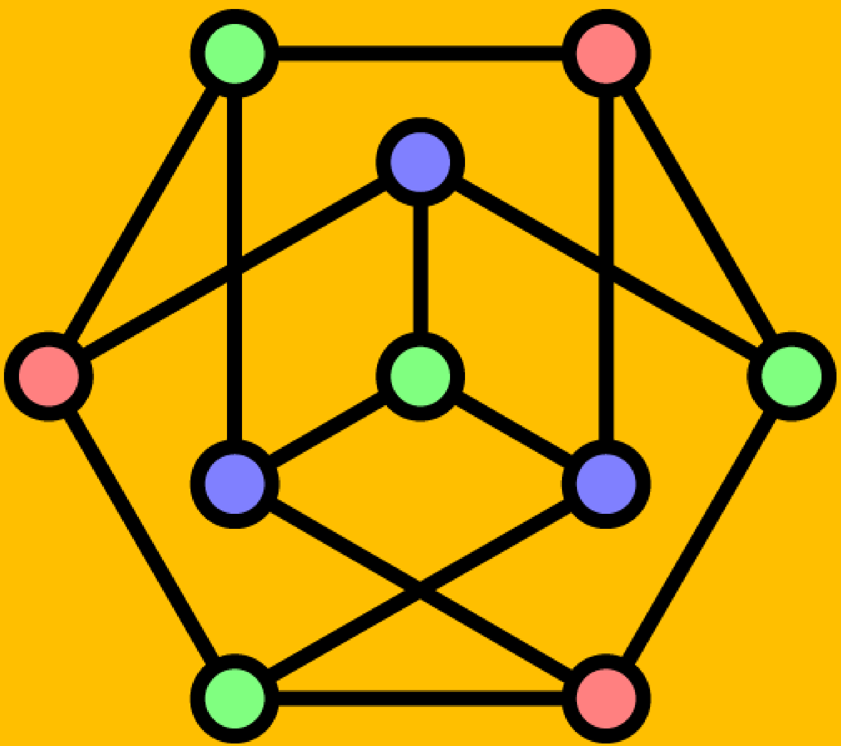
THEORETICAL COMPUTER SCIENCE

DISCRETE STRUCTURES FOR COMPUTER SCIENCE STUDENTS



UP FAMNIT

Fall 2024 – Version 0.1

CIP – Kataložni zapis o publikaciji
Narodna in univerzitetna knjižnica, Ljubljana

123.4(567)(8.901.2)

TEORETIČNE osnove računalništva – vaje / avtor M. Krnc; [urednik] M. Krnc. - Version 0.1. - El. knjiga. - Ljubljana : samozal. M. Krnc, 2020.

Način dostopa (URL):
https://github.com/mkrnc/TOR1-vaje---TCS1-exercises.git

ISBN 978-961-XXX-XXX-X (pdf)
123456789

PREFACE

Those notes are supposed to be parsed together with explanations from the lectures. Any questions or found errors should be addressed to matjaz.krnc@upr.si, or raised as an issue in our public repository

https://github.com/mkrnc/TOR1-vaje---TCS1-exercises.git.

Among most notable student contributors are:

Theoretical Computer Science
Discrete Structures for Computer Science Students

Authors:	Matjaž Krnc
Editor:	Matjaž Krnc
Cover photo:	R. A. Nonenmacher *Desargues Graph*
Design and self-publishing:	Matjaž Krnc

ISBN:	978-961-XXX-XXX-X

Ljubljana, Fall 2024

CONTENTS

1 MATHEMATICAL LOGIC

1.1 BASIC EXERCISES

Exercise 1.1. *The following two propositions are given: A: "It is cold outside." B: "It is raining outside."*

Write the following compound propositions in natural language:

- *(a)* $\neg A$
- *(b)* $A \wedge B$
- *(c)* $A \vee B$
- *(d)* $B \vee \neg A$

Exercise 1.2. *The following two propositions are given: A: "Janez is rich." B: "Janez is happy."*

Write the following propositions symbolically:

- *(a) If Janez is rich, then he is unhappy.*
- *(b) Janez is neither happy nor rich.*
- *(c) Janez is happy only if he is poor.*
- *(d) Janez is poor if and only if he is unhappy.*

Exercise 1.3. *Find the truth tables for the examples from the previous task.*

Exercise 1.4. *Is the following reasoning correct?*

- *Premise 1: "I think, therefore I am."*
- *Premise 2: "I think, therefore I reason."*
- *Conclusion: "I am, therefore I reason."*

Exercise 1.5. *The following two propositions are given: A: "Andrej speaks French." and B: "Andrej speaks Danish." Write the following compound propositions in natural language:*

(a) $A \lor B$
(b) $A \land B$
(c) $A \land \neg B$
(d) $\neg A \lor \neg B$
(e) $\neg\neg A$
(f) $\neg(\neg A \land \neg B)$

Exercise 1.6. *Given the propositions:*
A : *"John reads The New York Times."*
B : *"John reads The Wall Street Journal."*
C : *"John reads The Daily Mail."*

Transcribe the following statements into symbolic propositions:

1. *John reads The New York Times, but not The Wall Street Journal.*
2. *Either John reads both The New York Times and The Wall Street Journal, or he does not read The New York Times and The Wall Street Journal.*
3. *It is not true that John reads The New York Times, and does not read The Daily Mail.*
4. *It is not true that John reads The Daily Mail or The Wall Street Journal, and not The New York Times.*

Exercise 1.7. *Find the truth tables for the symbolic propositions from Exercise 1.6.*

Exercise 1.8. *For three lines p, q, r we may construct also geometric propositions. Suppose that the following is true:*

$$(p||q) \land (p \cap q \neq \emptyset) \land (q \cap r \neq \emptyset).$$

What can you say about the lines p, q, r?

Exercise 1.9. *Express the propositions below with connectives $\land$ and $\neg$ only!*

1. $A \lor B$
2. $A \Rightarrow B$
3. $A \Leftrightarrow B$

1.2 KNIGHTS AND SERVANTS (KNEVES)

Knights always tell the truth, while servants always lie.

Exercise 1.10. *Artur: "It is not true that Cene is a servant."*
Bine: "Cene is a knight or I am a knight."
Cene: "Bine is a servant."
For each of them, determine whether they are knights or servants!

Exercise 1.11. *Artur: "Cene is a servant or Bine is a servant."*
Bine: "Cene is a knight and Artur is a knight."
For each of them, determine whether they are knights or servants!

Exercise 1.12. *Let us analyze the statements made by A, B, C, and D:*

- *A: "D is a servant and C is a servant."*
- *B: "If A and D are servants, then C is a servant."*
- *C: "If B is a servant, then A is a knight."*
- *D: "If E is a servant, then both C and B are servants."*

Exercise 1.13. *Solve the following exercises about knights and servants:*

- *Arthur: "It is not true that Bine is a servant."*
- *Bine: "We are not both of the same kind."*

Exercise 1.14. *Now Arthur and Bine say the following:*

- *Arthur: "Me and Bine are not of the same kind."*
- *Bine: "Exactly one of us is a knight."*

Exercise 1.15. *Knights and servants!*

1. *Arthur: Chloe or Bob are servants.*
2. *Bob: Cene and Arthur are knights.*

1.3 CANONICAL FORMS

Exercise 1.16. *Find the canonical disjunctive normal form (DNF) and the canonical conjunctive normal form (CNF) for the following propositions:*

(i) $\neg(A \wedge B) \Rightarrow (\neg B \Rightarrow A)$

(ii) $\neg(A \vee B) \wedge (A \Rightarrow B)$

(iii) $(A \vee \neg B) \wedge (B \Rightarrow A)$

(iv) $(\neg A \vee B) \Rightarrow (\neg B \vee A)$

(v) $\neg((A \vee B) \wedge (\neg A \vee \neg C))$

Exercise 1.17. *For the following compound proposition find a truth table, determine DNF, CNF and draw the corresponding circuit.*

$$(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)).$$

Exercise 1.18. *For the following compound proposition, create a truth table, determine the DNF and CNF, and draw the corresponding circuit diagram:*

$$\neg((A \wedge B) \Rightarrow (\neg C \vee D)).$$

Exercise 1.19. *Determine if the following logical equivalences hold. Justify your answers by transforming each side into its canonical form:*

(i) $(A \Rightarrow B) \vee (\neg A \wedge \neg B) \sim \neg B \Rightarrow \neg A$

(ii) $(A \wedge (B \vee C)) \Rightarrow (A \wedge B) \sim (\neg A \vee B)$

Exercise 1.20. *Prove or disprove the validity of the following compound proposition using truth tables, DNF, or CNF:*

$$(\neg A \vee B) \wedge (B \Rightarrow C) \sim (\neg A \vee C).$$

Exercise 1.21. *Simplify the following compound proposition using the laws of logic, and then find its canonical DNF and CNF:*

$$(\neg A \wedge (B \vee C)) \vee (A \wedge \neg B \wedge \neg C).$$

Exercise 1.22. *Find DNF and CNF (if they exist) for the following proposition:*

$$(A \vee B) \wedge (\neg A \vee C) \wedge (\neg B \vee \neg C).$$

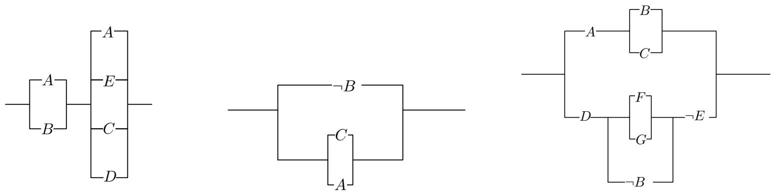


Figure 1: Switching circuits for Exercise 1.23.

1.4 SWITCHING CIRCUITS

Exercise 1.23. *For the circuits in Figure 1, find the corresponding compound propositions.*

Exercise 1.24. *For the following compound proposition, find a truth table, determine DNF, CNF, and draw the corresponding circuit.*

$$(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$$

Exercise 1.25. *Find a compound proposition I such that*

$$(A \Rightarrow (I \Rightarrow \neg B)) \Rightarrow (A \wedge B) \vee I$$

is a tautology.

1.5 LOGICAL IMPLICATIONS

Exercise 1.26. *Prove the following logical equivalences:*

(1) $(A \Rightarrow B) \wedge (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$

(2) $(A \Rightarrow B) \Rightarrow ((C \Rightarrow A) \Rightarrow (C \Rightarrow B))$

(3) $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$

(4) $(A \Rightarrow B) \Rightarrow (A \wedge C \Rightarrow B \wedge C)$

(5) $(A \Rightarrow B) \Rightarrow (A \vee C \Rightarrow B \vee C)$

(6) $(A \Leftrightarrow B) \wedge (B \Leftrightarrow C) \Rightarrow (A \Leftrightarrow C)$

(7) $(A \Leftrightarrow B) \Rightarrow (A \Rightarrow B)$

(8) $(A \Leftrightarrow B) \Rightarrow (B \Rightarrow A)$

(9) $A \wedge (A \Leftrightarrow B) \Rightarrow B$

(10) $\neg A \wedge (A \Leftrightarrow B) \Rightarrow \neg B$

(11) $B \Rightarrow (A \Leftrightarrow A \wedge B)$

(12) $\neg B \Rightarrow (A \Leftrightarrow A \vee B)$

(13) $(A \Rightarrow (B \wedge \neg B)) \Rightarrow \neg A$

Exercise 1.27. *Simplify the following logical proposition:*

$$(A \Rightarrow B) \vee (B \Rightarrow C)$$

1.6 PROOFS

Exercise 1.28. *Use direct or indirect proof to show every property from Exercise 1.26.*

Exercise 1.29. *Show that the following propositions are logical implications (i.e., tautologies where the main connective is implication):*

(i) $A \wedge (A \Rightarrow B) \Rightarrow B$

(ii) $\neg B \wedge (A \Rightarrow B) \Rightarrow \neg A$

(iii) $\neg A \wedge (A \vee B) \Rightarrow B$

(iv) $(A \Rightarrow B) \wedge (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$

(v) $A \wedge (A \Leftrightarrow B) \Rightarrow B$

Exercise 1.30. *Are the following propositions logical implications?*

(i) $(A \Rightarrow B) \wedge (A \Rightarrow C) \wedge A \Rightarrow B \wedge C$

(ii) $\neg(A \vee B) \wedge (A \vee C) \wedge (D \Rightarrow C) \Rightarrow D$

(iii) $(A \Rightarrow B) \wedge (A \Rightarrow C) \wedge (D \wedge E \Rightarrow F) \wedge (C \Rightarrow E) \Rightarrow F$

Exercise 1.31. *Are the following propositions logical implications?*

(i) $(A \Rightarrow B) \wedge (A \Rightarrow C) \wedge A \Rightarrow B \wedge C$

(ii) $\neg(A \vee B) \wedge (A \vee C) \wedge (D \Rightarrow C) \Rightarrow D$

(iii) $(A \Rightarrow B) \wedge (A \Rightarrow C) \wedge (D \wedge E \Rightarrow F) \wedge (C \Rightarrow E) \Rightarrow F$

Exercise 1.32. *With a direct proof show:*

If n is even, then so is $n^2 + 3n$.

Is the converse also true?

Exercise 1.33. *Use a direct proof of the implication to show: If a real number x is non-negative, then the sum of the number x and its reciprocal is greater than or equal to 2.*

Exercise 1.34. *Use contradiction to show that there are infinitely many prime numbers.*

Exercise 1.35. *Find the error in the following proof.*

Statement: 1 *is the largest natural number.*

Proof *(by contradiction): Suppose the opposite. Let $n > 1$ be the largest natural number. Since n is positive, we can multiply the inequality $n > 1$ by n, giving*

$$n > 1 \Leftrightarrow n^2 > n.$$

We have found that n^2 is greater than n, which contradicts the assumption that n is the largest natural number. Therefore, the assumption was incorrect, and 1 is the largest natural number.

Exercise 1.36. *Let x and y be real numbers such that $x < 2y$. By an indirect proof show:*

If $7xy \leq 3x^2 + 2y^2$, then $3x \leq y$.

