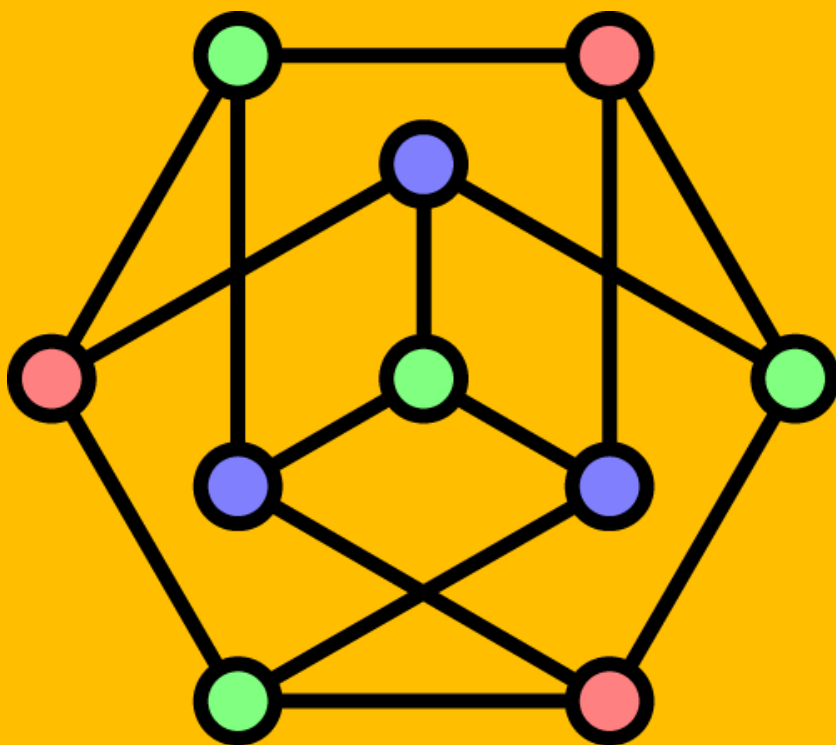


THEORETICAL COMPUTER SCIENCE

DISCRETE STRUCTURES FOR COMPUTER SCIENCE STUDENTS



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PREFACE

Those notes are supposed to be parsed together with explanations from the lectures. Any questions or found errors should be addressed to matjaz.krnc@upr.si, or raised as an issue in our public repository

<https://github.com/mkrnc/TOR1-vaje---TCS1-exercises.git>.

Among most notable student contributors are:

Theoretical Computer Science

Discrete Structures for Computer Science Students

Authors: Matjaž Krnc

Editor: Matjaž Krnc

Cover photo: R. A. Nonenmacher
Desargues Graph

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1

MATHEMATICAL LOGIC

1.1 BASIC EXERCISES

Exercise 1.1. *The following two propositions are given: A: "It is cold outside." B: "It is raining outside."*

Write the following compound propositions in natural language:

- (a) $\neg A$
- (b) $A \wedge B$
- (c) $A \vee B$
- (d) $B \vee \neg A$

Exercise 1.2. *The following two propositions are given: A: "Janez is rich." B: "Janez is happy."*

Write the following propositions symbolically:

- (a) *If Janez is rich, then he is unhappy.*
- (b) *Janez is neither happy nor rich.*
- (c) *Janez is happy only if he is poor.*
- (d) *Janez is poor if and only if he is unhappy.*

Exercise 1.3. *Find the truth tables for the examples from the previous task.*

Exercise 1.4. *Is the following reasoning correct?*

- *Premise 1: "I think, therefore I am."*
- *Premise 2: "I think, therefore I reason."*
- *Conclusion: "I am, therefore I reason."*

Exercise 1.5. The following two propositions are given: A : “Andrej speaks French.” and B : “Andrej speaks Danish.” Write the following compound propositions in natural language:

- (a) $A \vee B$
- (b) $A \wedge B$
- (c) $A \wedge \neg B$
- (d) $\neg A \vee \neg B$
- (e) $\neg \neg A$
- (f) $\neg(\neg A \wedge \neg B)$

Exercise 1.6. Given the propositions:

A : “John reads The New York Times.”

B : “John reads The Wall Street Journal.”

C : “John reads The Daily Mail.”

Transcribe the following statements into symbolic propositions:

1. John reads The New York Times, but not The Wall Street Journal.
2. Either John reads both The New York Times and The Wall Street Journal, or he does not read The New York Times and The Wall Street Journal.
3. It is not true that John reads The New York Times, and does not read The Daily Mail.
4. It is not true that John reads The Daily Mail or The Wall Street Journal, and not The New York Times.

Exercise 1.7. Find the truth tables for the symbolic propositions from Exercise 1.6.

Exercise 1.8. For three lines p, q, r we may construct also geometric propositions. Suppose that the following is true:

$$(p \parallel q) \wedge (p \cap q \neq \emptyset) \wedge (q \cap r \neq \emptyset).$$

What can you say about the lines p, q, r ?

Exercise 1.9. Express the propositions below with connectives \wedge and \neg only!

1. $A \vee B$

2. $A \Rightarrow B$

3. $A \Leftrightarrow B$

1.2 KNIGHTS AND SERVANTS (KNEVES)

Knights always tell the truth, while servants always lie.

Exercise 1.10. *Artur: "It is not true that Cene is a servant."*

Bine: "Cene is a knight or I am a knight."

Cene: "Bine is a servant."

