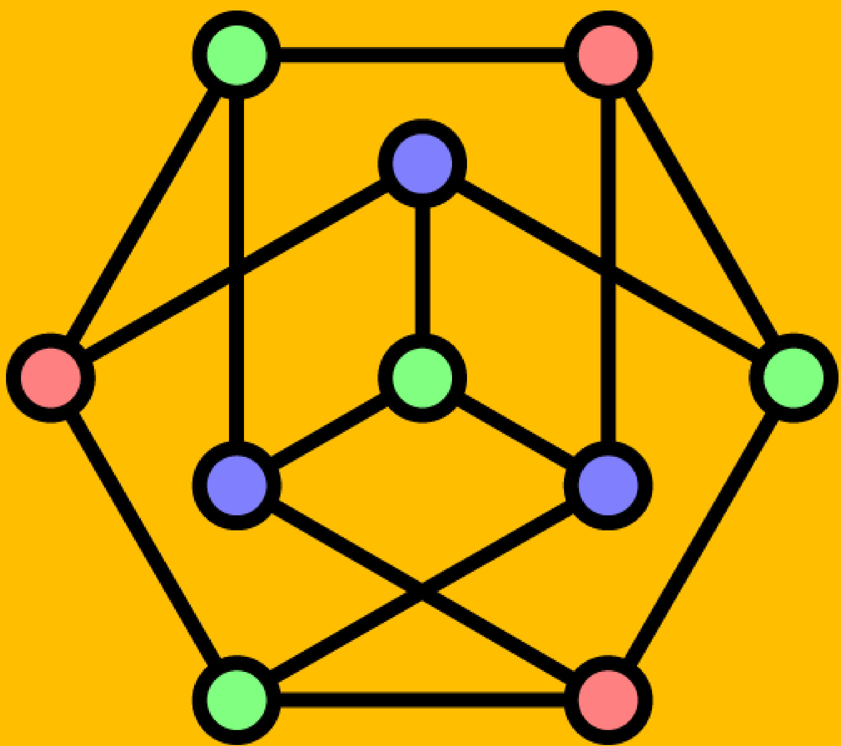
THEORETICAL COMPUTER SCIENCE

DISCRETE STRUCTURES FOR COMPUTER SCIENCE STUDENTS



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PREFACE

Those notes are supposed to be parsed together with explanations from the lectures. Any questions or found errors should be addressed to `matjaz.krnc@upr.si`, or raised as an issue in our public repository

`https://github.com/mkrnc/TOR1-vaje---TCS1-exercises.git`.

Among most notable student contributors are:

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CONTENTS

1 MATHEMATICAL LOGIC

1.1 BASIC EXERCISES

Problem 1. The following two propositions are given: A: "It is cold outside." B: "It is raining outside."

Write the following compound propositions in natural language:

- (a) $\neg A$
- (b) $A \wedge B$
- (c) $A \vee B$
- (d) $B \vee \neg A$

Problem 2. The following two propositions are given: A: "Janez is rich." B: "Janez is happy."

Write the following propositions symbolically:

- (a) If Janez is rich, then he is unhappy.
- (b) Janez is neither happy nor rich.
- (c) Janez is happy only if he is poor.
- (d) Janez is poor if and only if he is unhappy.

Problem 3. Find the truth tables for the examples from the previous task.

Problem 4. Is the following reasoning correct?

- Premise 1: "I think, therefore I am."
- Premise 2: "I think, therefore I reason."
- Conclusion: "I am, therefore I reason."

Solution: The logical implication is:

$$(A_1 \Rightarrow A_2) \wedge (A_1 \Rightarrow A_3) \Rightarrow (A_2 \Rightarrow A_3)$$

For $A_1(d) = 0$, $A_2(d) = 1$, $A_3(d) = 0$, this implication is false. Therefore, the reasoning is incorrect.

Problem 5. The following two propositions are given: A: "Andrej speaks French." and B: "Andrej speaks Danish." Write the following compound propositions in natural language:

(a) $A \ \vee \ B$

(b) $A \ \wedge \ B$

(c) $A \ \wedge \ \neg B$

(d) $\neg A \ \vee \ \neg B$

(e) $\neg\neg A$

(f) $\neg(\neg A \ \wedge \ \neg B)$

Problem 6. Given the propositions:
A : "John reads The New York Times."
B : "John reads The Wall Street Journal."
C : "John reads The Daily Mail."