Exercise 1.37. *Prove the following equivalence in two parts: Let m and n be integers. Then m and n have different parities if and only if $m^2 - n^2$ is odd.*

Exercise 1.38. *Using an "if and only if" proof, show that $ac \mid bc \Leftrightarrow a \mid b$.*

Exercise 1.39. *Is the following inference correct?*

(i) *If today is Wednesday, I will have a tutorial. Today is Wednesday. Conclusion: I will have a tutorial.*

(ii) *If I study, I will pass the exam. I did not study. Conclusion: I will not pass the exam.*

(iii) *A student took the city bus to the exam. He thought, "If the next traffic light is green, I will pass the exam." When the bus reached the next light, it was not green, so the student said to himself, "Darn, I'll fail again."*

(iv) *An engineer who understands theory always designs a good circuit. A good circuit is economical. Therefore, an engineer who designs an uneconomical circuit does not understand theory.*

Exercise 1.40. *Which of the following propositions are correct where the language of the conversation are real numbers?*

(i) $(\forall x)(\exists y)(x + y = 0)$.

(ii) $(\exists x)(\forall y)(x + y = 0)$.

(iii) $(\exists x)(\exists y)(x^2 + y^2 = -1)$.

(iv) $(\forall x)[x > 0 \Rightarrow (\exists y)(y < 0 \wedge xy > 0)]$.

1.7 PROOFS BY INDUCTION

Exercise 1.41. *Prove each using induction:*

(a) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

(b) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

(c) $\sum_{i=1}^{n} 2^{i-1} = 2^n - 1$

(d) $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

(e) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

(f) $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$

(g) $\sum_{i=1}^{n}(2i - 1) = n^2$

(h) $n! > 2^n$ *for* $n \geq 4$.

(i) $2^{n+1} > n^2$ *for all positive integers.*

Exercise 1.42. *This exercise refers to the Fibonacci sequence:*

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

The sequence is defined recursively by $f_1 = 1$*,* $f_2 = 1$*, then* $f_{n+1} = f_n + f_{n-1}$ *for each* $n > 2$*. As before, prove each of the following using induction. You might investigate each with several examples before you start.*

(a) $f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$

(b) $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$

(c) $f_1 + f_3 + f_5 + \cdots + f_{2n-1} = f_{2n}$

2 SET THEORY

Exercise 2.1. *Let* $A = \{x \in \mathbb{N}; x < 7\}, B = \{x \in \mathbb{Z}; |x - 2| < 4\}$ *and* $C = \{x \in \mathbb{R}; x^3 - 4x = 0\}$.

(i) *Write down the elements for all three sets.*

(ii) *Find* $A \cup C, B \cap C, B \setminus C, (A \setminus B) \setminus C$ *and* $A \setminus (B \setminus C)$.

Exercise 2.2. *Let* $\mathbb{Z}$ *be a universal set and let* P *denote the set of all prime numbers, and* S *the set of all even integers. Write the following propositions in terms of set theory:*

(i) *There exists an even prime number.*

(ii) 0 *is an integer, but it is not natural number.*

(iii) *Every natural number is an integer.*

(iv) *Not every integer is a natural number.*

(v) *Every prime number except 2 is odd.*

(vi) *2 is an even prime number.*

Exercise 2.3. *Let* A, B, C *and* D *be subsets of some universal set* U. *Simplify the following expression*

$$\overline{(\overline{(A \cup B)} \cap \overline{(\overline{A} \cup C)})} \setminus \overline{D}.$$

Exercise 2.4. *Show that* $(A \cup C) \cap (B \setminus C) = (A \cap B) \setminus C$.

Exercise 2.5. *Prove that* $A \subseteq C \wedge B \subseteq C \Rightarrow A \cup B \subseteq C$.

Exercise 2.6. *Prove that* $A \subseteq B \Leftrightarrow A \cap B = A$.

Exercise 2.7. *Prove that* $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Exercise 2.8. *Prove that* $(A \cap B) \setminus B = \emptyset$.

Exercise 2.9. *Determine the following sets:*

(i) $\{\emptyset, \{\emptyset\}\} \setminus \emptyset$

(ii) $\{\emptyset, \{\emptyset\}\} \setminus \{\emptyset\}$

(iii) $\{\emptyset, \{\emptyset\}\} \setminus \{\emptyset\}$

(iv) $\{1, 2, 3, \{1\}, \{5\}\} \setminus \{2, \{3\}, 5\}$

Exercise 2.10. *Which of the following propositions are correct for arbitrary sets A, B and C:*

1. *If $A \in B$ and $B \in C$, then $A \in C$.*

2. *If $A \subseteq B$ and $B \in C$, then $A \in C$.*

3. *If $A \cap B \subseteq \overline{C}$ and $A \cup C \subseteq B$, then $A \cap C = \emptyset$.*

4. *If $A \neq B$ and $B \neq C$, then $A \neq C$.*

5. *If $A \subseteq \overline{(B \cup C)}$ and $B \subseteq \overline{(A \cup C)}$, then $B = \emptyset$.*

Exercise 2.11. *Find $\mathcal{P}(A)$, where $A = \{a, b, c, d\}$.*

Exercise 2.12. *Let $A = \{\{1, 2, 3\}, \{4, 5\}, \{6, 7, 8\}\}$.*

(i) *Write down the elements of A.*

(ii) *Is it true?*
(a) $1 \in A$ *(b)* $\{1, 2, 3\} \subseteq A$ *(c)* $\{6, 7, 8\} \in A$ *(d)* $\{\{4, 5\}\} \subseteq A$
(e) $\emptyset \in A$ *(f)* $\emptyset \subseteq A$

Exercise 2.13. *Show that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.*

Exercise 2.14. *Let A and B be arbitrary subsets of the universal set $U = A \cup B$. Show that $A \setminus B \subseteq \overline{B}$.*

Exercise 2.15. *Let A, B in C be arbitrary subsets of the universal set $U = A \cup B \cup C$. Show that $A \cap B \subseteq C \Leftrightarrow A \subseteq \overline{B} \cup C$.*

Exercise 2.16. *Show that $(A \setminus B) \cap B = \emptyset$.*

Exercise 2.17. *Show that $(A \setminus B) \cup B = A \Leftrightarrow B \subseteq A$.*

Exercise 2.18. *Show that, if $B \subseteq A$, then $B \times B = (B \times A) \cap (A \times B)$.*

Exercise 2.19. *Let A be a nonempty set. Which of the following sets*

$$\emptyset, \{\emptyset\}, A, \{A\}, \{A, \emptyset\}$$

are elements and which are subsets of (i) $\mathcal{P}(A)$ and (ii) $\mathcal{P}(\mathcal{P}(A))$?

Exercise 2.20. *Is it true that $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$?*

Exercise 2.21. *Let A, B and C be arbitrary sets. Show the following propositions:*

1. $A \subseteq B \Leftrightarrow A \cap \overline{B} = \emptyset$.
2. $A \setminus B = \overline{B} \setminus \overline{A}$.

3 RELATIONS

Exercise 3.1. *Let* $S = \{1,2,3,4,5\}$.

1. *Is* $R = \{(1,2),(2,3),(3,5),(2,4),(5,1)\}$ *a binary relation?*
2. *Find the domain* $\mathcal{D}R$ *and the range* $\mathcal{Z}R$ *of* R.
3. *Determine the inverse relation* R^{-1} *and* $\mathcal{D}R^{-1}$ *and* $\mathcal{Z}R^{-1}$.

Exercise 3.2. *Define binary relations:*

$$R = \{(1,1),(2,1),(3,3),(1,5)\},$$
$$T = \{(1,4),(2,1),(2,2),(2,5)\}.$$

1. *Determine the compositions* $R \circ T$ *and* $T \circ R$.
2. *Is it true that* $R \circ T = T \circ R$*?*

Exercise 3.3. *Let* $S = \{1,2,3,4,5,6,7\}$. *Define*

$$R = \{(x,y) \mid x-y \text{ is divisible by } 3\} \quad \text{in} \quad T = \{(x,y) \mid x-y \geq 3\}.$$

Determine $R, T, R \circ R$.

Exercise 3.4. *Determine the Domain and Range of the following relations:*

1. $R = \{(1,2),(2,3),(3,4)\}$
2. $R = \{(1,5),(2,3),(3,3),(4,9)\}$
3. $R = \{(1,2),(3,5),(4,5)\}$
4. $R = \{(-1,-1),(2,2),(3,3)\}$
5. $R = \{(2,0),(9,0)\}$

Exercise 3.5. *For relations from Exercise 3.4 determine their inverses.*

Exercise 3.6. *Let* $R_1 = \{(1,2),(2,3),(3,4)\}$ *and* $R_2 = \{(2,4),(3,5),(4,6)\}$. *Compute* $R_2 \circ R_1$ *and* $R_1 \circ R_2$. *Are they equal?*

Exercise 3.7. *For relations below, compute the corresponding compositions.*

1. $R_1 = \{(1,2),(2,3),(3,4)\}$, $R_2 = \{(1,4),(2,6),(3,4)\}$, $R_2 \circ R_1 =?$

2. $R_1 = \{(1,2),(2,3),(3,4)\}$, $R_2 = \{(2,4),(3,6),(4,4)\}$, $R_2 \circ R_1 =?$

3. $R_1 = \{(4,3),(5,4),(6,5)\}$, $R_2 = \{(3,4),(4,5),(5,6)\}$, $R_2 \circ R_1 =?$

4. $R_1 = \{(4,3),(5,4),(6,5)\}$, $R_1 \circ R_1 =?$

Exercise 3.8. *In the universe $S = \mathbb{Z}$, we are given $R_1 = \{(x,y) \mid x+1=y\}$ and $R_2 = \{(y,z) \mid y+2=z\}$. Find $R_2 \circ R_1$.*

Exercise 3.9. *Let $S = \{1,2,3,4\}$ and $R = \{(1,2),(2,3),(3,4)\}$. Compute $R \circ R$ and $R \circ R \circ R$, and $R \circ R \circ R \circ R \circ R$.*

Exercise 3.10. *Given $S = \{a,b,c,d\}$, let $R = \{(a,b),(b,c),(c,d),(d,a)\}$.*

(i) Determine $R \circ R$ and $R \circ R \circ R$.

(ii) Prove that $R \circ R \circ R \circ R = \{(x,x) \mid x \in S\}$.

Exercise 3.11. *Let $P(A) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ be the power set of $A = \{1,2\}$. Define a relation $R \subseteq P(A) \times P(A)$ such that $(X,Y) \in R$ if and only if $X \subseteq Y$. Compute $R \circ R$ and interpret the result in terms of subset inclusion.*

Exercise 3.12. *Let R_1 and R_2 be binary relations. Prove that*

$$(R_1 \circ R_2)^{-1} = R_2^{-1} \circ R_1^{-1}.$$

3.1 PROPERTIES OF RELATIONS

Exercise 3.13. *Prove or disprove: If R is symmetric, then $R \circ R$ is also symmetric.*

Exercise 3.14. *For $S = \{1,2,3\}$, $R = \{(1,1),(1,2),(2,3)\}$. Is R transitive? Explain.*

Exercise 3.15. *Let $S = \mathbb{R}$. On S we define the relation R as follows*

$$(\forall x)(\forall y)(xRy \Leftrightarrow y \geq x + 3).$$

Is R reflexive, symmetric, transitive or strict total?

Exercise 3.16. *Let $S = \{1,2,3,4\}$. We have the following relations*

(i) $R_1 = \{(1,1),(1,2),(2,3),(1,3),(4,4)\}$,

(ii) $R_2 = \{(1,1),(1,2),(2,1),(2,2),(3,3),(4,4)\}$,

(iii) $R_3 = \{(1,3),(2,1)\}$,

(iv) $R_4 = \emptyset$,

(v) $R_5 = S \times S$.

Which of the following properties hold for each relation: reflexive, symmetric, antisymmetric, transitive?

Exercise 3.17. *Let R and S be symmetric relations. Show: $R \circ S$ symmetric $\Leftrightarrow R \circ S = S \circ R$.*

3.2 EQUIVALENCES

Exercise 3.18. *Let $S = \mathbb{Z} \times \mathbb{Z} \setminus (0,0)$ and define the relation R as follows*

$$(a,b)R(c,d) \Leftrightarrow ad = bc.$$

Show that R is an equivalence relation and find the corresponding equivalence classes.

Exercise 3.19. *Let $S = \mathbb{R}^2$ and define the relation R as follows*

$$(x_1,y_1)R(x_2,y_2) \Leftrightarrow x_1^2 + y_1^2 = x_2^2 + y_2^2.$$

Show that R is an equivalence relation and find the equivalenec class $R[(7,1)]$.

Exercise 3.20. *Let $S = \{m \in \mathbb{N} \,|\, 1 \leq n \leq 10\}$ in $R = \{(m,n) \in S \times S \,|\, 3|m-n\}$. Is R an equivalence relation? If yes, determine the corresponding equivalence classes and the factor set.*

Exercise 3.21. *For each relation R on the set $A = \{1, 2, 3, 4\}$, determine if R is an equivalence relation. If it is, identify the equivalence classes.*

1. $R_1 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$
2. $R_2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4)\}$
3. $R_3 = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (3, 4), (4, 3)\}$

Exercise 3.22. *Let S be the set of all people, and define a relation R on S by setting $(a, b) \in R$ if a and b have the same birth month. Prove that R is an equivalence relation and determine the equivalence classes.*

Exercise 3.23. *Consider the set $X = \mathbb{Z}$ (integers), and let R be the relation defined by aRb if $a - b$ is divisible by 5. Show that R is an equivalence relation and find the equivalence classes.*

Exercise 3.24. *Define a relation R on the set of all triangles in the plane by setting $\triangle ABC \, R \, \triangle DEF$ if the triangles are similar. Prove that R is an equivalence relation and determine the equivalence classes.*

Exercise 3.25. *Let X be a set, and $P(X)$ be the power set of X (the set of all subsets of X). Define a relation R on $P(X)$ by setting ARB if $|A \Delta B|$ is even, where Δ is the symmetric difference. Prove that R is an equivalence relation.*

Exercise 3.26. *For $a, b \in \mathbb{R}$, define $a \sim b$ to mean $ab = 0$. Prove or disprove each of the following:*

(a) The relation $\sim$ is reflexive.

(b) The relation $\sim$ is symmetric.

(c) The relation $\sim$ is transitive.

Exercise 3.27. *For $a, b \in \mathbb{R}$, define $a \sim b$ to mean that $ab \neq 0$. Prove or disprove each of the following:*

(a) The relation $\sim$ is reflexive.

