviation of log (inverse) labor shares across entering firms. Given φ^h/φ^ℓ, the dispersion of log labor shares at entry depends on the share of product lines operated by high-type firms (an endogenous equilibrium object) and the share of high-type firms among entrants (the parameter p^h). The dispersion of inverse labor shares across entrants, hence, disciplines p^h. The standard deviation of log (inverse) labor shares of entering firms averaged over 1997-2005 equals 0.053.[18]

The estimation follows a two-step approach. In the first (global) step, the algorithm computes the sum of squared percentage deviations from the targeted moments for a large Sobol sequence of parameter vectors. In the second (local) step, I take the best candidates from the first step and perform a local search using the Nelder-Mead algorithm. The local search, again, minimizes the distance from the targets. All targets receive equal weights. The best parameter vectors from the second step converge to the same parameter values.

Table 2 shows the estimation results. The model replicates all targeted moments well. The

[17]FRED series RTFPNASEA632NRUG. The labor share is obtained from FRED, series LABSHPSEA156NRUG, averaged over 1997–2005

[18]For firms with a low wage bill, inverse labor shares explode. Therefore, I focus on firms with a sales-to-wage bill ratio between one and three (model implied markups between 0% and 200%). Further, sales relative to the wage bill in the data may vary for reasons outside the model. I bin firms into equally sized groups based on their capital and intermediate inputs and compute the dispersion of log inverse labor shares across firms within these groups.

Table 2: Initial balanced growth path. Moments and parameters

	Data	Model
Moments		
Sales growth by age 8 in % (cohorts 1997–2000)	55.9	55.8
Employment growth by age 8 in % (cohorts 1997–2000)	28.8	28.8
Cross-sectional SD of log labor shares across entrants (1997–2005)	0.053	0.053
Agg. productivity growth g in % (1997–2005; FRED)	3.02	3.02
Entry rate in % (1997–2005)	14.3	14.3
Agg. markup μ in % (Sandström, 2020; De Loecker and Eeckhout, 2018)	7.5	7.5
Parameters		
ψ_I *Internal R&D efficiency*		0.144
ψ_x *Expansion R&D efficiency*		0.282
ψ_z *Entry R&D efficiency*		1.483
λ *Step size of innovation*		1.136
φ^h/φ^ℓ *Productivity gap*		1.091
p^h *Share of high type among entrants*		0.683
Set exogenously		
ρ *Discount rate*		0.02
ζ *R&D cost curvature*		2

Notes: except for aggregate productivity (TFP) growth and μ, the moments are computed using Swedish registry data. TFP growth is obtained from Federal Reserve Economic Data (FRED), series RTFPNASEA632NRUG, in labor augmenting terms (the labor share is obtained from FRED, series LABSHPSEA156NRUG, averaged over the same period 1997–2005).

estimated parameters can be interpreted as follows: successful innovation increases product quality by 13.6%. High and low-type firms' productivity differs by 9.1%, and 68.3% of firms enter the economy as high-type firms.

Along the balanced growth path, the constant share of high-productivity type firms in the cross-section, S_{M^h} in eq. (27), equals 74%. This number is larger than their share at entry ($p^h = 0.683$) due to high-type firms choosing higher expansion R&D rates than low-type firms: $x^h - x^\ell = 0.075$. This is reflected in their expected life cycle growth. Over the first eight years of the firm, sales grow by 63% for high-type firms relative to 37% for low-type firms. Weighted by the share of firms of each type at age eight as in eq. (24), this results in sales growth of 55.8% as shown in Table 2.

Sterk, Sedláček and Pugsley (2021) emphasize the importance of ex-ante heterogeneity in firm life cycle trajectories. In this model, ex-ante heterogeneity in firm life cycle trajectories arises from heterogeneous expansion R&D rates (x^h and x^ℓ) related to the firm's permanent productivity. I provide suggestive evidence that permanent firm productivity relates to firm life cycle growth in the data, see Section 6.2.

4.2 New balanced growth path

This section estimates the model on a new balanced growth path that replicates the observed changes in firm life cycle growth in the data vis-a-vis the initial balanced growth path. To replicate the changes in firm sales and employment growth, I re-estimate two parameters, particularly the internal R&D efficiency ψ_I and the entry efficiency ψ_z. These two parameters are promising candidates because one affects sales and employment growth jointly, whereas the other moves employment relative to sales growth. Rising entry costs shield incumbents from creative destruction. A lower replacement rate, ceteris paribus, increases the value of a product line, eq. (13). This incentivizes incumbents to expand into new product lines, increasing their sales and employment life cycle growth. In contrast, the rising internal R&D costs increase firm employment growth relative to sales growth: internal R&D costs govern firms' markup growth, which drives a wedge between sales and employment growth. Slower markup growth, ceteris paribus, increases firm employment growth.[19]

Table 3: New balanced growth path. Moments and parameters

	Data (%)	Model (%)	ΔBGPs (pp)
Moments			
Sales growth by age 8 (cohorts 2009–2012)	67.4	67.4	+11.5
Employment growth by age 8 (cohorts 2009–2012)	46.6	46.6	+17.8
Parameters			
ψ_I *Internal R&D efficiency* (Δ in %)			-51.0
ψ_z *Entry R&D efficiency* (Δ in %)			-22.0

Notes: the column ΔBGPs reports the difference between ending and initial balanced growth path moments (in percentage points) and parameters (in percent).

Table 3 shows the changes in the targeted moments and the estimated parameters. For the cohorts 2009 to 2012, sales growth over the first eight years equals 67.4% (an increase of 11.5pp relative to the cohorts 1997 to 2000) and employment growth 46.6% (an increase of 17.8pp). The model matches these changes by lowering the internal R&D efficiency by 51% and the entry efficiency by 22%. The estimated fall in the internal R&D efficiency relates to Bloom, Jones, Van Reenen and Webb (2020), who find that research productivity has declined in the U.S. over time. Their main example of declining research productivity of (incumbent) firms in the semiconductor industry can be interpreted as declining internal R&D productivity.[20] The estimated fall in the entry efficiency is further in line with Davis (2017)

[19] The sign of the parameter change is not restricted in the estimation.

[20] A decline in the expansion R&D efficiency ψ_x would also be consistent with declining research productivity. However, in the estimated model, declining expansion R&D efficiency counterfactually leads to falling concentration, rising entry, and a slowdown in both sales and employment life cycle growth. In contrast, declining internal R&D efficiency results in rising concentration, falling entry, and an acceleration in employment relative to sales life cycle growth, as I document in the Swedish data. Declining internal R&D efficiency further results in increasing within-firm labor shares that Autor, Dorn, Katz, Patterson and Van Reenen (2020) find for most U.S. sectors.

and Gutiérrez and Philippon (2018), who argue that the increasing complexity of regulatory requirements and the tax system, as well as rising lobbying expenditures, disproportionately affect entrants.

How does the estimated rise in internal R&D and entry costs affect firm size growth? Table 4 shows the effect of each parameter change on firm sales and employment growth in isolation relative to the initial balanced growth path.[21] The acceleration in firm size growth is mainly due to rising entry costs, which increase sales and employment growth by age eight by 12.94pp and 14.8pp, respectively. The acceleration in employment growth relative to sales growth is mainly due to rising internal R&D costs that lower sales growth by 1.78pp and increase employment growth by 2.23pp. The rise in internal R&D costs slows markup growth, thereby raising employment relative to sales growth.

Table 4: Contributions to changes in firm size growth

	Initial BGP (%)	$\psi_I \downarrow$ (pp)	$\psi_z \downarrow$ (pp)	$\psi_I \downarrow, \psi_z \downarrow$ (pp)
Sales growth by age 8	55.8	-1.78	+12.94	+11.5
Employment growth by age 8	28.8	+2.23	+14.80	+17.8

Notes: the table shows the change in firm sales and employment growth relative to the initial balanced growth path in percentage points (pp). $\psi_I \downarrow$ denotes the 51% fall in the internal R&D efficiency and $\psi_z \downarrow$ the 22% fall in the entry efficiency.

The rise in the cost of entry and incumbent R&D causes a substantial change in the composition of firm types. The share of high-type firms in the cross-section, S_{M^h}, increases by 12pp across the balanced growth paths. Recall that, for the aggregate growth rate g, the composition of firm types at the product level matters. Changes in the composition of firm types at the product level are even larger than at the firm level. The sales share of high-type firms, S, increases by 17pp. The sales share of high-type firms increases by more than their share in the cross-section of firms (S_{M^h}), as low-type firms with more than one product lose sales shares without exiting the economy. That high-productivity type firms gain sales shares at the expense of low-type firms is consistent with a rise in sales concentration that I document within Swedish industries in the Appendix, Section B.2. Further, the reallocation of sales shares to firms with relatively low labor shares is qualitatively consistent with Kehrig and Vincent (2021). Similarly, De Loecker, Eeckhout and Unger (2020) and Baqaee and Farhi (2020) document a reallocation of sales shares to firms with a high sales-to-cost-of-goods-sold ratio in Compustat data.[22]

The implied changes for economic growth and firm entry align with recent macroeconomic trends: the aggregate (long-run) growth rate declines by 0.6pp, and the firm entry rate drops by 8pp. In Sweden, aggregate TFP growth, measured from 2010 to 2015, declined by about

[21] I report the effect relative to the initial BGP for all exercises to ease the comparison with Section 6.1, where I estimate an alternative ending BGP.

[22] In the estimated model, differences in markup growth are minor compared to the difference in markup levels at birth between high- and low-productivity firms, so high-productivity firms remain high-markup (low-labor share) firms throughout.

1pp relative to 1997–2005. Further, Engbom (2023) documents a fall in the entry rate by about 10pp from the early 1990s to the mid-2010s in the Swedish economy. The comparative statics exercise, therefore, accounts for roughly 60 percent of the fall in economic growth and 80 percent of the decline in firm entry.

The acceleration of firm life cycle growth points to rising costs of entry and internal R&D as the driving force behind the recent macroeconomic trends. These forces rationalize the changes in firm life cycle growth and, at the aggregate level, cause a decline in economic growth, a fall in firm entry, and rising concentration in line with the data.

4.3 Incumbent innovation, reallocation, entry and growth

How does the reallocation of sales shares across firm types affect long-run growth? How much of the fall in growth is due to entrants and incumbents? This section quantifies the relative importance of these channels. The aggregate growth rate g naturally lends itself to such decomposition. Along a balanced growth path, the aggregate growth rate defined in eq. (15), in a slightly rewritten formulation, reads

$$g = Sg^h + (1-S)g^\ell + g^z,$$

where $g^h \equiv (I + x^h)\ln(\lambda)$, $g^\ell \equiv (I + x^\ell)\ln(\lambda)$ and $g^z \equiv z\ln(\lambda)$ denote the contributions to economic growth by high type incumbents, low type incumbents and entrants. Note that for the total contribution by incumbents, their innovation rates and the share of products operated by each type matter. Using a shift-share decomposition, I decompose changes in the growth rate across balanced growth paths, $\Delta g \equiv g_{new} - g_{old}$, as follows

$$\Delta g = \underbrace{S_{old}\Delta g^h + (1-S_{old})\Delta g^\ell}_{\Delta\text{Within}} + \underbrace{g^h_{old}\Delta S - g^\ell_{old}\Delta S}_{\Delta\text{Between}} + \underbrace{\Delta g^h\Delta S - \Delta g^\ell\Delta S}_{\Delta\text{Cross}} + \underbrace{\Delta g^z}_{\Delta\text{Entry}}, \tag{29}$$

where *old* and *new* index balanced growth path variables before and after the parameter change. Changes in the aggregate growth rate are due to changes in innovation rates holding the distribution of sales shares constant (ΔWithin), due to changes in the distribution of sales shares holding innovation rates constant (ΔBetween), due to changes in both innovation rates and sales shares (ΔCross) as well as due to changes in firm entry (Δg^z). The ΔWithin, ΔBetween, and ΔCross terms capture changes due to incumbents, whereas Δg^z captures by definition changes due to entrants. Because the ΔCross term is absent without firm type heterogeneity, I group the ΔBetween and ΔCross-term into a common ΔReallocation term.[23]

[23] In the decomposition, the ΔCross-term compares small to the ΔBetween term.