Transcribe the following statements into symbolic propositions:

a) John reads The New York Times, but not The Wall Street Journal.

b) Either John reads both The New York Times and The Wall Street Journal, or he does not read The New York Times and The Wall Street Journal.

c) It is not true that John reads The New York Times, and does not read The Daily Mail.

d) It is not true that John reads The Daily Mail or The Wall Street Journal, and not The New York Times.

Problem 7. Find the truth tables for the symbolic propositions from Exercise 6.

Problem 8. For three lines p, q, r we may construct also geometric propositions. Suppose that the following is true:

$$(p||q) \wedge (p \cap q \neq \emptyset) \wedge (q \cap r \neq \emptyset).$$

What can you say about the lines p, q, r?

1.2 KNIGHTS AND SERVANTS (KNEVES)

Knights always tell the truth, while servants always lie.

Problem 1. Artur: "It is not true that Cene is a servant."

Bine: "Cene is a knight or I am a knight."

Cene: "Bine is a servant."

For each of them, determine whether they are knights or servants!

Solution: Let us denote: A – Arthur is a knight, B – Bine is a knight, C – Cene is a knight.

(i) The following compound statement holds:

$$(A \iff C) \wedge (B \iff C \vee B) \wedge (C \iff \neg B)$$

With the help of the truth table, we see that the statement is true only for the set $A = 1$ and $B = C = 0$.

Problem 2. Artur: "Cene is a servant or Bine is a servant." Bine: "Cene is a knight and Artur is a knight."

For each of them, determine whether they are knights or servants!

Solution: Let us denote: A – Arthur is a knight, B – Bine is a knight, C – Cene is a knight.

The following compound statement holds:

$$(A \iff \neg C \vee \neg B) \wedge (B \iff C \wedge A)$$

With the help of the truth table, we see that the statement is true only for the set $A = 1$ and $B = C = 0$.

Problem 3. Let us analyze the statements made by A, B, C, and D:

- A: "D is a servant and C is a servant."
- B: "If A and D are servants, then C is a servant."
- C: "If B is a servant, then A is a knight."
- D: "If E is a servant, then both C and B are servants."

Solution: Let A represent "A is a knight", etc. We seek the only solution d such that the following is true:

$$A_1 \wedge B_1 \wedge C_1 \wedge D_1$$

where:

$$\begin{gathered} A_1 : A \iff (\neg D \wedge \neg C) \\ B_1 : B \iff (\neg A \wedge \neg D \Rightarrow \neg C) \\ C_1 : C \iff (\neg B \Rightarrow A) \\ D_1 : D \iff (\neg E \Rightarrow \neg C \wedge \neg B) \end{gathered}$$

Since the truth table would contain 32 rows, we solve this by analyzing cases.

Case 1: $A(d) = 1$

Given A_1, $D(d) = 0$ and $C(d) = 0$. Substituting into C_1 with $A(d) = 1$ and $C(d) = 0$, we get:

$$\neg(\neg B \Rightarrow 1) \Rightarrow \neg(B \vee 1) \Rightarrow \neg 1$$

which is a false statement. Hence, this case is not possible.

Case 2: $A(d) = 0$

Given A_1, either $C(d) = 1$ or $D(d) = 1$.

Case 2.1: $C(d) = 1$

Since C_1, $\neg B \Rightarrow 0$, implies $\neg B = 0$, hence $B(d) = 1$.

Substituting into B_1 with $A(d) = 0$, $B(d) = 1$, and $C(d) = 1$, we get:

$$1 \wedge \neg D \Rightarrow 0 \Rightarrow \neg D = 0$$

thus, $D(d) = 1$.

Substituting into D_1, we get:

$$\neg E \Rightarrow 0 \wedge 0$$

Hence $E(d) = 1$.

Case 2.2: $C(d) = 0$ **and** $D(d) = 1$

From B_1, we get $B(d) = 1$, but the statement C_1 becomes false: $0 \iff (0 \Rightarrow 1)$.

Thus, B, C, D, and E are knights, while A is a servant.

Problem 4. Solve the following exercises about knights and servants:

- Arthur: “It is not true that Bine is a servant."
- Bine: “We are not both of the same kind.”