(b) The relation $\sim$ is symmetric.

(c) The relation $\sim$ is transitive.

Exercise 3.28. *For $a, b \in \mathbb{R}$, define $a \sim b$ to mean that $|a - b| < 5$. Prove or disprove each of the following:*

(a) The relation $\sim$ is reflexive.

(b) The relation $\sim$ is symmetric.

(c) The relation $\sim$ is transitive.

Exercise 3.29. *Define a function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 + 1$. For $a, b \in \mathbb{R}$, define $a \sim b$ to mean that $f(a) = f(b)$.*

(a) Prove that $\sim$ is an equivalence relation on $\mathbb{R}$.

(b) List all elements in the set $\{x \in \mathbb{R} \mid x \sim 3\}$.

Exercise 3.30. *For points $(a, b), (c, d) \in \mathbb{R}^2$, define $(a, b) \sim (c, d)$ to mean that $a^2 + b^2 = c^2 + d^2$.*

(a) Prove that $\sim$ is an equivalence relation on $\mathbb{R}^2$.

(b) List all elements in the set $\{(x, y) \in \mathbb{R}^2 \mid (x, y) \sim (0, 0)\}$.

(c) List five distinct elements in the set $\{(x, y) \in \mathbb{R}^2 \mid (x, y) \sim (1, 0)\}$.

Exercise 3.31. *Recall that for $a, b \in \mathbb{Z}$, $a \equiv b \pmod 8$ means that $a - b$ is divisible by 8.*

(a) Find all integers x such that $0 \leq x < 8$ and $2x \equiv 6 \pmod 8$.

(b) Use the Division Algorithm to prove that for every integer m, there exists an integer r such that $m \equiv r \pmod 8$ and $0 \leq r < 8$.

(c) Use the Division Algorithm to find integers r_1 and r_2 such that $0 \leq r_1 < 8$, $0 \leq r_2 < 8$, $1038 \equiv r_1 \pmod 8$, and $-1038 \equiv r_2 \pmod 8$.

Exercise 3.32. *For what positive integers $n > 1$ is:*

(a) $30 \equiv 6 \pmod n$

(b) $30 \equiv 7 \pmod n$

Exercise 3.33. *Let m and n be positive integers such that m divides n. Prove that for all integers a and b, if $a \equiv b \pmod n$, then $a \equiv b \pmod m$.*

Exercise 3.34. *(a) Prove or disprove: For all positive integers n and for all integers a and b, if $a \equiv b \pmod n$, then $a^2 \equiv b^2 \pmod n$.*

(b) Prove or disprove: For all positive integers n and for all integers a and b, if $a^2 \equiv b^2 \pmod n$, then $a \equiv b \pmod n$.

3.3 FUNCTIONS

Exercise 3.35. *Let $A = \{1, 2, 3, 4\}$, $B = \{x, y, z\}$, $C = \{a, b\}$. You are given functions $f : A \to B$ and $g : B \to C$.*

$$f = \{(1, x), (2, y), (3, y), (4, x)\}$$

$$g = \{(x, a), (y, b), (z, b)\}$$

(a) Is f injective?
(b) Is f surjective?
(c) Is g injective?
(d) Is g surjective?
(e) Is $g \circ f$ surjective?

Exercise 3.36. *Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$, $C = \{x, y\}$. You are given functions $f : A \to B$ and $g : B \to C$.*

$$f = \{(a, 1), (b, 3), (c, 2)\}$$

$$g = \{(1, x), (2, y), (3, x)\}$$

(a) Is f injective?
(b) Is f surjective?
(c) Is g injective?
(d) Is g surjective?
(e) Is $g \circ f$ surjective?

Exercise 3.37. *Let $A = \{x, y, z\}$, $B = \{1, 2, 3\}$, $C = \{a, b, c\}$. You are given functions $f : A \to B$ and $g : B \to C$.*

$$f = \{(x, 2), (y, 1), (z, 3)\}$$

$$g = \{(1, a), (2, b), (3, c)\}$$

(a) Is f injective?
(b) Is f surjective?
(c) Is g injective?
(d) Is g surjective?
(e) Is $g \circ f$ surjective?

Exercise 3.38. *Injective functions are also said to be* one-to-one, *while surjective functions are said to be* onto. *Find an example of:*

(a) A one-to-one function $f : \mathbb{N} \to \mathbb{N}$ which is not onto.

(b) A function $f : \mathbb{N} \to \mathbb{N}$ which is onto but not one-to-one.

Exercise 3.39. *Show that if X is a finite set, then a function $f : X \to X$ is one-to-one if and only if it is onto.*

Exercise 3.40. *Decide which of the following functions $\mathbb{Z} \to \mathbb{Z}$ are injective and which are surjective:*

$$x \mapsto 1 + x, \quad x \mapsto 1 + x^2, \quad x \mapsto 1 + x^3, \quad x \mapsto 1 + x^2 + x^3.$$

Does anything in the answer change if we consider them as functions $\mathbb{R} \to \mathbb{R}$? (You may want to sketch their graphs and/or use some elementary calculus methods.)

Exercise 3.41. *For a set X, let $id_X : X \to X$ denote the function defined by $id_X(x) = x$ for all $x \in X$ (the identity function). Let $f : X \to Y$ be some function. Prove:*

(a) A function $g : Y \to X$ such that $g \circ f = id_X$ exists if and only if f is one-to-one.

(b) A function $g : Y \to X$ such that $f \circ g = id_Y$ exists if and only if f is onto.

(c) A function $g : Y \to X$ such that both $f \circ g = id_Y$ and $g \circ f = id_X$ exist if and only if f is a bijection.

(d) If $f : X \to Y$ is a bijection, then the following three conditions are equivalent for a function $g : Y \to X$:

   (i) $g = f^{-1}$,

   (ii) $g \circ f = id_X$, and

   (iii) $f \circ g = id_Y$.

Exercise 3.42. *(a) If $g \circ f$ is an onto function, does g have to be onto? Does f have to be onto?*

(b) If $g \circ f$ is a one-to-one function, does g have to be one-to-one? Does f have to be one-to-one?

Exercise 3.43. *Prove that the following two statements about a function $f : X \to Y$ are equivalent (X and Y are some arbitrary sets):*

(i) *f is one-to-one.*

(ii) *For any set Z and any two distinct functions $g_1 : Z \to X$ and $g_2 : Z \to X$, the composed functions $f \circ g_1$ and $f \circ g_2$ are also distinct.*

(First, make sure you understand what it means that two functions are equal and what it means that they are distinct.)

Exercise 3.44. *In everyday mathematics, the number of elements of a set is understood in an intuitive sense and no definition is usually given. In the logical foundations of mathematics, however, the number of elements is defined via bijections: $|X| = n$ means that there exists a bijection from X to the set $\{1, 2, ..., n\}$.*[1]

1. *Prove that if X and Y have the same size according to this definition, then there exists a bijection from X to Y.*

2. *Prove that if X has size n according to this definition, and there exists a bijection from X to Y, then Y has size n too.*

3. *(*) Prove that a set cannot have two different sizes m and n, $m \neq n$, according to this definition. Be careful not to use the intuitive notion of "size" but only the definition via bijections. Proceed by induction.*

3.4 GRAPH THEORY

Exercise 3.45. ***Degrees and Handshaking***
Let G be a graph with vertex set $V(G) = \{1, 2, 3, 4, 5\}$ and edge set

$$E(G) = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{3,4\}, \{4,5\}\}.$$

1. *Calculate the degree $deg(v)$ for every vertex $v \in V(G)$.*

2. *Determine $\Delta(G)$ (maximum degree) and $\delta(G)$ (minimum degree).*

1 Also other alternative definitions of set size exist but we will consider only this one here.

3. *Verify the basic degree sum property (Handshaking Lemma).*

Exercise 3.46. ***Graph Operations: Removal***
Consider the cycle graph C_4 with vertices labeled $1, 2, 3, 4$ and edges $\{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\}$.

1. *Let $G' = C_4 - v$ where $v = 1$. Identify the resulting graph.*

2. *Let $G'' = C_4 - e$ where $e = \{1, 2\}$. Is G'' connected?*

Exercise 3.47. ***Special Graphs***
Consider the complete graph K_5.

1. *Find the size (number of edges) of K_5.*

2. *Determine the clique number $\omega(K_5)$ and independence number $\alpha(K_5)$.*

Exercise 3.48. ***Subgraphs vs. Induced Subgraphs***
The lecture notes distinguish between subgraphs (vertex/edge removal) and induced subgraphs (only vertex removal).

1. *Let $G = C_5$ (Cycle: $1 - 2 - 3 - 4 - 5 - 1$). Does G contain P_3 (Path: $a - b - c$) as an **induced** subgraph?*

2. *Let $G = K_4$. Is C_4 (Cycle of 4) a subgraph of K_4? Is it an **induced** subgraph?*

Exercise 3.49. ***Isomorphism and Invariants***
Let $G = C_6$ (cycle of length 6) and $H = 2K_3$ (two disjoint triangles).

1. *Calculate the number of vertices and degrees for both G and H.*

2. *Determine the girth $g(G)$ and $g(H)$.*

3. *Are G and H isomorphic?*

Exercise 3.50. ***Automorphisms***
An automorphism is an isomorphism from a graph to itself. Consider the path graph P_3 with vertices $V = \{1, 2, 3\}$ and edges $\{\{1, 2\}, \{2, 3\}\}$.

1. *Find the degree of each vertex.*

2. *List all automorphisms $\phi : V \to V$.*

Exercise 3.51. ***Girth and Trees***
The girth $g(G)$ is the size of the shortest cycle contained in G as a subgraph.

1. *Construct a graph with $\Delta(G) = 2$ and $g(G) = 4$.*
2. *What is the girth of a Tree graph (a connected graph with no cycles)?*

Exercise 3.52. *Let G be any graph from Section A. Determine:*

1. *Order and size of the graph, that is, number of vertices and number of edges in G;*
2. *The maximum and minimum degree, i.e., $\Delta(G)$, $\delta(G)$*
3. *The size of the largest clique, i.e., $\omega(G)$,*
4. *The girth of the graph, i.e., $g(G)$*
5. *The size of the largest independent set, i.e., $\alpha(G)$, and*
6. *The minimum number of colors needed for coloring the graph, i.e., $\chi(G)$.*

Exercise 3.53. *Let $n \geq 3$. Recall the definition of cycles and complete graphs:*

$$C_n = \{[n], E_1\}$$

$$K_n = \{[n], E_2\}$$

and define

$$G_n = \{[n], E_2 \setminus E_1\}$$

- *Draw $H, G_4, G_5, G_6, C_5, C_6, \overline{C_i}$*
- *For all the above graphs, determine $\Delta(G_i), \delta(G_i), \alpha(G_i), \omega(G_i), \chi(G_i), g(G_i)$*
- *Prove $(\forall i \geq 3)(G_i \simeq \overline{C_i})$*

Exercise 3.54. *Let $G = ([n], E)$ be a graph.*

- *Prove: $\chi(G) \geq \omega(G)$*
- *Prove: $\chi(G) \geq \frac{n}{\alpha(G)}$*

Exercise 3.55. ***The Pigeonhole Principle in Graphs***
Let G be a simple graph with n vertices ($n \geq 2$). Prove that there must exist at least two vertices $u, v \in V(G)$ such that $deg(u) = deg(v)$. (Hint: Consider the possible values the degree of a vertex can take. Can a graph contain both a vertex of degree 0 and a vertex of degree $n - 1$?)

Exercise 3.56. ***Structural Existence Proof (The Longest Path Argument)***
Let G be a finite graph where every vertex has a degree of at least 2 (i.e., $\delta(G) \geq 2$). Prove that G must contain a cycle. (Hint: Consider the longest path $P = v_1 - v_2 - ... - v_k$ in the graph. Look at the neighbors of the endpoint v_k.)

Exercise 3.57. ***Extremal Graph Theory: Triangle-Free Graphs***
A graph is called "triangle-free" if it contains no K_3 (complete graph of size 3) as a subgraph. Equivalently, $\omega(G) < 3$.

1. *Draw a graph with 5 vertices that is triangle-free but has as many edges as possible. How many edges does it have?*
2. *Prove that if a graph G on n vertices is bipartite, it is triangle-free.*

Exercise 3.58. ***Automorphism Groups of Cycles vs. Paths***
Let C_4 be the cycle graph with 4 vertices, and P_4 be the path graph with 4 vertices.

1. *Determine the number of automorphisms for P_4.*
2. *Determine the number of automorphisms for C_4.*
3. *Explain why the symmetry (number of automorphisms) increases when we add an edge to P_4 to turn it into C_4.*

3.5 ORDER STRUCTURES

Exercise 3.59. *Let $P(A) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ be the power set of $A = \{1,2\}$. Define a relation $R \subseteq P(A) \times P(A)$ such that $(X,Y) \in R$ if and only if $X \subseteq Y$. Prove that R is a partial order.*

Exercise 3.60. *A **preorder** is a relation that is **reflexive** and **transitive**. A **partial order** is a relation that is **reflexive**, **anti-symmetric**, and **transitive**.*

Based on the definitions:

1. *Show that every partial order is a preorder.*
2. *Give an example of a preorder that is NOT a partial order.*

Exercise 3.61. *The diagram suggests that a **strict partial order** is characterized only by the **transitivity** property and an additional property.*

1. *What is the standard additional property that defines a strict partial order? (Hint: It is a property that ensures (a,a) is never in the relation.)*
2. *Show that a relation $\mathcal{R}$ is a strict partial order if and only if it is **transitive** and **irreflexive**.*

Exercise 3.62. *A **linear order** (or **total order**) is a partial order that is also **total**. A **strict linear order** (or **strict total order**) is a strict partial order that is also **total** on all distinct pairs (i.e., for $a \neq b$, we must have $(a,b) \in \mathcal{R}$ or $(b,a) \in \mathcal{R}$).*

Show that for any set A, there is a one-to-one correspondence between the set of linear orders on A and the set of strict linear orders on A.

