OPTI Assignment-2
Mathematical Model for Optimally Refilling ATMs

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March 9, 2025

1 Sets

- A: Set of ATMs ($a \in A$)
- Va: Available vehicles
- Vr: Rented vehicles
- V: Set of vehicles ($v \in V$, $V = Va \cup Vr$)
- D: Set of days in the planning horizon ($d \in D$)
- T: Set of denominations available ($t \in T$)

2 Parameters

- C_a: Capacity of ATM a (maximum cash that can be stored)
- H_a: Minimum required cash level for ATM a
- $D_{a,d}$: Demand (expected withdrawal) at ATM a on day d ($D_{a,d} \geq 0$)
- L_v: Maximum cash limit that vehicle v can carry for security reasons (It is less than the capacity of the vehicle)
- O_v: Maximum operational cost of vehicle v
- N_d: Maximum number of ATM vehicle v can visit in a day (20 on weekdays, 30 on weekends)
- X: Security constraint on revisiting an ATM by the same vehicle (minimum days before revisiting)
- P_t: Preferred proportion of denomination t
- T_d: Time required to service ATM a (30 min weekdays, 20 min weekends)
- W: Maximum working hours in a day
- R_v: Rental cost for hiring additional vehicles if needed

3 Decision Variables

- $x_{a,v,d}$: Binary variable; 1 if vehicle v visits ATM a on day d, 0 otherwise
- $y_{a,d}$: Cash deposited at ATM a on day d (Non-negative real number)
- $k_{a,d}$: Non-negative integer variable; determines the amount cash deposited in multiples of 10000 to ATM a on the day d
- $z_{a,t,d}$: Cash deposited in denomination t at ATM a on day d

- $u_{v,d}$: Binary variable; 1 if vehicle v is used on day d, 0 otherwise
- $h_{v,d}$: Binary variable; 1 if rented vehicle v is used on day d, 0 otherwise
- $sr_{a,t,d}$: Surplus variable for deviation from preferred denomination distribution
- $sl_{a,t,d}$: Slack variable for deviation from preferred denomination distribution

4 Objective Function

4.1 Minimize total operational cost

$$Cost_{d,1} = \sum_{v \in V} O_v u_{v,d} + \sum_{v \in V} R_v h_{v,d} \quad \forall d \in D \tag{1}$$

- The first term is the operation cost of the vehicles we have.
- The second term is the rental cost of the vehicles we need to rent.

4.2 Minimize the total cash deposited in a day

This ensures that there is not too much cash lying idle:

$$Cost_{d,2} = \sum_{a \in A} y_{a,d} \quad \forall d \in D \tag{2}$$

4.3 Workload balance across vehicles in terms of ATMs visits

$$Cost_{d,3} = \sum_{v \in V} s^N_{v,d} \tag{3}$$

Let $\bar{N}_d$ be the average number of ATMs visited by vehicles on day d:

$$\bar{N}_d = \frac{\sum_{v \in V} \sum_{a \in A} x_{a,v,d}}{|V|}$$

$s^N_{v,d}$ be deviation variable for the number of ATMs visited.

$$s^N_{v,d} \geq \sum_{a \in A} x_{a,v,d} - \bar{N}_d, \quad \forall v \in V, d \in D$$

$$s^N_{v,d} \geq \bar{N}_d - \sum_{a \in A} x_{a,v,d}, \quad \forall v \in V, d \in D$$

4.4 Workload balance across vehicles in terms of cash deposited

$$Cost_{d,4} = \sum_{v \in V} s^Y_{v,d} \tag{4}$$

Let $\bar{Y}_d$ average cash carried by vehicles on day d.

$$\bar{Y}_d = \sum_{a \in A} Y_{a,v,d}$$

Note: Here $Y_{a,v,d}$ is the amount of money deposited in ATM a by vehicle v on day d. Since there is only one vehicle visiting ATM a there is no need to sum the variable $Y_{a,v,d}$ over v.