Problem 5. Solve the following exercises about knights and servants:

- Arthur: “It is not true that Cene is servant."
- Bine: “Cene is a knight or I am a knight."
- Cene: “Bine is a servant."

Problem 6. Now Arthur and Bine say the following:

- Arthur: "Me and Bine are not of the same kind."
- Bine: "Exactly one of us is a knight."

Problem 7. Knights and servants! For both cases below (separately) determine the roles.

a) Arthur: It's not true that Chloe is a servant.
   Bob: Chloe is a knight, or I am a knight.
   Chloe: Bob is a servant.

b) Arthur: Chloe or Bob are servants.
   Bob: Cene and Arthur are knights.

Problem 8. Express the propositions below with connectives $\wedge$ and $\neg$ only!

a) $A \vee B$

b) $A \Rightarrow B$

c) $A \Leftrightarrow B$

1.3 OTHER EXERCISES

(possibly in Slovene)

1. (Sklepanje) Ali je naslednje sklepanje pravilno?

   Dojenčki se obnašajo nelogično. Kdor je sposoben ukrotiti krokodila, je spoštovanja vreden. Kdor se obnaša nelogično, ni spoštovanja vreden. Sklep: Dojenčki niso sposobni ukrotiti krokodila.

   Rešitev:

   A_1: Sem dojenček.

   A_2: Obnašam se nelogično.

   A_3: Sposoben sem ukrotiti krokodila.

   A_4: Vreden sem spoštovanja.

   $(A_1 \Rightarrow A_2) \wedge (A_3 \Rightarrow A_4) \wedge (A_2 \Rightarrow \neg A_4) \Rightarrow (A_1 \Rightarrow \neg A_3)$

   Pa recimo, da je sklep napačen. Tedaj obstaja določilo d, da velja

(1) $(A_1(d) \Rightarrow \neg A_3(d)) = 0$

(2) $(A_1(d) \Rightarrow A_2(d)) = 1$

(3) $(A_3(d) \Rightarrow A_4(d)) = 1$

(4) $(A_2(d) \Rightarrow \neg A_4(d)) = 1$

Torej je, zaradi (1), $A_1(d) = 1$ in $A_3(d) = 1$. Zaradi (2) je $A_2(d) = 1$. Zaradi (4) je $A_4(d) = 0$. To pa je protislovje s (3).

Torej je sklepanje pravilno. □

Rešitev domače naloge:

Dejstva:

Barona je umoril eden izmed njegovega osebja: kuharica, strežnik ali šofer.

Če je morilka kuharica, je zastrupila hrano.

Če je morilec šofer, mu je postavil bombo v avto.

Hrana ni bila zastrupljena in strežnik ni morilec.

Sklep: Morilec je šofer.

Rešitev:

K: Morilka je kuharica.

S: Morilec je strežnik.

Š: Morilec je šofer.

H: Kuharica je zastrupila hrano.

B: Šofer je postavil bombo v avto.

Ali je naslednja implikacija tavtologija?

$(K \vee S \vee \text{Š}) \wedge (K \Rightarrow H) \wedge (\text{Š} \Rightarrow B) \wedge (\neg H \wedge \neg S) \Rightarrow \text{Š}$

Recimo, da ni.

(1) $(K \vee S \vee \text{Š})(d) = 1$

(2) $(K \Rightarrow H)(d) = 1$

(3) (Š $\Rightarrow$ $B)(d) = 1$

(4) $(\neg H \ \wedge \ \neg S)(d) = 1$

(5) $\check{S}(d) = 0$

Iz (4) sledi:

(5) $H(d) = S(d) = 0.$

Iz (2) in (5) potem sledi

(6) $K(d) = 0.$

Iz (1), (5) in (6) potem sledi $\check{S}(d) = 1$. Protislovje. Izjava je tavtologija in sklepanje je pravilno.