Exercise 3.63. *Choose a diagram from Section B.1, and arbitrarily label its universe S with labels $S = \{1,2,\dots\}$. Let $A = \{1,3\}$, $B = \{1,2,3\}$, $C = S$, and $D = \emptyset$.*

1. *For the sets A, B, C and D, fill in the following table:*

	R-lower bounds	*R-upper bounds*	*R-infimum*	*R-supremum*
A				
B				
C				
D				

2. *Detemine whether the poset is a lattice. If not, determine the corresponding set which does not admit infimum or supremum.*

Exercise 3.64. *Choose a diagram from Appendix B.2, with universe $S = \{a, b, \dots\}$, and set $A = \{a, c\}$, $B = \{b, d, e\}$, $C = S$, and $D = \emptyset$.*

1. *For the sets A, B, C and D, fill in the following table:*

	R-lower bounds	*R-upper bounds*	*R-infimum*	*R-supremum*
A				
B				
C				
D				

2. *Detemine whether the poset is a lattice. If not, determine the corresponding set which does not admit infimum or supremum.*

Exercise 3.65. *Let $S = \{(x, y) : x, y \in \mathbb{R}$ and $y \leq 0\}$ and let R be the relation on S defined by*

$$(x_1, y_1)\, R\, (x_2, y_2) \Leftrightarrow x_1 = x_2 \text{ and } y_1 \leq y_2.$$

(i) *Show that R is a partial order on S.*

(ii) *Find all R-minimal elements.*

Exercise 3.66. *Let $S = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots, 1\}$, and let $R = \{(a, b) \mid a < b\}$ the relation on S. Does R well order S?*

Exercise 3.67. *Let $S = \mathbb{N} \setminus \{0\}$ and let R be the relation on S defined by*

$$m\, R\, n \Leftrightarrow\ p < u \text{ or } (p = u \text{ and } q < v),$$

where $m = 2^p(2q + 1)$ and $n = 2^u(2v + 1)$ with p and u maximal possible exponents.

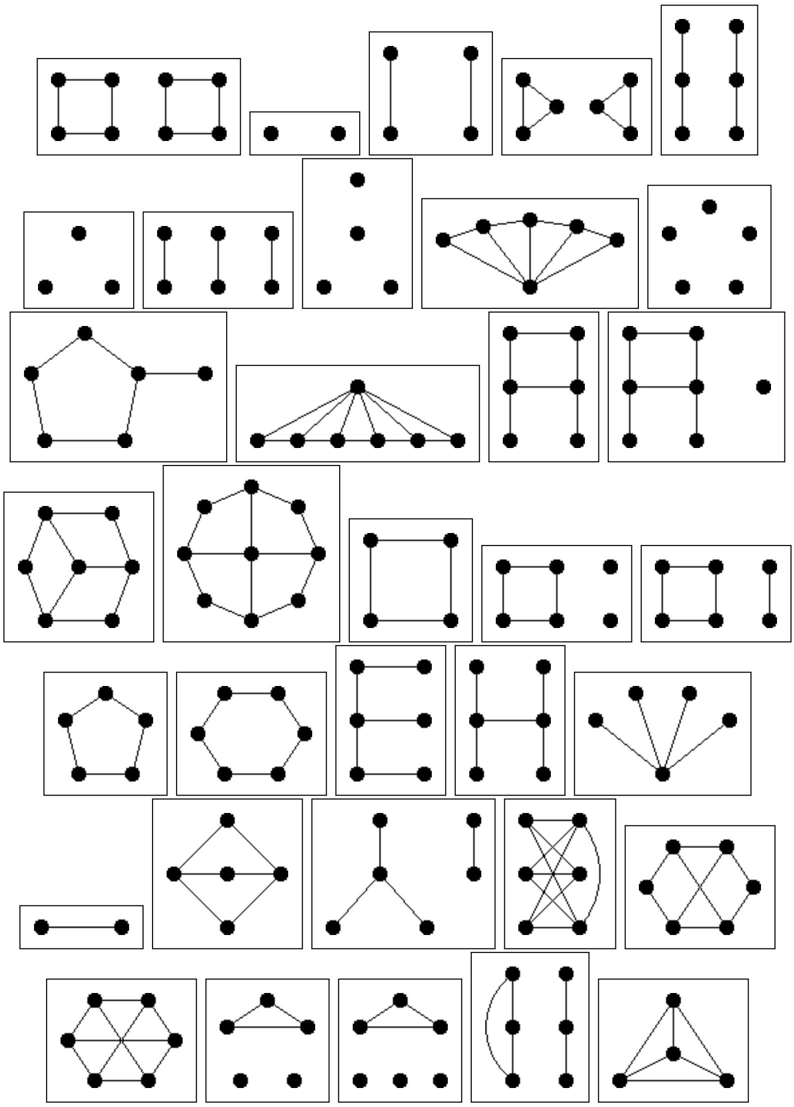
(i) *Show that R well orders S.*

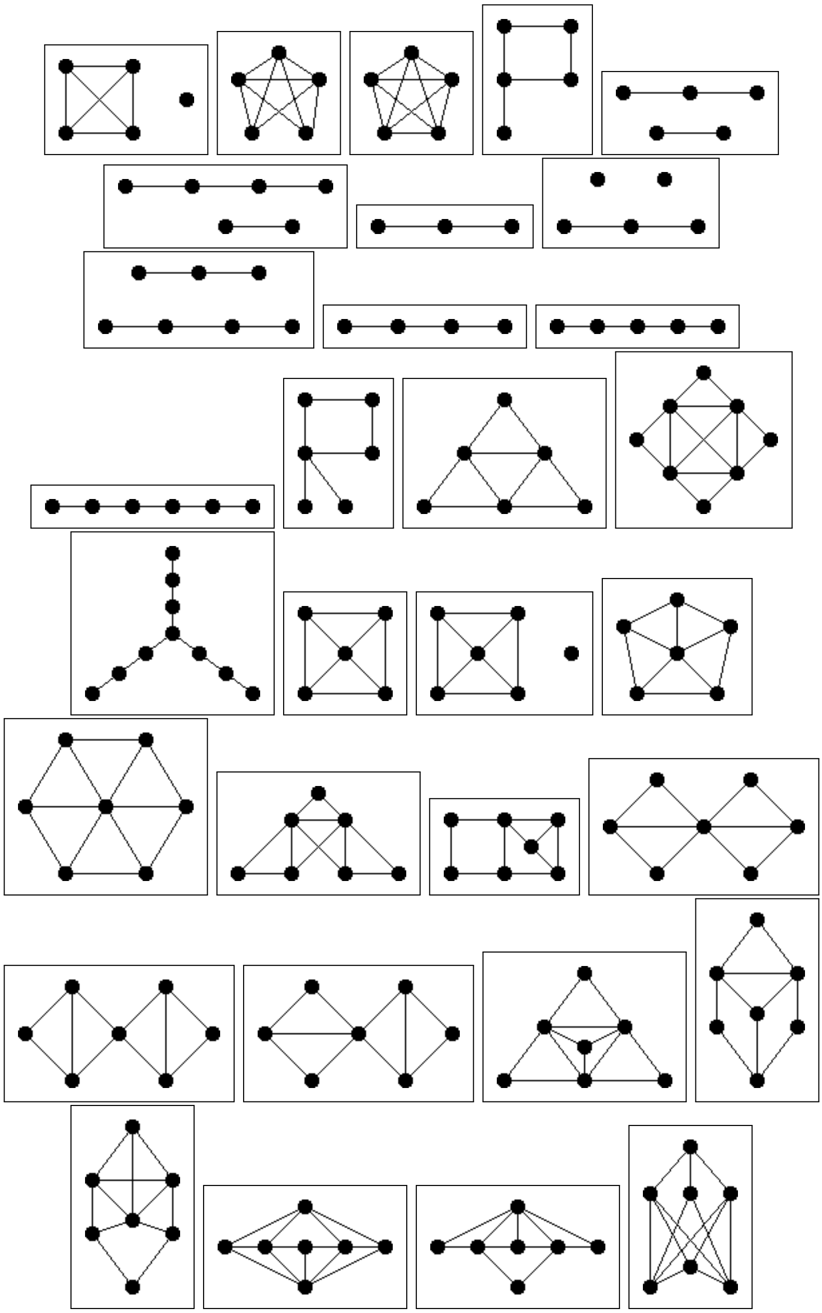
(ii) *Order the set $\{1, 2 \dots, 10\}$ with respect to R.*

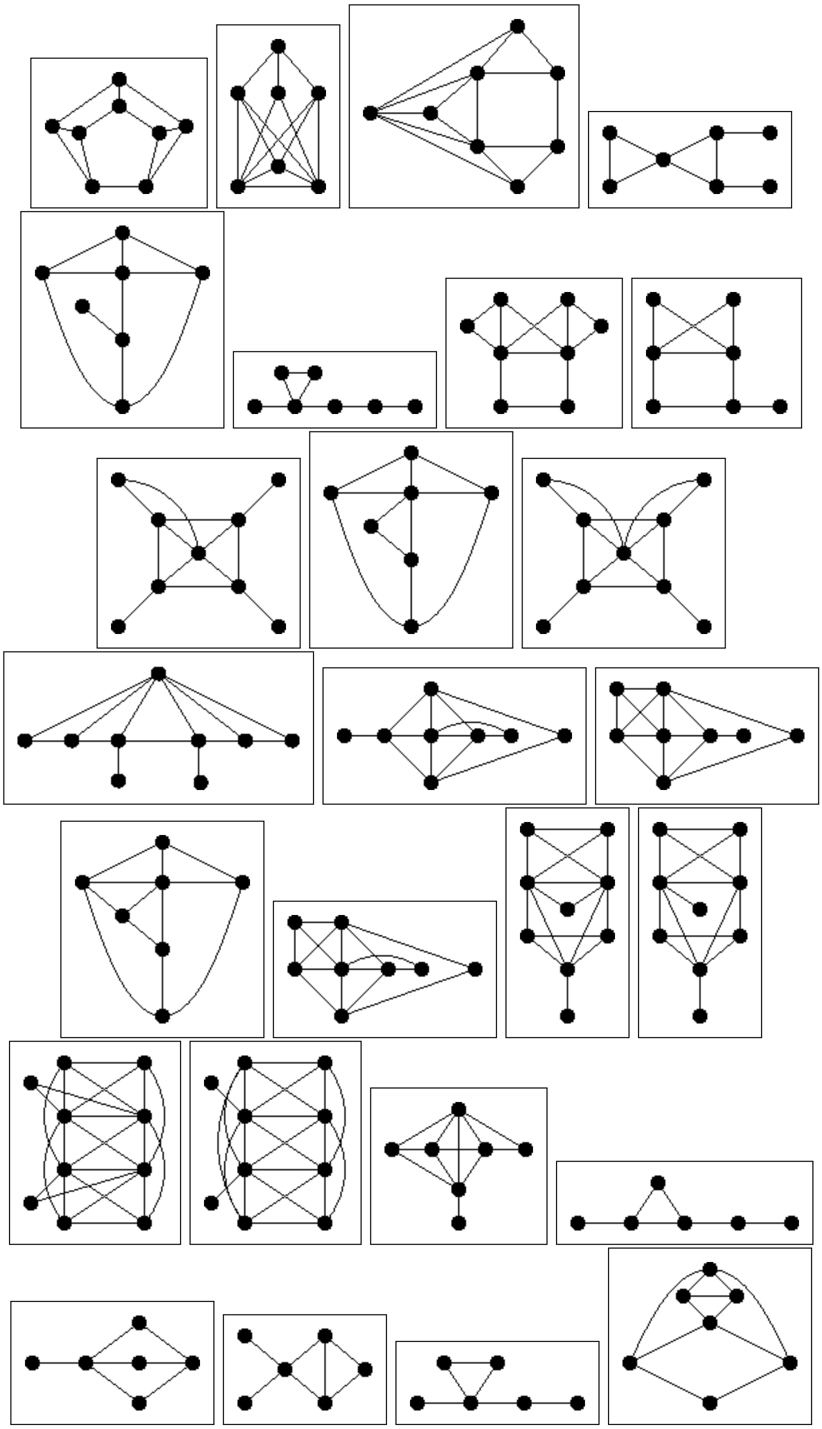
(iii) *Let $C = \{50, 51, 52, \dots\}$. Find R-minimal element of C.*

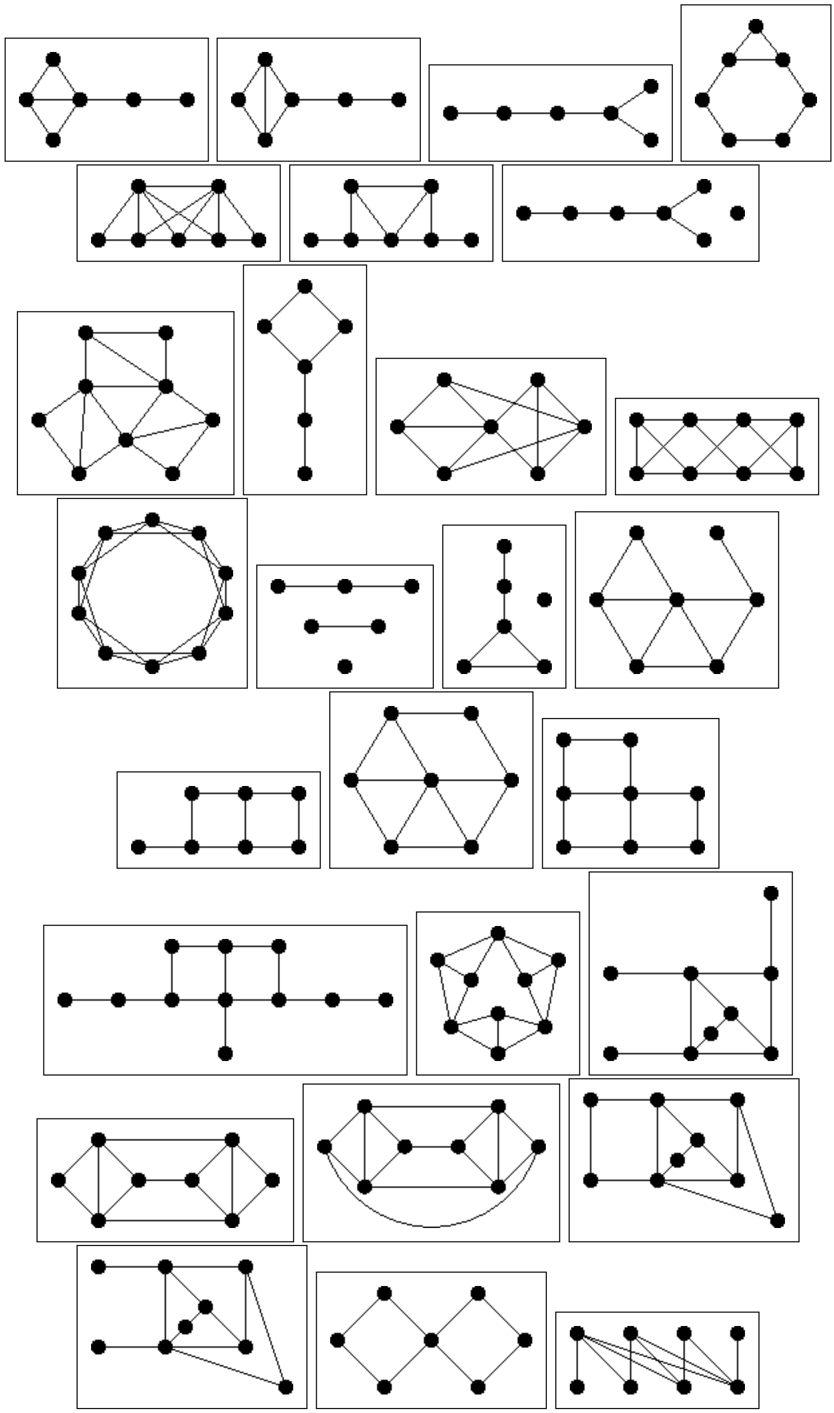
(iv) *Find an immediate successor of 96.*