For each of them, determine whether they are knights or servants!

Exercise 1.11. *Artur: "Cene is a servant or Bine is a servant."*

Bine: "Cene is a knight and Artur is a knight."

For each of them, determine whether they are knights or servants!

Exercise 1.12. *Let us analyze the statements made by A, B, C, and D:*

- *A: "D is a servant and C is a servant."*
- *B: "If A and D are servants, then C is a servant."*
- *C: "If B is a servant, then A is a knight."*
- *D: "If E is a servant, then both C and B are servants."*

Exercise 1.13. *Solve the following exercises about knights and servants:*

- *Arthur: "It is not true that Bine is a servant."*
- *Bine: "We are not both of the same kind."*

Exercise 1.14. *Now Arthur and Bine say the following:*

- *Arthur: "Me and Bine are not of the same kind."*
- *Bine: "Exactly one of us is a knight."*

Exercise 1.15. *Knights and servants!*

1. *Arthur: Chloe or Bob are servants.*
2. *Bob: Cene and Arthur are knights.*

1.3 CANONICAL FORMS

Exercise 1.16. Find the canonical disjunctive normal form (DNF) and the canonical conjunctive normal form (CNF) for the following propositions:

(i) $\neg(A \wedge B) \Rightarrow (\neg B \Rightarrow A)$

(ii) $\neg(A \vee B) \wedge (A \Rightarrow B)$

(iii) $(A \vee \neg B) \wedge (B \Rightarrow A)$

(iv) $(\neg A \vee B) \Rightarrow (\neg B \vee A)$

(v) $\neg((A \vee B) \wedge (\neg A \vee \neg C))$

Exercise 1.17. For the following compound proposition find a truth table, determine DNF, CNF and draw the corresponding circuit.

$$(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)).$$

Exercise 1.18. For the following compound proposition, create a truth table, determine the DNF and CNF, and draw the corresponding circuit diagram:

$$\neg((A \wedge B) \Rightarrow (\neg C \vee D)).$$

Exercise 1.19. Determine if the following logical equivalences hold. Justify your answers by transforming each side into its canonical form:

(i) $(A \Rightarrow B) \vee (\neg A \wedge \neg B) \sim \neg B \Rightarrow \neg A$

(ii) $(A \wedge (B \vee C)) \Rightarrow (A \wedge B) \sim (\neg A \vee B)$

Exercise 1.20. Prove or disprove the validity of the following compound proposition using truth tables, DNF, or CNF:

$$(\neg A \vee B) \wedge (B \Rightarrow C) \sim (\neg A \vee C).$$

Exercise 1.21. Simplify the following compound proposition using the laws of logic, and then find its canonical DNF and CNF:

$$(\neg A \wedge (B \vee C)) \vee (A \wedge \neg B \wedge \neg C).$$

Exercise 1.22. Find DNF and CNF (if they exist) for the following proposition:

$$(A \vee B) \wedge (\neg A \vee C) \wedge (\neg B \vee \neg C).$$

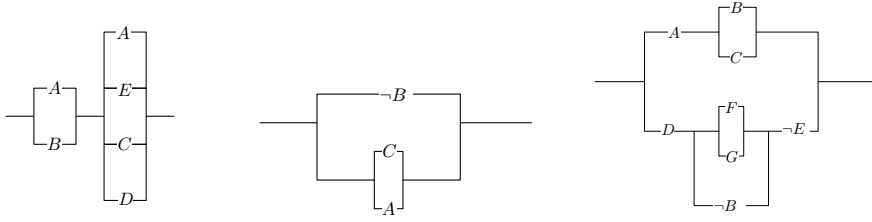


Figure 1: Switching circuits for Exercise 1.23.

1.4 SWITCHING CIRCUITS

Exercise 1.23. For the circuits in Figure 1, find the corresponding compound propositions.

Exercise 1.24. For the following compound proposition, find a truth table, determine DNF, CNF, and draw the corresponding circuit.

$$(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$$

Exercise 1.25. Find a compound proposition I such that

$$(A \Rightarrow (I \Rightarrow \neg B)) \Rightarrow (A \wedge B) \vee I$$

is a tautology.

1.5 LOGICAL IMPLICATIONS

Exercise 1.26. Prove the following logical equivalences:

- (1) $(A \Rightarrow B) \wedge (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$
- (2) $(A \Rightarrow B) \Rightarrow ((C \Rightarrow A) \Rightarrow (C \Rightarrow B))$
- (3) $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$
- (4) $(A \Rightarrow B) \Rightarrow (A \wedge C \Rightarrow B \wedge C)$
- (5) $(A \Rightarrow B) \Rightarrow (A \vee C \Rightarrow B \vee C)$

$$(6) (A \Leftrightarrow B) \wedge (B \Leftrightarrow C) \Rightarrow (A \Leftrightarrow C)$$

$$(7) (A \Leftrightarrow B) \Rightarrow (A \Rightarrow B)$$

$$(8) (A \Leftrightarrow B) \Rightarrow (B \Rightarrow A)$$

$$(9) A \wedge (A \Leftrightarrow B) \Rightarrow B$$

$$(10) \neg A \wedge (A \Leftrightarrow B) \Rightarrow \neg B$$

$$(11) B \Rightarrow (A \Leftrightarrow A \wedge B)$$

$$(12) \neg B \Rightarrow (A \Leftrightarrow A \vee B)$$

$$(13) (A \Rightarrow (B \wedge \neg B)) \Rightarrow \neg A$$

Exercise 1.27. Simplify the following logical proposition:

$$(A \Rightarrow B) \vee (B \Rightarrow C)$$

1.6 PROOFS

Exercise 1.28. Use direct or indirect proof to show every property from Exercise 1.26.

