Value at Risk

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This is a report of *Testing and comparing Value-at-Risk measures* (2001) by Christoffersen, P.F., J. Hahn, and A. Inoue.

VaR Definition

Let $\epsilon_1, \epsilon_2, \ldots$ be a sequence of daily returns.

Mathematically, the value at risk VaR_t at day t given some confidence level $0 < p < 1$ is defined by the equation

$$\Pr(\epsilon_t \leq \mathrm{VaR}_t | \mathcal{F}_{t-1}) = p.$$

Here, $\mathcal{F}_{t-1}$ is the available information available up to and including time $t-1$ (Christofferson, 2).

Assuming normal returns $\epsilon_t | \mathcal{F}_{t-1} \backsim \mathcal{N}(0, \sigma_t)$, the value at risk may be calculated as $\mathrm{VaR}_t | \mathcal{F}_{t-1} = \phi^{-1}(p)\sigma_t$ if the volatility parameter σ_t is known. Different volatility measures give different VaR measures.

As long as ϵ_t/σ_t are i.i.d., the VaR will be linear in σ_t:

$$\mathrm{VaR}_t(\beta) = \beta\sigma_t.$$

(Cristoffersen, 4)

Here β replaces $\phi^{-1}(p)$. β can be estimated which will enable distributional assumptions for ϵ_t to be removed.

VaR Measures

Measuring VaR depends on how the return volatilty series σ_t is modeled. Some models include:

Implied Volatility

Volatility may be deduced from known option prices. For example, given the price C_t of a european call option at day t, the volatility parameter σ_t may be solved from the Black-Scholes formula (Christofferson, 4):

$$C_t = S_t \cdot \Phi(d_+) - \exp(-r(T-t))\Phi(d_-)$$
$$d_\pm = \frac{\log \frac{S_t}{K} + \left(r \pm \frac{\sigma_t^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

Here, S_t is the value of the underlying asset at time t.

Stochastic Volatility

The dynamics of the observed asset price S_t depends on the dynamics of the underlying volatility σ_t. One such volatility model is Heston's model (Christofferson, 4):

$$dS_t = \mu S_t dt = \sigma_t S_t dW_t^1$$
$$d\sigma_t^2 = \kappa(\theta - \sigma_t^2)dt + \xi\sigma_t dW_t^2.$$

Here, W_t^1 and W_t^2 are independent Wiener processes. The constant parameters are μ, κ, θ, ξ, and may be callibrated using maximum likelihood.

GARCH(1,1) and RiskMetrics

Alternatively, σ_t may be modeled using a discrete data generating process. Such models include GARCH(1,1) and JP Morgan's RiskMetrics (Christofferson, 3).

A GARCH(1,1) model takes the form of

$$\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2.$$

The parameters can be estimated using maximum likelihood.

A RiskMetrics model takes the form of

$$\sigma_{t-1}^2 = (1-\lambda)\epsilon_{t-1}^2 + \lambda\sigma_{t-1}^2.$$

Here, $0 < \lambda < 1$ and is generally set to a predetermined value, for example .94 for daily returns.

Specification Testing

Recall

$$\mathrm{VaR}_t(\beta) = \beta\sigma_t.$$

The condition

$$\Pr\left(\epsilon_t \le \mathrm{VaR}_t(\beta)|\mathcal{F}_{t-1}\right) = p$$

is equivalent to

$$E(\mathbf{1}_{\epsilon_t \le \mathrm{VaR}_t(\beta)} - p|\mathcal{F}_{t-1}) = 0.$$

This is because for a large sample, the percentage of asset returns less than or equal to the predicted $VaR_t(\beta)$ should converge to p if it is correctly specified. Also,

$$E(f(x,\beta)] = 0$$

where

$$\mathbf{f}(x,\beta) = (\mathbf{1}_{\epsilon_t \le \mathrm{VaR}_t(\beta)} - p) \times k(z_{t-1}).$$

Here, z_{t-1} is any $\mathcal{F}_{t-1}$-measurable instrument and k is any measurable vector-valued function. x is a vector whose elements contain ϵ_t, z_t and σ_t. (Christofferson, 6-7)

For $VaR_t(\beta)$ calculated under σ_t, the R estimators of the volalility at t-1 placed in a vector $\sigma_{r,t-1}$ should be independent of $\mathbf{1}_{\epsilon_t \le \mathrm{VaR}_t(\beta)} - p$. Therefore $\sigma_{r,t-1}$ is considered the vector $k(z_{t-1})$. Hence $E[(\mathbf{f}(x,\beta))] = \mathbf{0}$ is an equation of vectors corresponding to the number of volatility estimators R.

While the Generalized Method of Moments could be used to estimate β under these moment constraints, the information theoretic equivalent Kullback-Liebler Information Criterion is used as below. The power of these estimators is that they do not require distributional assumptions.

For a given vector γ of length r, define

$$M_T(\beta,\gamma) = \frac{1}{T}\sum_{t=1}^{T} \exp \gamma^{'}\mathbf{f}(x_t,\beta).$$

And then define the vectors $\hat{\gamma}$ and $\hat{\beta}$ by

$$(\hat{\beta},\hat{\gamma}) = \max_{\beta} \min_{\gamma} M_T(\beta,\gamma).$$

Then define the test statistic $\hat{\kappa} = -2T \log M_T(\hat{\beta},\hat{\gamma})$. As $T \to \infty$, $\hat{\kappa}$ approaches a χ^2-distribution under the hypothesis that the model is correctly specified. The degrees of freedom are R - m, where m is the length of β. This is similar to the J-test in the GMM.

Comparison Testing

Suppose we are testing a VaR measure with an alternative measure, derived from a different volatility measure:

$$\text{alternative VaR}_t(\theta) = \theta_2 \sigma_t',$$

where σ_t' comes from the alternative volatility model. And define

$$\begin{aligned}
\mathbf{g}(x_t, \theta) &= (\mathbf{1}_{\epsilon_t \leq \text{altVaR}_t(\theta))} - p) \times k(z_{t-1}) \\
N_T(\lambda, \theta) &= \frac{1}{T} \sum_t \exp \lambda' \mathbf{g}(x_t, \theta) \\
(\hat{\lambda}, \hat{\theta}) &= \max_\theta \min_\lambda N_T(\lambda, \theta).
\end{aligned}$$

Under the hypothesis that both VaR measures are equally effective, the test statistic

$$t = (M_T(\hat{\beta}, \hat{\gamma}) - N_T(\hat{\theta}, \hat{\lambda}))$$

should approach a normal distribution. (Christofferson, 8).

If the test statistic is 0, both are equally good estimators, if positive first VaR is preferred and the alternative VaR if the test statistic is negative. The mean of the normal distribution is 0 and the variance can be approximated by computing $E[(M_T(\hat{\beta}, \hat{\gamma}) - N_T(\hat{\theta}, \hat{\lambda}))^2]$ from the sample.

References/Further Reading

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