Table 5: Decomposing the fall in economic growth

	$\psi_I \downarrow, \psi_z \downarrow$	$\psi_I \downarrow$	$\psi_z \downarrow$
ΔWithin	+0.22	-0.23	+0.47
ΔReallocation	+0.27	+0.01	+0.20
ΔEntry	-1.10	-0.11	-0.93
Δg	-0.62	-0.33	-0.26

Notes: the table shows the contributions to the change in the aggregate growth rate g according to the balanced growth path decomposition in eq. (29) in percentage points. ΔReallocation is the sum of the ΔBetween and ΔCross terms. g in the initial balanced growth path is equal to 3.02%. $\psi_I \downarrow$ denotes the 51% fall in the internal R&D efficiency and $\psi_z \downarrow$ the 22% fall in the entry efficiency.

Table 5 quantifies the different contributions to the fall in the aggregate growth rate. First, the ΔWithin term is positive at 0.22pp, indicating that incumbents' innovation rates increased. Second, the reallocation of sales shares to more productive firms that (endogenously) feature higher innovation rates contributes positively to economic growth. The ΔReallocation term is positive at 0.27pp. Therefore, changes in incumbent innovation (ΔWithin plus ΔReallocation) *raised* the aggregate growth rate by a total of 0.49pp. Note that ΔReallocation accounts for 55% (0.27/0.49) of the total contribution by incumbent firms. Thus, incumbents mainly contributed to changes in long-run growth through the reallocation of sales shares to more innovative firms. This channel is absent in standard models of creative destruction where firms innovate at identical rates. Lastly, falling firm entry lowers the aggregate growth rate substantially by 1.1pp. The fall in firm entry dominates the positive contribution by incumbents, resulting in a total decline of the growth rate of 0.62pp. Falling firm entry squares the acceleration of incumbents' life cycle growth with a fall in aggregate economic growth.

That the ΔWithin term is positive may be surprising given that R&D costs of incumbents have increased. Columns 3 and 4 of Table 5 repeat the decomposition for each parameter change in isolation. The ΔWithin effect of a rise in the internal R&D costs is negative (-0.23pp). At the same time, the rise in the entry costs generates a positive ΔWithin effect. Rising barriers to entry incentivize incumbent firms to innovate faster. Overall, the positive ΔWithin effect following the rise in the entry costs outweighs the negative ΔWithin effect of the rising internal R&D. Note also that the rise in the entry costs drives the positive ΔReallocation effect. A fall in firm entry reallocates market shares across incumbents towards the relatively faster-growing high-type firms.

The results of the decomposition complement the findings in Akcigit and Kerr (2018), Garcia-Macia, Hsieh and Klenow (2019), and Peters (2020). These studies show that economic growth is mainly due to incumbent firms.[24] The decomposition in this paper suggests that entrants play a more prominent role when explaining *changes* in economic growth. That

[24] Decomposing the level of the aggregate growth rate shows that this is also the case in this model in both balanced growth paths.

falling firm entry drives the slowdown in economic growth is consistent with the observation in Garcia-Macia, Hsieh and Klenow (2019) that the relative contribution to economic growth by entrants has declined over time.

5 Transition dynamics

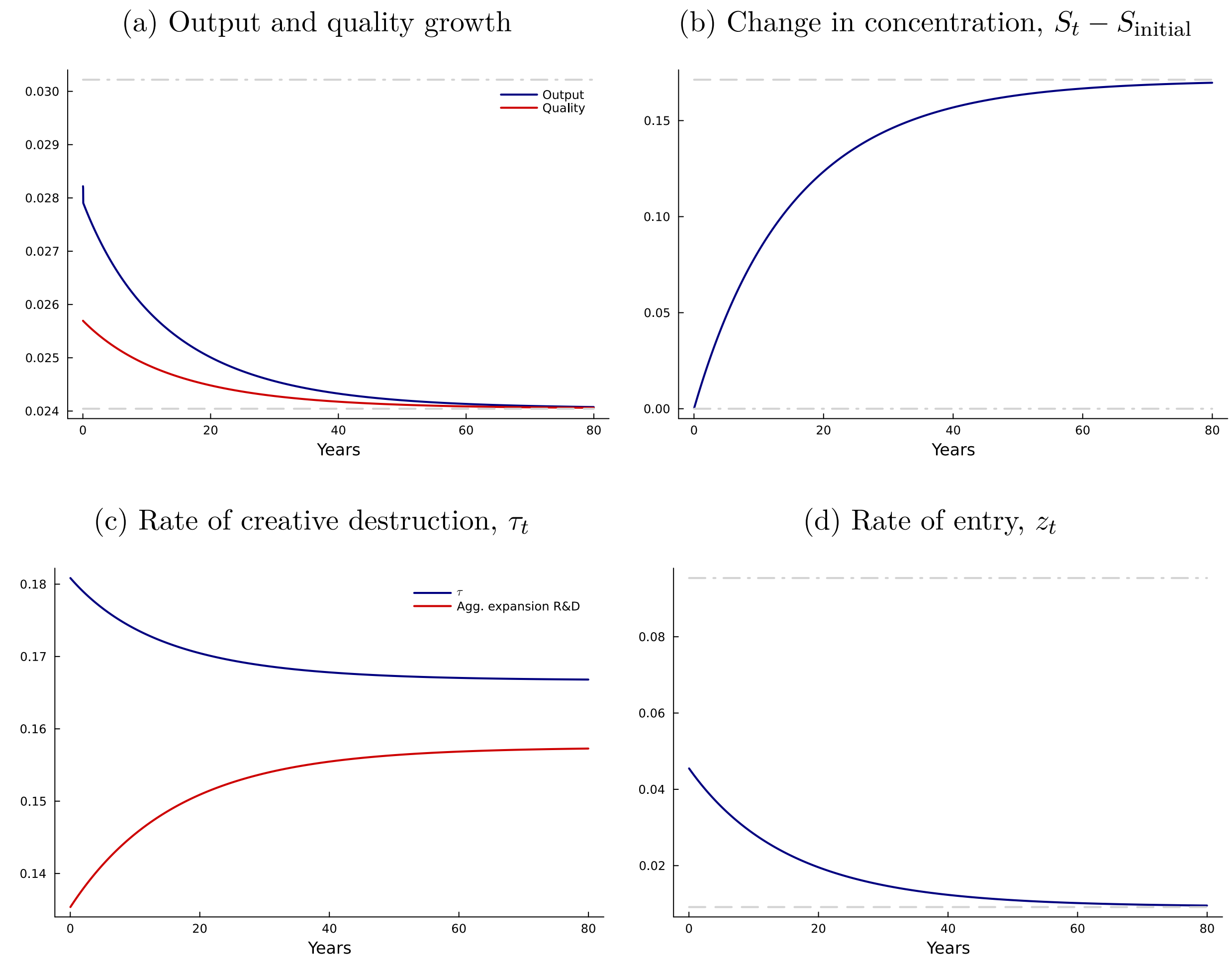
The share of high-productivity firms increases across the balanced growth paths. Over the transition, rising average productivity counteracts falling firm entry. This section's subject is how these counteracting forces affect aggregate output and, hence, welfare. To this end, I numerically solve for the equilibrium outcomes during the transition period using a backward iteration algorithm. I solve for policy and value functions from the ending balanced growth path backward for a guessed sequence of wage growth, interest rates, and firm productivity types over the product distribution (S_t). I then use the obtained policy functions over the transition period to simulate the two-dimensional distribution of quality and productivity gaps forward, starting from the initial balanced growth path. Using the evolution of this distribution over the transition, I back out the implied sequences of wage growth, interest rates, and S_t. The algorithm finds the fixed point between the guessed and implied sequences. The detailed algorithm is outlined in the Appendix, Section D.

Starting from the initial balanced growth path, I introduce the estimated rise in entry and internal R&D costs (Table 3) as shocks in period zero, after which no further parameter changes occur. Figure 3 shows the paths of output (Y_t) growth (in %), quality (Q_t) growth (in %), changes in the sales share of high productivity type firms S_t with respect to the initial balanced growth path (in pp), the rate of creative destruction (τ_t), and the rate of entry (z_t) over the transition period. Convergence is relatively quick. Most changes in equilibrium outcomes occur over the first 20 years of the transition. Both output and quality growth decline on impact in period zero and converge quickly after to their new long-run values, as shown in Panel (a). Along a balanced growth path, quality and output grow at the same rate. Over the transition, aggregate quality growth differs from output growth with growth in average productivity, markup dispersion, and production labor, explaining the residual according to eq. (6). Output growth declines by less than quality growth on impact as, e.g., a rising sales share by high productivity firms, S_t, as shown in Panel (b), contributes positively to growth in average productivity and hence aggregate output. Over the entire transition period, S_t increases by 17pp. This rise does not suffice to counteract the fall in quality growth. Panel (a) shows that output growth follows the declining pattern of quality growth over the transition.[25]

That quality growth continuously declines over the transition period is not self-evident as contrasting forces are at play. On the one hand, firm entry declines over the transition, as shown in Panel (d), which lowers quality growth. On the other hand, external and internal R&D efforts by incumbents are also subject to change over the transition. Figure 4 shows

[25] Changes in misallocation, $\mathcal{M}_t$, also have a negligible effect on output during the transition.

Figure 3: Transition dynamics



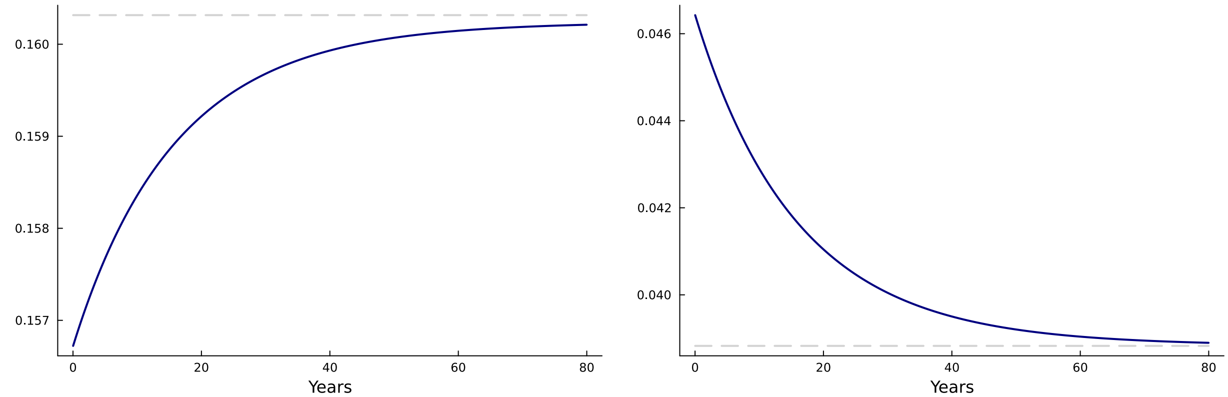
Notes: the figure shows the response in equilibrium outcomes following the increase in entry costs and fall in R&D productivity of incumbents as in Table 3 (period zero). Output and quality growth (Panel a) refer to the growth rate of Y_t and Q_t in percent. The change in concentration refers to the change in the sales share of high-productivity type firms relative to the initial balanced growth path in percentage points. The gray dashed and dash-dotted lines indicate the ending and initial balanced growth paths, respectively. Aggregate expansion R&D in panel (c) is computed as $S_t \times x_t^h + (1 - S_t) \times x_t^l$.

the evolution of expansion R&D rates by high- and low-type firms. In line with the rise in concentration, expansion rates of high-type firms increase while the ones of low-type firms decline over the transition. Aggregate expansion R&D rates (productivity-type specific R&D rates weighted by their respective sales shares S_t) are, in fact, increasing over the transition as shown in Panel (c) of Figure 3. That falling entry outweighs the rise in aggregate expansion R&D becomes evident after looking at the path of the rate of aggregate creative destruction τ_t, also shown in Panel (c). The rate of creative destruction is the sum of the aggregate expansion R&D rate and the firm entry rate z_t. The rate of creative destruction is strictly falling over the transition, highlighting that the fall in quality growth is due to falling firm entry dominating rising aggregate expansion R&D.[26] As the decline in firm entry exceeds

[26]Internal R&D also declines over the transition period (not shown). However, this effect is small.