Exercise 1.29. Show that the following propositions are logical implications (i.e., tautologies where the main connective is implication):

$$(i) A \wedge (A \Rightarrow B) \Rightarrow B$$

$$(ii) \neg B \wedge (A \Rightarrow B) \Rightarrow \neg A$$

$$(iii) \neg A \wedge (A \vee B) \Rightarrow B$$

$$(iv) (A \Rightarrow B) \wedge (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$$

$$(v) A \wedge (A \Leftrightarrow B) \Rightarrow B$$

Exercise 1.30. Are the following propositions logical implications?

$$(i) (A \Rightarrow B) \wedge (A \Rightarrow C) \wedge A \Rightarrow B \wedge C$$

$$(ii) \neg(A \vee B) \wedge (A \vee C) \wedge (D \Rightarrow C) \Rightarrow D$$

$$(iii) (A \Rightarrow B) \wedge (A \Rightarrow C) \wedge (D \wedge E \Rightarrow F) \wedge (C \Rightarrow E) \Rightarrow F$$

Exercise 1.31. Show that the following propositions are logical implications (a tautology where the main connective is implication).

$$(i) A \wedge (A \Rightarrow B) \Rightarrow B$$

$$(ii) \neg B \wedge (A \Rightarrow B) \Rightarrow \neg A$$

$$(iii) \neg A \wedge (A \vee B) \Rightarrow B$$

$$(iv) (A \Rightarrow B) \wedge (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$$

$$(v) A \wedge (A \Leftrightarrow B) \Rightarrow B$$

Exercise 1.32. Are the following propositions logical implications?

$$(i) (A \Rightarrow B) \wedge (A \Rightarrow C) \wedge A \Rightarrow B \wedge C$$

$$(ii) \neg(A \vee B) \wedge (A \vee C) \wedge (D \Rightarrow C) \Rightarrow D$$

$$(iii) (A \Rightarrow B) \wedge (A \Rightarrow C) \wedge (D \wedge E \Rightarrow F) \wedge (C \Rightarrow E) \Rightarrow F$$

Exercise 1.33. With a direct proof show:

If n is even, then so is $n^2 + 3n$.

Is the converse also true?

Exercise 1.34. Use a direct proof of the implication to show: If a real number x is non-negative, then the sum of the number x and its reciprocal is greater than or equal to 2.

Exercise 1.35. Use contradiction to show that there are infinitely many prime numbers.

Exercise 1.36. Find the error in the following proof.

Statement: 1 is the largest natural number.

Proof (by contradiction): Suppose the opposite. Let $n > 1$ be the largest natural number. Since n is positive, we can multiply the inequality $n > 1$ by n , giving

$$n > 1 \Leftrightarrow n^2 > n.$$

We have found that n^2 is greater than n , which contradicts the assumption that n is the largest natural number. Therefore, the assumption was incorrect, and 1 is the largest natural number.

Exercise 1.37. Let x and y be real numbers such that $x < 2y$. By an indirect proof show:

$$\text{If } 7xy \leq 3x^2 + 2y^2, \text{ then } 3x \leq y.$$

Exercise 1.38. Prove the following equivalence in two parts: Let m and n be integers. Then m and n have different parities if and only if $m^2 - n^2$ is odd.

Exercise 1.39. Using an "if and only if" proof, show that $ac \mid bc \Leftrightarrow a \mid b$.

Exercise 1.40. Is the following inference correct?

- (i) If today is Wednesday, I will have a tutorial. Today is Wednesday. Conclusion: I will have a tutorial.
- (ii) If I study, I will pass the exam. I did not study. Conclusion: I will not pass the exam.
- (iii) A student took the city bus to the exam. He thought, "If the next traffic light is green, I will pass the exam." When the bus reached the next light, it was not green, so the student said to himself, "Darn, I'll fail again."
- (iv) An engineer who understands theory always designs a good circuit. A good circuit is economical. Therefore, an engineer who designs an uneconomical circuit does not understand theory.

Exercise 1.41. Which of the following propositions are correct where the language of the conversation are real numbers?

- (i) $(\forall x)(\exists y)(x + y = 0)$.
- (ii) $(\exists x)(\forall y)(x + y = 0)$.
- (iii) $(\exists x)(\exists y)(x^2 + y^2 = -1)$.
- (iv) $(\forall x)[x > 0 \Rightarrow (\exists y)(y < 0 \wedge xy > 0)]$.

1.7 PROOFS BY INDUCTION

Exercise 1.42. *Prove each using induction:*

(a) $\sum_{i=1}^n i = \frac{n(n+1)}{2}$

(b) $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$

(c) $\sum_{i=1}^n 2^{i-1} = 2^n - 1$

(d) $\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$

(e) $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

(f) $\sum_{i=1}^n \frac{1}{i(i+1)} = \frac{n}{n+1}$

(g) $\sum_{i=1}^n (2i - 1) = n^2$

(h) $n! > 2^n$ for $n \geq 4$.

(i) $2^{n+1} > n^2$ for all positive integers.

Exercise 1.43. *This exercise refers to the Fibonacci sequence:*

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \dots$$

The sequence is defined recursively by $f_1 = 1$, $f_2 = 1$, then $f_{n+1} = f_n + f_{n-1}$ for each $n > 2$. As before, prove each of the following using induction. You might investigate each with several examples before you start.