Formulation of $Y_{a,v,d}$ is given in subsection 5.6. $s^Y_{v,d}$ be deviation variables for cash carried.

$$s^Y_{v,d} \geq \sum_{a \in A} Y_{a,v,d} - \bar{Y}_d, \quad \forall v \in V, d \in D$$

$$s^Y_{v,d} \geq \bar{Y}_d - \sum_{a \in A} Y_{a,v,d}, \quad \forall v \in V, d \in D$$

4.5 Minimize the surplus and slack variables

$$Cost_{t,4} = \sum_{a \in A} \sum_{d \in D} sr_{a,t,d} + \sum_{a \in A} \sum_{d \in D} sl_{a,t,d} \quad \forall t \in T \tag{5}$$

Final Cost function:

$$\text{Minimize} \quad \left[\left(\sum_{d \in D} (\lambda_1 \cdot Cost_{d,1} + \lambda_2 \cdot Cost_{d,2} + \lambda_3 \cdot Cost_{d,3} + \lambda_4 \cdot Cost_{d,4}) \right) + \sum_{t \in T} \lambda_t Cost_{t,4} \right] \tag{6}$$

The parameters λ_1 and λ_2 control the trade-off between the cost of vehicle usage and the amount of cash stored in ATMs. Their values can be adjusted based on the desired priority. A higher λ_1 emphasizes minimizing vehicle costs, leading to a single shipment of 150,000 in one day. Conversely, a higher λ_2 prioritizes reducing excess cash in ATMs, resulting in three separate shipments of 50,000 over three days.

The parameters λ_3 and λ_4 allow adjusting the weightage of workload balancing.

Since lower denomination is preferred, λ_t can be set accordingly to give more or less preference to respective surplus and slack variables.

These λ parameters can be tuned through historical data or sensitivity analysis.

5 Constraints

5.1 ATM Demand Satisfaction

Ensures that the cash deposited at an ATM meets or exceeds its expected demand.

$$y_{a,d} \geq D_{a,d}, \quad \forall a \in A, d \in D \tag{7}$$

5.2 ATM Visit Constraints

Ensures that an ATM is visited by at most one vehicle on a given day if cash is deposited.

$$\sum_{v \in V} x_{a,v,d} \leq 1, \quad \forall a \in A, d \in D \tag{8}$$

5.3 ATM Cash Limits

If cash is deposited then the cash is within the minimum and maximum limit

$$H_a \sum_{v \in V} x_{a,v,d} \leq y_{a,d} \leq C_a \sum_{v \in V} x_{a,v,d}, \quad \forall a \in A, d \in D \tag{9}$$

Explanation: If no vehicle visit ATM a on day d then $\sum_{v \in V} x_{a,v,d} = 0 \implies 0 \leq y_{a,d} \leq 0 \implies y_{a,d} = 0$ that is cash deposited at ATM a on day d is 0.

And if vehicle v deposits the cash in ATM a on the day d then $\sum_{v \in V} x_{a,v,d} = 1 \implies H_a \leq y_{a,d} \leq C_a$ that is cash deposited is within the limit.

5.4 Security Constraint on Revisiting ATMs

Enforces a security rule that prevents a vehicle from revisiting the same ATM within a minimum number of days.

$$x_{a,v,d} + x_{a,v,d+1} + \cdots + x_{a,v,d+X} \leq 1, \quad \forall a \in A, v \in V, d \in D \tag{10}$$

5.5 Rental Vehicles

Ensures that rental vehicles are used only if all available vehicles are already in use and demand exceeds capacity.

$$|Va| - \sum_{v \in Va} u_{v,d} \leq M_1 \cdot k, \quad \sum_{v \in Vr} h_{v,d} \leq M_1 \cdot (1-k), \quad \forall d \in D \tag{11}$$

Where M_1 is a large constant and k is a binary variable.

