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Introduction to Mathematical Logic

Unit – VIII

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Syllabus

Unit 8: Introduction to Mathematical Logic

Sets and Subsets, set operations and the Laws of Set theory, Counting and Venn diagram, Fundamentals of Logic, Basic connectives- Truth Tables, logical equivalences.

Set: It is a well-defined collection of objects.

There are two methods of representing a set:

1. Roster form or tabular form
2. Set-builder form

$A = \{1, 2, 3, 4\}$

$A = \{x \mid x \text{ is a natural no} \leq 4\}$

(i) In roster form, all the elements of a set are listed, the elements are being separated by commas and are enclosed within braces { }. For example, the set of all even positive integers less than 7 is described in roster form as {2, 4, 6}. Some more examples of representing a set in roster form are given below :

(a) The set of all natural numbers which divide 42 is {1, 2, 3, 6, 7, 14, 21, 42}.

(b) The set of all vowels in the English alphabet is $\{a, e, i, o, u\}$.

(c) The set of odd natural numbers is represented by $\{1, 3, 5, \ldots\}$. The dots tell us that the list of odd numbers continue indefinitely.

(ii) In set-builder form, all the elements of a set possess a single common property which is not possessed by any element outside the set. For example, in the set $\{a, e, i, o, u\}$, all the elements possess a common property, namely, each of them is a vowel in the English alphabet, and no other letter possess this property. Denoting this set by V, we write

V = $\{x : x$ is a vowel in English alphabet$\}$

$x^2 - 2 = 0$

$\Rightarrow x^2 = 2$

$\Rightarrow x = \pm\sqrt{2}$

Empty Set (Null Set): A set containing no elements is called a null set and is usually denoted by $\emptyset$.

$\phi = \{\ \}$

(i) Let A = $\{x : 1 < x < 2,\ x$ is a natural number$\}$. Then A is the empty set, because there is no natural number between 1 and 2.

(ii) B = $\{x : x^2 - 2 = 0$ and x is rational number$\}$. Then B is the empty set because the equation $x^2 - 2 = 0$ is not satisfied by any rational value of x.

Finite and infinite sets: A set containing finite number of elements is called a finite set. Otherwise it is an infinite set.

For example

Set of all days of a week is finite and set of all points on a line is an infinite set.

Cardinality = Total no. of elements

Equal Sets: Two sets A and B are said to be equal if they have precisely same elements. We denote it by $A = B$. Otherwise we write $A \neq B$.

Let $A = \{1,2,3,4\}, B = \{2,3,1,4\}$ *and* $C = \{5,6,7,8\}$

We note that $A = B$ *and* $B \neq C$

$D = \{1,2,3,5\}$

$E = \{1,1,2,2,3,3,4\}$

$A \neq D$

Subset: A set A is called a sub set of another set B if every element of A belongs to B.

$$i.e \text{ if } a \in A \text{ then } a \in B$$

We also call B as super set of A. We denote it by

$$A \subseteq B \text{ or } B \supseteq A$$

Example:

Let $A = \{1,2,3,4\}$, $B = \{1,2,3,4,5\}$ and $C = \{1,2,3,4\}$

We note that $A \subseteq B, A \subseteq C$ and $C \subseteq A$

We note that A is a proper sub-set of B and A is not a proper sub-set of C.

$\in$ belongs to

$A \subseteq B$ $\quad$ $A \subset B$

Note:

1. Null set is a sub-set of every set.
2. Every set is a sub-set of itself.

$A \subseteq B$ and $B \subseteq A$

then $A = B$

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Power set: The set of all sub-sets of a set A is called power set of the set A and is usually denoted by $P(A)$.

Let $A = \{1,2,3\}$

$$P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

Note: If A has n elements then $P(A)$ will have 2^n elements.

Universal Set: In a discussion, if all sets under consideration are sub-sets of a set U, then U is called universal set for that discussion.

Venn diagram: Relationship between sets are represented pictorially using diagrams called Venn diagrams.