(a) $f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$

(b) $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$

(c) $f_1 + f_3 + f_5 + \cdots + f_{2n-1} = f_{2n}$

2

SET THEORY

Exercise 2.1. Let $A = \{x \in \mathbb{N}; x < 7\}$, $B = \{x \in \mathbb{Z}; |x - 2| < 4\}$ and $C = \{x \in \mathbb{R}; x^3 - 4x = 0\}$.

- (i) Write down the elements for all three sets.
- (ii) Find $A \cup C$, $B \cap C$, $B \setminus C$, $(A \setminus B) \setminus C$ and $A \setminus (B \setminus C)$.

Exercise 2.2. Let \mathbb{Z} be a universal set and let P denote the set of all prime numbers, and S the set of all even integers. Write the following propositions in terms of set theory:

- (i) There exists an even prime number.
- (ii) 0 is an integer, but it is not natural number.
- (iii) Every natural number is an integer.
- (iv) Not every integer is a natural number.
- (v) Every prime number except 2 is odd.
- (vi) 2 is an even prime number.

Exercise 2.3. Let A, B, C and D be subsets of some universal set U . Simplify the following expression

$$\overline{((A \cup B) \cap (\overline{A \cup C})) \setminus \overline{D}}.$$

Exercise 2.4. Show that $(A \cup C) \cap (B \setminus C) = (A \cap B) \setminus C$.

Exercise 2.5. Prove that $A \subseteq C \wedge B \subseteq C \Rightarrow A \cup B \subseteq C$.

Exercise 2.6. Prove that $A \subseteq B \Leftrightarrow A \cap B = A$.

Exercise 2.7. Prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Exercise 2.8. Prove that $(A \cap B) \setminus B = \emptyset$.

Exercise 2.9. Determine the following sets:

- (i) $\{\emptyset, \{\emptyset\}\} \setminus \emptyset \quad I\{\emptyset, \{\emptyset\}\}I$
- (ii) $\{\emptyset, \{\emptyset\}\} \setminus \{\emptyset\}$
- (iii) $\{\emptyset, \{\emptyset\}\} \setminus \{\}\emptyset\}$
- (iv) $\{1, 2, 3, \{1\}, \{5\}\} \setminus \{2, \{3\}, 5\}$

Exercise 2.10. Which of the following propositions are correct for arbitrary sets A, B and C :

- 1. If $A \in B$ and $B \in C$, then $A \in C$.
- 2. If $A \subseteq B$ and $B \in C$, then $A \in C$.
- 3. If $A \cap B \subseteq \overline{C}$ and $A \cup C \subseteq B$, then $A \cap C = \emptyset$.
- 4. If $A \neq B$ and $B \neq C$, then $A \neq C$.
- 5. If $A \subseteq \overline{(B \cup C)}$ and $B \subseteq \overline{(A \cup C)}$, then $B = \emptyset$.

Exercise 2.11. Find $\mathcal{P}(A)$, where $A = \{a, b, c, d\}$.

Exercise 2.12. Let $A = \{\{1, 2, 3\}, \{4, 5\}, \{6, 7, 8\}\}$.

- (i) Write down the elements of A .
- (ii) Is it true?
 - (a) $1 \in A$ (b) $\{1, 2, 3\} \subseteq A$ (c) $\{6, 7, 8\} \in A$ (d) $\{\{4, 5\}\} \subseteq A$
 - (e) $\emptyset \in A$ (f) $\emptyset \subseteq A$

Exercise 2.13. Show that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Exercise 2.14. Let A and B be arbitrary subsets of the universal set $U = A \cup B$. Show that $A \setminus B \subseteq \overline{B}$.

Exercise 2.15. Let A, B in C be arbitrary subsets of the universal set $U = A \cup B \cup C$. Show that $A \cap B \subseteq C \Leftrightarrow A \subseteq \overline{B} \cup C$.

Exercise 2.16. Show that $(A \setminus B) \cap B = \emptyset$.

Exercise 2.17. Show that $(A \setminus B) \cup B = A \Leftrightarrow B \subseteq A$.

Exercise 2.18. Show that, if $B \subseteq A$, then $B \times B = (B \times A) \cap (A \times B)$.

Exercise 2.19. Let A be a nonempty set. Which of the following sets

$$\emptyset, \{\emptyset\}, A, \{A\}, \{A, \emptyset\}$$

are elements and which are subsets of (i) $\mathcal{P}(A)$ and (ii) $\mathcal{P}(\mathcal{P}(A))$?

Exercise 2.20. Is it true that $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$?

Exercise 2.21. Let A, B and C be arbitrary sets. Show the following propositions:

1. $A \subseteq B \Leftrightarrow A \cap \overline{B} = \emptyset$.
2. $A \setminus B = \overline{B} \setminus \overline{A}$.

3

RELATIONS

Exercise 3.1. Let $S = \{1, 2, 3, 4, 5\}$.

1. Is $R = \{(1, 2), (2, 3), (3, 5), (2, 4), (5, 1)\}$ a binary relation?
2. Find the domain $\mathcal{D}R$ and the range $\mathcal{Z}R$ of R .
3. Determine the inverse relation R^{-1} and $\mathcal{D}R^{-1}$ and $\mathcal{Z}R^{-1}$.