Figure 4: Expansion R&D rates over the transition

(a) High-type firms, x_t^h (b) Low-type firms, x_t^l



Notes: the figure shows the evolution of the optimal expansion R&D rates by high- and low-type firms following the increase in entry costs and fall in R&D productivity of incumbents as in Table 3 in period zero.

the rise in expansion R&D, quality and output growth do not display hump-shaped profiles over the transition. Falling firm entry drives a fall in growth, both over the short and long run.

What are the effects on welfare? There are contrasting forces at play. On the one hand, average productivity and aggregate expansion R&D increase over the transition. On the other hand, firm entry declines. The path of output growth over the transition in Figure 3 trades off these forces. As output growth gradually declines right from the shock period, the net effect on welfare is negative. To quantify the change in welfare, I compute the permanent consumption change (in percent) along the initial balanced growth path that makes the consumer as well off as with the obtained consumption stream over the transition towards the new balanced growth path. I find that welfare decreases by 23.3%. This number is sizable and should be interpreted with substantial caution. The initial balanced growth path matches macroeconomic conditions (and firm growth) during the late 1990s. Aggregate productivity growth averaged about 3% during this period in Sweden. Therefore, the transition path is compared to a scenario in which the high growth period of the late 1990s would have continued forever. Targeting a lower aggregate growth rate in the initial balanced growth path that reflects average growth before the 1990s boom, as in Aghion, Bergeaud, Boppart, Klenow and Li (2023) or De Ridder (2024), would result in a lower welfare loss. However, this would introduce an inconsistency in targeted moments: targeted firm growth reflects conditions during the late 1990s, while aggregate growth refers to an earlier period. Note also that the decline in output growth is monotone during the transition, i.e., no initial burst in growth exists. Given that the initial balanced growth path already reflects the high growth period of the late 1990s, it is consistent with the data that the transition does not feature a further burst in growth. This does, however, translate into a larger welfare loss.

If one were to compare welfare along two balanced growth paths that grow at the rates of the estimated initial and ending balanced growth path (without taking the transition nor any level effects into account) the consumption equivalent change (in percent) ξ is determined by $\ln(1+\xi) = (g^{\text{ending}} - g^{\text{initial}})/\rho$, where g^{ending} and g^{initial} refer to the growth rates of the initial and ending balanced growth paths. Given that the growth rate declines by roughly six percentage points across the balanced growth paths and ρ equals 0.02, this loss in welfare amounts to 26.6% ($\xi = -0.266$). Comparing this number to the above 23.3% welfare loss shows again that the fall in output growth during the transition is mainly driven by declining quality growth and that the transition to the new balanced growth path is fast.

6 Robustness and further model validation

This section provides robustness for the results of the growth decomposition and tests other model predictions.

6.1 Reallocation, entry and economic growth revisited

One could argue that the strength of the reallocation effect and the relative contributions of entrants and incumbents in the decomposition in Table 5 are sensitive to the parameter change in question. In this section, I show that the findings of the growth decomposition are robust to alternative explanations behind the macroeconomic trends. However, these alternative explanations are only qualitatively (not quantitatively) consistent with the observed changes in firm life cycle growth.

Aghion, Bergeaud, Boppart, Klenow and Li (2023) explain the fall in economic growth and the rise in concentration in the U.S. economy through changes in the R&D efficiency and rising productivity dispersion of incumbents. In line with their story, I estimate an alternative ending balanced growth path where the parameters subject to change are the productivity gap φ^h/φ^ℓ (instead of the entry efficiency) and the internal R&D efficiency ψ_I (as in the previous estimation).

Table 6: Alternative new balanced growth path. Moments and parameters

	ΔData (pp)	ΔModel (pp)
Moments		
Sales growth by age 8 (cohorts 2009–2012)	+11.5	+2.1
Employment growth by age 8 (cohorts 2009–2012)	+17.8	+7.4
Parameters		
ψ_I *Internal R&D efficiency* (Δ in %)		-54
φ^h/φ^ℓ *Productivity gap* (Δ in %)		+6

Notes: the table shows changes in moments (in percentage points) and parameters (in percent) with respect to the initial balanced growth path.

Table 6 shows the estimation results. The internal R&D efficiency falls by 54% (compared to 51% in the previous estimation), and the productivity gap increases by 6%.[27] The implied changes in firm sales and employment growth are qualitatively in line with the data, yet fall short in explaining them quantitatively.[28] Therefore, changes in the productivity gap cannot fully account for the changes in firm growth. Nevertheless, changes in economic aggregates match recent trends in the macroeconomy: the long-run aggregate growth rate falls by 0.49pp, concentration rises, and the firm entry rate declines by 3pp. Therefore, the estimated fall in the R&D efficiency and increase in the productivity gap give rise to a similar fall in the aggregate growth rate as the one targeted in Aghion, Bergeaud, Boppart, Klenow and Li (2023) (-0.42pp).

Table 7: Decomposing the fall in economic growth revisited

	$\psi_I \downarrow, \varphi^h/\varphi^\ell \uparrow$	$\psi_I \downarrow$	$\varphi^h/\varphi^\ell \uparrow$
ΔWithin	-0.13	-0.24	+0.11
ΔReallocation	+0.18	+0.01	+0.13
ΔEntry	-0.53	-0.12	-0.35
Δg	-0.49	-0.35	-0.11

Notes: the table shows the contributions to the change in the aggregate growth rate g according to the balanced growth path decomposition in eq. (29) in percentage points. ΔReallocation is the sum of the ΔBetween and ΔCross terms. g in the initial balanced growth path is equal to 3.02%. $\psi_I \downarrow$ denotes the 54% fall in the internal R&D efficiency and $\varphi^h/\varphi^\ell \uparrow$ the 6% rise in the productivity gap.

I decompose the fall in the aggregate growth rate for the alternative ending balanced growth path according to eq. (29.) as before. First, changes in incumbent innovation rates, ΔWithin, lower the growth rate slightly (-0.13pp), whereas the reallocation of sales shares, ΔReallocation, towards the more productive firms with higher innovation rates generates a positive growth effect (+0.18pp), shown in Table 7. ΔReallocation outweighs ΔWithin as in the previous estimation. Second, the fall in firm entry more than explains the fall in the aggregate growth rate: -0.53pp compared to -0.49pp. Therefore, the two findings that incumbent firms have mainly contributed to changes in long-run growth through reallocation effects and that the decline in the aggregate growth rate is driven by a fall in firm entry even hold for an alternative mechanism that keeps the entry technology constant. Comparing the last column of Table 5 and Table 7 shows that the rising productivity gap works similarly as rising entry costs on growth: both generate positive ΔWithin and ΔReallocation effects that are dominated by a negative ΔEntry effect. The ΔWithin, ΔReallocation, and ΔEntry contributions resulting from a fall in the internal R&D efficiency are quantitatively almost identical to the previous estimation.

[27] For this estimation, I assume that entrants always replace incumbents as the estimated productivity gap exceeds the step size of innovation λ. Estimating the parameters with the constraint $\varphi^h/\varphi^\ell < \lambda$ results in the constraint binding at $\varphi^h/\varphi^\ell = 1.136$, which is the value of λ.

[28] For a large enough productivity disadvantage, low-type firms stop expanding into new product markets and remain one-product firms, which reduces the degrees of freedom in the model to match the increase in sales and employment life cycle growth.

In Aghion, Bergeaud, Boppart, Klenow and Li (2023), all firms innovate at the same rate, and there is no firm entry such that changes in within-firm innovation rates, ΔWithin, fully explain the decline in the aggregate growth rate. Table 7 suggests that reallocation effects and firm entry matter for changes in long-run growth. The ΔReallocation effect outweighs the ΔWithin effect, and ΔEntry dominates both.

Would the role of entry change when relaxing the assumption of a unitary demand elasticity? With a demand elasticity greater than one, firms also gain market shares through successful internal R&D. This suggests that, ceteris paribus, an even larger rise in firm entry costs would be required to offset the negative size-growth effect from rising internal R&D costs when matching the increase in firm life cycle growth.

6.2 Firm productivity and life cycle growth

Sterk, Sedláček and Pugsley (2021) highlight the importance of ex-ante heterogeneity of firm life cycle trajectories. In this model, heterogeneity in expected life cycle profiles arises through permanent differences in firm productivity that, in equilibrium, give rise to heterogeneous innovation rates. This section provides suggestive evidence that firm life cycle growth relates to permanent firm productivity in the data.

Firm productivity is generally unobserved in the data. I use a model-based approach to infer the firms' productivity. As firms enter the model economy with one product, eq. (3) captures firm markups upon entry. Their productivity advantage allows more productive firms to charge higher markups in equilibrium. Guided by the theory, I proxy firm productivity by its markup (sales relative to wage bill) at age zero, and regress observed firm life cycle growth on the productivity proxy.

$$\ln \text{Size}_{\text{Age}_{j,t}=a_f} - \ln \text{Size}_{\text{Age}_{j,t}=0} = \beta_0 + \beta_1 \log\left(\frac{py}{wl}\right)_{\text{Age}_{j,t}=0} + \theta_c + \theta_k + \epsilon_{j,t} \tag{30}$$

py/wl denotes inverse labor shares. Otherwise, the notation follows eq. (1). As in the model estimation, I focus on firm size growth over the first eight years, i.e., $a_f = 8$. I use employment as the measure of firm size to avoid sales at age zero on both sides of eq. (30).

