Solving Equations of Motion: Numerical Integration Algorithms

Maria Kondzielska

Overview

1. Integration algorithms
2. Euler method
3. Verlet method
4. Leapfrog method
5. Chenciner infinity

Algorithms

Case: 2-body system, the heavier body is motionless.

$$\frac{d\mathbf{r}}{dt} = \frac{\mathbf{p}}{m} \qquad \mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \frac{GMm}{r^3}\mathbf{r}$$

To integrate Newton's equations of motion we can use one of the following algorithms:

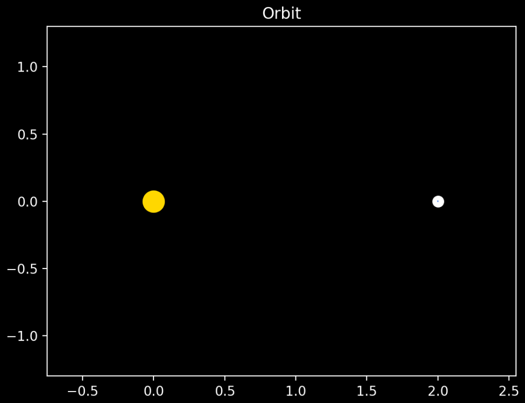
- Euler
- Verlet
- Leapfrog

Euler method

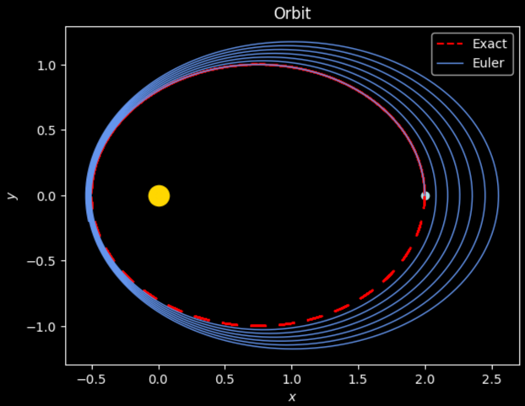
$$\mathbf{r}_{n+1} = \mathbf{r}_n + \mathbf{p}_n \frac{\delta t}{m} + \frac{1}{2m} \mathbf{F}_n \delta t^2$$

$$\mathbf{p}_{n+1} = \mathbf{p}_n + \mathbf{F}_n \delta t$$

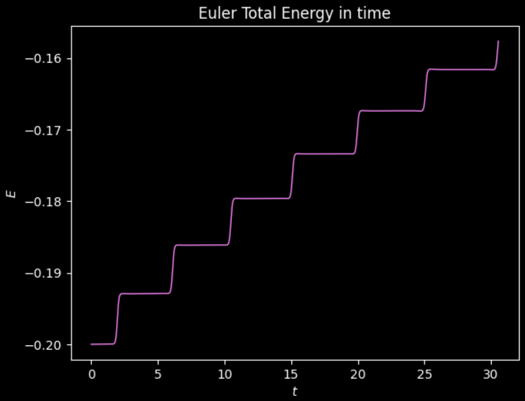
When the planet is the closest to the Sun,
the algorithm loses its accuracy.



Euler method

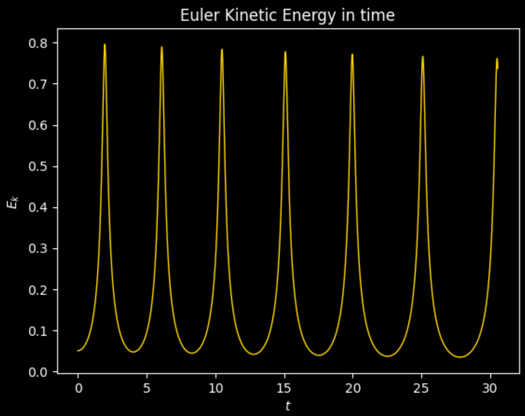


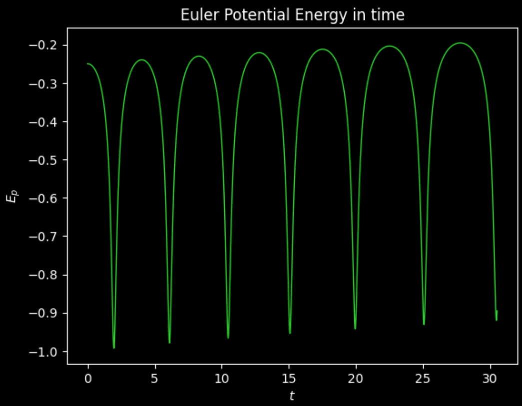
Euler orbit in case of two-body problem.



Total energy is not conserved!