Exercise 3.2. Define binary relations:

$$R = \{(1, 1), (2, 1), (3, 3), (1, 5)\},$$
$$T = \{(1, 4), (2, 1), (2, 2), (2, 5)\}.$$

1. Determine the compositions $R \circ T$ and $T \circ R$.
2. Is it true that $R \circ T = T \circ R$?

Exercise 3.3. Let $S = \{1, 2, 3, 4, 5, 6, 7\}$. Define

$$R = \{(x, y) \mid x - y \text{ is divisible by } 3\} \quad \text{in} \quad T = \{(x, y) \mid x - y \geq 3\}.$$

Determine $R, T, R \circ R$.

Exercise 3.4. Determine the Domain and Range of the following relations:

1. $R = \{(1, 2), (2, 3), (3, 4)\}$
2. $R = \{(1, 5), (2, 3), (3, 3), (4, 9)\}$
3. $R = \{(1, 2), (3, 5), (4, 5)\}$
4. $R = \{(-1, -1), (2, 2), (3, 3)\}$
5. $R = \{(2, 0), (9, 0)\}$

Exercise 3.5. For relations from Exercise 3.4 determine their inverses.

Exercise 3.6. Let $R_1 = \{(1,2), (2,3), (3,4)\}$ and $R_2 = \{(2,4), (3,5), (4,6)\}$. Compute $R_2 \circ R_1$ and $R_1 \circ R_2$. Are they equal?

Exercise 3.7. For relations below, compute the corresponding compositions.

1. $R_1 = \{(1,2), (2,3), (3,4)\}$, $R_2 = \{(1,4), (2,6), (3,4)\}$,
 $R_2 \circ R_1 = ?$
2. $R_1 = \{(1,2), (2,3), (3,4)\}$, $R_2 = \{(2,4), (3,6), (4,4)\}$,
 $R_2 \circ R_1 = ?$
3. $R_1 = \{(4,3), (5,4), (6,5)\}$, $R_2 = \{(3,4), (4,5), (5,6)\}$,
 $R_2 \circ R_1 = ?$
4. $R_1 = \{(4,3), (5,4), (6,5)\}$,
 $R_1 \circ R_1 = ?$

Exercise 3.8. In the universe $S = \mathbb{Z}$, we are given $R_1 = \{(x,y) \mid x+1=y\}$ and $R_2 = \{(y,z) \mid y+2=z\}$. Find $R_2 \circ R_1$.

Exercise 3.9. Let $S = \{1,2,3,4\}$ and $R = \{(1,2), (2,3), (3,4)\}$. Compute $R \circ R$ and $R \circ R \circ R$, and $R \circ R \circ R \circ R \circ R$.

Exercise 3.10. Given $S = \{a,b,c,d\}$, let $R = \{(a,b), (b,c), (c,d), (d,a)\}$.

(i) Determine $R \circ R$ and $R \circ R \circ R$.

(ii) Prove that $R \circ R \circ R \circ R = \{(x,x) \mid x \in S\}$.

Exercise 3.11. Let $P(A) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ be the power set of $A = \{1,2\}$. Define a relation $R \subseteq P(A) \times P(A)$ such that $(X,Y) \in R$ if and only if $X \subseteq Y$. Compute $R \circ R$ and interpret the result in terms of subset inclusion.

Exercise 3.12. Let R_1 and R_2 be binary relations. Prove that

$$(R_1 \circ R_2)^{-1} = R_2^{-1} \circ R_1^{-1}.$$

3.1 PROPERTIES OF RELATIONS

Exercise 3.13. Prove or disprove: If R is symmetric, then $R \circ R$ is also symmetric.

Exercise 3.14. For $S = \{1, 2, 3\}$, $R = \{(1, 1), (1, 2), (2, 3)\}$. Is R transitive? Explain.

Exercise 3.15. Let $S = \mathbb{R}$. On S we define the relation R as follows

$$(\forall x)(\forall y)(xRy \Leftrightarrow y \geq x + 3).$$

Is R reflexive, symmetric, transitive or strict total?

Exercise 3.16. Let $S = \{1, 2, 3, 4\}$. We have the following relations

- (i) $R_1 = \{(1, 1), (1, 2), (2, 3), (1, 3), (4, 4)\}$,
- (ii) $R_2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$,
- (iii) $R_3 = \{(1, 3), (2, 1)\}$,
- (iv) $R_4 = \emptyset$,
- (v) $R_5 = S \times S$.

Which of the following properties hold for each relation: reflexive, symmetric, antisymmetric, transitive?

Exercise 3.17. Let R and S be symmetric relations. Show: $R \circ S$ symmetric $\Leftrightarrow R \circ S = S \circ R$.

3.2 EQUIVALENCES

Exercise 3.18. Let $S = \mathbb{Z} \times \mathbb{Z}$ and define the relation R as follows

$$(a, b)R(c, d) \Leftrightarrow ad = bc.$$

Show that R is an equivalence relation and find the corresponding equivalence classes.

Exercise 3.19. Let $S = \mathbb{R}^2$ and define the relation R as follows

$$(x_1, y_1)R(x_2, y_2) \Leftrightarrow x_1^2 + y_1^2 = x_2^2 + y_2^2.$$

Show that R is an equivalence relation and find the equivalence class $R[(7, 1)]$.

Exercise 3.20. Let $S = \{m \in \mathbb{N} \mid 1 \leq m \leq 10\}$ in $R = \{(m, n) \in S \times S \mid 3 \mid m - n\}$. Is R an equivalence relation? If yes, determine the corresponding equivalence classes and the factor set.

