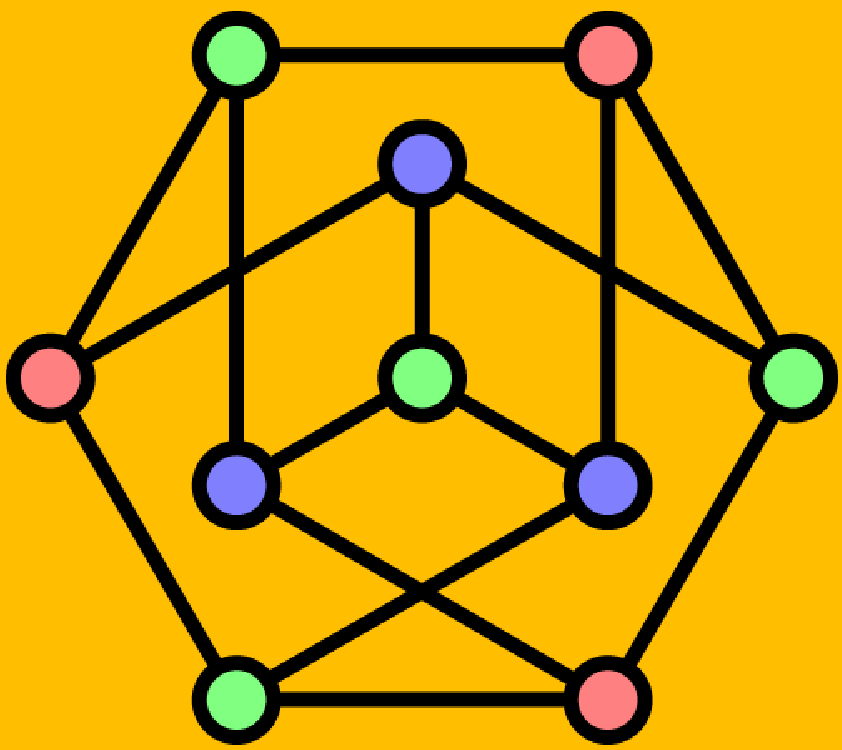
THEORETICAL COMPUTER SCIENCE

DISCRETE STRUCTURES FOR COMPUTER SCIENCE STUDENTS



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PREFACE

Those notes are supposed to be parsed together with explanations from the lectures. Any questions or found errors should be addressed to `matjaz.krnc@upr.si`, or raised as an issue in our public repository

`https://github.com/mkrnc/TOR1-vaje---TCS1-exercises.git`.

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CONTENTS

1 MATHEMATICAL LOGIC

1.1 BASIC EXERCISES

Exercise 1.1. *The following two propositions are given: A: "It is cold outside." B: "It is raining outside."*

Write the following compound propositions in natural language:

- *(a)* $\neg A$
- *(b)* $A \wedge B$
- *(c)* $A \vee B$
- *(d)* $B \vee \neg A$

Exercise 1.2. *The following two propositions are given: A: "Janez is rich." B: "Janez is happy."*

Write the following propositions symbolically:

- *(a) If Janez is rich, then he is unhappy.*
- *(b) Janez is neither happy nor rich.*
- *(c) Janez is happy only if he is poor.*
- *(d) Janez is poor if and only if he is unhappy.*

Exercise 1.3. *Find the truth tables for the examples from the previous task.*

Exercise 1.4. *Is the following reasoning correct?*

- *Premise 1: "I think, therefore I am."*
- *Premise 2: "I think, therefore I reason."*
- *Conclusion: "I am, therefore I reason."*

Exercise 1.5. *The following two propositions are given: A: "Andrej speaks French." and B: "Andrej speaks Danish." Write the following compound propositions in natural language:*

(a) $A \lor B$
(b) $A \land B$
(c) $A \land \neg B$
(d) $\neg A \lor \neg B$
(e) $\neg\neg A$
(f) $\neg(\neg A \land \neg B)$

Exercise 1.6. *Given the propositions:*
A : *"John reads The New York Times."*
B : *"John reads The Wall Street Journal."*
C : *"John reads The Daily Mail."*

Transcribe the following statements into symbolic propositions:

1. *John reads The New York Times, but not The Wall Street Journal.*
2. *Either John reads both The New York Times and The Wall Street Journal, or he does not read The New York Times and The Wall Street Journal.*
3. *It is not true that John reads The New York Times, and does not read The Daily Mail.*
4. *It is not true that John reads The Daily Mail or The Wall Street Journal, and not The New York Times.*

Exercise 1.7. *Find the truth tables for the symbolic propositions from Exercise 1.6.*

Exercise 1.8. *For three lines p, q, r we may construct also geometric propositions. Suppose that the following is true:*

$$(p||q) \land (p \cap q \neq \emptyset) \land (q \cap r \neq \emptyset).$$

What can you say about the lines p, q, r?