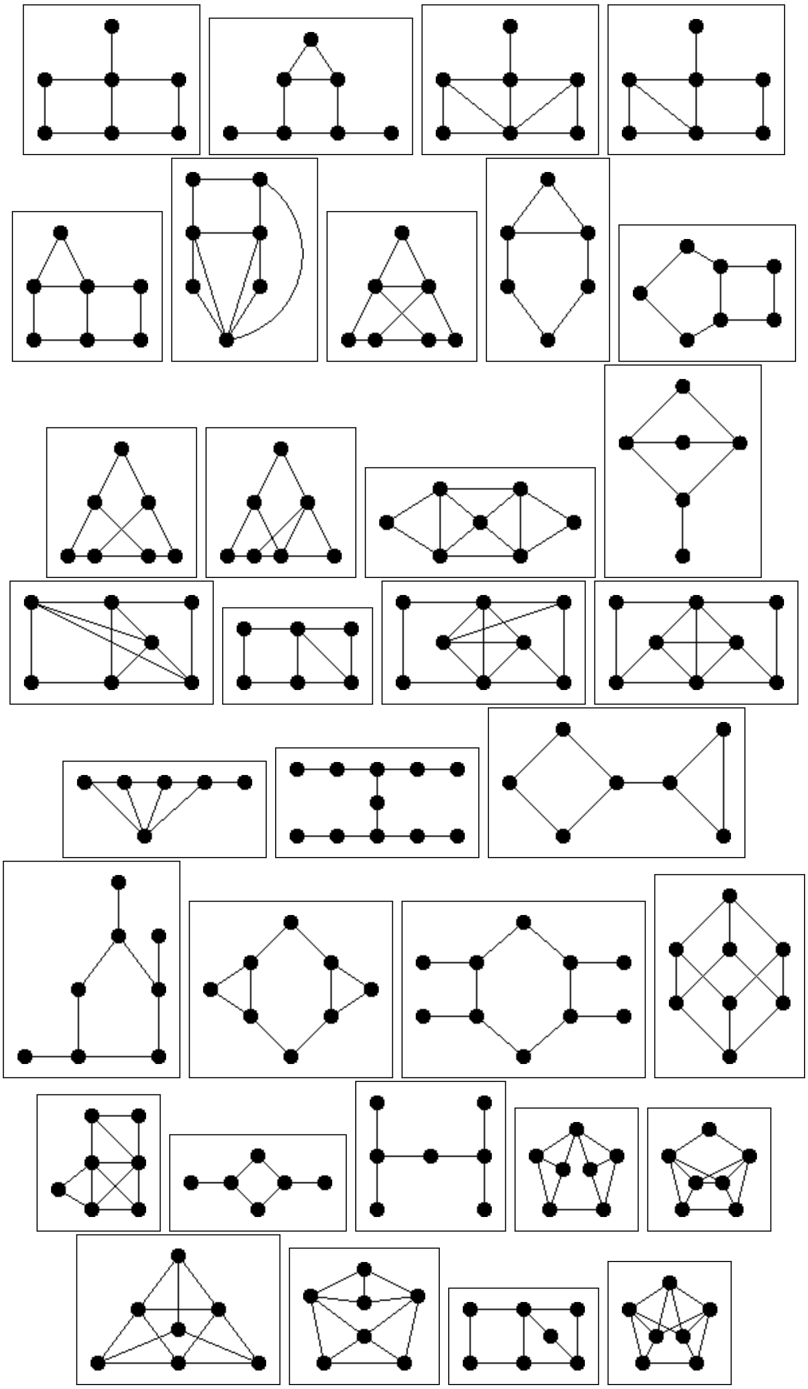
A LIST OF GRAPHS

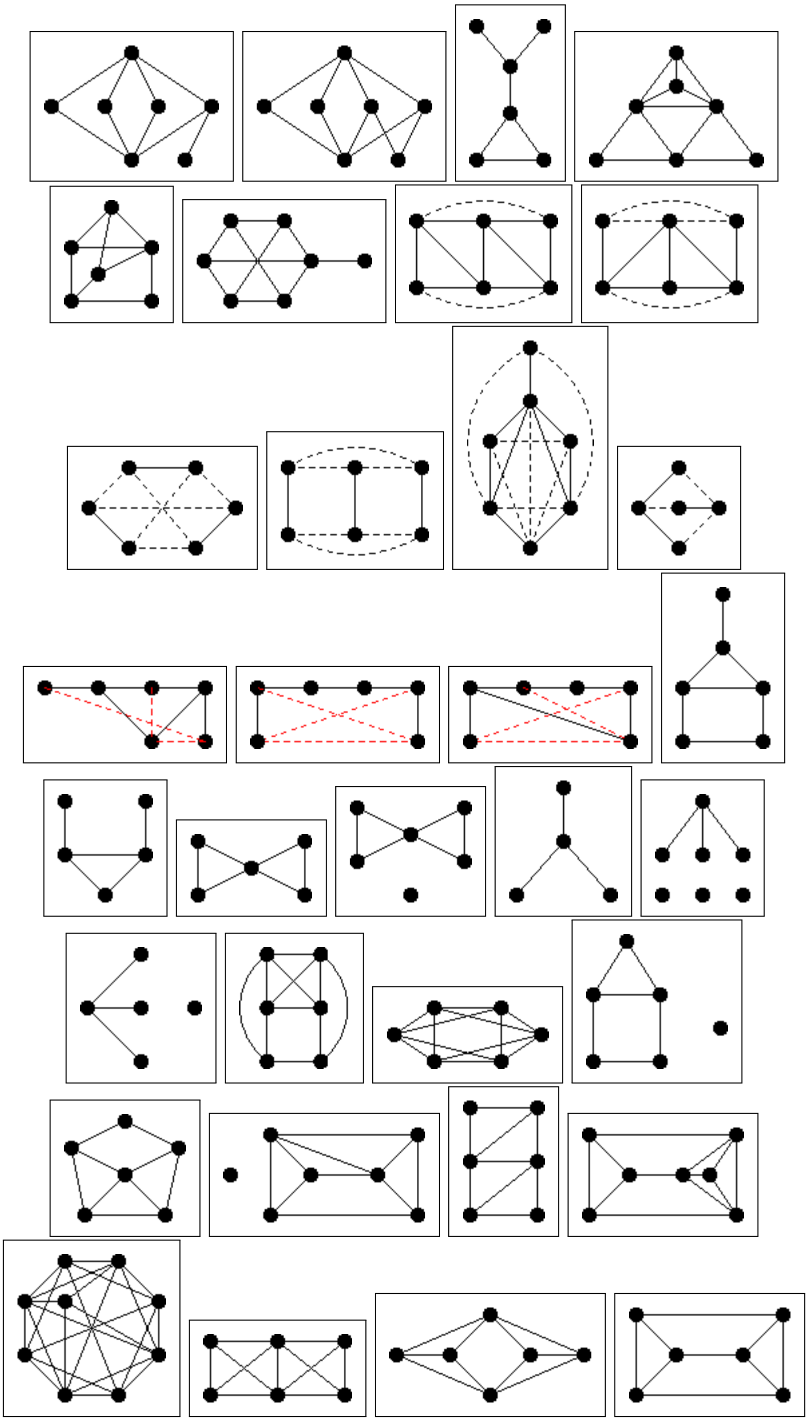


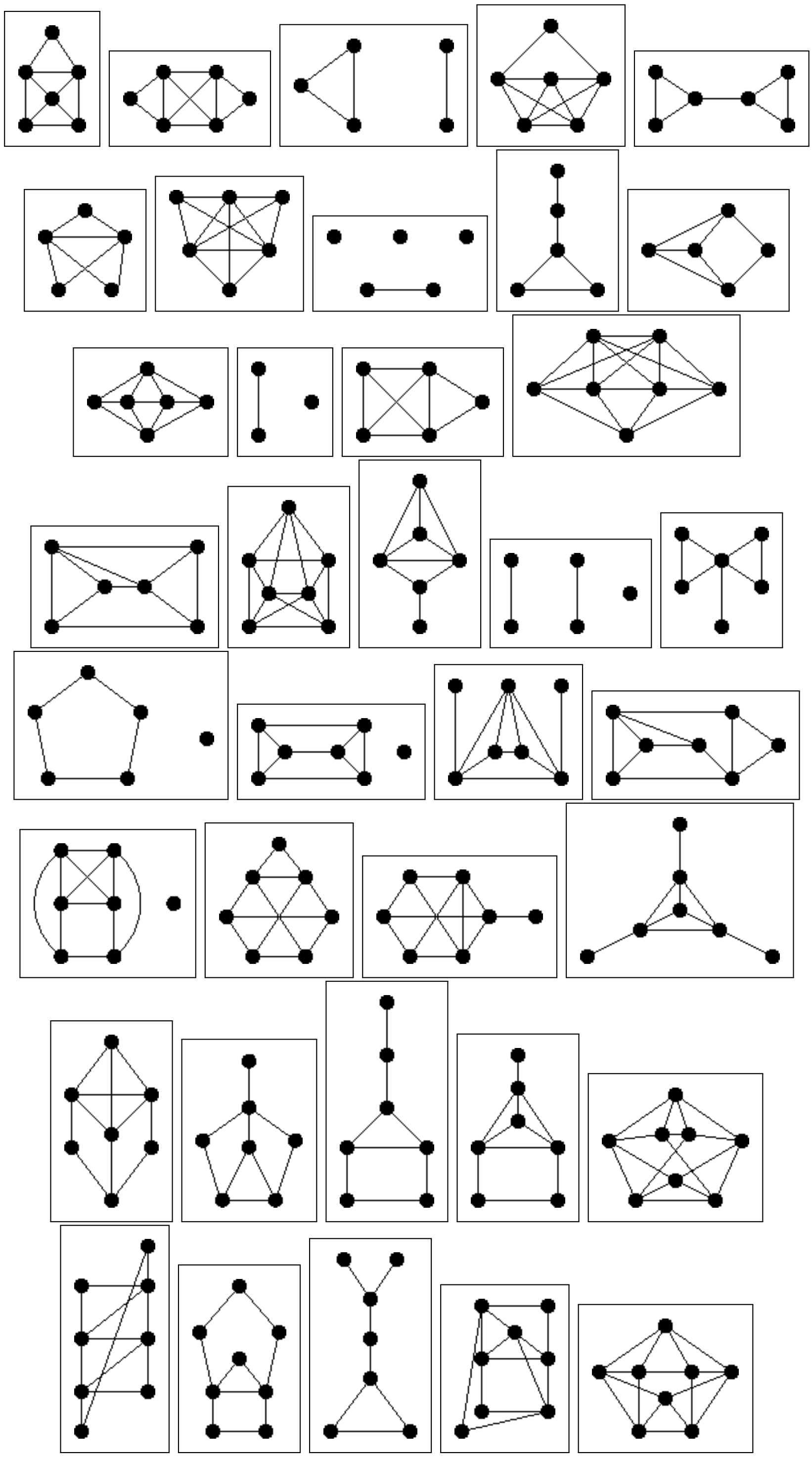


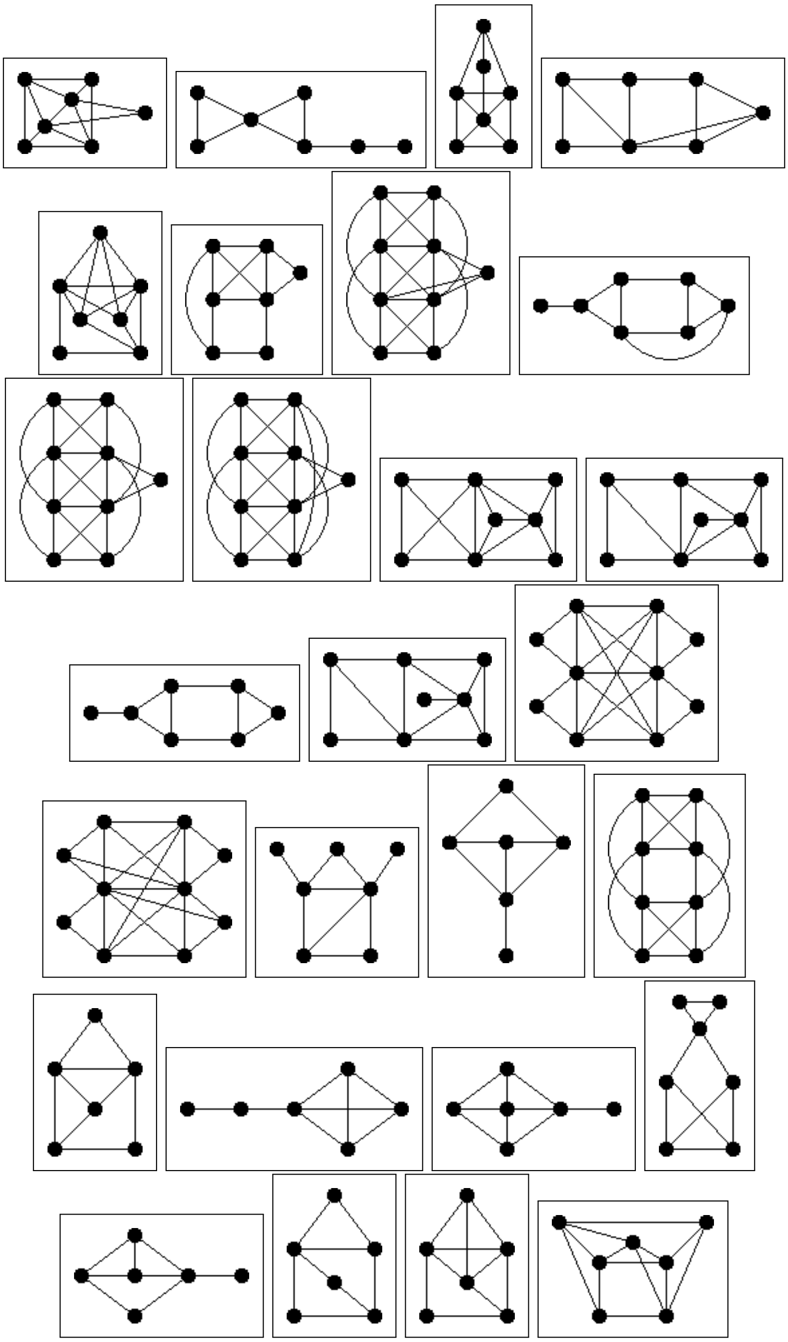


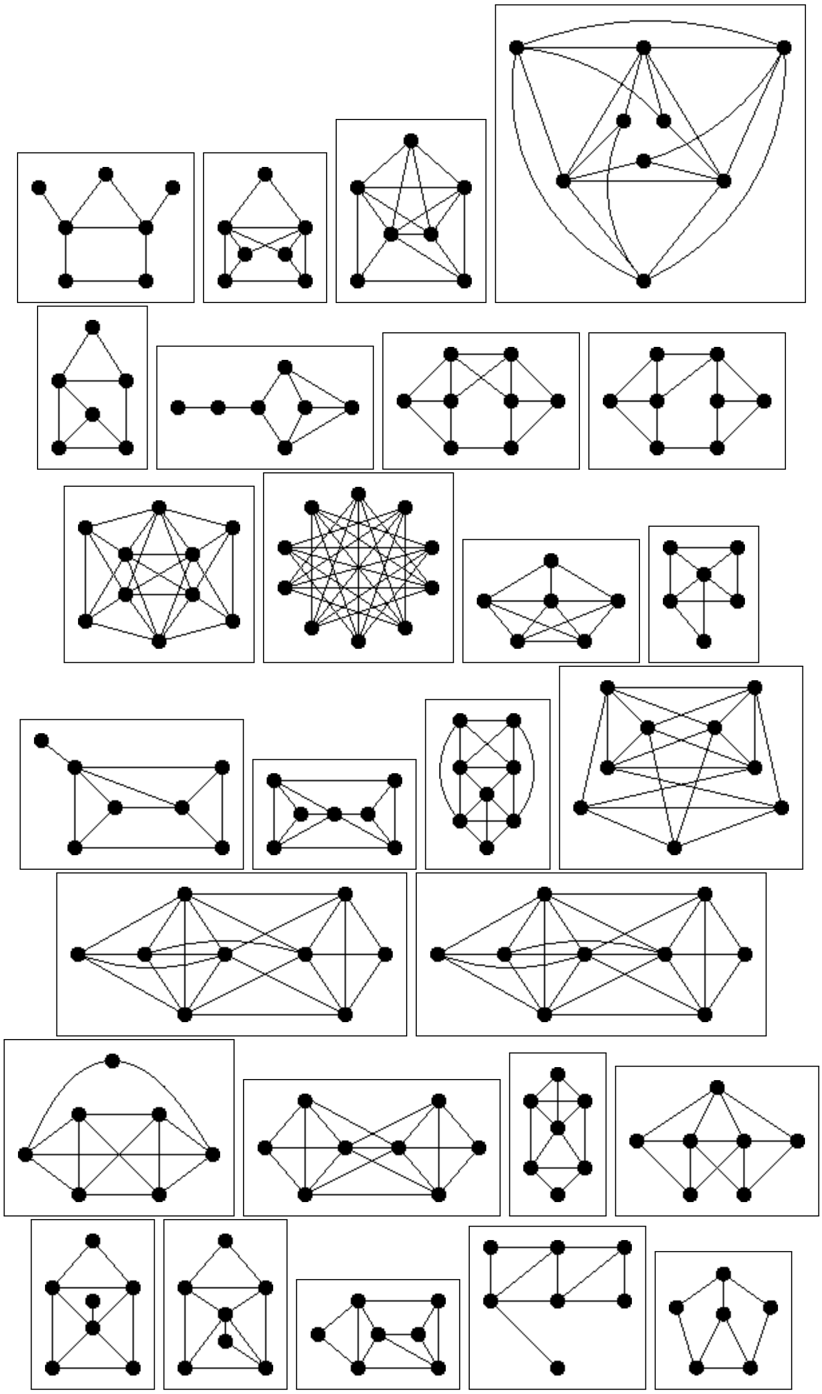


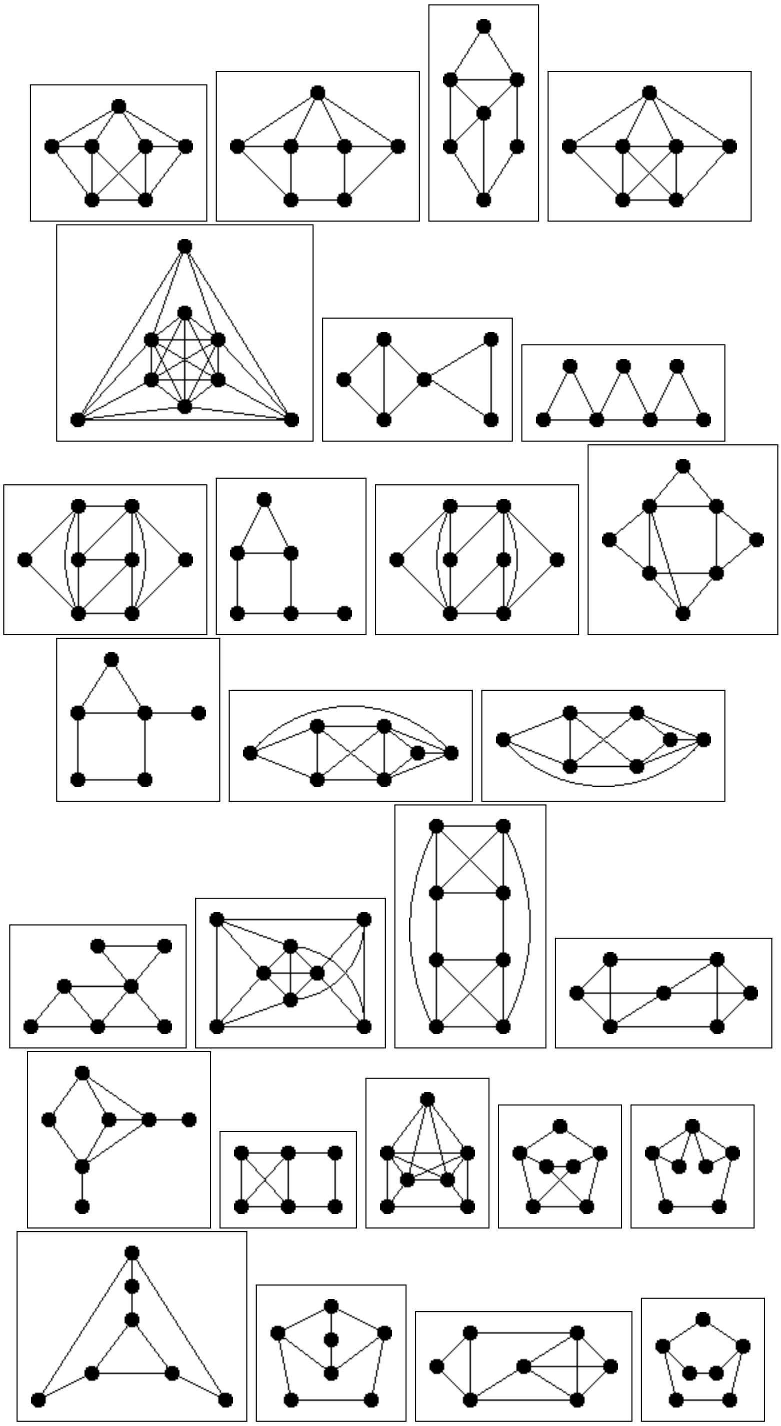


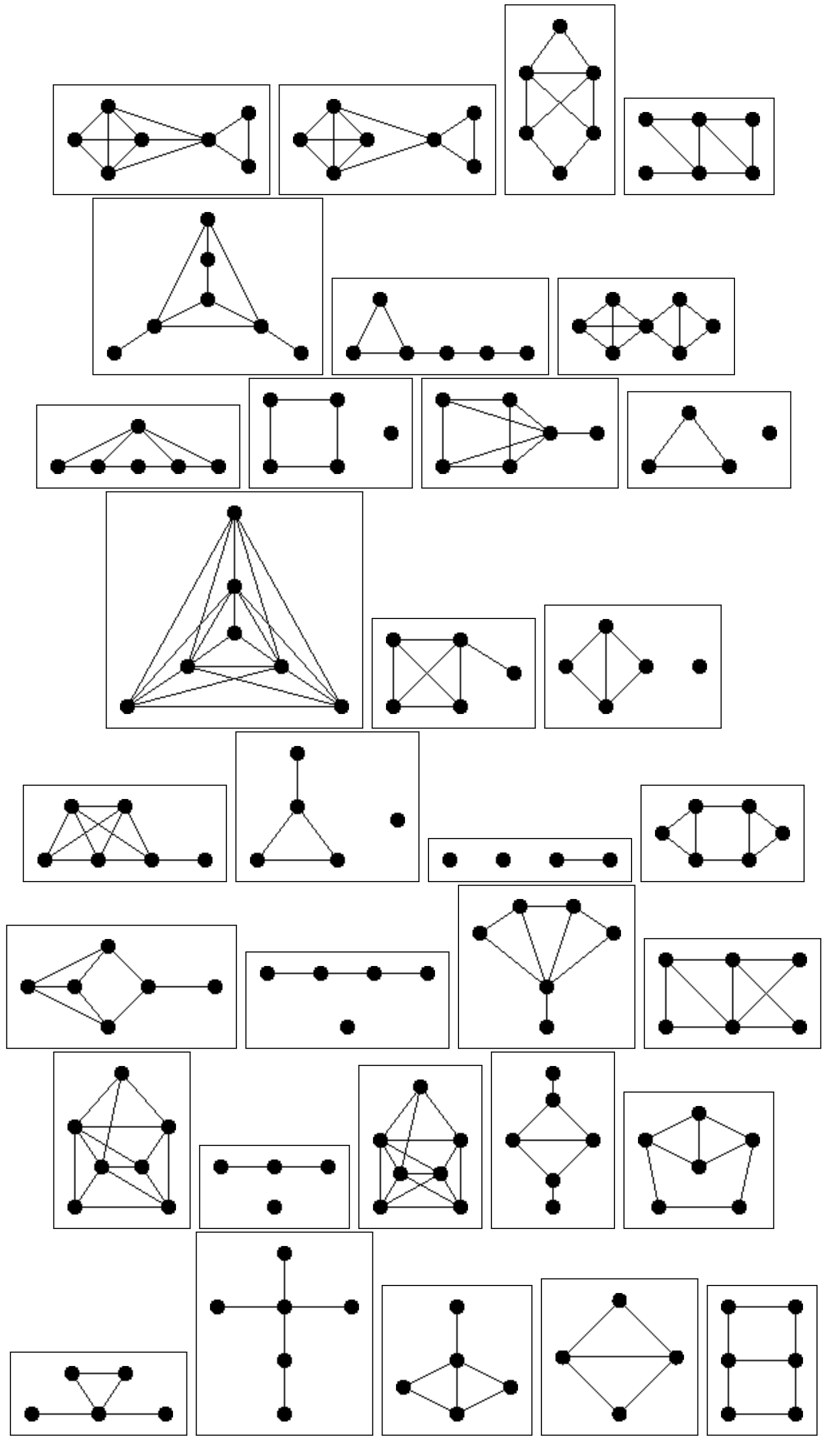


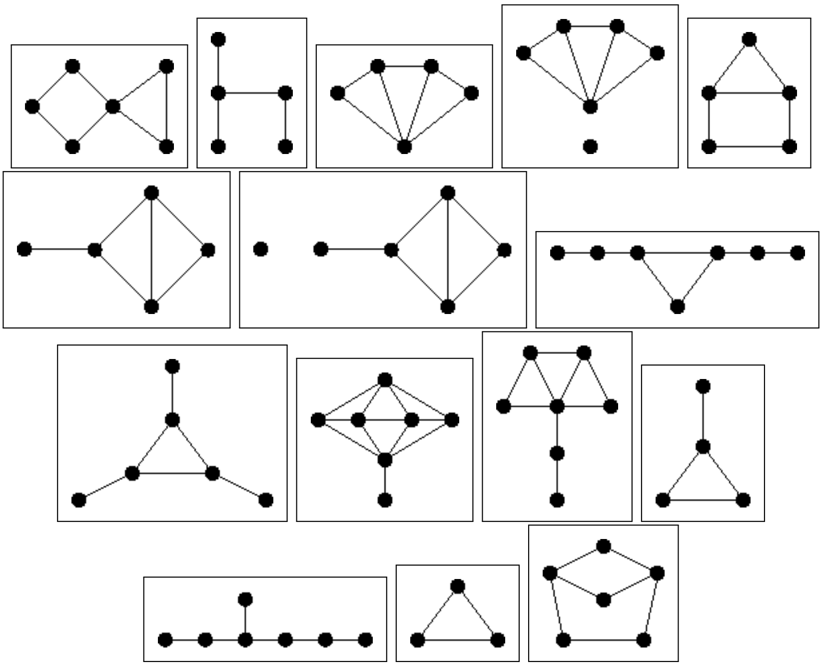






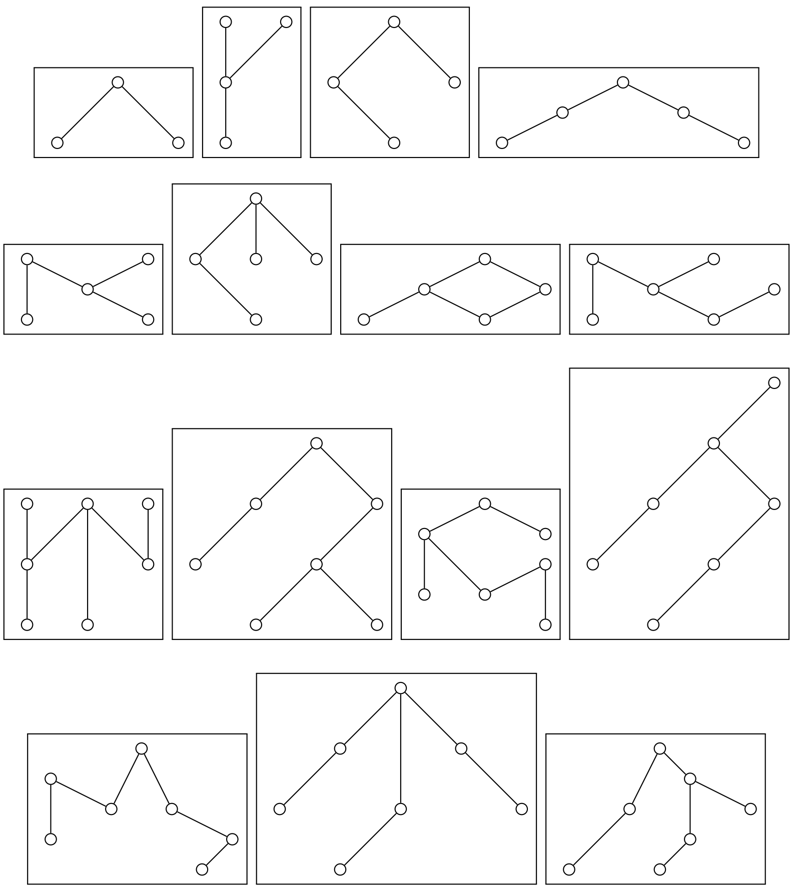






B LIST OF HASSE DIAGRAMS

B.1 UNLABELED DIAGRAMS



B.2 LABELED DIAGRAMS

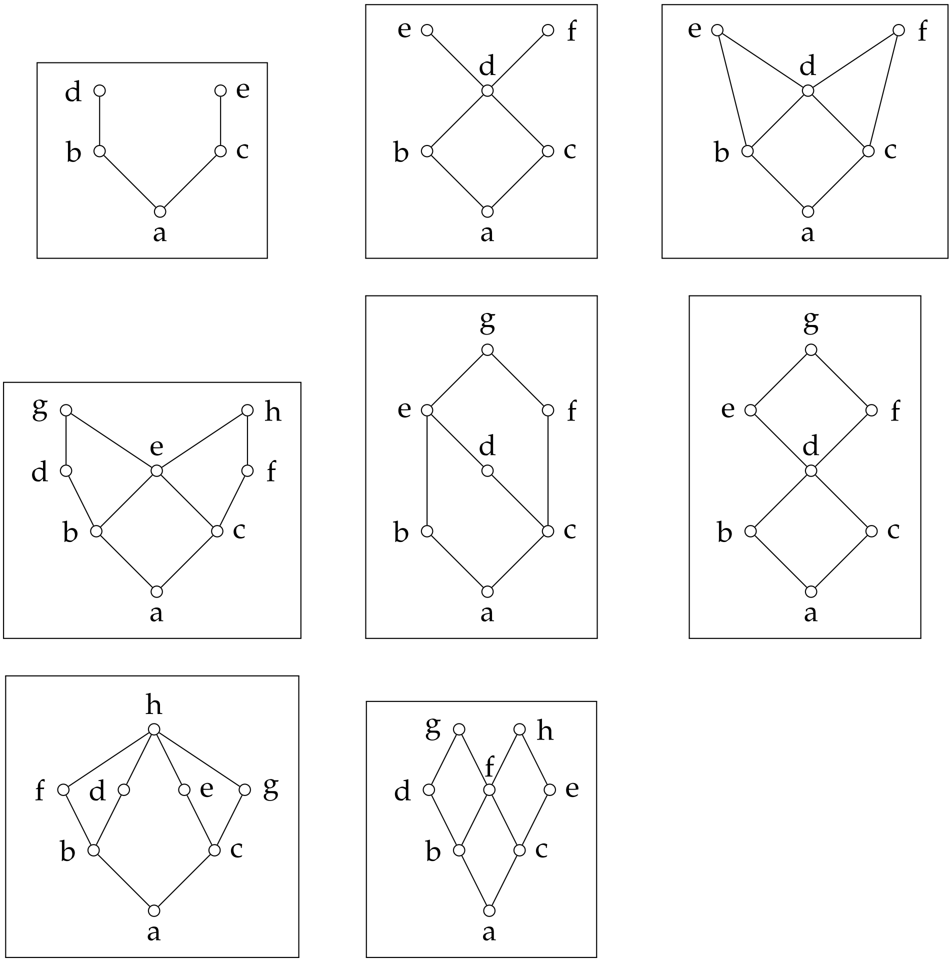


Table 1: List of Hasse diagrams.

C SOLUTIONS

Answer for exercise 1.4: The logical implication is:

$$(A_1 \Rightarrow A_2) \wedge (A_1 \Rightarrow A_3) \Rightarrow (A_2 \Rightarrow A_3)$$

For $A_1(d) = 0$, $A_2(d) = 1$, $A_3(d) = 0$, this implication is false. Therefore, the reasoning is incorrect.

Answer for exercise 1.11:
Let us denote: A – Arthur is a knight, B – Bine is a knight, C – Cene is a knight.
The following compound statement holds:

$$(A \iff \neg C \vee \neg B) \wedge (B \iff C \wedge A)$$

With the help of the truth table, we see that the statement is true only for the set $A = 1$ and $B = C = 0$.

Answer for exercise 1.12: Let A represent "A is a knight", etc. We seek the only solution d such that the following is true:

$$A_1 \wedge B_1 \wedge C_1 \wedge D_1$$

where:

$$\begin{aligned}
A_1 &: A \iff (\neg D \wedge \neg C) \\
B_1 &: B \iff (\neg A \wedge \neg D \Rightarrow \neg C) \\
C_1 &: C \iff (\neg B \Rightarrow A) \\
D_1 &: D \iff (\neg E \Rightarrow \neg C \wedge \neg B)
\end{aligned}$$

Since the truth table would contain 32 rows, we solve this by analyzing cases.

Case 1: $A(d) = 1$

Given A_1, $D(d) = 0$ and $C(d) = 0$. Substituting into C_1 with $A(d) = 1$ and $C(d) = 0$, we get:

$$\neg(\neg B \Rightarrow 1) \Rightarrow \neg(B \vee 1) \Rightarrow \neg 1$$

which is a false statement. Hence, this case is not possible.

Case 2: $A(d) = 0$

Given A_1, either $C(d) = 1$ or $D(d) = 1$.

Case 2.1: $C(d) = 1$

Since C_1, $\neg B \Rightarrow 0$, implies $\neg B = 0$, hence $B(d) = 1$. Substituting into B_1 with $A(d) = 0$, $B(d) = 1$, and $C(d) = 1$, we get:

$$1 \wedge \neg D \Rightarrow 0 \Rightarrow \neg D = 0$$

thus, $D(d) = 1$.
Substituting into D_1, we get:

$$\neg E \Rightarrow 0 \wedge 0$$

Hence $E(d) = 1$.

Case 2.2: $C(d) = 0$ **and** $D(d) = 1$