2. Find the canonical disjunctive normal form (DNF) and the canonical conjuctive normal form (CNF) for the following propositions:

   (i) $\neg(A \wedge B) \Rightarrow (\neg B \Rightarrow A)$

   (ii) $\neg(A \vee B) \wedge (A \Rightarrow B)$

   Rešitev. (i) Napiši pravilnostno tabelo. DNO: vzemi vrstice z enicami (poveži jih med sabo s konjunkcijo) in jih poveži med sabo z disjunkcijo $(A \wedge B) \vee (A \wedge \neg B) \vee (\neg A \wedge B)$. KNO: vzemi vrstice z ničlami (vzemi nasprotne vrednosti in jih poveži med sabo z disjunkcijo) in jih poveži med sabo s konjunkcijo $(A \vee B)$. (ii) Podobno.

3. For the following compound proposition find a truth table, determine DNF, CNF and draw the corresponding circuit.

$$(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)).$$

4. Find a compound proposition $\mathcal{I}$ such that

$$(A \Rightarrow (\mathcal{I} \Rightarrow \neg B)) \Rightarrow (A \wedge B) \vee \mathcal{I}$$

   is tautology.

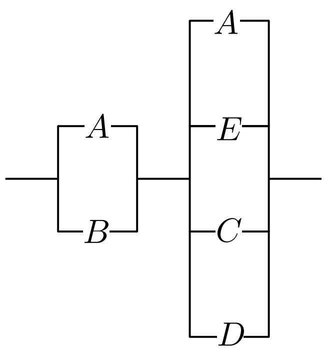
   Rešitev. Napiši pravilnostno tabelo za osnovni izjave A, B skupaj s (sestavljeno) izjavo $\mathcal{I}$. Iz nje razberi, da je pravilnostna tabela za $\mathcal{I}$ enaka

   Torej je $\mathcal{I} \Leftrightarrow \neg A \vee \neg B$ v KNO.

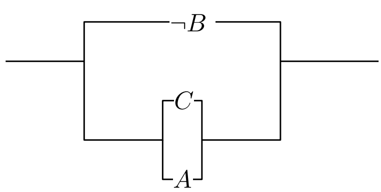
A	B	$\mathcal{I}$
1	1	0
1	0	1
0	1	1
0	0	1

5. For the following circuits find the corresponding compound propositions

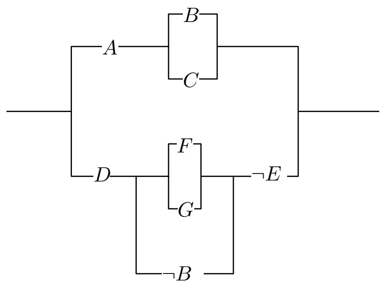
(i)



(ii)



(iii)



6. Simplify the following logical equivalence
$$(A \Rightarrow B) \vee (B \Rightarrow C).$$

Rešitev.
$$\begin{aligned}
(A \Rightarrow B) \vee (B \Rightarrow C) &\Leftrightarrow (\neg A \vee B) \vee (\neg B \vee C) \\
&\Leftrightarrow \neg A \vee B \vee \neg B \vee C \\
&\Leftrightarrow \neg A \vee (B \vee \neg B) \vee C \\
&\Leftrightarrow \neg A \vee 1 \vee C \\
&\Leftrightarrow 1.
\end{aligned}$$

7. Show that the following propositions are logical implications (a tautology where the main connective is implication).
   (i) $A \wedge (A \Rightarrow B) \Rightarrow B$
   (ii) $\neg B \wedge (A \Rightarrow B) \Rightarrow \neg A$
   (iii) $\neg A \wedge (A \vee B) \Rightarrow B$
   (iv) $(A \Rightarrow B) \wedge (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$
   (v) $A \wedge (A \Leftrightarrow B) \Rightarrow B$

   Rešitev. (i) Recimo $A \wedge (A \Rightarrow B)$ pravilna, B pa nepravilna. Potem je A pravilna in $A \Rightarrow B$ pravilna. Sledi B pravilna. Protislovje.

8. Are the following propositions logical implications?
   (i) $(A \Rightarrow B) \wedge (A \Rightarrow C) \wedge A \Rightarrow B \wedge C$
   (ii) $\neg(A \vee B) \wedge (A \vee C) \wedge (D \Rightarrow C) \Rightarrow D$

(iii) $(A \Rightarrow B) \wedge (A \Rightarrow C) \wedge (D \wedge E \Rightarrow F) \wedge (C \Rightarrow E) \Rightarrow F$

9. With a direct proof show:

   If n is even, then so is $n^2 + 3n$.