Explanation:

- If $k = 0$ then $|Va| \leq \sum_{v \in Va} u_{v,d} \implies \sum_{v \in Va} u_{v,d} = |Va|$ and $\sum_{v \in Vr} h_{v,d} \leq M_1$. We are using all available vehicles and rented vehicles are also being used.
- If $k = 1$ then $|Va| \leq \sum_{v \in Va} u_{v,d} + M_1$ and $\sum_{v \in Vr} h_{v,d} \leq 0 \implies \sum_{v \in Vr} h_{v,d} = 0$. We are not using all available vehicles and no vehicles are rented.

5.6 Vehicle Cash Carrying Limit

Ensures a vehicle does not carry more cash than its allowed security limit.

$$\sum_{a \in A} y_{a,d} x_{a,v,d} \leq L_v, \quad \forall v \in V, d \in D \tag{12}$$

Since this equation is **non-linear**, this can be written as following **linear equations**:

$$\sum_{a \in A} Y_{a,v,d} \leq L_v, \quad \forall v \in V, d \in D \tag{13}$$

Where $Y_{a,v,d} = y_{a,d} x_{a,v,d}$ and can be expressed as:

$$Y_{a,v,d} \leq M_2 x_{a,v,d}, \quad Y_{a,v,d} \leq y_{a,d} + M_2(1 - x_{a,v,d}), \quad Y_{a,v,d} \geq y_{a,d} - M_2(1 - x_{a,v,d})$$

Where M_2 is a large constant.

Explanation:

- If $x_{a,v,d} = 0$, then $Y_{a,v,d}$ must be 0 (enforced by the first constraint).
- If $x_{a,v,d} = 1$, then $Y_{a,v,d}$ must be equal to $y_{a,d}$ (enforced by the second and third constraints).

5.7 Vehicle Visit Limit

Restricts the number of ATMs a vehicle can visit in a single day.

$$\sum_{a \in A} x_{a,v,d} \leq N_d, \quad \forall v \in V, d \in D \tag{14}$$

5.8 Vehicle Time Limit

Ensures that a vehicle's total working hours, including travel and refilling time, do not exceed the daily limit.

$$T_d \sum_{a \in A} x_{a,v,d} \leq W, \quad \forall v \in V, d \in D \tag{15}$$

5.9 Denomination Preference

Encourages the deposited cash to match the preferred denomination distribution while allowing deviations via surplus and slack variables.

$$z_{a,t,d} - sr_{a,t,d} + sl_{a,t,d} = P_t y_{a,d}, \quad \forall a \in A, t \in T, d \in D \tag{16}$$

5.10 Denomination Availability Constraint

Ensures that the total cash deposited at an ATM matches the sum of its denomination-based deposits.

$$\sum_{t \in T} z_{a,t,d} = y_{a,d}, \quad \forall a \in A, d \in D \tag{17}$$

5.11 Minimum and Multiples of Cash

To ensure that cash deposited at a particular ATM is either 0 or at least $50K$, and is in multiples of $10K$.

$$y_{a,d} = 10000k_{a,d} \quad \forall a \in A, d \in D \tag{18}$$

$$k_{a,d} \leq M_3 b_{a,d}, \quad k_{a,d} \geq 5b_{a,d}, \quad \forall a \in A, d \in D \tag{19}$$

Where $b_{a,d}$ is a binary variable and M_3 is a large constant.

Explanation:

- If $b_{a,d} = 0$, then $k_{a,d}$ must be 0.
- If $b_{a,d} = 1$, then $k_{a,d} \geq 5$.

5.12 Binary Constraints

Enforces binary values for decision variables related to vehicle assignments and ATM visits.

$$x_{a,v,d} \in \{0,1\}, \quad u_{v,d} \in \{0,1\}, \quad h_{v,d} \in \{0,1\} \tag{20}$$

6 Assumptions

- The planning horizon is 1 week.
- Vehicles cater to the needs of only one bank.
- Replenishment vehicles can only be refilled at the beginning of the day.
- Scheduling and routing are not considered in this model.
- Bank does not have cash limitation, i.e. It can provide any amount of cash required based on the demand.