Exercise 3.21. For each relation R on the set $A = \{1, 2, 3, 4\}$, determine if R is an equivalence relation. If it is, identify the equivalence classes.

1. $R_1 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$
2. $R_2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4)\}$
3. $R_3 = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (3, 4), (4, 3)\}$

Exercise 3.22. Let S be the set of all people, and define a relation R on S by setting $(a, b) \in R$ if a and b have the same birth month. Prove that R is an equivalence relation and determine the equivalence classes.

Exercise 3.23. Consider the set $X = \mathbb{Z}$ (integers), and let R be the relation defined by aRb if $a - b$ is divisible by 5. Show that R is an equivalence relation and find the equivalence classes.

Exercise 3.24. Define a relation R on the set of all triangles in the plane by setting $\triangle ABC R \triangle DEF$ if the triangles are similar. Prove that R is an equivalence relation and determine the equivalence classes.

Exercise 3.25. Let X be a set, and $P(X)$ be the power set of X (the set of all subsets of X). Define a relation R on $P(X)$ by setting ARB if $|A \Delta B|$ is even, where Δ is the symmetric difference. Prove that R is an equivalence relation.

Exercise 3.26. For $a, b \in \mathbb{R}$, define $a \sim b$ to mean $ab = 0$. Prove or disprove each of the following:

- (a) The relation \sim is reflexive.
- (b) The relation \sim is symmetric.
- (c) The relation \sim is transitive.

Exercise 3.27. For $a, b \in \mathbb{R}$, define $a \sim b$ to mean that $ab \neq 0$. Prove or disprove each of the following:

- (a) The relation \sim is reflexive.
- (b) The relation \sim is symmetric.

(c) The relation \sim is transitive.

Exercise 3.28. For $a, b \in \mathbb{R}$, define $a \sim b$ to mean that $|a - b| < 5$. Prove or disprove each of the following:

(a) The relation \sim is reflexive.

(b) The relation \sim is symmetric.

(c) The relation \sim is transitive.

Exercise 3.29. Define a function $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = x^2 + 1$. For $a, b \in \mathbb{R}$, define $a \sim b$ to mean that $f(a) = f(b)$.

(a) Prove that \sim is an equivalence relation on \mathbb{R} .

(b) List all elements in the set $\{x \in \mathbb{R} \mid x \sim 3\}$.

Exercise 3.30. For points $(a, b), (c, d) \in \mathbb{R}^2$, define $(a, b) \sim (c, d)$ to mean that $a^2 + b^2 = c^2 + d^2$.

(a) Prove that \sim is an equivalence relation on \mathbb{R}^2 .

(b) List all elements in the set $\{(x, y) \in \mathbb{R}^2 \mid (x, y) \sim (0, 0)\}$.

(c) List five distinct elements in the set $\{(x, y) \in \mathbb{R}^2 \mid (x, y) \sim (1, 0)\}$.

Exercise 3.31. Recall that for $a, b \in \mathbb{Z}$, $a \equiv b \pmod{8}$ means that $a - b$ is divisible by 8.

(a) Find all integers x such that $0 \leq x < 8$ and $2x \equiv 6 \pmod{8}$.

(b) Use the Division Algorithm to prove that for every integer m , there exists an integer r such that $m \equiv r \pmod{8}$ and $0 \leq r < 8$.

(c) Use the Division Algorithm to find integers r_1 and r_2 such that $0 \leq r_1 < 8$, $0 \leq r_2 < 8$, $1038 \equiv r_1 \pmod{8}$, and $-1038 \equiv r_2 \pmod{8}$.

Exercise 3.32. For what positive integers $n > 1$ is:

(a) $30 \equiv 6 \pmod{n}$

(b) $30 \equiv 7 \pmod{n}$

Exercise 3.33. Let m and n be positive integers such that m divides n . Prove that for all integers a and b , if $a \equiv b \pmod{n}$, then $a \equiv b \pmod{m}$.

Exercise 3.34. (a) Prove or disprove: For all positive integers n and for all integers a and b , if $a \equiv b \pmod{n}$, then $a^2 \equiv b^2 \pmod{n}$.

(b) Prove or disprove: For all positive integers n and for all integers a and b , if $a^2 \equiv b^2 \pmod{n}$, then $a \equiv b \pmod{n}$.

3.3 FUNCTIONS

Exercise 3.35. Let $A = \{1, 2, 3, 4\}$, $B = \{x, y, z\}$, $C = \{a, b\}$. You are given functions $f : A \rightarrow B$ and $g : B \rightarrow C$.

$$f = \{(1, x), (2, y), (3, y), (4, x)\}$$

$$g = \{(x, a), (y, b), (z, b)\}$$

- (a) Is f injective?
- (b) Is f surjective?
- (c) Is g injective?
- (d) Is g surjective?
- (e) Is $g \circ f$ surjective?

Exercise 3.36. Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$, $C = \{x, y\}$. You are given functions $f : A \rightarrow B$ and $g : B \rightarrow C$.

$$f = \{(a, 1), (b, 3), (c, 2)\}$$

$$g = \{(1, x), (2, y), (3, x)\}$$

- (a) Is f injective?
- (b) Is f surjective?
- (c) Is g injective?
- (d) Is g surjective?
- (e) Is $g \circ f$ surjective?