Exercise 1.9. *Express the propositions below with connectives $\land$ and $\neg$ only!*

1. $A \lor B$

2. $A \Rightarrow B$

3. $A \Leftrightarrow B$

1.2 Knights and servants (kneves)

Knights always tell the truth, while servants always lie.

Exercise 1.10. *Artur: "It is not true that Cene is a servant."*
Bine: "Cene is a knight or I am a knight."
Cene: "Bine is a servant."
For each of them, determine whether they are knights or servants!

Exercise 1.11. *Artur: "Cene is a servant or Bine is a servant."*
Bine: "Cene is a knight and Artur is a knight."
For each of them, determine whether they are knights or servants!

Exercise 1.12. *Let us analyze the statements made by A, B, C, and D:*

- *A: "D is a servant and C is a servant."*
- *B: "If A and D are servants, then C is a servant."*
- *C: "If B is a servant, then A is a knight."*
- *D: "If E is a servant, then both C and B are servants."*

Exercise 1.13. *Solve the following exercises about knights and servants:*

- *Arthur: "It is not true that Bine is a servant."*
- *Bine: "We are not both of the same kind."*

Exercise 1.14. *Now Arthur and Bine say the following:*

- *Arthur: "Me and Bine are not of the same kind."*
- *Bine: "Exactly one of us is a knight."*

Exercise 1.15. *Knights and servants!*

1. *Arthur: Chloe or Bob are servants.*
2. *Bob: Cene and Arthur are knights.*

1.3 CANONICAL FORMS

Exercise 1.16. *Find the canonical disjunctive normal form (DNF) and the canonical conjunctive normal form (CNF) for the following propositions:*

(i) $\neg(A \wedge B) \Rightarrow (\neg B \Rightarrow A)$

(ii) $\neg(A \vee B) \wedge (A \Rightarrow B)$

(iii) $(A \vee \neg B) \wedge (B \Rightarrow A)$

(iv) $(\neg A \vee B) \Rightarrow (\neg B \vee A)$

(v) $\neg((A \vee B) \wedge (\neg A \vee \neg C))$

Exercise 1.17. *For the following compound proposition find a truth table, determine DNF, CNF and draw the corresponding circuit.*

$$(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)).$$

Exercise 1.18. *For the following compound proposition, create a truth table, determine the DNF and CNF, and draw the corresponding circuit diagram:*

$$\neg((A \wedge B) \Rightarrow (\neg C \vee D)).$$

Exercise 1.19. *Determine if the following logical equivalences hold. Justify your answers by transforming each side into its canonical form:*

(i) $(A \Rightarrow B) \vee (\neg A \wedge \neg B) \sim \neg B \Rightarrow \neg A$

(ii) $(A \wedge (B \vee C)) \Rightarrow (A \wedge B) \sim (\neg A \vee B)$

Exercise 1.20. *Prove or disprove the validity of the following compound proposition using truth tables, DNF, or CNF:*

$$(\neg A \vee B) \wedge (B \Rightarrow C) \sim (\neg A \vee C).$$

Exercise 1.21. *Simplify the following compound proposition using the laws of logic, and then find its canonical DNF and CNF:*

$$(\neg A \wedge (B \vee C)) \vee (A \wedge \neg B \wedge \neg C).$$

Exercise 1.22. *Find DNF and CNF (if they exist) for the following proposition:*

$$(A \vee B) \wedge (\neg A \vee C) \wedge (\neg B \vee \neg C).$$

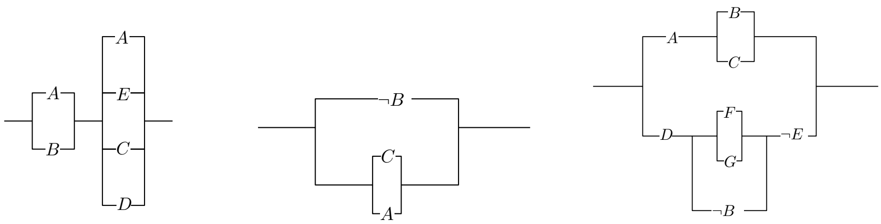

Figure 1: Switching circuits for Exercise 1.23.

1.4 SWITCHING CIRCUITS

Exercise 1.23. *For the circuits in Figure 1, find the corresponding compound propositions.*

Exercise 1.24. *For the following compound proposition, find a truth table, determine DNF, CNF, and draw the corresponding circuit.*

$$(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$$

Exercise 1.25. *Find a compound proposition I such that*

$$(A \Rightarrow (I \Rightarrow \neg B)) \Rightarrow (A \land B) \lor I$$

is a tautology.