P1) $U = \{1,2,3,4,5\}$

$A = \{1,2,3\}$

$B = \{3,4,5\}$

$C = \{1,4,5\}$

P2) $U = \{1,2,3,4,5,6\}$

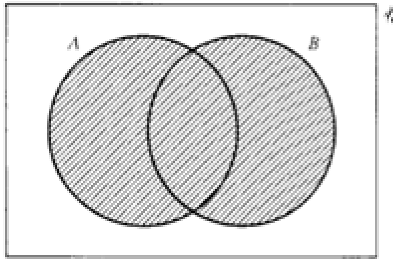
$A = \{1,2,3\}$

$B = \{3,4,5,6\}$

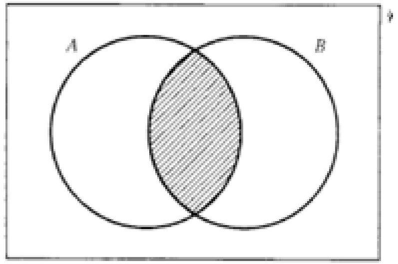
$C = \{4,5,6\}$

Operations on Sets:

Union: $A \cup B = \{x : x \in A \text{ or } x \in B\}$



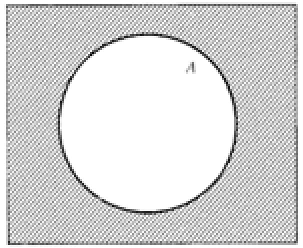
Intersection: $A \cap B = \{x : x \in A \text{ and } x \in B\}$



$\cup$ = OR $+$

$\cap$ = AND $\times$

Complement of a set: $\bar{A} = \{x : x \notin A\}$



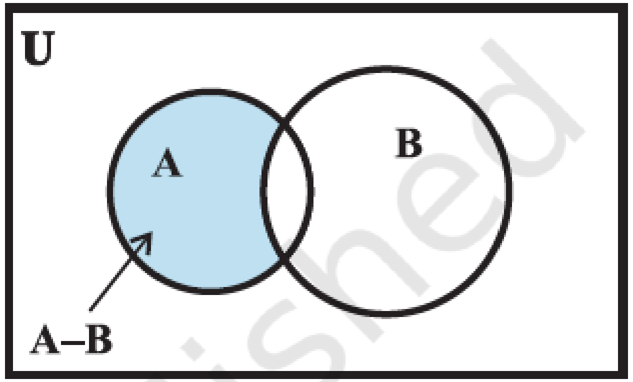
$U = \{1, 2, 3, 4, 5\}$

$A = \{1, 2, 3\}$ $\bar{A} = \{4, 5\}$

Relative Complement:

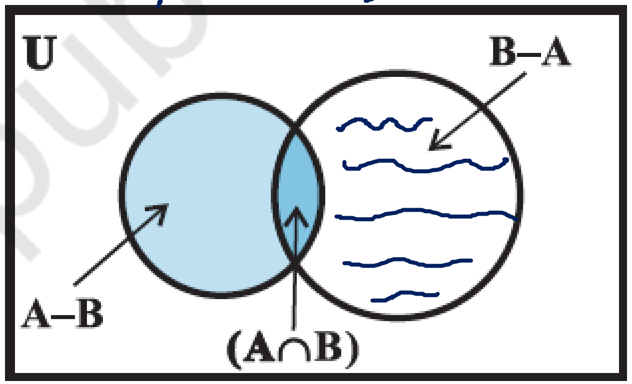
$A - B = \{x : x \in A \text{ and } x \notin B\}$

Set of elements in A not in B



$B - A = \{x : x \in B \text{ and } x \notin A\}$

Set of elements in B not in A

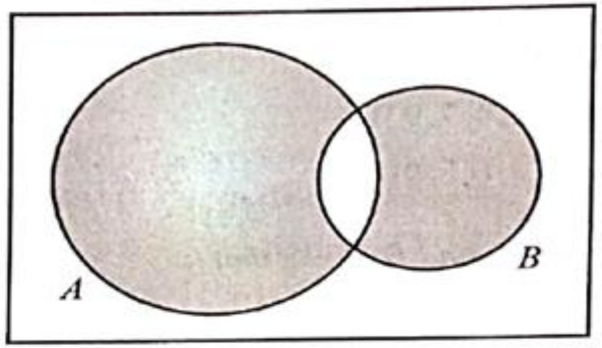


$(A \Delta B) = (A \cup B) - (A \cap B)$

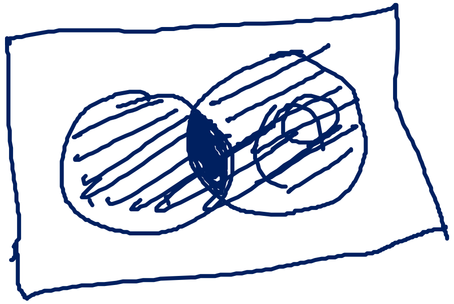
Symmetric Difference:

$A \,\Delta\, B = (A - B) \cup (B - A)$

$= \{x : x \in A \cup B \text{ and } x \notin A \cap B\}$



$A\Delta B$ (shaded)



Example. Let $A = \{2,4,6,8\}$ and $B = \{6,8,10,12\}$

Then $A \cup B = \{2,4,6,8,10,12\}$

$A \cap B = \{6,8\}$

Example. Let $A = \{1,2,3,4,5,6,\}$ and $B = \{2,4,6,8\}$

Then $A - B = \{1,3,5\}$

$B - A = \{8\}$

$A\Delta B = (A - B) \cup (B - A) = \{1,3,5,8\}$

Addition Principle:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$|A| =$ number of elements

$n =$ number of elements

Q1. f X and Y are two sets such that $n(X) = 17$, $n(Y) = 23$ and $n(X \cup Y) = 38$, find $n(X \cap Y)$.

$n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$

$38 = 17 + 23 - n(X \cap Y) \Rightarrow n(X \cap Y) = 2$

Q2. In a group of 400 people, 250 can speak Hindi and 200 can speak English. How many people can speak both Hindi and English?

A C B

Logic is the science dealing with the methods of reasoning. Reasoning plays a very important role in every area of knowledge, particularly mathematics. A symbolic language has been developed over the past two centuries to express the principles of logic in precise and unambiguous terms. Logic expressed in such a language has come to be called "Symbolic Logic" or "Mathematical logic". We shall discuss some basic notations of this subject.

Proposition: A proposition is a statement(declaration) which, in a given context, can be said to be either true or false, but not both. The truth or falsity of a proposition is called its truth value. If a proposition is true, we will indicate its truth value by the symbol 1 and if it is false by the symbol 0.

Logical Connectives and Truth Tables: New propositions are obtained by starting with given propositions with the aid of words or phrases like *'not'* , *'and'* *'or'* , *'if then'* and *'iff'*. Such words or phrases are called **logical connectives.** New proposition thus obtained is called a compound proposition.

$A \longrightarrow B$

iff : if and only if

p $\neg p$

Negation: A proposition obtained by inserting the word '*not*' at an appropriate place in a given proposition is called the negation of the given proposition. If p is a proposition then its negation is denoted by $\sim p$.

Negation

$\sim$ tilde

Truth table for Negation

p	$\sim p$
0	1
1	0

$p : 5+3=8$

$\sim p : 5+3 \neq 8$

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Conjunction: The conjunction of two propositions $p\ and\ q$ is denoted by $p \wedge q$ and is true when both propositions are true and is false otherwise.

Truth table for Conjunction

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

p : 5 is an integer T

q : $\sqrt{2}$ is irrational T

$p \wedge q$ = 5 is an integer & $\sqrt{2}$ is irrational

$p \wedge q$

p Conjunction q

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Disjunction: The disjunction of two propositions p ~~and~~ OR q is denoted by $p \lor q$ and is false when both propositions are false and is true otherwise.

Truth table for Disjunction

p	q	$p \lor q$
0	0	0
0	1	1
1	0	1
1	1	1

p
1) 8 + 2 = 10 T 1
2) 8 + 2 = 16 F 0
q

p or q

$p \lor q$

Conditional: The compound proposition ′ *if p then q* ′ is called conditional denoted by $p \to q$ is false if p is true and q is false.

Truth table for Conditional

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

$1 \longrightarrow 0 = 0$

Biconditional: The compound proposition $'\ p\ iff\ q\ '$ is called biconditional denoted by $p \leftrightarrow q$ is false if one of the two propositions is false. Otherwise, it is true.

Truth table for Biconditional

p	q	$p \leftrightarrow q$
0	0	1
0	1	0
1	0	0
1	1	1

$p \leftrightarrow q$

$= (p \rightarrow q) \wedge (q \rightarrow p)$

Tautology: A compound proposition which is true for all possible truth values of its components is called a tautology (or a logical truth or a universally valid statement).

Contradiction: A compound proposition which is false for all possible truth values of its components is called a contradiction or absurdity.

Contingency: A compound proposition which is neither a tautology nor a contradiction is called a contingency.

Logical Equivalence: Two propositions u and v are said to be logically equivalent whenever u and v have the same truth value, or equivalently, the biconditional $u \leftrightarrow v$ is a tautology. Then we write $u \Leftrightarrow v$. Here the symbol $\Leftrightarrow$ *or* $\equiv$ stands for "logically equivalent to".

Q & A

THANK YOU