Table 8 shows the regression results. The regression coefficient of interest, β_1, stands at 0.13, i.e., within the same 5-digit industry and cohort, firms with 1% higher inverse labor shares at entry are associated with approximately 0.13pp faster employment growth over the first eight years. For the model-relevant sub-sample of firms with positive markups (firms with inverse labor shares larger than one), the regression coefficient increases to 0.198 (column two). One strength of the Swedish data is that it contains information on the capital stock and intermediate input usage. Higher inverse labor shares at entry are positively related to

Table 8: Firm productivity and size growth

	$\Delta \ln \text{Size}_{\text{Age}=8}$	$\Delta \ln \text{Size}_{\text{Age}=8}$	$\Delta \ln \text{Size}_{\text{Age}=8}$	$\Delta \ln \text{Size}_{\text{Age}=8}$
$\log\left(\frac{py}{wl}\right)_{\text{Age}=0}$	0.130	0.198	0.222	0.237
	(0.006)	(0.005)	(0.005)	(0.006)
$\log K_{\text{Age}=0}$			-0.041	0.003
			(0.003)	(0.003)
$\log M_{\text{Age}=0}$				-0.107
				(0.004)
Cohort fixed effects	✓	✓	✓	✓
Industry fixed effects	✓	✓	✓	✓
$\log\left(\frac{py}{wl}\right)_{\text{Age}=0} > 0$		✓	✓	✓
N	66,817	65,875	60,950	60,832
R^2	0.06	0.08	0.08	0.10

Notes: the table reports the regression coefficient β_1 of eq. (30). Firm size growth over the first eight years, $\Delta \ln \text{Size}_{\text{Age}_{j,t}=8} \equiv \ln \text{Size}_{\text{Age}_{j,t}=8} - \ln \text{Size}_{\text{Age}_{j,t}=0}$, is measured using firm employment. $\log(py/wl)_{\text{Age}_{j,t}=0}$ denotes the log inverse labor share at age zero, the proxy of firm productivity, as explained in the main text. $\log K$ and $\log M$ denote the firm's capital stock and intermediate inputs, respectively. Robust standard errors are in parentheses.

firm life cycle growth, even when controlling for capital and intermediate inputs. Including capital or intermediate inputs at age zero in the regression increases β_1 to 0.222 and 0.237, respectively (third and fourth column).[29] Across all specifications, the coefficient of interest remains highly significant, with an almost constant (robust) standard error of 0.005. The data confirms that firms with relatively higher inverse labor shares at entry, perhaps due to systematically higher productivity as suggested by the model, display faster life cycle growth.

7 Conclusion

Sales and employment growth over the firm's life cycle has accelerated. For firms established in the late 1990s, sales grew by 55.9 percent over the first eight years compared to 67.4 percent for firms established in the early 2010s. Similarly, employment growth increased from 28.8 percent to 46.6 percent. I view these trends at the firm level through the lens of a model of creative destruction to study their implications for economic aggregates, in particular productivity growth. The acceleration of sales and employment life cycle growth points towards changes in the cost of firm entry and incumbent R&D as the cause behind recent macroeconomic trends. At the aggregate level, these forces result in a slowdown in productivity growth, a fall in firm entry, and a rise in concentration.

The changes in firm life cycle growth inform about the contribution of incumbent firms to the changes in productivity growth since the 1990s. A growth decomposition shows that

[29]I obtain similar results when using $TFPR$ at age zero instead of labor productivity as a proxy for firm productivity, where $TFPR \equiv \frac{py}{K^{\alpha}(wl)^{1-\alpha}}$ with α estimated at the industry level using cost shares.

incumbent firms have mainly contributed to long-run productivity growth since the 1990s through the reallocation of sales shares to more innovative incumbents. This highlights the importance of long-run growth effects associated with changes in industry concentration. Policymakers should trade off these dynamic effects of reallocation with the usual static efficiency losses when evaluating antitrust policies. However, falling firm entry rather than incumbent R&D accounts for the recent slowdown in productivity growth. This suggests a promising role for policies that support new firm formation to reverse the slowdown in productivity growth.

How does the positive reallocation effect on the long-run growth rate compare to other, more severe, episodes of reallocation? Over the last decades, many Western economies privatized their education, health care, transportation, or communication sectors. It would be interesting to decompose changes in long-run growth following these events into changes in innovation rates, reallocation, and firm entry, as in this paper. To disentangle how reallocation ultimately affects short and long-run economic growth, one could further compare the effect of reallocation on quality growth to the effects of reallocation on average productivity and misallocation over the transition. The quantitative framework in this paper, disciplined by changes in firm dynamics, could separate these forces.

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Appendices

A Data

The main data set, *Företagens Ekonomi* (FEK), covers information from balance sheets and profit and loss statements for the universe of Swedish firms. From this data, I obtain the main variables of interest, namely sales (*Nettoomsättning*, variable name: *Nettoomsattning*) and employment (*Antal anställda*, variable name: *MedelantalAnstallda*). In the FEK codebook by Statistics Sweden, these variables are defined as follows.[30] Sales refer to income from the companies' main business for goods sold and provided services. Employment refers to the average number of employees in full-time units in accordance with the company's annual report. As described in the main text, I focus on firms in the private sector. These firms have a legal type (variable name: *JurForm*) less than 50 or equal to 96.

The 5-digit industry classification (SNI codes) changed twice between 1997 and 2017, once in 2002 and once in 2007. I ensure a consistent industry classification using the following steps. During the year of the change, I observe both the old and the new industry classifications. For the firms present in the data this year, extending the new industry classification further back in time before the change in the classification is straightforward. This way, the industry

[30] https://www.scb.se/contentassets/9dd20ce462644cc19f6f04eb2edbbe28/nv0109_kd_2017_bv_190508_v2.pdf, accessed 07.02.2024.

codes of almost all firms are updated. A firm might be in the data before and after the cutoff year but not at the cutoff year. For these firms, the above method does not work. If the firm appears in the data one year after the classification change, I use the observed classification after the change and update the classification before the change accordingly. For firms that are absent for several years around the year of change, I use industry mappings provided by Statistics Sweden. These mappings do not always provide a 1:1 mapping between industries before and after the classification change, so I use the most common transitions for the m:m mappings.

One concern is that changes in the firm structure, e.g., when firms merge with other firms, change the firm ID. To address this concern, I impute changes in firm IDs using worker flows between firms. The auxiliary data set *Registerbaserad Arbetsmarknadsstatistik* (RAMS) contains the universe of employer-employee matches. I impute changes in the firm ID of firms with at least five employees as follows: if more than 50% of the workforce of firm A in year t makes up for more than 50% of the workforce of firm B in year $t+1$, I substitute firm B's firm ID by firm A's firm ID following $t+1$. The results remain virtually unchanged when excluding firms for whom the imputed firm ID differs from the observed firm ID.

B Trends in the Swedish economy

B.1 Changes in firm life cycle growth

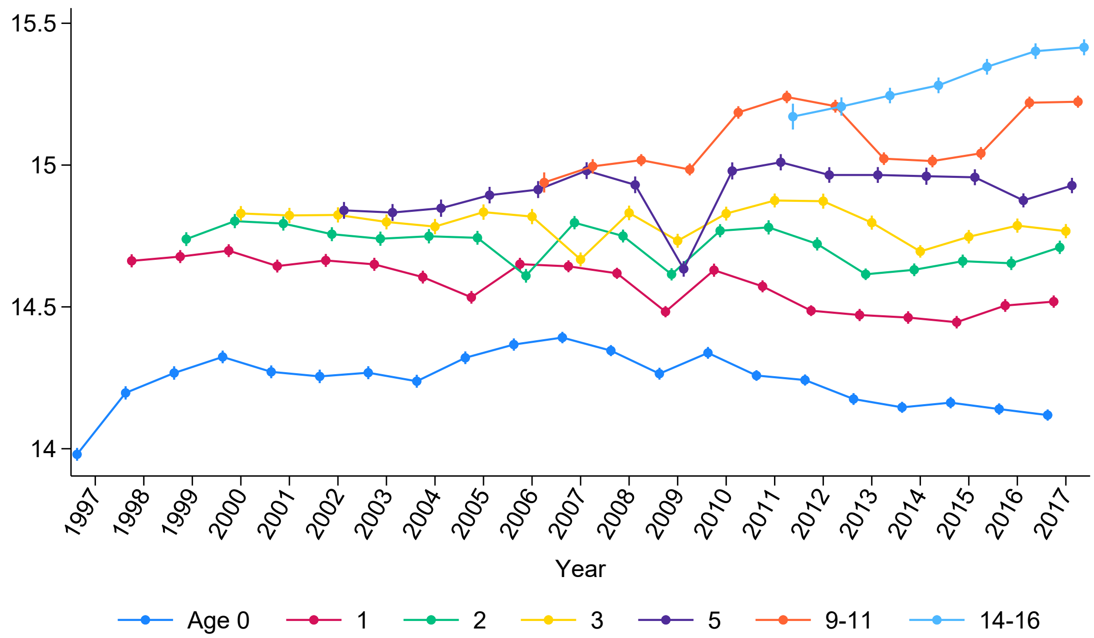
Figure 5 shows the age-conditional average firm size patterns for sales as the measure of firm size. Similarly to the patterns for employment, average firm sales are relatively stable for young firms, whereas a positive trend is apparent for older firms.

I run regression (1) using sales as the firm size measure to quantify the changes in firm life cycle growth. Figure 6 plots the age coefficients for the different cohorts. As for employment, sales life cycle growth accelerates over time.

B.2 Changes in industry concentration

I compute the standard deviation of log sales within industries to measure industry concentration. Note that this measure coincides with the standard deviation of log sales shares. The more dispersed sales (or sales shares), the more concentrated the industry. I filter firm sales at the 1% tails for each year and compute the standard deviation for industries with at least 50 firms to avoid changes in industry size affecting the concentration measure. Figure 7 shows the standard deviation, averaged across all industries. Concentration displays positive

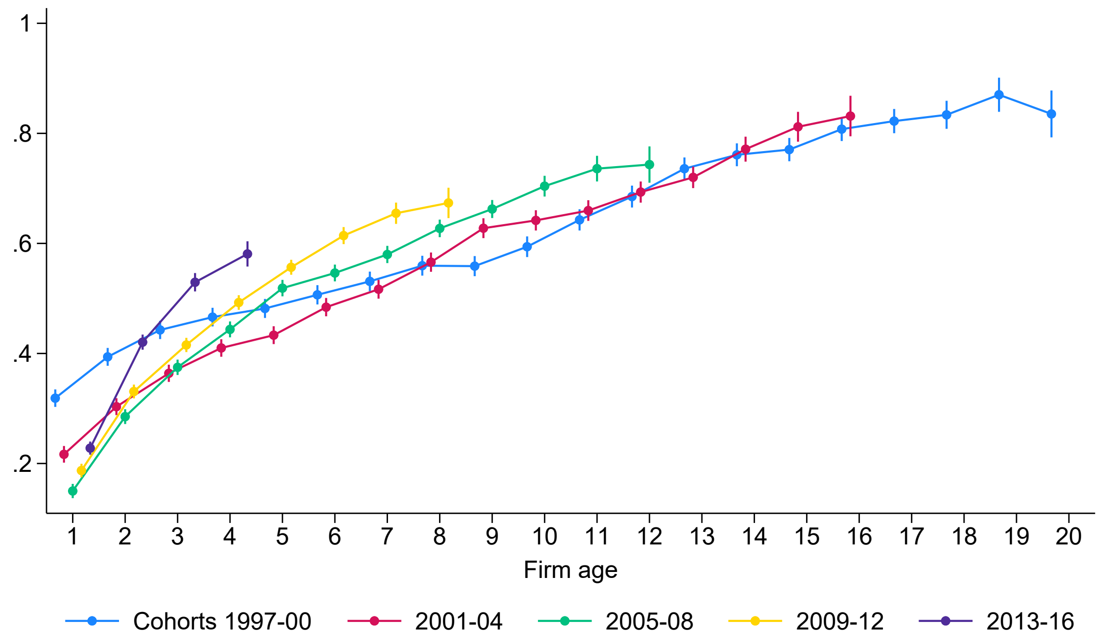
Figure 5: Average firm size (log sales) conditional on age



Notes: the figure shows avg. firm size (log sales) conditional on firm age over time. Sales are deflated to 2017 Swedish Krona (SEK) using the GDP deflator. 95% confidence intervals are shown.

