Ming Kong
Week 2 Homework

Question 5:
a)
1. Exercise 1.12.2, sections b, e

Section b:

$p \to (q \land r)$
$\neg q$
$\therefore \neg p$

1	$p \to (q \land r)$	Hypothesis
2	$p \to q$	Simplification, 1
3	$-q$	Hypothesis
4	$-p$	Modus Tollens, 2, 3

Section e:

$p \lor q$
$\neg p \lor r$
$\neg q$
$\therefore r$

1	$p \lor q$	Hypothesis
2	$\neg p \lor r$	Hypothesis
3	$q \lor r$	Resolution, 1, 2
4	$-q$	Hypothesis
5	r	Disjunctive syllogism, 3, 4

2. Exercise 1.12.3, section c:

Section c:

$p \lor q$
$\neg p$
$\therefore q$

1	$p \lor q$	Hypothesis
2	$-p \to q$	Conditional Identity
3	$-p$	Hypothesis
4	q	Modus tollens, 2, 3

3. Exercise 1.12.5, sections c, d

Section c:

I will buy a new car and a new house only if I get a job.
I am not going to get a job.
∴ I will not buy a new car.

c: I will buy a new car
j: I will get a job
h: I will get a new house

$(c \land h) \to j$
$\neg j$
$\therefore \neg c$

The argument is invalid. Both c and h if False can make hypothesis ¬j True.

Section d:

I will buy a new car and a new house only if I get a job.
I am not going to get a job.
I will buy a new house.
∴ I will not buy a new car.

c: I will buy a new car
j: I will get a job
h: I will get a new house

$(c \land h) \to j$
$\neg j$
h
$\therefore \neg c$

The argument is valid

1	$(c \land h) \to j$	Hypothesis
2	$\neg j$	Hypothesis
3	$\neg(c \land h)$	Modus Tollens, 1, 2
4	$\neg c \lor \neg h$	DeMorgan's Law
5	h	Hypothesis
6	$\neg c$	Disjunctive syllogism, 4, 5

b)
1. Exercise 1.13.3, section b

$\exists x\, (P(x) \vee Q(x))$
$\exists x\, \neg Q(x)$
$\therefore\ \exists x\, P(x)$

	P	Q
a	F	T
b	F	F

$\exists x\, (P(x) \vee Q(x))$ is true because there exists Q(a) which causes the hypothesis to be true. $\exists x\, \neg Q(x)$ is true because $\neg Q(b)$ is true. But the conclusion is False because both P(a) and P(b) are false.

2. Exercise 1.13.5, sections d, e

Section d:

Every student who missed class got a detention.
Penelope is a student in the class.
Penelope did not miss class.

Penelope did not get a detention

M(x): x missed class
D(x): x got a detention.

$\forall x\, (M(x) \rightarrow D(x))$
Penelope, a student in the class
$\neg M(\text{Penelope})$

$\neg D(\text{Penelope})$

Invalid, because Penelope could not miss class and still get detention and it would cause the conditional hypothesis to be true. This causes the hypotheses to be true and the conclusion false.

Section e:

Every student who missed class or got a detention did not get an A.
Penelope is a student in the class.
Penelope got an A.

Penelope did not get a detention.

M(x): x missed class
A(x): x received an A.
D(x): x got a detention.

$\forall x\ [(M(x) \lor D(x)) \to \neg A(x)]$
Penelope, a student in the class
A(Penelope)

$\neg D(\text{Penelope})$

1	$\forall x\ [(M(x) \lor D(x)) \to \neg A(x)]$	Hypothesis
2	$\forall x\ [\neg(M(x) \lor D(x)) \lor \neg A(x)]$	Conditional Identity
3	$\forall x[\neg M(x) \land \neg D(x) \lor \neg A(x)]$	DeMorgan’s Law
4	Penelope, a student in the class	Element definition
5	$\neg M(\text{Penelope}) \land \neg D(\text{Penelope}) \lor \neg A(\text{Penelope})$	Universal Instantiation, 3, 4
6	A(Penelope)	Hypothesis
7	$\neg M(\text{Penelope}) \land \neg D(\text{Penelope}) \lor \text{True}$	Complement Law
8	$\neg D(\text{Penelope}) \lor \text{True}$	Simplification
9	$\neg D(\text{Penelope})$	Identity Law

Question 6:

Exercise 2.2.1, sections d, c

Section d:

The product of two odd integers is an odd integer.

Let x, y be odd integers.
Assume k is an integer.
Assume $x = (2k + 1)$
Assume $y = (2k + 3)$

$(2k + 1)(2k+3) = 4k^2 + 8k+3$
$2(2k^2 + 4k) + 3$
Since k is an integer, $(2k^2 + 4k)$ is also an integer.
Let $c = 2k^2 + 4k$, then $2(c) + 3$ is odd.
Therefore the product of two odd integers is an odd integer.