From B_1, we get $B(d) = 1$, but the statement C_1 becomes false: $0 \iff (0 \Rightarrow 1)$.
Thus, B, C, D, and E are knights, while A is a servant.

Answer for exercise 1.27:

$$\begin{aligned}
(A \Rightarrow B) \vee (B \Rightarrow C) &\Leftrightarrow (\neg A \vee B) \vee (\neg B \vee C) \\
&\Leftrightarrow \neg A \vee B \vee \neg B \vee C \\
&\Leftrightarrow \neg A \vee (B \vee \neg B) \vee C \\
&\Leftrightarrow \neg A \vee 1 \vee C \\
&\Leftrightarrow 1.
\end{aligned}$$

Answer for exercise 1.33: We need to show that $x + \frac{1}{x} \geq 2$. Since $x \geq 0$, we can multiply the inequality by x to get

$$x^2 + 1 \geq 2x$$

or equivalently,

$$(x-1)^2 \geq 0.$$

This is obviously always true.

Answer for exercise 1.34: Suppose there are only finitely many primes $p_1, p_2, \ldots, p_n$. Then the number $p = p_1 p_2 \cdots p_n + 1$ is not divisible by any prime p_i and $p_i \neq p$ for each i. By definition, p is therefore a prime number that is different from each of the previous ones. This is a contradiction.

Answer for exercise 1.35: The opposite statement is: there exists a natural number greater than 1.

Answer for exercise 1.37: We prove both directions separately:
($\Rightarrow$) Assume that m and n have different parities. Write $m = 2k$ and $n = 2l + 1$, substitute into the expression $m^2 - n^2$, and the result follows.
($\Leftarrow$) We show indirectly: If m and n have the same parity, then $m^2 - n^2$ is even. Consider both cases.

Answer for exercise 1.39:

(i) $(A \Rightarrow B) \wedge A \Rightarrow B$. This is true.

(ii) $(A \Rightarrow B) \wedge \neg A \Rightarrow \neg B$. This is not necessarily true.

(iii) $((A \Rightarrow B) \wedge \neg A) \Rightarrow \neg B$. This is not necessarily true.

(iv) $((A \Rightarrow B) \wedge (B \Rightarrow C)) \Rightarrow (\neg C \Rightarrow \neg A)$. This is true.

Answer for exercise 2.2:

(i) $P \cap S \neq \emptyset$

(ii) $0 \in \mathbb{Z} \setminus \mathbb{N}$

(iii) $\mathbb{N} \subseteq \mathbb{Z}$

(iv) $\mathbb{Z} \nsubseteq \mathbb{N}$

(v) $P \setminus \{2\} \subseteq \overline{S}$

(vi) $2 \in S \cap P$

Answer for exercise 2.4:

$$\begin{aligned} x \in (A \cup C) \cap (B \setminus C) &\Leftrightarrow (x \in A \vee x \in C) \wedge (x \in B \wedge x \notin C) \\ &\Leftrightarrow ((x \in A \vee x \in C) \wedge (x \notin C)) \wedge x \notin B \\ &\Leftrightarrow ((x \in A \wedge x \notin C) \vee (x \in C \wedge x \notin C)) \wedge x \in B \\ &\Leftrightarrow x \in A \wedge x \notin C \wedge x \in B \\ &\Leftrightarrow x \in A \wedge x \in B \wedge x \notin C \\ &\Leftrightarrow x \in (A \cap B) \setminus C. \end{aligned}$$

Answer for exercise 2.5: Use direct proof.

Answer for exercise 2.6: Prove separately each direction.

Answer for exercise 2.7:

$$\begin{aligned}(x,y)\in A\times(B\cap C) &\Leftrightarrow x\in A\wedge y\in B\cap C\\ &\Leftrightarrow x\in A\wedge y\in B\wedge y\in C\\ &\Leftrightarrow x\in A\wedge x\in A\wedge y\in B\wedge y\in C\\ &\Leftrightarrow x\in A\wedge y\in B\wedge x\in A\wedge y\in C\\ &\Leftrightarrow (x,y)\in A\times B\wedge(x,y)\in A\times C\\ &\Leftrightarrow (x,y)\in(A\times B)\cap(A\times C).\end{aligned}$$

Answer for exercise 2.8:

$$\begin{aligned}x\in(A\cap B)\setminus B &\Leftrightarrow x\in(A\cap B)\wedge x\notin B\\ &\Leftrightarrow (x\in A\wedge x\in B)\wedge x\notin B\\ &\Leftrightarrow x\in A\wedge(x\in B\wedge x\notin B)\\ &\Leftrightarrow x\in\emptyset.\end{aligned}$$

Answer for exercise 2.10: (In slovene.)

1. **Incorrect.** Take $A=\emptyset$, $B=\{\emptyset\}$, $C=\{\{\emptyset\}\}$.
2. **Incorrect.** Take the same example as in (a).
3. **Correct.** Proof by contradiction. Suppose the statement is not correct. Let $A\cap B\subseteq\overline{C}$, $A\cup C\subseteq B$ and let there exist $x\in A\cap C$. Thus, $x\in A$ and $x\in C$. Since $A\cup C\subseteq B$ by the second assumption, $x\in B$. It follows that $x\in A\cap B$. Since $A\cap B\subseteq\overline{C}$ by the first assumption, $x\in\overline{C}$. Contradiction, since $x\in C$.
4. **Incorrect.** Take $A=C\neq B$.
5. **Incorrect.** Take three pairwise disjoint non-empty sets.

Answer for exercise 2.21:

$\rightarrow$ Let $A \subseteq B$. We will show that $A \cap \overline{B} = \emptyset$ holds. By assumption, we have

$$(\forall x)(x \in A \Rightarrow x \in B). \tag{1}$$

Suppose that there exists $x \in A \cap \overline{B}$. Then

$$\begin{aligned} x \in A \cap \overline{B} &\Rightarrow x \in A \wedge x \in \overline{B} \\ &\Rightarrow x \in A \wedge x \notin B \\ &\Rightarrow x \in B \wedge x \notin B \quad \text{(due to (1))}, \end{aligned}$$

a contradiction. Therefore, $A \cap \overline{B} = \emptyset$.