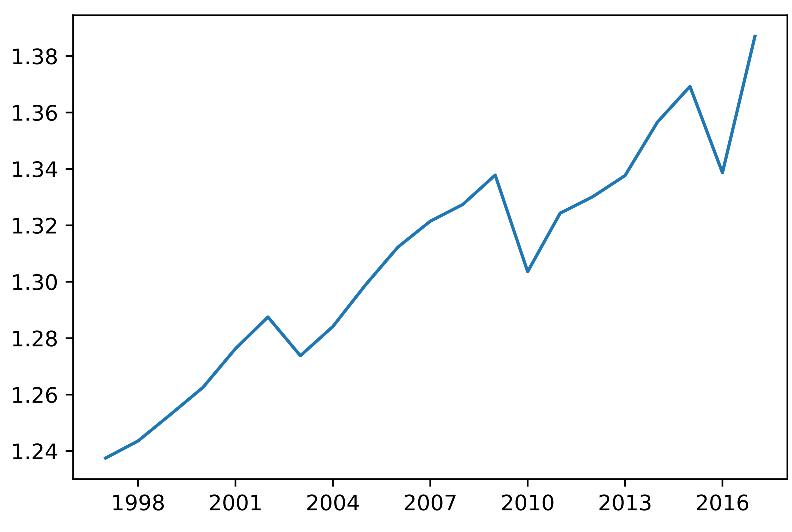
trend growth. Only the crisis episodes of the early 2000s and the financial crisis temporarily put rising concentration on hold.

Figure 6: Log sales relative to age zero (by cohort)



Notes: the figure shows cumulative sales growth over the firm's life cycle, measured as the difference between average log sales at age a_f and age zero according to eq. (1). Cohorts are pooled as indicated in the legend. Firm sales are filtered at their 1% tails. The figure includes 95% confidence intervals.

Figure 7: Within-industry sales concentration



Notes: the figure shows the within-industry standard deviation of log sales, averaged over all industries with at least 50 firms.

C Model

C.1 Solving the dynamic firm problem

The HJB for a high productivity-type firm h reads[31]

$$
\begin{aligned}
&r_t V_t^h(n,[\mu_i],S_t) - \dot{V}_t^h(n,[\mu_i],S_t) = \\
&\quad \sum_{k=1}^{n} \pi(\mu_k) + \sum_{k=1}^{n} \tau_t \Big[V_t^h\left(n-1,[\mu_i]_{i\neq k},S_t\right) - V_t^h(n,[\mu_i],S_t) \Big] \\
&\quad + \max_{[x_k,I_k]} \Bigg\{ \sum_{k=1}^{n} I_k \Big[V_t^h\left(n,\left[[\mu_i]_{i\neq k},\mu_k\times\lambda\right],S_t\right) - V_t^h(n,[\mu_i],S_t) \Big] \\
&\quad + \sum_{k=1}^{n} x_k \Big[S_t V_t^h\left(n+1,\left[[\mu_i],\lambda\right],S_t\right) + (1-S_t) V_t^h\left(n+1,\left[[\mu_i],\lambda\times\varphi^h/\varphi^l\right],S_t\right) - V_t^h(n,[\mu_i],S_t) \Big] \\
&\quad - w_t \left[\mu_k^{-1} \frac{1}{\psi_I} (I_k)^\zeta + \frac{1}{\psi_x} (x_k)^\zeta \right] \Bigg\}
\end{aligned}
$$

The HBJ for a low productivity-type firm l reads

$$
\begin{aligned}
&r_t V_t^l(n,[\mu_i],S_t) - \dot{V}_t^l(n,[\mu_i],S_t) = \\
&\quad \sum_{k=1}^{n} \pi(\mu_k) + \sum_{k=1}^{n} \tau_t \Big[V_t^l\left(n-1,[\mu_i]_{i\neq k},S_t\right) - V_t^l(n,[\mu_i],S_t) \Big] \\
&\quad + \max_{[x_k,I_k]} \Bigg\{ \sum_{k=1}^{n} I_k \Big[V_t^l\left(n,\left[[\mu_i]_{i\neq k},\mu_k\times\lambda\right],S_t\right) - V_t^l(n,[\mu_i],S_t) \Big] \\
&\quad + \sum_{k=1}^{n} x_k \Big[S_t V_t^l\left(n+1,\left[[\mu_i],\lambda\times\varphi^l/\varphi^h\right],S_t\right) + (1-S_t) V_t^l\left(n+1,\left[[\mu_i],\lambda\right],S_t\right) - V_t^l(n,[\mu_i],S_t) \Big] \\
&\quad - w_t \left[\mu_k^{-1} \frac{1}{\psi_I} (I_k)^\zeta + \frac{1}{\psi_x} (x_k)^\zeta \right] \Bigg\}.
\end{aligned}
$$

I solve for the value function of a high-type firm, however the steps for the low-type firm are equivalent. For clarity, I suppress the dependence of the value function on S_t in the following. Guess that the value function of the firm consists of a component that is common to all lines and a line-specific component

$$
V_t^h(n,[\mu_i]) = V_{t,P}^h(n) + \sum_{k=1}^{n} V_{t,M}^h(\mu_k).
$$

[31] The notation follows Peters (2020) where possible.

Substituting the guess into the HJB, $V^h_{t,P}(n)$ and $V^h_{t,M}(\mu_k)$ solve the following differential equations

$$r_t V^h_{t,M}(\mu_i) - \dot{V}^h_{t,M}(\mu_i) = \pi(\mu_i) - \tau_t V^h_{t,M}(\mu_i) + \max_{I_i}\left\{I_i\Big[V^h_{t,M}(\mu_i \times \lambda) - V^h_{t,M}(\mu_i)\Big] - w_t \mu_i^{-1}\frac{1}{\psi_I}(I_i)^\zeta\right\}. \tag{31}$$

and

$$r_t V^h_{t,P}(n) - \dot{V}^h_{t,P}(n) = \sum_{k=1}^{n}\tau_t\Big[V^h_{t,P}(n-1) - V^h_{t,P}(n)\Big] + \max_{[x_k]}\left\{\sum_{k=1}^{n} x_k\Big[V^h_{t,P}(n+1) - V^h_{t,P}(n) + S_t V^h_{t,M}(\lambda) + (1-S_t)V^h_{t,M}(\lambda \times \varphi^h/\varphi^l)\Big] - w_t\frac{1}{\psi_x}(x_k)^\zeta\right\} \tag{32}$$

Assume that in steady-state $V^h_{t,P}$ and $V^h_{t,M}$ grow at the constant rate g. Using this guess in eq. (31) and following Peters (2020), we obtain for $V^h_{t,M}(\mu_i)$

$$V^h_{t,M}(\mu_i) = \frac{\pi(\mu_i) + \frac{\zeta-1}{\psi_I}(I_i)^\zeta w_t \mu_i^{-1}}{\rho+\tau},$$

where I_i solves

$$I_i = \left(\left(\frac{Y_t}{w_t} - \frac{\zeta-1}{\psi_I}(I_i)^\zeta\right)\left(1-\frac{1}{\lambda}\right)\frac{\psi_I}{\zeta(\rho+\tau)}\right)^{\frac{1}{\zeta-1}}. \tag{33}$$

Eq. (33) shows that internal innovation rates I_i are time invariant, and independent of the product line and the productivity type of the firm, $I \equiv I^h = I^l$.

With this at hand, we can turn back to the differential equation for $V^h_{t,P}(n)$ in eq. (32). In addition to the guess that $V^h_{t,P}(n)$ grows at rate g, conjecture that $V^h_{t,P}(n) = n \times v^h_t$. Combined with the Euler we get

$$(\rho+\tau)nv^h_t = \max_{[x_k]}\left\{\sum_{k=1}^{n} x_k\Big[v^h_t + S_t V^h_{t,M}(\lambda) + (1-S_t)V^h_{t,M}(\lambda \times \varphi^h/\varphi^l)\Big] - w_t\frac{1}{\psi_x}(x_k)^\zeta\right\}. \tag{34}$$

The optimality condition for x_k is given by

$$v^h_t + S_t V^h_{t,M}(\lambda) + (1-S_t)V^h_{t,M}(\lambda \times \varphi^h/\varphi^l) = w_t\frac{\zeta}{\psi_x}(x_k)^{\zeta-1}. \tag{35}$$

Several observations are noteworthy. First, the FOC shows that optimal expansion rates are independent of quality and productivity gaps in line k. We can hence drop the item indexation: $x_k = x^d$, where $d \in \{h, \ell\}$. Second, $v_t, V^h_{t,M}, w_t$ all grow at the same rate g, which implies that expansion rates are constant over time. We can hence write eq. (34) as

$$v^h_t = \frac{1}{(\rho+\tau)} \frac{\zeta - 1}{\psi_x} (x^h)^\zeta w_t.$$

Gathering all terms, the value function is given by

$$\begin{aligned} V^h_t(n, [\mu_i]) =& V^h_{t,P}(n) + \sum_{k=1}^{n} V_{t,M}(\mu_k) \\ =& n v^h_t + \sum_{k=1}^{n} V_{t,M}(\mu_k) \\ =& n \frac{1}{(\rho+\tau)} \frac{\zeta-1}{\psi_x} (x^h)^\zeta w_t + \sum_{k=1}^{n} \frac{\pi(\mu_k) + \frac{\zeta-1}{\psi_I} I^\zeta w_t \mu_k^{-1}}{\rho+\tau}, \end{aligned} \tag{36}$$

which is the expression for the value function stated in the main text, Proposition 1. To see that high-type firms expand at different rates than low-type firms, assume that $x^h = x^\ell$. In this case, $v^h_t = v^\ell_t$, however $E[V^h_t(1, \mu_i)] > E[V^\ell_t(1, \mu_i)]$, because the value function is increasing in the markup. This is true because $Y - \frac{\zeta-1}{\psi_I} I^\zeta w > 0$, otherwise the optimal internal R&D rate defined in eq. (33) would be negative (or zero). The optimality condition for expansion R&D in eq. (35) relates the expected value of expanding into a new product market to the marginal cost of expanding. Given $E[V^h_t(1, \mu_i)] > E[V^\ell_t(1, \mu_i)]$, the cost of expansion R&D (the right hand side of eq. (35)) must be larger for high-type than for low-type firms, which implies $x^h > x^\ell$; contradicting the initial assumption. As in Lentz and Mortensen (2008), the fact that the marginal value of a product line increases in profits per line implies that firms' expansion rates increase with profitability (productivity).