   Is the converse also true?

10. Z direktnim dokazom implikacije pokaži: Če je realno število x nenegtivno, potem je vsota števila x in njegove obratne vrednosti večja ali enaka 2.

    Rešitev. Pokažimo $x + \frac{1}{x} \geq 2$. Ker $x \geq 0$, pomnožimo neenakost z x in dobimo $x^2 + 1 \geq 2x$ oziroma $(x-1)^2 \geq 0$. Slednje je očitno vedno res.

11. S protislovjem pokaži, da je praštevil neskončno.

    Rešitev. Recimo, da jih je končno mnogo $p_1, p_2, \ldots, p_n$. Potem $p = p_1 p_2 \cdots p_n + 1$ ni deljivo z nobenim praštevilom p_i in $p_i \neq p$ za vsak i. Po definiciji je torej p praštevilo, ki ni enako nobenemu prejšnjemu. Protislovje.

12. Poišči napako v naslednjem dokazu.

    Trditev: 1 je največje naravno število.

    Dokaz (s protislovjem): Predpostavimo nasprotno. Naj bo $n > 1$ največje naravno število. Ker je n pozitivno, lahko neenakost $n > 1$ pomnožimo z n. Torej $n > 1 \Leftrightarrow n^2 > n$. Dobili smo, da je n^2 večje od n, kar je v protislovju s predpostavko, da je n največje naravno število. Torej je bila predpostavka napačna in je 1 največje naravno število.

    Rešitev. Nasporotna trdtev je: obstaja naravno število, ki je večje od 1.

13. Let x and y be real numbers such that $x < 2y$. By an indirect proof show:

    If $7xy \leq 3x^2 + 2y^2$, then $3x \leq y$.

    Solution (in slovene). Naj bo $x < 2y$, to je, $2y - x > 0$. Pokazali bomo: če je $3x > y$, potem je $7xy > 3x^2 + 2y^2$. Predpostavimo torej, da je

$3x - y > 0$. Potem je $(2y - x)(3x - y) = 7xy - 3x^2 - 2y^2 > 0$, to je, $7xy > 3x^2 + 2y^2$.

14. Dokaži naslednjo ekvivalenco v dveh delih: Naj bosta m in n celi števili. Tedaj sta števili m in n različnih parnosti natanko tedaj, ko je število $m^2 - n^2$ liho.

   Rešitev. ($\Rightarrow$) Predpostavimo, da sta različnih parnosti. Pišimo $m = 2k$ in $n = 2l + 1$, vstavimo v izraz $m^2 - n^2$ in rezultat sledi.

   ($\Leftarrow$) Pokažemo indirektno in sicer: Če sta m in n iste parnosti, potem je $m^2 - n^2$ sodo. Obravnavaj oba primera.

15. Z uporabo če in samo če dokaza pokaži: $ac \mid bc \Leftrightarrow a \mid b$.

16. Ali je naslednji sklep pravilen?

   (i) Če je danes sreda bom imel vaje. Danes je sreda. Sklep: Imel bom vaje.

   Rešitev. $(A \Rightarrow B) \wedge A \Rightarrow B$. Res je.

   (ii) Če se učim, bom opravil izpit. Nisem se učil. Sklep: Ne bom opravil izpita.

   Rešitev. $(A \Rightarrow B) \wedge \neg A \Rightarrow \neg B$. Ni nujno res.

17. Ali je naslednji premislek pravilen?

   (i) Študent se je z mestni avtobusom odpravil na izpit. Rekel si je: Če bo na naslednjem semaforju zelena luč, bom naredil izpit. No, ko je avtobus pripeljal na naslednji semafor, na semaforju ni svetila zelena luč, študent pa si je dejal: Presneto, spet bom padel.

   Rešitev. $((A \Rightarrow B) \wedge \neg A) \Rightarrow \neg B$. Ni nujno res.

   (ii) Inženir, ki obvlada teorijo, vedno načrta dobro vezje. Dobro vezje je ekonomično. Torej, inženir, ki načrta neekonomično vezje, ne obvlada teorije.

   Rešitev. $((A \Rightarrow B) \wedge (B \Rightarrow C)) \Rightarrow (\neg C \Rightarrow \neg A)$. Res je.