1.5 LOGICAL IMPLICATIONS

Exercise 1.26. *Prove the following logical equivalences:*

(1) $(A \Rightarrow B) \land (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$

(2) $(A \Rightarrow B) \Rightarrow ((C \Rightarrow A) \Rightarrow (C \Rightarrow B))$

(3) $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$

(4) $(A \Rightarrow B) \Rightarrow (A \land C \Rightarrow B \land C)$

(5) $(A \Rightarrow B) \Rightarrow (A \lor C \Rightarrow B \lor C)$

(6) $(A \Leftrightarrow B) \wedge (B \Leftrightarrow C) \Rightarrow (A \Leftrightarrow C)$

(7) $(A \Leftrightarrow B) \Rightarrow (A \Rightarrow B)$

(8) $(A \Leftrightarrow B) \Rightarrow (B \Rightarrow A)$

(9) $A \wedge (A \Leftrightarrow B) \Rightarrow B$

(10) $\neg A \wedge (A \Leftrightarrow B) \Rightarrow \neg B$

(11) $B \Rightarrow (A \Leftrightarrow A \wedge B)$

(12) $\neg B \Rightarrow (A \Leftrightarrow A \vee B)$

(13) $(A \Rightarrow (B \wedge \neg B)) \Rightarrow \neg A$

Exercise 1.27. *Simplify the following logical proposition:*

$$(A \Rightarrow B) \vee (B \Rightarrow C)$$

1.6 PROOFS

Exercise 1.28. *Show that the following propositions are logical implications (i.e., tautologies where the main connective is implication):*

(i) $A \wedge (A \Rightarrow B) \Rightarrow B$

(ii) $\neg B \wedge (A \Rightarrow B) \Rightarrow \neg A$

(iii) $\neg A \wedge (A \vee B) \Rightarrow B$

(iv) $(A \Rightarrow B) \wedge (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$

(v) $A \wedge (A \Leftrightarrow B) \Rightarrow B$

Exercise 1.29. *Are the following propositions logical implications?*

(i) $(A \Rightarrow B) \wedge (A \Rightarrow C) \wedge A \Rightarrow B \wedge C$

(ii) $\neg(A \vee B) \wedge (A \vee C) \wedge (D \Rightarrow C) \Rightarrow D$

(iii) $(A \Rightarrow B) \wedge (A \Rightarrow C) \wedge (D \wedge E \Rightarrow F) \wedge (C \Rightarrow E) \Rightarrow F$

Exercise 1.30. *Show that the following propositions are logical implications (a tautology where the main connective is implication).*

(i) $A \wedge (A \Rightarrow B) \Rightarrow B$

(ii) $\neg B \wedge (A \Rightarrow B) \Rightarrow \neg A$

(iii) $\neg A \wedge (A \vee B) \Rightarrow B$

(iv) $(A \Rightarrow B) \wedge (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$

(v) $A \wedge (A \Leftrightarrow B) \Rightarrow B$

Exercise 1.31. *Are the following propositions logical implications?*

(i) $(A \Rightarrow B) \wedge (A \Rightarrow C) \wedge A \Rightarrow B \wedge C$

(ii) $\neg(A \vee B) \wedge (A \vee C) \wedge (D \Rightarrow C) \Rightarrow D$

(iii) $(A \Rightarrow B) \wedge (A \Rightarrow C) \wedge (D \wedge E \Rightarrow F) \wedge (C \Rightarrow E) \Rightarrow F$

Exercise 1.32. *With a direct proof show:*

> *If n is even, then so is $n^2 + 3n$.*

Is the converse also true?

Exercise 1.33. *Use a direct proof of the implication to show: If a real number x is non-negative, then the sum of the number x and its reciprocal is greater than or equal to* 2*.*

Exercise 1.34. *Use contradiction to show that there are infinitely many prime numbers.*

Exercise 1.35. *Find the error in the following proof.*

Statement: 1 *is the largest natural number.*

Proof *(by contradiction): Suppose the opposite. Let $n > 1$ be the largest natural number. Since n is positive, we can multiply the inequality $n > 1$ by n, giving*

$$n > 1 \Leftrightarrow n^2 > n.$$

We have found that n^2 is greater than n, which contradicts the assumption that n is the largest natural number. Therefore, the assumption was incorrect, and 1 *is the largest natural number.*

Exercise 1.36. *Let x and y be real numbers such that $x < 2y$. By an indirect proof show:*

> *If $7xy \leq 3x^2 + 2y^2$, then $3x \leq y$.*

Exercise 1.37. *Prove the following equivalence in two parts: Let m and n be integers. Then m and n have different parities if and only if $m^2 - n^2$ is odd.*

Exercise 1.38. *Using an "if and only if" proof, show that $ac \mid bc \Leftrightarrow a \mid b$.*

Exercise 1.39. *Is the following inference correct?*

(i) *If today is Wednesday, I will have a tutorial. Today is Wednesday. Conclusion: I will have a tutorial.*

(ii) *If I study, I will pass the exam. I did not study. Conclusion: I will not pass the exam.*

(iii) *A student took the city bus to the exam. He thought, "If the next traffic light is green, I will pass the exam." When the bus reached the next light, it was not green, so the student said to himself, "Darn, I'll fail again."*

(iv) *An engineer who understands theory always designs a good circuit. A good circuit is economical. Therefore, an engineer who designs an uneconomical circuit does not understand theory.*

Exercise 1.40. *Which of the following propositions are correct where the language of the conversation are real numbers?*

(i) $(\forall x)(\exists y)(x + y = 0)$.