Using the expression for v^h_t, write the optimality condition in eq. (35) as

$$\begin{aligned} \frac{\zeta-1}{\psi_x}(x^h)^\zeta + S_t \left(\frac{Y_t}{w_t} \left(1 - \frac{1}{\lambda}\right) + \frac{\zeta-1}{\psi_I} I^\zeta \lambda^{-1} \right) + (1 - S_t) \left(\frac{Y_t}{w_t} \left(1 - \frac{\varphi^l}{\varphi^h} \frac{1}{\lambda}\right) + \frac{\zeta-1}{\psi_I} I^\zeta \lambda^{-1} \frac{\varphi^l}{\varphi^h} \right) \\ = (\rho+\tau) \frac{\zeta}{\psi_x} (x^h)^{\zeta-1}. \end{aligned}$$

Following the same steps for low-productivity firms, we obtain the optimality condition

$$\frac{\zeta-1}{\psi_x}(x^l)^\zeta + S_t\left(\frac{Y_t}{w_t}\left(1-\frac{1}{\lambda}\frac{\varphi^h}{\varphi^l}\right)+\frac{\zeta-1}{\psi_I}I^\zeta\lambda^{-1}\frac{\varphi^h}{\varphi^l}\right)+(1-S_t)\left(\frac{Y_t}{w_t}\left(1-\frac{1}{\lambda}\right)+\frac{\zeta-1}{\psi_I}I^\zeta\lambda^{-1}\right)$$
$$=(\rho+\tau)\frac{\zeta}{\psi_x}(x^l)^{\zeta-1}.$$

C.2 Joint distribution of quality and productivity gaps

I characterize the two-dimensional distribution of quality and productivity gaps along the BGP as a function of firm policies. This allows for optimal policies and the distribution to be solved jointly. I solve for the steady state distribution over quality and productivity gaps by setting the differential equations characterizing the law-of-motion in eq. (9) and (10) equal to zero. From this, one obtains the stationary mass of product lines with quality gap λ^Δ and productivity gap φ^i/φ^j

$$\nu\left(\Delta,\frac{\varphi^l}{\varphi^h}\right)=\left(\frac{I}{I+\tau}\right)^\Delta\frac{(1-S)x^lS+z(1-p^h)S}{I}$$
$$\nu\left(\Delta,\frac{\varphi^l}{\varphi^l}\right)=\left(\frac{I}{I+\tau}\right)^\Delta\frac{(1-S)x^l(1-S)+z(1-p^h)(1-S)}{I}$$
$$\nu\left(\Delta,\frac{\varphi^h}{\varphi^h}\right)=\left(\frac{I}{I+\tau}\right)^\Delta\frac{Sx^hS+zp^hS}{I}$$
$$\nu\left(\Delta,\frac{\varphi^h}{\varphi^l}\right)=\left(\frac{I}{I+\tau}\right)^\Delta\frac{Sx^h(1-S)+zp^h(1-S)}{I}.$$

Summing over all Δ for a given productivity gap gives $S_{\varphi^l,\varphi^h}, S_{\varphi^l,\varphi^l}, S_{\varphi^h,\varphi^h}, S_{\varphi^h,\varphi^l}$ as stated in Proposition 1 in main text. It follows that

$$\Pr\left(\Delta\le d,\frac{\varphi^l}{\varphi^h}\right)=\sum_{i=1}^{d}\nu\left(i,\frac{\varphi^l}{\varphi^h}\right)=S_{\varphi^l,\varphi^h}\left(1-\left(\frac{I}{I+\tau}\right)^d\right)$$
$$\Pr\left(\Delta\le d,\frac{\varphi^l}{\varphi^l}\right)=\sum_{i=1}^{d}\nu\left(i,\frac{\varphi^l}{\varphi^l}\right)=S_{\varphi^l,\varphi^l}\left(1-\left(\frac{I}{I+\tau}\right)^d\right)$$
$$\Pr\left(\Delta\le d,\frac{\varphi^h}{\varphi^h}\right)=\sum_{i=1}^{d}\nu\left(i,\frac{\varphi^h}{\varphi^h}\right)=S_{\varphi^h,\varphi^h}\left(1-\left(\frac{I}{I+\tau}\right)^d\right)$$
$$\Pr\left(\Delta\le d,\frac{\varphi^h}{\varphi^l}\right)=\sum_{i=1}^{d}\nu\left(i,\frac{\varphi^h}{\varphi^l}\right)=S_{\varphi^h,\varphi^l}\left(1-\left(\frac{I}{I+\tau}\right)^d\right).$$

Focusing on product lines where a low-productivity incumbent faces a high-productivity second-best firm:

$$P\left(\Delta \leq d, \frac{\varphi^l}{\varphi^h}\right) = S_{\varphi^l,\varphi^h}\left(1 - e^{-d[\ln(I+\tau)-\ln I]}\right)$$

or

$$P\left(\ln\left(\lambda^\Delta\right) \leq d, \frac{\varphi^l}{\varphi^h}\right) = S_{\varphi^l,\varphi^h}\left(1 - e^{-\frac{\ln(I+\tau)-\ln I}{\ln(\lambda)}d}\right).$$

Conditional on the productivity gap, $\ln\left(\lambda^\Delta\right)$ is exponentially distributed with parameter $\frac{\ln(I+\tau)-\ln I}{\ln(\lambda)}$. Further

$$P\left(\lambda^\Delta \leq d, \frac{\varphi^l}{\varphi^h}\right) = S_{\varphi^l,\varphi^h}\left(1 - d^{-\frac{\ln(I+\tau)-\ln I}{\ln(\lambda)}}\right).$$

Conditional on the productivity gap, quality gaps follow a Pareto distribution with parameter $\frac{\ln(I+\tau)-\ln I}{\ln(\lambda)}$. Denote $\theta = \frac{\ln(I+\tau)-\ln I}{\ln(\lambda)}$. We then have

$$P\left(\lambda^\Delta \leq m, \frac{\varphi^l}{\varphi^h}\right) = S_{\varphi^l,\varphi^h}\left(1 - m^{-\theta}\right).$$

Conditional on the productivity gap, quality gaps follow a Pareto distribution with parameter θ. As in Peters (2020), the Pareto shape parameter is affected by the rate of internal R&D I relative to creative destruction τ. The more internal R&D, the more mass is in the tail of the quality gap distribution. In Peters (2020), the distribution of quality gaps follows a Pareto distribution. In this model, quality gaps *conditional* on the productivity gap are Pareto distributed.

After repeating the same steps for lines with different productivity gaps, we obtain the aggregate labor share as follows

$$\begin{aligned}\Lambda =& \sum_{k\in\{h,l\}} \sum_{n\in\{h,l\}} \int_1^\infty \frac{1}{\varphi_k/\varphi_n} \frac{1}{m} S_{\varphi_k,\varphi_n} \theta m^{-(\theta+1)} dm \\ =& \frac{\theta}{\theta+1} \sum_{k\in\{h,l\}} \sum_{n\in\{h,l\}} \frac{1}{\varphi_k/\varphi_n} S_{\varphi_k,\varphi_n}.\end{aligned}$$

The TFP misallocation statistic $\mathcal{M}$ is then given by

$$\mathcal{M} = \frac{e^{\sum_{k\in\{h,l\}}\sum_{n\in\{h,l\}}\int\left[\ln\left(\frac{1}{\frac{\varphi_k}{\varphi_n}}\frac{1}{m}\right)S_{\varphi_k,\varphi_n}\theta m^{-(\theta+1)}\right]dm}}{\Lambda}$$
$$= \frac{e^{\sum_{k\in\{h,l\}}\sum_{n\in\{h,l\}}\left[S_{\varphi_k,\varphi_n}\left(\ln\left(\frac{1}{\frac{\varphi_k}{\varphi_n}}\right)-\frac{1}{\theta}\right)\right]}}{\Lambda},$$

where I have made use of

$$\int_1^\infty \ln\left(\frac{1}{m}\right)S_{\varphi_k,\varphi_n}\theta m^{-(\theta+1)}dm = \left[\frac{\theta\ln(m)+1}{\theta m^\theta}+C\right]_1^\infty = -\frac{1}{\theta}.$$

Alternatively note that this expression is equal to $-S_{\varphi_k,\varphi_n}E[\ln(\lambda^\Delta)|\varphi_k,\varphi_n]$. I have shown above that $\ln(\lambda^\Delta)$ conditional on the productivity gap is exponentially distributed with parameter θ. From the characteristics of an exponential distribution, its expected value is $1/\theta$.

The aggregate markup is then given by

$$E[\mu] = \sum_{k\in\{h,l\}}\sum_{n\in\{h,l\}}\int_1^\infty \frac{\varphi_k}{\varphi_n}mS_{\varphi_k,\varphi_n}\theta m^{-(\theta+1)}dm$$
$$=\frac{\theta}{\theta-1}\sum_{k\in\{h,l\}}\sum_{n\in\{h,l\}}\frac{\varphi_k}{\varphi_n}S_{\varphi_k,\varphi_n}.$$

These are the expressions for the aggregate labor income share, TFP misallocation statistic and aggregate markup stated in the main text in Proposition 2.

C.3 Deriving the steady-state growth rate of aggregate variables

The growth rate of Q_t determines the growth rate of aggregate variables.

$$g = \frac{\dot{Q}_t}{Q_t} = \frac{\partial\ln(Q_t)}{\partial t}$$

Quality of a product in a given product line increases through internal R&D, expansion R&D or firm entry. For the growth rate of Q_t we have

$$\ln(Q_{t+\Delta}) = \int_0^1\left[(\Delta I + \Delta S x^h + \Delta(1-S)x^l + \Delta z)\ln(\lambda) + \ln(q_{t,i})\right]di$$

so that

$$\frac{\ln(Q_{t+\Delta}) - \ln(Q_t)}{\Delta} = \Big(I + Sx^h + (1-S)x^l + z\Big)\ln(\lambda).$$

For $\Delta \to 0$, $g = \Big(I + Sx^h + (1-S)x^l + z\Big)\ln(\lambda)$ as stated in Proposition 1.

C.4 Solving for the steady state equilibrium

In the model there are the seven unknown variables $x^h, x^l, I, z, \tau, \frac{Y_t}{w_t}, S$ and the markup distribution $\nu()$ in seven equations plus the system of differential equations characterizing $\nu()$.

Optimality condition for the internal innovation rate

$$I = \left(\left(\frac{Y_t}{w_t} - \frac{\zeta-1}{\psi_I} I^\zeta\right)\left(1 - \frac{1}{\lambda}\right)\frac{\psi_I}{\zeta(\rho+\tau)}\right)^{\frac{1}{\zeta-1}}$$

Optimality condition for high-productivity type expansion rate

$$\begin{aligned}\frac{\zeta-1}{\psi_x}(x^h)^\zeta + S\left(\frac{Y_t}{w_t}\left(1-\frac{1}{\lambda}\right) + \frac{\zeta-1}{\psi_I} I^\zeta \lambda^{-1}\right) + (1-S)\left(\frac{Y_t}{w_t}\left(1 - \frac{\varphi^l}{\varphi^h}\frac{1}{\lambda}\right) + \frac{\zeta-1}{\psi_I} I^\zeta \lambda^{-1}\frac{\varphi^l}{\varphi^h}\right) \\ = (\rho+\tau)\frac{\zeta}{\psi_x}(x^h)^{\zeta-1}\end{aligned}$$

Optimality condition for low-productivity type expansion rate

$$\begin{aligned}\frac{\zeta-1}{\psi_x}(x^l)^\zeta + S\left(\frac{Y_t}{w_t}\left(1-\frac{1}{\lambda}\frac{\varphi^h}{\varphi^l}\right) + \frac{\zeta-1}{\psi_I l} I^\zeta \lambda^{-1}\frac{\varphi^h}{\varphi^l}\right) + (1-S)\left(\frac{Y_t}{w_t}\left(1 - \frac{1}{\lambda}\right) + \frac{\zeta-1}{\psi_I l} I^\zeta \lambda^{-1}\right) \\ = (\rho+\tau)\frac{\zeta}{\psi_x}(x^l)^{\zeta-1}\end{aligned}$$

Free entry condition

$$p^h\Big(SV_t^h(1,\lambda) + (1-S)V_t^h(1, \lambda \times \varphi^h/\varphi^l)\Big) + (1-p^h)\Big(SV_t^l(1,\lambda\times\varphi^l/\varphi^h) + (1-S)V_t^l(1,\lambda)\Big) = \frac{1}{\psi_z}w_t,$$

where

$$V_t^d(1,\mu) = \frac{1}{(\rho+\tau)}\frac{\zeta-1}{\psi_x}(x^d)^\zeta w_t + \frac{Y_t\left(1-\frac{1}{\mu}\right) + \frac{\zeta-1}{\psi_I}I^\zeta w_t \mu^{-1}}{\rho+\tau}$$