18. Which of the following propositions are correct where the language of the conversation are real numbers?

(i) $(\forall x)(\exists y)(x + y = 0)$.

(ii) $(\exists x)(\forall y)(x + y = 0)$.

(iii) $(\exists x)(\exists y)(x^2 + y^2 = -1)$.

(iv) $(\forall x)[x > 0 \Rightarrow (\exists y)(y < 0 \wedge xy > 0)]$.

2 SET THEORY

1. Let $A = \{x \in \mathbb{N}; x < 7\}, B = \{x \in \mathbb{Z}; |x - 2| < 4\}$ and $C = \{x \in \mathbb{R}; x^3 - 4x = 0\}$.
   (i) Write down the elements for all three sets.
   (ii) Find $A \cup C, B \cap C, B \setminus C, (A \setminus B) \setminus C$ and $A \setminus (B \setminus C)$.

2. Let $\mathbb{Z}$ be a universal set and let P denote the set of all prime numbers, and S the set of all even integers. Write the following propositions in terms of set theory:
   (i) There exists an even prime number. $[P \cap S \neq \emptyset]$
   (ii) 0 is an integer, but it is not natural number. $[0 \in \mathbb{Z} \setminus \mathbb{N}]$
   (iii) Every natural number is an integer. $[\mathbb{N} \subseteq \mathbb{Z}]$
   (iv) Not every integer is a natural number. $[\mathbb{Z} \nsubseteq \mathbb{N}]$
   (v) Every prime number except 2 is odd. $[P \setminus \{2\} \subseteq \overline{S}]$
   (vi) 2 is an even prime number. $[2 \in S \cap P]$

3. Let A, B, C and D be subsets of some universal set U. Simplify the following expression
$$\overline{(\overline{(A \cup B)} \cap \overline{(\overline{A} \cup C)})} \setminus \overline{D}.$$

4. Show that $(A \cup C) \cap (B \setminus C) = (A \cap B) \setminus C$.

   Rešitev.
$$\begin{aligned} x \in (A \cup C) \cap (B \setminus C) \quad &\Leftrightarrow \quad (x \in A \vee x \in C) \wedge (x \in B \wedge x \notin C) \\ &\Leftrightarrow \quad ((x \in A \vee x \in C) \wedge (x \notin C)) \wedge x \notin B \\ &\Leftrightarrow \quad ((x \in A \wedge x \notin C) \vee (x \in C \wedge x \notin C)) \wedge x \in B \\ &\Leftrightarrow \quad x \in A \wedge x \notin C \wedge x \in B \\ &\Leftrightarrow \quad x \in A \wedge x \in B \wedge x \notin C \\ &\Leftrightarrow \quad x \in (A \cap B) \setminus C. \end{aligned}$$

5. (Zadnja lastnost pri uniji) Prove that $A \subseteq C \land B \subseteq C \Rightarrow A \cup B \subseteq C$.

   Rešitev. Direktno.

6. (Predzadnja lastnost pri preseku) Prove that $A \subseteq B \Leftrightarrow A \cap B = A$.

   Rešitev. V dveh delih.

7. (Predzadnja lastnost pri kartezičnemu produktu) Prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

   Rešitev.

$$
\begin{aligned}
(x,y) \in A \times (B \cap C) \quad &\Leftrightarrow \quad x \in A \land y \in B \cap C \\
&\Leftrightarrow \quad x \in A \land y \in B \land y \in C \\
&\Leftrightarrow \quad x \in A \land x \in A \land y \in B \land y \in C \\
&\Leftrightarrow \quad x \in A \land y \in B \land x \in A \land y \in C \\
&\Leftrightarrow \quad (x,y) \in A \times B \land (x,y) \in A \times C \\
&\Leftrightarrow \quad (x,y) \in (A \times B) \cap (A \times C).
\end{aligned}
$$

8. (Predzadnja lastnost pri razliki) Prove that $(A \cap B) \setminus B = \emptyset$. *Rešitev.*

$$
\begin{aligned}
x \in (A \cap B) \setminus B \quad &\Leftrightarrow \quad x \in (A \cap B) \land x \notin B \\
&\Leftrightarrow \quad (x \in A \land x \in B) \land x \notin B \\
&\Leftrightarrow \quad x \in A \land (x \in B \land x \notin B) \\
&\Leftrightarrow \quad x \in \emptyset.
\end{aligned}
$$