(ii) $(\exists x)(\forall y)(x + y = 0)$.

(iii) $(\exists x)(\exists y)(x^2 + y^2 = -1)$.

(iv) $(\forall x)[x > 0 \Rightarrow (\exists y)(y < 0 \wedge xy > 0)]$.

1.7 PROOFS BY INDUCTION

Exercise 1.41. *Prove each using induction:*

(a) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

(b) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

(c) $\sum_{i=1}^{n} 2^{i-1} = 2^n - 1$

(d) $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

(e) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

(f) $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$

(g) $\sum_{i=1}^{n}(2i-1) = n^2$

(h) $n! > 2^n$ *for* $n \geq 4$.

(i) $2^{n+1} > n^2$ *for all positive integers.*

Exercise 1.42. *This exercise refers to the Fibonacci sequence:*

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

The sequence is defined recursively by $f_1 = 1$*,* $f_2 = 1$*, then* $f_{n+1} = f_n + f_{n-1}$ *for each* $n > 2$*. As before, prove each of the following using induction. You might investigate each with several examples before you start.*

(a) $f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$

(b) $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$

(c) $f_1 + f_3 + f_5 + \cdots + f_{2n-1} = f_{2n}$

2 | SET THEORY

Exercise 2.1. *Let* $A = \{x \in \mathbb{N}; x < 7\}, B = \{x \in \mathbb{Z}; |x-2| < 4\}$ *and* $C = \{x \in \mathbb{R}; x^3 - 4x = 0\}$.

(i) Write down the elements for all three sets.

(ii) Find $A \cup C, B \cap C, B \setminus C, (A \setminus B) \setminus C$ *and* $A \setminus (B \setminus C)$.

Exercise 2.2. *Let* $\mathbb{Z}$ *be a universal set and let* P *denote the set of all prime numbers, and* S *the set of all even integers. Write the following propositions in terms of set theory:*

(i) There exists an even prime number.

(ii) 0 *is an integer, but it is not natural number.*

(iii) Every natural number is an integer.

(iv) Not every integer is a natural number.

(v) Every prime number except 2 is odd.

(vi) 2 is an even prime number.

Exercise 2.3. *Let* A, B, C *and* D *be subsets of some universal set* U. *Simplify the following expression*

$$\overline{(\overline{(A \cup B)} \cap \overline{(\overline{A} \cup C)})} \setminus \overline{D}.$$

Exercise 2.4. *Show that* $(A \cup C) \cap (B \setminus C) = (A \cap B) \setminus C$.

Exercise 2.5. *Prove that* $A \subseteq C \wedge B \subseteq C \Rightarrow A \cup B \subseteq C$.

Exercise 2.6. *Prove that* $A \subseteq B \Leftrightarrow A \cap B = A$.

Exercise 2.7. *Prove that* $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Exercise 2.8. *Prove that* $(A \cap B) \setminus B = \emptyset$.

Exercise 2.9. *Determine the following sets:*

(i) $\{\emptyset, \{\emptyset\}\} \setminus \emptyset$ $\quad$ [$\{\emptyset, \{\emptyset\}\}$]

(ii) $\{\emptyset, \{\emptyset\}\} \setminus \{\emptyset\}$

(iii) $\{\emptyset, \{\emptyset\}\} \setminus \{\}\emptyset\}\}$

(iv) $\{1, 2, 3, \{1\}, \{5\}\} \setminus \{2, \{3\}, 5\}$

Exercise 2.10. *Which of the following propositions are correct for arbitrary sets A, B and C:*

1. *If $A \in B$ and $B \in C$, then $A \in C$.*

2. *If $A \subseteq B$ and $B \in C$, then $A \in C$.*

3. *If $A \cap B \subseteq \overline{C}$ and $A \cup C \subseteq B$, then $A \cap C = \emptyset$.*

4. *If $A \neq B$ and $B \neq C$, then $A \neq C$.*

5. *If $A \subseteq \overline{(B \cup C)}$ and $B \subseteq \overline{(A \cup C)}$, then $B = \emptyset$.*