$\leftarrow$ Let $A \cap \overline{B} = \emptyset$. We will show that $A \subseteq B$. Take any $x \in A$. Then $x \notin \overline{B}$, since $A \cap \overline{B} = \emptyset$. Thus, $x \in B$. Since x was arbitrary, it follows that $A \subseteq B$.

1.

$$\begin{aligned} A \setminus B &= \{x; x \in A \wedge x \notin B\} \\ &= \{x; x \notin \overline{A} \wedge x \in \overline{B}\} \\ &= \{x; x \in \overline{B} \wedge x \notin \overline{A}\} \\ &= \overline{B} \setminus \overline{A}. \end{aligned}$$

Answer for exercise 3.4:

1. Domain: $\{1,2,3\}$, Range: $\{2,3,4\}$
2. Domain: $\{1,2,3,4\}$, Range: $\{5,3,9\}$
3. Domain: $\{1,3,4\}$, Range: $\{2,5\}$
4. Domain: $\{-1,2,3\}$, Range: $\{-1,2,3\}$
5. Domain: $\{2,9\}$, Range: $\{0\}$

Answer for exercise 3.6: For $R_1 = \{(1,2),(2,3),(3,4)\}$ and $R_2 = \{(2,4),(3,5),(4,6)\}$:

$$R_2 \circ R_1 = \{(1,4),(2,5),(3,6)\}, \quad R_1 \circ R_2 = \{(2,4),(3,5)\}.$$

They are **not equal** because composition is not commutative.

Answer for exercise 3.7:

1. $R_2 \circ R_1 = \{(1,6), (2,4)\}$
2. $R_2 \circ R_1 = \{(1,4), (2,6), (3,4)\}$
3. $R_2 \circ R_1 = \{(4,5), (5,6), (6,4)\}$
4. $R_1 \circ R_1 = \{(4,4), (5,5), (6,6)\}$

Answer for exercise 3.8: $R_1 = \{(x,y) \mid y = x+1\}$, $R_2 = \{(y,z) \mid z = y+2\}$:

$$R_2 \circ R_1 = \{(x,z) \mid z = x+3,\ x,z \in \mathbb{Z}\}.$$

Answer for exercise 3.9: For $R = \{(1,2), (2,3), (3,4)\}$:

$$R \circ R = \{(1,3), (2,4)\}, \quad R \circ R \circ R = \{(1,4)\},$$

$$\text{while } R \circ R \circ R \circ R \circ R = \emptyset$$

Answer for exercise 3.14: To check transitivity: If $(x,y) \in R$ and $(y,z) \in R$, then $(x,z) \in R$. Here, $(1,2) \in R$ and $(2,3) \in R$, but $(1,3) \notin R$. Thus, R is **not transitive**.

Answer for exercise 3.45:

1. Using the definition of degree as the number of neighbors:
   - $deg(1) = 3$ (connected to 2, 3, 4)
   - $deg(2) = 2$ (connected to 1, 3)
   - $deg(3) = 3$ (connected to 1, 2, 4)
   - $deg(4) = 3$ (connected to 1, 3, 5)
   - $deg(5) = 1$ (connected to 4)
2. - $\Delta(G) = \max\{3,2,3,3,1\} = 3$.
   - $\delta(G) = \min\{3,2,3,3,1\} = 1$.
3. The sum of degrees is $3+2+3+3+1 = 12$. The number of edges $|E(G)|$ is 6. The property states $\sum deg(v) = 2|E(G)|$. Since $12 = 2(6)$, the property holds.

Answer for exercise 3.46:

1. Vertex removal removes the vertex and all incident edges. Removing vertex 1 removes edges $\{1,2\}$ and $\{1,4\}$. The remaining vertices are $\{2,3,4\}$ with edges $\{\{2,3\},\{3,4\}\}$. This is isomorphic to the Path graph P_3.

2. Edge removal removes only the edge, keeping vertices intact. Removing $\{1,2\}$ leaves vertices $\{1,2,3,4\}$ and edges $\{\{2,3\},\{3,4\},\{4,1\}\}$. This forms a path $2-3-4-1$ (which is P_4). Yes, it is still connected.

Answer for exercise 3.47:

1. A complete graph has all possible edges. For $n = 5$, size is $\binom{5}{2} = \frac{5\times 4}{2} = 10$.

2. - The clique number $\omega(G)$ is the size of the largest complete subgraph. Since K_5 is complete, $\omega(K_5) = 5$.
   - The independence number $\alpha(K_5)$ is the size of the largest empty induced subgraph. In a complete graph, no two vertices are non-adjacent, so $\alpha(K_5) = 1$.

Answer for exercise 3.48:

1. Yes. If we choose vertices $\{1,2,3\}$ from C_5, the induced edges are those in C_5 connecting these vertices: $\{1,2\}$ and $\{2,3\}$. The edge $\{1,3\}$ does not exist in C_5. Thus, the induced subgraph is $1-2-3$, which is P_3.

2. K_4 contains all possible edges between 4 vertices.
   - C_4 is a subgraph because we can select 4 vertices and remove the two "diagonal" edges.
   - C_4 is **not** an induced subgraph. If we induce on any 4 vertices in K_4, we must keep *all* edges between them. This yields K_4, not C_4.

Answer for exercise 3.49:

1. Both graphs have 6 vertices. Both are 2-regular (every vertex has degree 2).

2.
   - $g(G) = 6$ (the shortest cycle in C_6 is length 6).
   - $g(H) = 3$ (the shortest cycles are the triangles K_3).

3. No, they are not isomorphic. While they share the same number of vertices and degree sequence, their structural parameters differ. Specifically, $g(G) \neq g(H)$. Also, G is connected while H is disconnected.

Answer for exercise 3.50:

1. $deg(1) = 1$, $deg(2) = 2$, $deg(3) = 1$.

2. An automorphism must map a vertex to another vertex of the same degree.
   - Vertex 2 (unique degree 2) must map to itself: $\phi(2) = 2$.
   - Vertices 1 and 3 (degree 1) can be swapped or stay fixed.

   There are exactly 2 automorphisms:
   - Identity: $\{1 \to 1, 2 \to 2, 3 \to 3\}$
   - Flip: $\{1 \to 3, 2 \to 2, 3 \to 1\}$

Answer for exercise 3.51:

1. A graph with $\Delta(G) = 2$ consists of paths and cycles. To have girth 4, it must contain a cycle of length 4. The graph C_4 satisfies this: every vertex has degree 2, and the cycle length is 4.

2. By definition, a tree contains no cycles. Therefore, the set of cycle lengths is empty. Depending on convention, $g(Tree) = \infty$ or is undefined.

Answer for exercise 3.4: Let $V(G) = \{v_1, ..., v_n\}$. The possible values for the degree of any vertex are integers in the set $\{0, 1, 2, ..., n-1\}$.
However, it is impossible for a simple graph to contain both a vertex of degree 0 (connected to no one) and a vertex of degree $n-1$ (connected to everyone else).

- If there is a vertex of degree $n-1$, no vertex can have degree 0. The possible degree values are $\{1, 2, ..., n-1\}$.
- If there is a vertex of degree 0, no vertex can have degree $n-1$. The possible degree values are $\{0, 1, ..., n-2\}$.

In either case, the vertices can take values from a set of size $n-1$. Since there are n vertices (pigeons) and only $n-1$ possible degree values (holes), by the Pigeonhole Principle, at least two vertices must share the same degree.

Answer for exercise 3.4: Consider a path of maximum length in G, denoted $P = x_0, x_1, ..., x_k$. Since the path is maximal, x_k cannot be adjacent to any vertex outside the path (otherwise we could extend the path).
However, we know $deg(x_k) \geq 2$. Therefore, x_k must have at least two neighbors. Since these neighbors cannot be outside the path, they must be vertices already in the path (e.g., x_i where $0 \leq i < k-1$).
If x_k is connected to some x_i in the path, the edges (x_k, x_i) combined with the path segment $x_i, ..., x_k$ form a cycle. Thus, G contains a cycle.

Answer for exercise 3.4:

1. To maximize edges without creating a triangle, we should divide vertices into two sets and connect all vertices between the sets, but none within the sets (a complete bipartite graph). For $n = 5$, we split vertices into sets of size 2 and 3. The graph $K_{2,3}$ has $2 \times 3 = 6$ edges. (Note: C_5 has only 5 edges).

2. A bipartite graph divides $V(G)$ into two disjoint sets A and B where all edges go from A to B. A triangle requires 3 vertices connected in a cycle $u - v - w - u$. In a bipartite graph, a path of length 3 ($u \in A \to v \in B \to w \in A$) ends back in set A. For w to connect to u, there would need to be an edge within set A, which is forbidden. Thus, no odd cycles (including triangles) exist.

Answer for exercise 3.4:

1. **P_4 $(1 - 2 - 3 - 4)$:** The endpoints 1 and 4 have degree 1; the centers 2 and 3 have degree 2. An automorphism must map endpoints to endpoints. We can either fix everyone (Identity) or flip the whole path ($1 \leftrightarrow 4, 2 \leftrightarrow 3$). Total: 2 automorphisms.

2. **C_4 $(1 - 2 - 3 - 4 - 1)$:** Every vertex has degree 2, so we have more freedom. We can rotate the square $(0^\circ, 90^\circ, 180^\circ, 270^\circ)$—giving 4 rotations. We can also reflect it (flip horizontally, vertically, or diagonally). The automorphism group is the Dihedral group D_4, which has size $2n = 2(4) = 8$.

3. Adding the edge $(1, 4)$ to P_4 makes all vertices indistinguishable by degree (all become degree 2). This structural homogeneity allows any vertex to be mapped to any other vertex (vertex-transitivity), drastically increasing the number of symmetries from 2 to 8.

Answer for exercise 3.60:

1. Let $\mathcal{R}$ be a partial order on a set A. By definition, $\mathcal{R}$ is reflexive, anti-symmetric, and transitive. The definition of a preorder requires only that a relation be reflexive and transitive. Since $\mathcal{R}$ satisfies these two properties, $\mathcal{R}$ is a preorder.

2. Let $A = \{a, b\}$ and let $\mathcal{R} = \{(a, a), (b, b), (a, b), (b, a)\}$.
   - **Reflexivity:** $(a, a) \in \mathcal{R}$ and $(b, b) \in \mathcal{R}$, so it is reflexive.

- **Transitivity:** The only pairs that could fail transitivity are (a, b) and (b, a), which imply $(a, a) \in \mathcal{R}$ (satisfied); and (b, a) and (a, b), which imply $(b, b) \in \mathcal{R}$ (satisfied). So, it is transitive.
- **Anti-symmetry:** $(a, b) \in \mathcal{R}$ and $(b, a) \in \mathcal{R}$, but $a \neq b$. Thus, $\mathcal{R}$ is NOT anti-symmetric.

Since $\mathcal{R}$ is reflexive and transitive, it is a preorder. Since it is not anti-symmetric, it is not a partial order.

Answer for exercise 3.61:

1. The standard additional property is **irreflexivity** (or anti-reflexivity).

2. Proof:

   ($\Rightarrow$) Let $\mathcal{R}$ be a strict partial order. By definition, a strict partial order is an irreflexive and transitive relation. (The property of anti-symmetry is redundant given irreflexivity and transitivity: if $(a, b) \in \mathcal{R}$ and $(b, a) \in \mathcal{R}$, then by transitivity, $(a, a) \in \mathcal{R}$. But since $\mathcal{R}$ is irreflexive, this is impossible. Thus, we cannot have both (a, b) and (b, a), which vacuously satisfies anti-symmetry.)

   ($\Leftarrow$) Let $\mathcal{R}$ be a transitive and irreflexive relation. This is the standard definition of a strict partial order, so $\mathcal{R}$ is a strict partial order.

   (Alternatively, if we define a strict partial order as an irreflexive, anti-symmetric, and transitive relation, we only need to show anti-symmetry follows from the other two, as noted above).

Answer for exercise 3.62: Let $\mathcal{R}$ be a relation on A.

- **From Linear Order $\mathcal{R}$ to Strict Linear Order $\mathcal{R}^-$:** Define $\mathcal{R}^- = \mathcal{R} \setminus \{(a, a) \mid a \in A\}$. If $\mathcal{R}$ is a linear order, it is reflexive, anti-symmetric, transitive, and total. $\mathcal{R}^-$ is clearly irreflexive by construction. It is transitive because $\mathcal{R}$ is transitive, and removing the reflexive pairs does not create any new transitivity failures. $\mathcal{R}^-$ is total for distinct elements: since $\mathcal{R}$ is total, for $a \neq b$, we have $(a, b) \in \mathcal{R}$ or $(b, a) \in \mathcal{R}$. Since $a \neq b$, these are not reflexive pairs, so $(a, b) \in \mathcal{R}^-$ or $(b, a) \in \mathcal{R}^-$. Thus, $\mathcal{R}^-$ is a strict linear order.

- **From Strict Linear Order $\mathcal{S}$ to Linear Order $\mathcal{S}^+$:** Define $\mathcal{S}^+ = \mathcal{S} \cup \{(a, a) \mid a \in A\}$. If $\mathcal{S}$ is a strict linear order, it is irreflexive, transitive, and total for distinct elements. $\mathcal{S}^+$ is clearly reflexive by construction. It is transitive: any transitivity failure must involve at least one new reflexive pair, say $(a, a) \in \mathcal{S}^+$ and $(a, b) \in \mathcal{S}^+$. This implies $(a, b) \in \mathcal{S}$, and thus $(a, b) \in \mathcal{S}^+$. Similarly for $(b, a) \in \mathcal{S}^+$ and $(a, a) \in \mathcal{S}^+$. It is anti-symmetric: if $(a, b) \in \mathcal{S}^+$ and $(b, a) \in \mathcal{S}^+$, by irreflexivity of $\mathcal{S}$, this is only possible if $a = b$. It is total: for any a, b, if $a = b$, $(a, a) \in \mathcal{S}^+$. If $a \neq b$, then $(a, b) \in \mathcal{S}$ or $(b, a) \in \mathcal{S}$, which means $(a, b) \in \mathcal{S}^+$ or $(b, a) \in \mathcal{S}^+$. Thus, $\mathcal{S}^+$ is a linear order.

Since $(\mathcal{R}^-)^+ = \mathcal{R}$ and $(\mathcal{S}^+)^- = \mathcal{S}$, the maps are inverses, establishing a one-to-one correspondence.