Euler Energy





Verlet method

$$\boldsymbol{r}_{n+1} = 2\boldsymbol{r}_n - \boldsymbol{r}_{n-1} + \boldsymbol{F}_n \frac{\delta t^2}{m}$$

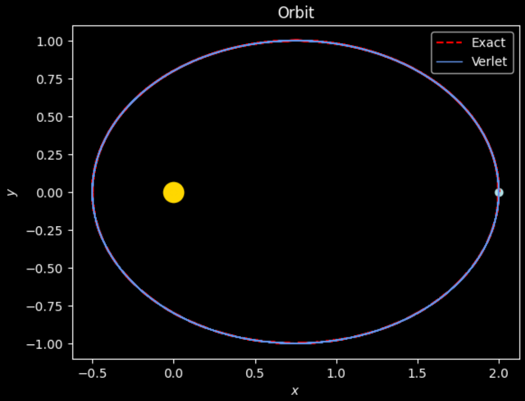
Previous step interpolation knowing the initial conditions $\boldsymbol{r}_1$ and $\boldsymbol{p}_1$:

$$\boldsymbol{r}_0 = \boldsymbol{r}_1 - \boldsymbol{p}_1 \frac{\delta t}{m}$$

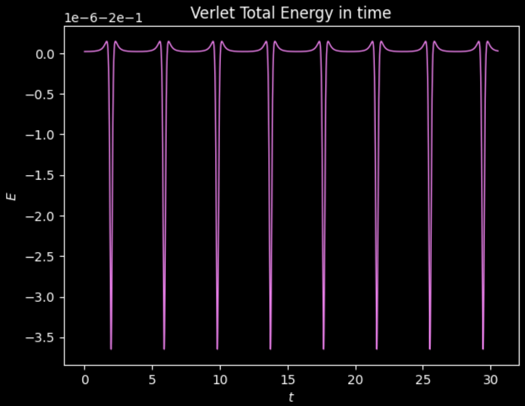
Additional, nonessential for the algorithm:

$$\boldsymbol{p}_{n+1} = \frac{m}{2\delta t}(\boldsymbol{r}_{n+1} - \boldsymbol{r}_{n-1})$$

Verlet method

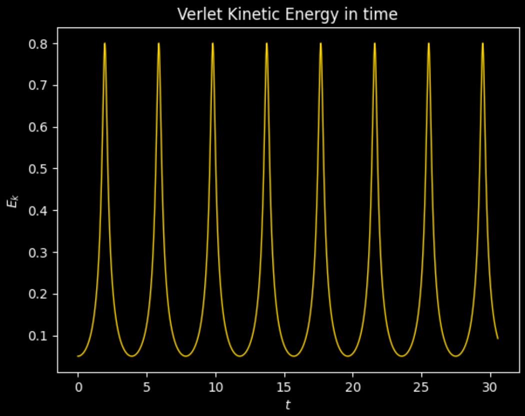


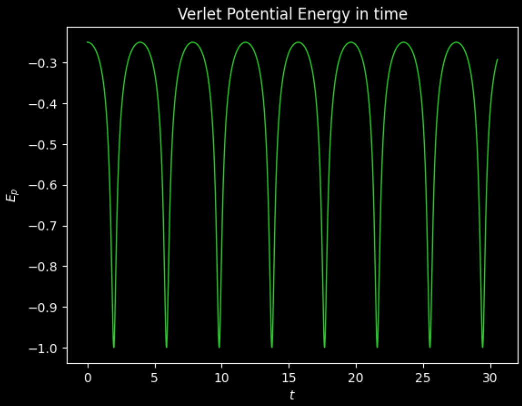
Verlet orbit in case of two-body problem.



Conservation of energy.

Verlet Energy





Leapfrog method

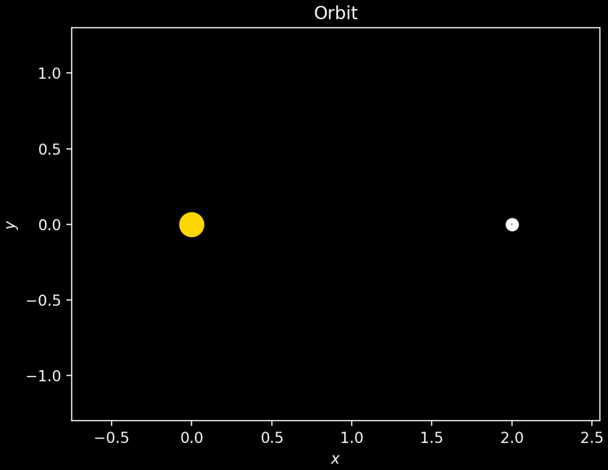
Previous half-step interpolation knowing the initial conditions $\boldsymbol{r}_1$ and $\boldsymbol{p}_1$:

$$\boldsymbol{r}_{1/2} = \boldsymbol{r}_1 - \boldsymbol{p}_1 \frac{\delta t}{2m}$$