Exercise 2.11. *Find $\mathcal{P}(A)$, where $A = \{a, b, c, d\}$.*

Exercise 2.12. *Let $A = \{\{1, 2, 3\}, \{4, 5\}, \{6, 7, 8\}\}$.*

(i) *Write down the elements of A.*

(ii) *Is it true?*
(a) $1 \in A$ $\quad$ *(b)* $\{1, 2, 3\} \subseteq A$ $\quad$ *(c)* $\{6, 7, 8\} \in A$ $\quad$ *(d)* $\{\{4, 5\}\} \subseteq A$
(e) $\emptyset \in A$ $\quad$ *(f)* $\emptyset \subseteq A$

Exercise 2.13. *Show that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.*

Exercise 2.14. *Let A and B be arbitrary subsets of the universal set $U = A \cup B$. Show that $A \setminus B \subseteq \overline{B}$.*

Exercise 2.15. *Let A, B in C be arbitrary subsets of the universal set $U = A \cup B \cup C$. Show that $A \cap B \subseteq C \Leftrightarrow A \subseteq \overline{B} \cup C$.*

Exercise 2.16. *Show that $(A \setminus B) \cap B = \emptyset$.*

Exercise 2.17. *Show that $(A \setminus B) \cup B = A \Leftrightarrow B \subseteq A$.*

Exercise 2.18. *Show that, if* $B \subseteq A$*, then* $B \times B = (B \times A) \cap (A \times B)$*.*

Exercise 2.19. *Let* A *be a nonempty set. Which of the following sets*

$$\emptyset, \{\emptyset\}, A, \{A\}, \{A, \emptyset\}$$

are elements and which are subsets of (i) $\mathcal{P}(A)$ *and (ii)* $\mathcal{P}(\mathcal{P}(A))$*?*

Exercise 2.20. *Is it true that* $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$*?*

3 RELATIONS

Exercise 3.1. *Let* $S = \{1, 2, 3, 4, 5\}$.

1. *Is* $R = \{(1, 2), (2, 3), (3, 5), (2, 4), (5, 1)\}$ *a binary relation?*
2. *Find the domain* $\mathcal{D}R$ *and the range* $\mathcal{Z}R$ *of* R.
3. *Determine the inverse relation* R^{-1} *and* $\mathcal{D}R^{-1}$ *and* $\mathcal{Z}R^{-1}$.

Exercise 3.2. *Define binary relations:*

$$R = \{(1, 1), (2, 1), (3, 3), (1, 5)\},$$
$$T = \{(1, 4), (2, 1), (2, 2), (2, 5)\}.$$

1. *Determine the compositions* $R \circ T$ *and* $T \circ R$.
2. *Is it true that* $R \circ T = T \circ R$*?*

Exercise 3.3. *Let* $S = \{1, 2, 3, 4, 5, 6, 7\}$. *Define*

$$R = \{(x, y) \mid x - y \text{ is divisible by } 3\} \quad \text{in} \quad T = \{(x, y) \mid x - y \geq 3\}.$$

Determine $R, T, R \circ R$.

Exercise 3.4. *Determine the Domain and Range of the following relations:*

1. $R = \{(1, 2), (2, 3), (3, 4)\}$
2. $R = \{(1, 5), (2, 3), (3, 3), (4, 9)\}$
3. $R = \{(1, 2), (3, 5), (4, 5)\}$
4. $R = \{(-1, -1), (2, 2), (3, 3)\}$
5. $R = \{(2, 0), (9, 0)\}$

Exercise 3.5. *For relations from Exercise 3.4 determine their inverses.*

Exercise 3.6. *Let* $R_1 = \{(1,2),(2,3),(3,4)\}$ *and* $R_2 = \{(2,4),(3,5),(4,6)\}$. *Compute* $R_2 \circ R_1$ *and* $R_1 \circ R_2$. *Are they equal?*

Exercise 3.7. *For relations below, compute the corresponding compositions.*

1. $R_1 = \{(1,2),(2,3),(3,4)\}$, $R_2 = \{(1,4),(2,6),(3,4)\}$,
   $R_2 \circ R_1 = ?$

2. $R_1 = \{(1,2),(2,3),(3,4)\}$, $R_2 = \{(2,4),(3,6),(4,4)\}$,
   $R_2 \circ R_1 = ?$

3. $R_1 = \{(4,3),(5,4),(6,5)\}$, $R_2 = \{(3,4),(4,5),(5,6)\}$,
   $R_2 \circ R_1 = ?$

4. $R_1 = \{(4,3),(5,4),(6,5)\}$,
   $R_1 \circ R_1 = ?$

Exercise 3.8. *In the universe* $S = \mathbb{Z}$, *we are given* $R_1 = \{(x,y) \mid x+1 = y\}$ *and* $R_2 = \{(y,z) \mid y+2 = z\}$. *Find* $R_2 \circ R_1$.

Exercise 3.9. *Let* $S = \{1,2,3,4\}$ *and* $R = \{(1,2),(2,3),(3,4)\}$. *Compute* $R \circ R$ *and* $R \circ R \circ R$, *and* $R \circ R \circ R \circ R \circ R$.