Exercise 3.37. Let $A = \{x, y, z\}$, $B = \{1, 2, 3\}$, $C = \{a, b, c\}$. You are given functions $f : A \rightarrow B$ and $g : B \rightarrow C$.

$$f = \{(x, 2), (y, 1), (z, 3)\}$$

$$g = \{(1, a), (2, b), (3, c)\}$$

- (a) Is f injective?
- (b) Is f surjective?

- (c) Is g injective?
- (d) Is g surjective?
- (e) Is $g \circ f$ surjective?

Exercise 3.38. Injective functions are also said to be one-to-one, while surjective functions are said to be onto. Find an example of:

- (a) A one-to-one function $f : \mathbb{N} \rightarrow \mathbb{N}$ which is not onto.
- (b) A function $f : \mathbb{N} \rightarrow \mathbb{N}$ which is onto but not one-to-one.

Exercise 3.39. Show that if X is a finite set, then a function $f : X \rightarrow X$ is one-to-one if and only if it is onto.

Exercise 3.40. Decide which of the following functions $\mathbb{Z} \rightarrow \mathbb{Z}$ are injective and which are surjective:

$$x \mapsto 1 + x, \quad x \mapsto 1 + x^2, \quad x \mapsto 1 + x^3, \quad x \mapsto 1 + x^2 + x^3.$$

Does anything in the answer change if we consider them as functions $\mathbb{R} \rightarrow \mathbb{R}$? (You may want to sketch their graphs and/or use some elementary calculus methods.)

Exercise 3.41. For a set X , let $\text{id}_X : X \rightarrow X$ denote the function defined by $\text{id}_X(x) = x$ for all $x \in X$ (the identity function). Let $f : X \rightarrow Y$ be some function. Prove:

- (a) A function $g : Y \rightarrow X$ such that $g \circ f = \text{id}_X$ exists if and only if f is one-to-one.
- (b) A function $g : Y \rightarrow X$ such that $f \circ g = \text{id}_Y$ exists if and only if f is onto.
- (c) A function $g : Y \rightarrow X$ such that both $f \circ g = \text{id}_Y$ and $g \circ f = \text{id}_X$ exist if and only if f is a bijection.
- (d) If $f : X \rightarrow Y$ is a bijection, then the following three conditions are equivalent for a function $g : Y \rightarrow X$:
 - (i) $g = f^{-1}$,
 - (ii) $g \circ f = \text{id}_X$, and
 - (iii) $f \circ g = \text{id}_Y$.

Exercise 3.42. (a) If $g \circ f$ is an onto function, does g have to be onto? Does f have to be onto?

(b) If $g \circ f$ is a one-to-one function, does g have to be one-to-one? Does f have to be one-to-one?

Exercise 3.43. Prove that the following two statements about a function $f : X \rightarrow Y$ are equivalent (X and Y are some arbitrary sets):

(i) f is one-to-one.

(ii) For any set Z and any two distinct functions $g_1 : Z \rightarrow X$ and $g_2 : Z \rightarrow X$, the composed functions $f \circ g_1$ and $f \circ g_2$ are also distinct.

(First, make sure you understand what it means that two functions are equal and what it means that they are distinct.)

Exercise 3.44. Prove that in everyday mathematics, the number of elements of a set is understood in an intuitive sense and no definition is usually given. In the logical foundations of mathematics, however, the number of elements is defined via bijections: $|X| = n$ means that there exists a bijection from X to the set $\{1, 2, \dots, n\}$.¹

1. Prove that if X and Y have the same size according to this definition, then there exists a bijection from X to Y .
2. Prove that if X has size n according to this definition, and there exists a bijection from X to Y , then Y has size n too.
3. (*) Prove that a set cannot have two different sizes m and n , $m \neq n$, according to this definition. Be careful not to use the intuitive notion of “size” but only the definition via bijections. Proceed by induction.

3.4 ORDER STRUCTURES

Exercise 3.45. Let R be as from from Exercise 3.11. Prove that R is a partial order.

¹ Also other alternative definitions of set size exist but we will consider only this one here.

1. Let $S = \{(x, y) : x, y \in \mathbb{R} \text{ and } y \leq 0\}$ and let R be the relation on S defined by

$$(x_1, y_1) R (x_2, y_2) \Leftrightarrow x_1 = x_2 \text{ and } y_1 \leq y_2.$$

- (i) Show that R is a partial order on S .
 - (ii) Find all R -minimal elements.
2. Let $S = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots, 1\}$, and let $R := <$ the relation on S . Does R well order S ?
 3. Let $S = \mathbb{N} \setminus \{0\}$ and let R be the relation on S defined by

$$m R n \Leftrightarrow p < u \text{ or } (p = u \text{ and } q < v),$$

where $m = 2^p(2q + 1)$ and $n = 2^u(2v + 1)$ with p and u maximal possible exponents.

- (i) Show that R well orders S .
- (ii) Order the set $\{1, 2, \dots, 10\}$ with respect to R .
- (iii) Let $C = \{50, 51, 52, \dots\}$. Find R -minimal element of C .
- (iv) Find an immediate successor of 96.