9. Determine the following sets:
   (i) $\{\emptyset, \{\emptyset\}\} \setminus \emptyset \quad [\{\emptyset, \{\emptyset\}\}]$
   (ii) $\{\emptyset, \{\emptyset\}\} \setminus \{\emptyset\}$
   (iii) $\{\emptyset, \{\emptyset\}\} \setminus \{\}\emptyset\}\}$
   (iv) $\{1, 2, 3, \{1\}, \{5\}\} \setminus \{2, \{3\}, 5\}$

10. Which of the following propositions are correct for arbitrary sets A, B and C:
    a) If $A \in B$ and $B \in C$, then $A \in C$.
    b) If $A \subseteq B$ and $B \in C$, then $A \in C$.

c) If $A \cap B \subseteq \overline{C}$ and $A \cup C \subseteq B$, then $A \cap C = \emptyset$.

d) If $A \neq B$ and $B \neq C$, then $A \neq C$.

e) If $A \subseteq \overline{(B \cup C)}$ and $B \subseteq \overline{(A \cup C)}$, then $B = \emptyset$.

Rešitev.

a) Napačna. Vzemi $A = \emptyset$, $B = \{\emptyset\}$, $B = \{\{\emptyset\}\}$.

b) Napačna. Vzemi isti primer kot v (a).

c) Pravilna. Dokaz s protislovjem. Recimo, da trditev ni pravilna. Naj bo $A \cap B \subseteq \overline{C}$, $A \cup C \subseteq B$ in naj obstaja $x \in A \cap C$. Torej je $x \in A$ in $x \in C$. Ker je po drugi predpostavki $A \cup C \subseteq B$, je $x \in B$. Sledi $x \in A \cap B$. Ker je po prvi predpostavki $A \cap B \subseteq \overline{C}$, je $x \in \overline{C}$. Protislovje, saj $x \in C$.

d) Napačna. Vzemi $A = C \neq B$.

e) Napačna. Vzemi tri paroma disjunktne neprazne množice.

11. Find $\mathcal{P}(A)$, where $A = \{a, b, c, d\}$.

12. Let $A = \{\{1, 2, 3\}, \{4, 5\}, \{6, 7, 8\}\}$.

    (i) Write down the elements of A.

    (ii) Is it true?
    (a) $1 \in A$ (b) $\{1, 2, 3\} \subseteq A$ (c) $\{6, 7, 8\} \in A$ (d) $\{\{4, 5\}\} \subseteq A$
    (e) $\emptyset \in A$ (f) $\emptyset \subseteq A$

13. Show that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

14. Let A, B in C be arbitrary subsets of the universal set $U = A \cup B \cup C$. Show the following propositions:

    a) $A \setminus B \subseteq \overline{B}$.

    b) $(A \setminus B) \cap B = \emptyset$.

    c) $A \cap B \subseteq C \Leftrightarrow A \subseteq \overline{B} \cup C$.

    d) $(A \setminus B) \cup B = A \Leftrightarrow B \subseteq A$.

    e) If $B \subseteq A$, then$B \times B = (B \times A) \cap (A \times B)$.

f) Let A be a nonempty set. Which of the following sets

$$\emptyset, \{\emptyset\}, A, \{A\}, \{A, \emptyset\}$$

are elements and which are subsets of (i) $\mathcal{P}(A)$ and (ii) $\mathcal{P}(\mathcal{P}(A))$?

g) Is it true that $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$?

3 RELATIONS

1. Let $S = \{1, 2, 3, 4, 5\}$.
   a) Is $R = \{(1, 2), (2, 3), (3, 5), (2, 4), (5, 1)\}$ a binary relation?
   b) Find the domain $\mathcal{D}R$ and the range $\mathcal{Z}R$ of R.
   c) Determine the inverse relation R^{-1} and $\mathcal{D}R^{-1}$ and $\mathcal{Z}R^{-1}$.
2. Let $R = \{(1, 1), (2, 1), (3, 3), (1, 5)\}$ and $T = \{(1, 4), (2, 1), (2, 2), (2, 5)\}$ be binary relations.
   a) Determine the compositions $R \circ T$ and $T \circ R$.
   b) Is it true that $R \circ T = T \circ R$?
3. Let $S = \{1, 2, 3, 4, 5, 6, 7\}$. Define

   $$R = \{(x, y) \mid x - y \text{ is divisible by } 3\} \quad \text{in} \quad T = \{(x, y) \mid x - y \geq 3\}.$$