Section c:

If x is a real number and $x \leq 3$, then $12 - 7x + x^2 \geq 0$.

Assume $x \leq 3$, then $x - 3 \leq 0$.
$12 - 7x + x^2 \geq 0$
$(x - 4)(x - 3) \geq 0$
Since we know $(x - 3) \leq 0$. and $(x - 4)$ is always negative when $x \leq 3$.
Therefore if x is a real number and $x \leq 3$, then $12 - 7x + x^2 \geq 0$.

Question 7:

Exercise 2.3.1, sections d, f, g, l

Section d:

For every integer n, if $n^2 - 2n + 7$ is even, then n is odd.

Proof by contrapositive. We assume n is even and show that $n^2 - 2n + 7$ is odd.
If n is an even integer, then n = 2k for some integer k. Plugging it into the expression gives us:
$(2k)^2 – 2(2k) + 7$
$4k^2-4k + 7$
$2(k^2-2k) + 7$
Since k is an integer, k^2-2k is also an integer. Can show that $c = k^2-2k$.
So 2(c) + 7.
2 times an integer is even. An even integer plus an odd integer is odd.
Therefore $n^2 - 2n + 7$ is odd.

Section f:

For every non-zero real number x, if x is irrational, then 1/x is also irrational.

Proof by contrapositive. We assume 1/x is rational and prove that x is rational.
If 1/x is rational that means x is a rational number.
By definition of a rational number x = a/b where a and b are integers and b ≠ 0. And if x is a non -zero number then a is also ≠ 0.
Then 1/x = 1/(a/b), which becomes 1/x = b/a, which is a rational number.
Therefore x is a rational number.

Section g:

For every pair of real numbers x and y, if $x^3+xy^2 \leq x^2y + y^3$, then $x \leq y$.

Proof by contrapositive. We assume x > y and show that $x^3+xy^2 > x^2y + y^{3.}$
Since $x^3+xy^2 > x^2y + y^3$ can be expressed as: $x(x^2+y^2) > y(x^2+y^2)$
We remove (x^2+y^2) from both sides, and are left with x > y.
Therefore $x^3+xy^2 > x^2y + y^3$

Section l:

For every pair of real numbers x and y, if x + y > 20, then x > 10 or y > 10.

Proof by contrapositive. We assume x ≤ 10 and y ≤ 10 and prove that x + y ≤ 20
Since x ≤ 10 and y ≤ 10, the maximum value of x + y is 20.
Therefore x + y ≤ 20.

Question 8:
Exercise 2.4.1, sections c, e

Section c:

The average of three real numbers is greater than or equal to at least one of the numbers.

Proof by contradiction. Assume that the average of three real numbers is less each of the three numbers.

Let a be the average, and x, y, z be real numbers.

By definition

$$a = (x + y + z) / 3$$

If we assume that the average of three real numbers is less each of the three numbers. That means:

$$x < a$$
$$y < a$$
$$z < a$$

Summing up the equations gets us:

$$x + y + z < 3a$$

Replacing ‘a’ with the definition of the average gets us:

$$x + y + z < 3 (x + y + z) / 3$$

Which causes a contradiction:

$$x + y + z < x + y + z$$

Therefore the average of three real numbers is greater than or equal to at least one of the numbers.

Section e:

There is no smallest integer.

Proof by contradiction. Assume there is a smallest integer.

Let r be the smallest integer. Let r which is an integer exists than there exists r -1

Contradiction: if r is the smallest integer, there also exists an integer which is r – 1.

Therefore there is no smallest integer.

Question 9:
Exercise 2.5.1, section c

If integers x and y have the same parity, then x + y is even.

The parity of a number tells whether the number is odd or even. If x and y have the same parity, they are either both even or both odd.

Proof by cases. We consider two cases.

Case 1:

x and y are both odd.
If x and y are odd they can be represented by some integer k.
With 2k + 1 as odd, or 2k + 3
Assuming x = 2k + 1, which is odd
Assuming y = 2k + 3, which is odd

$x + y = (2k+1) + (2k+3)$
$x + y = 4k + 4$
$x + y = 2(k + 2)$
We know that k is an integer, and k + 2 is an integer. Let c represent k + 2.
$x + y = 2c$
Two times an integer is even.
Case 1 is true.

Case 2:

x and y are both even.
If x and y are even then can be represented by some integer k.
With 2k or 2k + 2 as even.
Assuming x = 2k, which is even.
Assuming y = 2k + 2 which is even.

$x + y = 2k + 2k + 2$
$x + y = 4k + 2$
$x + y = 2(k + 1)$

We know that k is an integer and k + 1 is an integer. Let c represent k + 1.
$x + y = 2c$
Two times an integer is even.
Case 2 is true.