3.1 PROPERTIES OF RELATIONS

Exercise 3.10. *Prove or disprove: If* R *is symmetric, then* $R \circ R$ *is also symmetric.*

Exercise 3.11. *For* $S = \{1,2,3\}$, $R = \{(1,1),(1,2),(2,3)\}$. *Is* R *transitive? Explain.*

1. Let $S = \mathbb{R}$. On S we define the relation R as follows

   $$(\forall x)(\forall y)(xRy \Leftrightarrow y \geq x+3).$$

   Is R reflexive, symmetric, transitive or strict total?

2. Let $S = \{1,2,3,4\}$. We have the following relations
   (i) $R_1 = \{(1,1),(1,2),(2,3),(1,3),(4,4)\}$,
   (ii) $R_2 = \{(1,1),(1,2),(2,1),(2,2),(3,3),(4,4)\}$,

(iii) $R_3 = \{(1,3),(2,1)\}$,

(iv) $R_4 = \emptyset$,

(v) $R_5 = S \times S$.

Which of the following properties hold for each relation: reflexive, symmetric, antisymmetric, transitive?

3. Let R and S be symmetric relations. Show: $R \circ S$ symmetric $\Leftrightarrow$ $R \circ S = S \circ R$.

4. Let $S = \{m \in \mathbb{N} \mid 1 \leq n \leq 10\}$ in $R = \{(m,n) \in S \times S \mid 3|m-n\}$. Is R an equivalence relation? If yes, determine the corresponding equivalence classes and the factor set.

3.2 EQUIVALENCES

5. Let $S = \mathbb{Z} \times \mathbb{Z}$ and define the relation R as follows
$$(a,b)R(c,d) \Leftrightarrow ad = bc.$$
Show that R is an equivalence relation and find the corresponding equivalence classes.

6. Let $S = \mathbb{R}^2$ and define the relation R as follows
$$(x_1,y_1)R(x_2,y_2) \Leftrightarrow x_1^2 + y_1^2 = x_2^2 + y_2^2.$$
Show that R is an equivalence relation and find the equivalenec class $R[(7,1)]$.

3.3 FUNCTIONS

1. Let $A = \{1,2,3,4\}$, $B = \{x,y,z\}$, $C = \{a,b\}$. You are given functions $f : A \to B$ and $g : B \to C$.
$$f = \{(1,x),(2,y),(3,y),(4,x)\}$$

$$g = \{(x,a),(y,b),(z,b)\}$$

(a) Is f injective?

(b) Is f surjective?

(c) Is g injective?

(d) Is g surjective?

(e) Is $g \circ f$ surjective?

2. Let $A = \{a,b,c\}$, $B = \{1,2,3\}$, $C = \{x,y\}$. You are given functions $f : A \to B$ and $g : B \to C$.

$$f = \{(a,1),(b,3),(c,2)\}$$

$$g = \{(1,x),(2,y),(3,x)\}$$

(a) Is f injective?

(b) Is f surjective?

(c) Is g injective?

(d) Is g surjective?

(e) Is $g \circ f$ surjective?

3. Let $A = \{x,y,z\}$, $B = \{1,2,3\}$, $C = \{a,b,c\}$. You are given functions $f : A \to B$ and $g : B \to C$.

$$f = \{(x,2),(y,1),(z,3)\}$$

$$g = \{(1,a),(2,b),(3,c)\}$$

(a) Is f injective?

(b) Is f surjective?

(c) Is g injective?

(d) Is g surjective?

(e) Is $g \circ f$ surjective?

3.4 GRAPH THEORY

1. Let $n \geq 3$. Recall the definition of cycles and complete graphs:

$$C_n = \{[n], E_1\}$$

$$K_n = \{[n], E_2\}$$

and define

$$G_n = \{[n], E_2 \setminus E_1\}$$

   - Draw $H, G_4, G_5, G_6, C_5, C_6, \overline{C_i}$
   - For all the above graphs, determine $\Delta(G_i), \delta(G_i), \alpha(G_i), \omega(G_i), \chi(G_i), g(G_i)$
   - Prove $(\forall i \geq 3)(G_i \simeq \overline{C_i})$

2. Let $G = ([n], E)$ be a graph.
   - Prove: $\chi(G) \geq \omega(G)$
   - Prove: $\chi(G) \geq \frac{n}{\alpha(G)}$

4 SOLUTIONS

Answer for exercise 1.4: The logical implication is:

$$(A_1 \Rightarrow A_2) \wedge (A_1 \Rightarrow A_3) \Rightarrow (A_2 \Rightarrow A_3)$$

For $A_1(d) = 0$, $A_2(d) = 1$, $A_3(d) = 0$, this implication is false. Therefore, the reasoning is incorrect.

Answer for exercise 1.11:

Let us denote: A – Arthur is a knight, B – Bine is a knight, C – Cene is a knight.