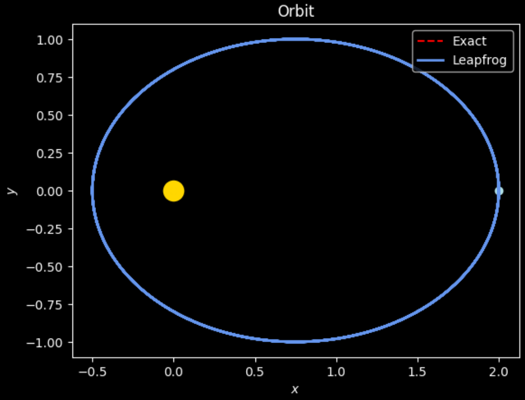
$$\boldsymbol{p}_{1/2} = \boldsymbol{p}_1 - \boldsymbol{F}_1 \frac{\delta t}{2}$$

$$\boldsymbol{p}_{n+1/2} = \boldsymbol{p}_{n-1/2} + \boldsymbol{F}_n \delta t$$

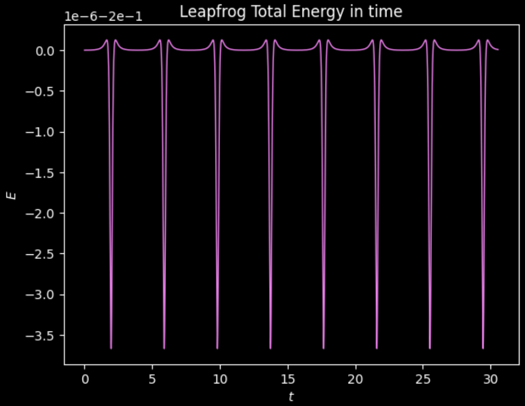
$$\boldsymbol{r}_{n+1} = \boldsymbol{r}_n + \boldsymbol{p}_{n+1/2} \frac{\delta t}{m}$$



Leapfrog method

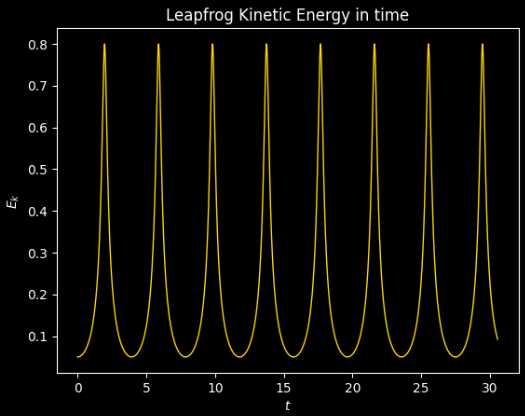


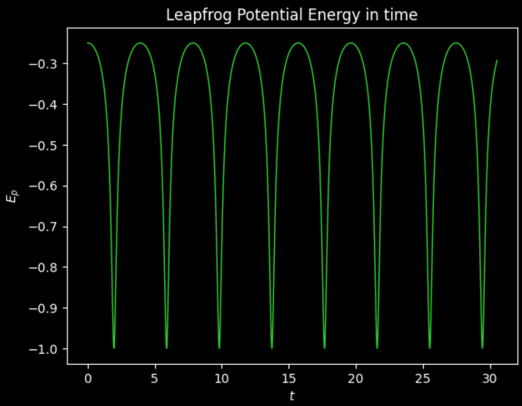
Leapfrog orbit in case of two-body problem.



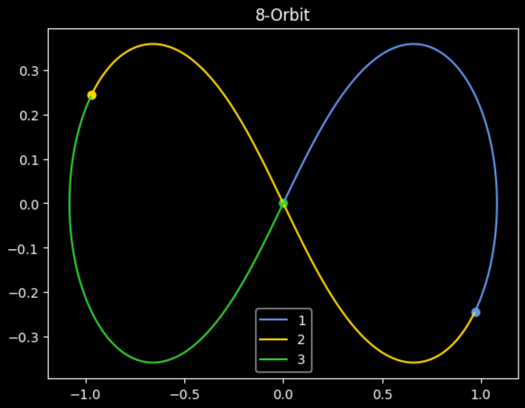
Conservation of energy.

Leapfrog Energy





Chenciner infinity



Initial conditions:

$$m_1 = m_2 = m_3$$

$$\boldsymbol{r}_1 = -\boldsymbol{r}_2$$

$$\boldsymbol{r}_3 = 0$$

$$\boldsymbol{v}_1 = \boldsymbol{v}_2 = -\frac{1}{2}\boldsymbol{v}_3$$

A. Chenciner, R. Montgomery. “A remarkable periodic solution of the three-body problem in the case of equal masses”, *Ann. of Math.* **152** (2000)

Chenciner infinity