Labor market clearing condition

$$1 = \frac{Y_t}{w_t} \sum_{\frac{\varphi_j}{\varphi_{j'}}} \sum_i \frac{1}{\lambda^i \frac{\varphi_j}{\varphi_{j'}}} \nu\left(i, \frac{\varphi_j}{\varphi_{j'}}\right) + \frac{1}{\psi_I} I^{\zeta} \sum_{\frac{\varphi_j}{\varphi_{j'}}} \sum_i \frac{1}{\lambda^i \frac{\varphi_j}{\varphi_{j'}}} \nu\left(i, \frac{\varphi_j}{\varphi_{j'}}\right) + S\frac{1}{\psi_x}(x^h)^{\zeta} + (1-S)\frac{1}{\psi_x}(x^l)^{\zeta} + \frac{z}{\psi_z}$$

Creative destruction

$$\tau = z + Sx^h + (1-S)x^l$$

Share of high productivity type

$$S = \sum_{i=1}^{\infty} \left[\nu\left(i, \frac{\varphi^h}{\varphi^h}\right) + \nu\left(i, \frac{\varphi^h}{\varphi^l}\right)\right],$$

where ν, the stationary distribution of quality and productivity gaps, is characterized by

$$0 = \dot{\nu}\left(\Delta, \frac{\varphi_j}{\varphi_{j'}}\right) = I\nu\left(\Delta - 1, \frac{\varphi_j}{\varphi_{j'}}\right) - \nu\left(\Delta, \frac{\varphi_j}{\varphi_{j'}}\right)(I+\tau) \quad \text{for} \quad \Delta \geq 2$$

and for the case of a unitary quality gap

$$\begin{aligned}
0 = \dot{\nu}\left(1, \frac{\varphi^l}{\varphi^h}\right) &= (1-S)x^l S + z_t(1-p^h)S - \nu\left(1, \frac{\varphi^l}{\varphi^h}\right)(I+\tau) \\
0 = \dot{\nu}\left(1, \frac{\varphi^l}{\varphi^l}\right) &= (1-S)x^l(1-S) + z_t(1-p^h)(1-S) - \nu\left(1, \frac{\varphi^l}{\varphi^l}\right)(I+\tau) \\
0 = \dot{\nu}\left(1, \frac{\varphi^h}{\varphi^h}\right) &= Sx^h S + z_t p^h S - \nu\left(1, \frac{\varphi^h}{\varphi^h}\right)(I+\tau) \\
0 = \dot{\nu}\left(1, \frac{\varphi^h}{\varphi^l}\right) &= Sx^h(1-S) + z_t p^h(1-S) - \nu\left(1, \frac{\varphi^h}{\varphi^l}\right)(I+\tau).
\end{aligned}$$

To simplify the system of equations, first rewrite the rate of creative destruction

$$z = (\tau - Sx^h - (1-S)x^l)$$

such that z can be substituted out from the remaining equations. Second, based on Proposition 1, we obtain

$$S = S_{\varphi^h,\varphi^h} + S_{\varphi^h,\varphi^l} = \frac{Sx^h + zp^h}{\tau}.$$

Third, the optimality conditions for expansion rates (multiplied by p^h and $(1-p^h)$) and the free entry condition together imply

$$\frac{1}{\psi_x}p^h(x^h)^{\zeta-1}+\frac{1}{\psi_x}(1-p^h)(x^l)^{\zeta-1}=\frac{1}{\psi_z\zeta}.$$

The system of equilibrium conditions can hence be reduced to:

Optimality condition for the internal innovation rate

$$I=\left(\left(\frac{Y_t}{w_t}\psi_I-(\zeta-1)I^\zeta\right)\frac{\left(1-\frac{1}{\lambda}\right)}{\zeta(\rho+\tau)}\right)^{\frac{1}{\zeta-1}}$$

Optimality condition for high-productivity type expansion rate

$$\frac{\zeta-1}{\psi_x}(x^h)^\zeta+S\left(\frac{Y_t}{w_t}\left(1-\frac{1}{\lambda}\right)+(\zeta-1)I^\zeta\lambda^{-1}\frac{1}{\psi_I}\right)+(1-S)\left(\frac{Y_t}{w_t}\left(1-\frac{\varphi^l}{\varphi^h}\frac{1}{\lambda}\right)+(\zeta-1)I^\zeta\lambda^{-1}\frac{\varphi^l}{\psi_I\varphi^h}\right)$$
$$=(\rho+\tau)\frac{\zeta}{\psi_x}(x^h)^{\zeta-1}$$

Optimality condition for low-productivity type expansion rate

$$\frac{\zeta-1}{\psi_x}(x^l)^\zeta+S\left(\frac{Y_t}{w_t}\left(1-\frac{1}{\lambda}\frac{\varphi^h}{\varphi^l}\right)+(\zeta-1)I^\zeta\lambda^{-1}\frac{\varphi^h}{\psi_I l\varphi^l}\right)+(1-S)\left(\frac{Y_t}{w_t}\left(1-\frac{1}{\lambda}\right)+(\zeta-1)I^\zeta\lambda^{-1}\frac{1}{\psi_I l}\right)$$
$$=(\rho+\tau)\frac{\zeta}{\psi_x}(x^l)^{\zeta-1}$$

Free entry

$$p^h\frac{(x^h)^{\zeta-1}}{\psi_x}+(1-p^h)\frac{(x^l)^{\zeta-1}}{\psi_x}=\frac{1}{\psi_z\zeta}$$

Labor market clearing condition

$$1=\frac{Y_t}{w_t}\Lambda+\Lambda_I+S\frac{1}{\psi_x}(x^h)^\zeta+(1-S)\frac{1}{\psi_x}(x^l)^\zeta+\frac{\tau-Sx^h-(1-S)x^l}{\psi_z},$$

where[32]

$$\begin{aligned}\Lambda=&\frac{\theta}{\theta+1}\sum_{k\in\{h,l\}}\sum_{n\in\{h,l\}}\frac{1}{\varphi_k/\varphi_n}S_{\varphi_k,\varphi_n}\\ \Lambda_I=&\frac{1}{\psi_I}I^\zeta\frac{\theta}{\theta+1}\sum_{k\in\{h,l\}}\sum_{n\in\{h,l\}}\frac{1}{\varphi_k/\varphi_n}S_{\varphi_k,\varphi_n}\\ \theta=&\frac{\ln(I+\tau)-\ln(I)}{\ln(\lambda)}\end{aligned}$$

[32]For the derivation of Λ I assume a continuous distribution of quality gaps.

Share of high productivity type

$$S = \frac{Sx^h + (\tau - Sx^h - (1-S)x^l)p^h}{\tau}$$

The expressions related to the labor market clearing condition are derived in Section C.2. This constitutes a system of seven equations in seven unknowns $(x^h, x^l, I, \tau, \frac{Y_t}{w_t}, S)$, which I solve using a root finder.

C.5 Firm markups

Firm markups are defined by $\mu_f = \frac{py_f}{wl_f} = \left(\frac{1}{n}\sum_{k=1}^{n} \mu_{kf}^{-1}\right)^{-1}$. Therefore

$$\ln \mu_f = -\ln\left(\frac{1}{n}\sum_{k=1}^{n} \mu_k^{-1}\right).$$

Rewrite the term in brackets (for a high-productivity firm) as

$$\frac{1}{n}\sum_{k=1}^{n} \mu_k^{-1} = \frac{1}{n}\sum_{k=1}^{n} e^{-\ln \mu_k} = \frac{1}{n}\left(\sum_{i=1}^{n_i} e^{-\ln\frac{\varphi^h}{\varphi^l} - \Delta_i \ln \lambda} + \sum_{j=1}^{n_j} e^{-\Delta_j \ln \lambda}\right), \tag{37}$$

where i indexes the product lines where the high productivity firm faces a low productivity second best producer, j the lines where it faces a high productivity second best producer and $n_i + n_j = n$. A two-dimensional linear Taylor expansion around $\ln \lambda = 0$ and $\ln \frac{\varphi^h}{\varphi^l} = 0$ gives

$$\frac{1}{n}\left(\sum_{i=1}^{n_i} e^{-\ln\frac{\varphi^h}{\varphi^l} - \Delta_i \ln \lambda} + \sum_{j=1}^{n_j} e^{-\Delta_j \ln \lambda}\right) \approx 1 - \left(\frac{1}{n}\sum_{k=1}^{n} \Delta_k\right)\ln \lambda - \frac{n_i}{n}\ln\left(\frac{\varphi^h}{\varphi^l}\right)$$

such that

$$E\left[\ln \mu_f | \text{firm age} = a_f, \varphi^h\right] \approx E\left[\frac{1}{n}\sum_{k=1}^{n} \Delta_k | \text{firm age} = a_f, \varphi^h\right]\ln \lambda + (1-S)\ln\left(\frac{\varphi^h}{\varphi^l}\right),$$

where I have used the fact that (in expectation) the share of the firm's product lines with a low productivity second best producer is equal to the aggregate share of product lines where the incumbent is of the low productivity type. From Peters (2020), we know that

$$E\left[\frac{1}{n}\sum_{k=1}^{n} \Delta_k | \text{firm age} = a_f, \varphi^h\right]\ln \lambda = \left(1 + I \times E[a_P^h | a_f]\right)\ln \lambda,$$

where $E[a_P^h|a_f]$ denotes the average product age of a high-productivity type firm conditional on firm age a_f and

$$E[a_P^h|a_f] = \frac{1}{x^h}\left(\frac{\frac{1}{\tau}\left(1-e^{-\tau a_f}\right)}{\frac{1}{x^h+\tau}\left(1-e^{-(x^h+\tau)a_f}\right)} - 1\right)\left(1-\phi^h(a_f)\right) + a_f\phi^h(a_f)$$

$$\phi^h(a) = e^{-x^h a}\frac{1}{\gamma^h(a)}\ln\left(\frac{1}{1-\gamma^h(a)}\right)$$

$$\gamma^h(a) = \frac{x^h\left(1-e^{-(\tau-x^h)a}\right)}{\tau - x^h e^{-(\tau-x^h)a}},$$

which gives the expression in the main text.

For a firm of the low-productivity type, the last term in eq. (37) reads

$$\frac{1}{n}\left(\sum_{i=1}^{n_i} e^{-\ln\Delta_i \ln\lambda} + \sum_{j=1}^{n_j} e^{-\ln\frac{\varphi^l}{\varphi^h} - \ln\Delta_j\ln\lambda}\right),$$

where i indexes the product lines where the low-productivity producer faces a low-productivity second best producer, j the lines where it faces a high-productivity second best producer and $n_i + n_j = n$. Following the same steps as for a high-productivity firm, this time linearizing around $\ln\frac{\varphi^l}{\varphi^h} = 0$ (and $\ln\lambda = 0$) gives

$$E\left[\ln\mu_f|\text{firm age} = a_f, \varphi^l\right] \approx \left(1 + I \times E[a_P^l|a_f]\right)\ln\lambda + S\ln\left(\frac{\varphi^l}{\varphi^h}\right),$$

where again I have made use of the fact that the share of the firm's product lines with a high-productivity second best producer is equal to the aggregate share of product lines where the incumbent is of the high-productivity type. $E[a_P^l|a_f]$ is exactly defined as $E[a_P^h|a_f]$ with x^h replaced by x^l in the above expressions.

D Computation of transition dynamics

In this section, I lay out the numerical procedure to solve for the transition path. Since time is continuous, I solve a discretized version of the model where the solution converges to the one in continuous time for small enough time intervals. As shown in Appendix C, value functions are additive across product lines. Therefore, I solve the problem of two representative one-product firms: one of the high productivity type and one of the low productivity type.