   Determine $R, T, R \circ R$.
4. Let $S = \mathbb{R}$. On S we define the relation R as follows

   $$(\forall x)(\forall y)(xRy \Leftrightarrow y \geq x + 3).$$

   Is R reflexive, symmetric, transitive or strict total?
5. Let $S = \{1, 2, 3, 4\}$. We have the following relations
   (i) $R_1 = \{(1, 1), (1, 2), (2, 3), (1, 3), (4, 4)\}$,
   (ii) $R_2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$,
   (iii) $R_3 = \{(1, 3), (2, 1)\}$,
   (iv) $R_4 = \emptyset$,
   (v) $R_5 = S \times S$.

   Which of the following properties hold for each relation: reflexive, symmetric, antisymmetric, transitive?

6. Let R and S be symmetric relations. Show: $R \circ S$ symmetric $\Leftrightarrow$ $R \circ S = S \circ R$.

7. Let $S = \{m \in \mathbb{N} \,|\, 1 \leq n \leq 10\}$ in $R = \{(m,n) \in S \times S \,|\, 3|m - n\}$. Is R an equivalence relation? If yes, determine the corresponding equivalence classes and the factor set.

3.1 EQUIVALENCES

8. Let $S = \mathbb{Z} \times \mathbb{Z}$ and define the relation R as follows

$$(a,b)R(c,d) \Leftrightarrow ad = bc.$$

Show that R is an equivalence relation and find the corresponding equivalence classes.

9. Let $S = \mathbb{R}^2$ and define the relation R as follows

$$(x_1,y_1)R(x_2,y_2) \Leftrightarrow x_1^2 + y_1^2 = x_2^2 + y_2^2.$$

Show that R is an equivalence relation and find the equivalenec class $R[(7,1)]$.

3.2 FUNCTIONS

1. Let $A = \{1,2,3,4\}$, $B = \{x,y,z\}$, $C = \{a,b\}$. You are given functions $f : A \to B$ and $g : B \to C$.

$$f = \{(1,x),(2,y),(3,y),(4,x)\}$$

$$g = \{(x,a),(y,b),(z,b)\}$$

(a) Is f injective?

(b) Is f surjective?

(c) Is g injective?

(d) Is g surjective?

(e) Is $g \circ f$ surjective?

2. Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$, $C = \{x, y\}$. You are given functions $f : A \to B$ and $g : B \to C$.

$$f = \{(a, 1), (b, 3), (c, 2)\}$$

$$g = \{(1, x), (2, y), (3, x)\}$$

(a) Is f injective?

(b) Is f surjective?

(c) Is g injective?

(d) Is g surjective?

(e) Is $g \circ f$ surjective?

3. Let $A = \{x, y, z\}$, $B = \{1, 2, 3\}$, $C = \{a, b, c\}$. You are given functions $f : A \to B$ and $g : B \to C$.

$$f = \{(x, 2), (y, 1), (z, 3)\}$$

$$g = \{(1, a), (2, b), (3, c)\}$$

(a) Is f injective?

(b) Is f surjective?

(c) Is g injective?

(d) Is g surjective?

(e) Is $g \circ f$ surjective?

3.3 GRAPH THEORY

1. Let $n \geq 3$. Recall the definition of cycles and complete graphs:

$$C_n = \{[n], E_1\}$$

$$K_n = \{[n], E_2\}$$

   and define

$$G_n = \{[n], E_2 \setminus E_1\}$$

   - Draw $H, G_4, G_5, G_6, C_5, C_6, \overline{C_i}$
   - For all the above graphs, determine $\Delta(G_i), \delta(G_i), \alpha(G_i), \omega(G_i), \chi(G_i), g(G_i)$
   - Prove $(\forall i \geq 3)(G_i \simeq \overline{C_i})$

2. Let $G = ([n], E)$ be a graph.
   - Prove: $\chi(G) \geq \omega(G)$
   - Prove: $\chi(G) \geq \frac{n}{\alpha(G)}$