The following compound statement holds:

$$(A \iff \neg C \vee \neg B) \wedge (B \iff C \wedge A)$$

With the help of the truth table, we see that the statement is true only for the set $A = 1$ and $B = C = 0$.

Answer for exercise 1.12: Let A represent "A is a knight", etc. We seek the only solution d such that the following is true:

$$A_1 \wedge B_1 \wedge C_1 \wedge D_1$$

where:

$$\begin{aligned}
A_1 &: A \iff (\neg D \wedge \neg C) \\
B_1 &: B \iff (\neg A \wedge \neg D \Rightarrow \neg C) \\
C_1 &: C \iff (\neg B \Rightarrow A) \\
D_1 &: D \iff (\neg E \Rightarrow \neg C \wedge \neg B)
\end{aligned}$$

Since the truth table would contain 32 rows, we solve this by analyzing cases.

Case 1: $A(d) = 1$

Given A_1, $D(d) = 0$ and $C(d) = 0$. Substituting into C_1 with $A(d) = 1$ and $C(d) = 0$, we get:

$$\neg(\neg B \Rightarrow 1) \Rightarrow \neg(B \vee 1) \Rightarrow \neg 1$$

which is a false statement. Hence, this case is not possible.

Case 2: $A(d) = 0$

Given A_1, either $C(d) = 1$ or $D(d) = 1$.

Case 2.1: $C(d) = 1$

Since C_1, $\neg B \Rightarrow 0$, implies $\neg B = 0$, hence $B(d) = 1$.

Substituting into B_1 with $A(d) = 0$, $B(d) = 1$, and $C(d) = 1$, we get:

$$1 \wedge \neg D \Rightarrow 0 \Rightarrow \neg D = 0$$

thus, $D(d) = 1$.

Substituting into D_1, we get:

$$\neg E \Rightarrow 0 \wedge 0$$

Hence $E(d) = 1$.

Case 2.2: $C(d) = 0$ **and** $D(d) = 1$

From B_1, we get $B(d) = 1$, but the statement C_1 becomes false: $0 \iff (0 \Rightarrow 1)$.

Thus, B, C, D, and E are knights, while A is a servant.

Answer for exercise 1.27:

$$
\begin{aligned}
(A \Rightarrow B) \vee (B \Rightarrow C) &\Leftrightarrow (\neg A \vee B) \vee (\neg B \vee C) \\
&\Leftrightarrow \neg A \vee B \vee \neg B \vee C \\
&\Leftrightarrow \neg A \vee (B \vee \neg B) \vee C \\
&\Leftrightarrow \neg A \vee 1 \vee C \\
&\Leftrightarrow 1.
\end{aligned}
$$

Answer for exercise 1.33: We need to show that $x + \frac{1}{x} \geq 2$. Since $x \geq 0$, we can multiply the inequality by x to get

$$x^2 + 1 \geq 2x$$

or equivalently,

$$(x-1)^2 \geq 0.$$

This is obviously always true.

Answer for exercise 1.34: Suppose there are only finitely many primes $p_1, p_2, \ldots, p_n$. Then the number $p = p_1 p_2 \cdots p_n + 1$ is not divisible by any prime p_i and $p_i \neq p$ for each i. By definition, p is therefore a prime number that is different from each of the previous ones. This is a contradiction.

Answer for exercise 1.35: The opposite statement is: there exists a natural number greater than 1.

Answer for exercise 1.37: We prove both directions separately:

($\Rightarrow$) Assume that m and n have different parities. Write $m = 2k$ and $n = 2l + 1$, substitute into the expression $m^2 - n^2$, and the result follows.

($\Leftarrow$) We show indirectly: If m and n have the same parity, then $m^2 - n^2$ is even. Consider both cases.

Answer for exercise 1.39:

(i) $(A \Rightarrow B) \wedge A \Rightarrow B$. This is true.

(ii) $(A \Rightarrow B) \wedge \neg A \Rightarrow \neg B$. This is not necessarily true.

(iii) $((A \Rightarrow B) \wedge \neg A) \Rightarrow \neg B$. This is not necessarily true.

(iv) $((A \Rightarrow B) \wedge (B \Rightarrow C)) \Rightarrow (\neg C \Rightarrow \neg A)$. This is true.