I normalize the value function by the wage w_t to obtain a stationary problem. The value function for the high-type firm (in discrete time) reads

$$\begin{aligned}
\frac{V_t^h(1,\mu_i,S_t)}{w_t} &= \frac{Y_t}{w_t}\left(1-\frac{1}{\mu_i}\right)dt \\
&- \tau_t \exp(-r_t dt)\frac{V_{t+dt}^h(1,\mu_i,S_{t+dt})}{w_{t+dt}}\frac{w_{t+dt}}{w_t}dt \\
&+ \max_{x_t^h}\left\{x_t^h \exp(-r_t dt)\left(S_{t+dt}\frac{V_{t+dt}^h(1,\lambda,S_{t+dt})}{w_{t+dt}} + (1-S_{t+dt})\frac{V_{t+dt}^h(1,\lambda\frac{\varphi^h}{\varphi^l},S_{t+dt})}{w_{t+dt}}\right)\frac{w_{t+dt}}{w_t}dt - \frac{1}{\psi_x}(x_t^h)^\zeta dt\right\} \\
&+ \max_{I_t^h}\left\{I_t^h \exp(-r_t dt)\left(\frac{V_{t+dt}^h(1,\mu_i\lambda,S_{t+dt})}{w_{t+dt}} - \frac{V_{t+dt}^h(1,\mu_i,S_{t+dt})}{w_{t+dt}}\right)\frac{w_{t+dt}}{w_t}dt - \frac{1}{\psi_I}\mu_i^{-1}(I_t^h)^\zeta dt\right\} \\
&+ \exp(-r_t dt)\frac{V_{t+dt}^h(1,\mu_i,S_{t+dt})}{w_{t+dt}}\frac{w_{t+dt}}{w_t}. \qquad (38)
\end{aligned}$$

The value function for the low-type firm reads

$$\begin{aligned}
\frac{V_t^l(1,\mu_i,S_t)}{w_t} &= \frac{Y_t}{w_t}\left(1-\frac{1}{\mu_i}\right)dt \\
&- \tau_t \exp(-r_t dt)\frac{V_{t+dt}^l(1,\mu_i,S_{t+dt})}{w_{t+dt}}\frac{w_{t+dt}}{w_t}dt \\
&+ \max_{x_t^l}\left\{x_t^l \exp(-r_t dt)\left(S_{t+dt}\frac{V_{t+dt}^l(1,\lambda\frac{\varphi^l}{\varphi^h},S_{t+dt})}{w_{t+dt}} + (1-S_{t+dt})\frac{V_{t+dt}^l(1,\lambda,S_{t+dt})}{w_{t+dt}}\right)\frac{w_{t+dt}}{w_t}dt - \frac{1}{\psi_x}(x_t^l)^\zeta dt\right\} \\
&+ \max_{I_t^l}\left\{I_t^l \exp(-r_t dt)\left(\frac{V_{t+dt}^l(1,\mu_i\lambda,S_{t+dt})}{w_{t+dt}} - \frac{V_{t+dt}^l(1,\mu_i,S_{t+dt})}{w_{t+dt}}\right)\frac{w_{t+dt}}{w_t}dt - \frac{1}{\psi_I}\mu_i^{-1}(I_t^h)^\zeta dt\right\} \\
&+ \exp(-r_t dt)\frac{V_{t+dt}^l(1,\mu_i,S_{t+dt})}{w_{t+dt}}\frac{w_{t+dt}}{w_t}. \qquad (39)
\end{aligned}$$

From this, one obtains the first order conditions for the policy functions. For the optimal expansion R&D rate of the high type firm x_t^h (again suppressing the dependence of the value function on S_t):

$$\exp(-r_t dt)\left(S_{t+dt}\frac{V_{t+dt}^h(1,\lambda)}{w_{t+dt}} + (1-S_{t+dt})\frac{V_{t+dt}^h(1,\lambda\frac{\varphi^h}{\varphi^l})}{w_{t+dt}}\right)\frac{w_{t+dt}}{w_t} = \frac{\zeta}{\psi_x}(x_t^h)^{\zeta-1} \qquad (40)$$

and for the low type firm x_t^l:

$$\exp(-r_t dt)\left(S_{t+dt}\frac{V_{t+dt}^l(1,\lambda\frac{\varphi^l}{\varphi^h})}{w_{t+dt}} + (1-S_{t+dt})\frac{V_{t+dt}^l(1,\lambda)}{w_{t+dt}}\right)\frac{w_{t+dt}}{w_t} = \frac{\zeta}{\psi_x}(x_t^l)^{\zeta-1}. \qquad (41)$$

Both are independent of the markup μ_i. For the optimal internal R&D rates of the high type, I_t^h, one obtains

$$\exp(-r_t dt)\left(\frac{V_{t+dt}^h(1,\mu_i\lambda)}{w_{t+dt}}-\frac{V_{t+dt}^h(1,\mu_i)}{w_{t+dt}}\right)\frac{w_{t+dt}}{w_t}=\frac{\zeta}{\psi_I}\mu_i^{-1}(I_t^h)^{\zeta-1} \tag{42}$$

and similarly for I_t^l

$$\exp(-r_t dt)\left(\frac{V_{t+dt}^l(1,\mu_i\lambda)}{w_{t+dt}}-\frac{V_{t+dt}^l(1,\mu_i)}{w_{t+dt}}\right)\frac{w_{t+dt}}{w_t}=\frac{\zeta}{\psi_I}\mu_i^{-1}(I_t^l)^{\zeta-1}. \tag{43}$$

Equations (38) to (43) characterize the firm problem in discrete time. These equations are supplemented by the law of motion for the two dimensional distribution of quality and productivity gaps

$$\nu_{t+dt}\left(\Delta,\frac{\varphi_j}{\varphi_{j'}}\right)-\nu_t\left(\Delta,\frac{\varphi_j}{\varphi_{j'}}\right)=dt\left[I_{\mu_i,t}\nu_t\left(\Delta-1,\frac{\varphi_j}{\varphi_{j'}}\right)-\nu_t\left(\Delta,\frac{\varphi_j}{\varphi_{j'}}\right)(I_{\mu_i,t}+\tau_t)\right]\quad\text{for}\quad\Delta\geq 2 \tag{44}$$

and for product lines with a unitary quality gap, $\Delta=1$,

$$\begin{aligned}
\nu_{t+dt}\left(1,\frac{\varphi^l}{\varphi^h}\right)-\nu_t\left(1,\frac{\varphi^l}{\varphi^h}\right)&=dt\left[(1-S_t)x_t^l S_t+z_t(1-p^h)S_t-\nu_t\left(1,\frac{\varphi^l}{\varphi^h}\right)(I_{\mu_i,t}+\tau_t)\right]\\
\nu_{t+dt}\left(1,\frac{\varphi^l}{\varphi^l}\right)-\nu_t\left(1,\frac{\varphi^l}{\varphi^l}\right)&=dt\left[(1-S_t)x_t^l(1-S_t)+z_t(1-p^h)(1-S_t)-\nu_t\left(1,\frac{\varphi^l}{\varphi^l}\right)(I_{\mu_i,t}+\tau_t)\right]\\
\nu_{t+dt}\left(1,\frac{\varphi^h}{\varphi^h}\right)-\nu_t\left(1,\frac{\varphi^h}{\varphi^h}\right)&=dt\left[S_t x_t^h S_t+z_t p^h S_t-\nu_t\left(1,\frac{\varphi^h}{\varphi^h}\right)(I_{\mu_i,t}+\tau_t)\right]\\
\nu_{t+dt}\left(1,\frac{\varphi^h}{\varphi^l}\right)-\nu_t\left(1,\frac{\varphi^h}{\varphi^l}\right)&=dt\left[S_t x_t^h(1-S_t)+z_t p^h(1-S_t)-\nu_t\left(1,\frac{\varphi^h}{\varphi^l}\right)(I_{\mu_i,t}+\tau_t)\right]
\end{aligned} \tag{45}$$

and a standard Euler equation

$$\frac{C_{t+dt}}{C_t}=\exp(-\rho dt)(1+r_{t+dt}dt). \tag{46}$$

Further, the (static) free entry and labor market clearing conditions remain unchanged and are characterized in the main text by equations (7) and (8).

The algorithm to compute the transition path assumes that an initial and ending balanced growth path has been solved for including the (stationary) two-dimensional distribution of quality and productivity gaps. I choose $dt = 0.02$ and set the transition period to 100 years (T), after which I assume the economy has reached its new balanced growth path. I further truncate the two dimensional distribution of quality and productivity gaps along the quality dimension at $\Delta = 30$, implying a maximum quality gap of λ^{30}. No mass reaches this state during the transition such that this assumption is satisfied. I then compute the transition path as follows:

1. Guess a path of interest rates r_t and wage growth $\frac{w_{d+dt}}{w_t}$ over the transition (equal to their values in the final balanced growth path)
   (a) Guess a path for S_t over the transition (equal to its value in the final balanced growth path).
      i. Starting backwards in period T, solve for optimal policy functions in $T - dt$ using equations (40)-(43).[33]
      ii. Solve for τ_{T-dt} that ensures that the free entry condition (7) holds.
      iii. Compute the value function in $T - dt$ using equations (38) and (39).
      iv. Iterate backwards until the first time period.
      v. Starting from the initial balanced growth path, simulate S_t forward using[34]

$$S_{t+dt} = S_t + dt \left[S_t x_t^h (1 - S_t) - (1 - S_t) x_t^l S_t + z_t (p^h (1 - S_t) - (1 - p^h) S_t) \right],$$

      where z_t can be substituted out by equation (12).

   (b) Update the guess for S_t from step v and go back to step i. Iterate until the guessed path for S_t converges to the implied one.
2. Starting from the initial balanced growth path, simulate the two dimensional distribution of quality and productivity gaps forward using equations 44 and 45.
3. Solve for the sequence of $\frac{Y_t}{w_t}$ from the labor market clearing condition.

[33]As shown in Appendix C, the optimal I_i is independent of the quality and productivity gap of the incumbent firm along a balanced growth path. This is not clear ex-ante during the transition period. For each productivity type, I solve for the optimal I_i (at each point in time) over a two-dimensional grid of quality and productivity gaps. It turns out that I_i is independent of the gaps also along the transition.

[34]One could already simulate the entire two-dimensional distribution forward here. However, for the inner loop, it is sufficient to iterate on S_t.

4. Compute the sequence of quality growth using

$$\frac{Q_{t+dt}}{Q_t} = \exp\left(\left[\int_0^1 I_{\mu_i,t}di + S_t x_t^h + (1 - S_t)x_t^l + z_t\right] dt \ln(\lambda)\right).$$

5. Compute the sequence of aggregate productivity growth using

$$\frac{\Phi_{t+dt}}{\Phi_t} = \left(\frac{\varphi^h}{\varphi^l}\right)^{S_{t+dt}-S_t}.$$

6. Using the two dimensional distribution of quality and productivity gaps, compute the sequence of $\mathcal{M}_t$ defined in equation (6).
7. Compute the sequence of production labor L_{Pt} using equation (4).
8. Compute the sequence of aggregate output growth $\frac{Y_{t+dt}}{Y_t}$ using equation (6).
9. With the path of aggregate output growth, obtain the implied path of interest rates from the Euler equation (46).
10. With the paths of aggregate output growth and Y_t/w_t, obtain the implied path of wage growth $\frac{w_{d+dt}}{w_t}$.
11. Update the guesses for the interest rate and wage growth and go back to step (a). Iterate until the guessed and implied paths converge.