Answer for exercise 2.2:

(i) $P \cap S \neq \emptyset$

(ii) $0 \in \mathbb{Z} \setminus \mathbb{N}$

(iii) $\mathbb{N} \subseteq \mathbb{Z}$

(iv) $\mathbb{Z} \nsubseteq \mathbb{N}$

(v) $P \setminus \{2\} \subseteq \overline{S}$

(vi) $2 \in S \cap P$

Answer for exercise 2.4:

$$\begin{aligned}
x \in (A \cup C) \cap (B \setminus C) &\Leftrightarrow (x \in A \vee x \in C) \wedge (x \in B \wedge x \notin C) \\
&\Leftrightarrow ((x \in A \vee x \in C) \wedge (x \notin C)) \wedge x \notin B \\
&\Leftrightarrow ((x \in A \wedge x \notin C) \vee (x \in C \wedge x \notin C)) \wedge x \in B \\
&\Leftrightarrow x \in A \wedge x \notin C \wedge x \in B \\
&\Leftrightarrow x \in A \wedge x \in B \wedge x \notin C \\
&\Leftrightarrow x \in (A \cap B) \setminus C.
\end{aligned}$$

Answer for exercise 2.5: Use direct proof.

Answer for exercise 2.6: Prove separately each direction.

Answer for exercise 2.7:

$$
\begin{aligned}
(x,y) \in A \times (B \cap C) &\Leftrightarrow x \in A \wedge y \in B \cap C \\
&\Leftrightarrow x \in A \wedge y \in B \wedge y \in C \\
&\Leftrightarrow x \in A \wedge x \in A \wedge y \in B \wedge y \in C \\
&\Leftrightarrow x \in A \wedge y \in B \wedge x \in A \wedge y \in C \\
&\Leftrightarrow (x,y) \in A \times B \wedge (x,y) \in A \times C \\
&\Leftrightarrow (x,y) \in (A \times B) \cap (A \times C).
\end{aligned}
$$

Answer for exercise 2.8:

$$
\begin{aligned}
x \in (A \cap B) \setminus B &\Leftrightarrow x \in (A \cap B) \wedge x \notin B \\
&\Leftrightarrow (x \in A \wedge x \in B) \wedge x \notin B \\
&\Leftrightarrow x \in A \wedge (x \in B \wedge x \notin B) \\
&\Leftrightarrow x \in \emptyset.
\end{aligned}
$$

Answer for exercise 2.10: (In slovene.)

1. Napačna. Vzemi $A = \emptyset$, $B = \{\emptyset\}$, $B = \{\{\emptyset\}\}$.
2. Napačna. Vzemi isti primer kot v (a).
3. Pravilna. Dokaz s protislovjem. Recimo, da trditev ni pravilna. Naj bo $A \cap B \subseteq \overline{C}$, $A \cup C \subseteq B$ in naj obstaja $x \in A \cap C$. Torej je $x \in A$ in $x \in C$. Ker je po drugi predpostavki $A \cup C \subseteq B$, je $x \in B$. Sledi $x \in A \cap B$. Ker je po prvi predpostavki $A \cap B \subseteq \overline{C}$, je $x \in \overline{C}$. Protislovje, saj $x \in C$.
4. Napačna. Vzemi $A = C \neq B$.
5. Napačna. Vzemi tri paroma disjunktne neprazne množice.

Answer for exercise 3.4:

1. Domain: $\{1,2,3\}$, Range: $\{2,3,4\}$
2. Domain: $\{1,2,3,4\}$, Range: $\{5,3,9\}$
3. Domain: $\{1,3,4\}$, Range: $\{2,5\}$
4. Domain: $\{-1,2,3\}$, Range: $\{-1,2,3\}$
5. Domain: $\{2,9\}$, Range: $\{0\}$

Answer for exercise 3.6: For $R_1 = \{(1,2),(2,3),(3,4)\}$ and $R_2 = \{(2,4),(3,5),(4,6)\}$:

$$R_2 \circ R_1 = \{(1,4),(2,5),(3,6)\}, \quad R_1 \circ R_2 = \{(2,4),(3,5)\}.$$

They are **not equal** because composition is not commutative.

Answer for exercise 3.7:

1. $R_2 \circ R_1 = \{(1,6),(2,4)\}$
2. $R_2 \circ R_1 = \{(1,4),(2,6),(3,4)\}$
3. $R_2 \circ R_1 = \{(4,5),(5,6),(6,4)\}$
4. $R_1 \circ R_1 = \{(4,4),(5,5),(6,6)\}$

Answer for exercise 3.8: $R_1 = \{(x,y) \mid y = x+1\}$, $R_2 = \{(y,z) \mid z = y+2\}$:

$$R_2 \circ R_1 = \{(x,z) \mid z = x+3,\ x,z \in \mathbb{Z}\}.$$

Answer for exercise 3.9: For $R = \{(1,2),(2,3),(3,4)\}$:

$$R \circ R = \{(1,3),(2,4)\}, \quad R \circ R \circ R = \{(1,4)\},$$

while $R \circ R \circ R \circ R \circ R = \emptyset$

Answer for exercise 3.11: To check transitivity: If $(x,y) \in R$ and $(y,z) \in R$, then $(x,z) \in R$. Here, $(1,2) \in R$ and $(2,3) \in R$, but $(1,3) \notin R$. Thus, R is **not transitive**.