3.5 GRAPH THEORY

1. Let $n \geq 3$. Recall the definition of cycles and complete graphs:

$$C_n = \{[n], E_1\}$$

$$K_n = \{[n], E_2\}$$

and define

$$G_n = \{[n], E_2 \setminus E_1\}$$

- Draw $H, G_4, G_5, G_6, C_5, C_6, \overline{C_i}$
- For all the above graphs, determine $\Delta(G_i), \delta(G_i), \alpha(G_i), \omega(G_i), \chi(G_i), g(G_i)$
- Prove $(\forall i \geq 3)(G_i \simeq \overline{C_i})$

2. Let $G = ([n], E)$ be a graph.

- Prove: $\chi(G) \geq \omega(G)$
- Prove: $\chi(G) \geq \frac{n}{\alpha(G)}$

4

SOLUTIONS

Answer for exercise 1.4: The logical implication is:

$$(A_1 \Rightarrow A_2) \wedge (A_1 \Rightarrow A_3) \Rightarrow (A_2 \Rightarrow A_3)$$

For $A_1(d) = 0, A_2(d) = 1, A_3(d) = 0$, this implication is false. Therefore, the reasoning is incorrect.

Answer for exercise 1.11:

Let us denote: A – Arthur is a knight, B – Bine is a knight, C – Cene is a knight.

The following compound statement holds:

$$(A \iff \neg C \vee \neg B) \wedge (B \iff C \wedge A)$$

With the help of the truth table, we see that the statement is true only for the set $A = 1$ and $B = C = 0$.

Answer for exercise 1.12: Let A represent "A is a knight", etc. We seek the only solution d such that the following is true:

$$A_1 \wedge B_1 \wedge C_1 \wedge D_1$$

where:

$$A_1 : A \iff (\neg D \wedge \neg C)$$

$$B_1 : B \iff (\neg A \wedge \neg D \Rightarrow \neg C)$$

$$C_1 : C \iff (\neg B \Rightarrow A)$$

$$D_1 : D \iff (\neg E \Rightarrow \neg C \wedge \neg B)$$

Since the truth table would contain 32 rows, we solve this by analyzing cases.

Case 1: $A(d) = 1$

Given A_1 , $D(d) = 0$ and $C(d) = 0$. Substituting into C_1 with $A(d) = 1$ and $C(d) = 0$, we get:

$$\neg(\neg B \Rightarrow 1) \Rightarrow \neg(B \vee 1) \Rightarrow \neg 1$$

which is a false statement. Hence, this case is not possible.

Case 2: $A(d) = 0$

Given A_1 , either $C(d) = 1$ or $D(d) = 1$.

Case 2.1: $C(d) = 1$

Since C_1 , $\neg B \Rightarrow 0$, implies $\neg B = 0$, hence $B(d) = 1$.

Substituting into B_1 with $A(d) = 0$, $B(d) = 1$, and $C(d) = 1$, we get:

$$1 \wedge \neg D \Rightarrow 0 \Rightarrow \neg D = 0$$

thus, $D(d) = 1$.

Substituting into D_1 , we get:

$$\neg E \Rightarrow 0 \wedge 0$$

Hence $E(d) = 1$.

Case 2.2: $C(d) = 0$ and $D(d) = 1$

From B_1 , we get $B(d) = 1$, but the statement C_1 becomes false: $0 \iff (0 \Rightarrow 1)$.

Thus, B , C , D , and E are knights, while A is a servant.

Answer for exercise 1.27:

$$\begin{aligned}
(A \Rightarrow B) \vee (B \Rightarrow C) &\Leftrightarrow (\neg A \vee B) \vee (\neg B \vee C) \\
&\Leftrightarrow \neg A \vee B \vee \neg B \vee C \\
&\Leftrightarrow \neg A \vee (B \vee \neg B) \vee C \\
&\Leftrightarrow \neg A \vee 1 \vee C \\
&\Leftrightarrow 1.
\end{aligned}$$

Answer for exercise 1.34: We need to show that $x + \frac{1}{x} \geq 2$. Since $x \geq 0$, we can multiply the inequality by x to get

$$x^2 + 1 \geq 2x$$

or equivalently,

$$(x - 1)^2 \geq 0.$$

This is obviously always true.

Answer for exercise 1.35: Suppose there are only finitely many primes p_1, p_2, \dots, p_n . Then the number $p = p_1 p_2 \cdots p_n + 1$ is not divisible by any prime p_i and $p_i \neq p$ for each i . By definition, p is therefore a prime number that is different from each of the previous ones. This is a contradiction.

Answer for exercise 1.36: The opposite statement is: there exists a natural number greater than 1.

Answer for exercise 1.38: We prove both directions separately:

(\Rightarrow) Assume that m and n have different parities. Write $m = 2k$ and $n = 2l + 1$, substitute into the expression $m^2 - n^2$, and the result follows.

(\Leftarrow) We show indirectly: If m and n have the same parity, then $m^2 - n^2$ is even. Consider both cases.

Answer for exercise 1.40:

- (i) $(A \Rightarrow B) \wedge A \Rightarrow B$. This is true.
- (ii) $(A \Rightarrow B) \wedge \neg A \Rightarrow \neg B$. This is not necessarily true.
- (iii) $((A \Rightarrow B) \wedge \neg A) \Rightarrow \neg B$. This is not necessarily true.
- (iv) $((A \Rightarrow B) \wedge (B \Rightarrow C)) \Rightarrow (\neg C \Rightarrow \neg A)$. This is true.

Answer for exercise 2.2:

- (i) $P \cap S \neq \emptyset$
- (ii) $0 \in \mathbb{Z} \setminus \mathbb{N}$
- (iii) $\mathbb{N} \subseteq \mathbb{Z}$
- (iv) $\mathbb{Z} \not\subseteq \mathbb{N}$
- (v) $P \setminus \{2\} \subseteq \bar{S}$
- (vi) $2 \in S \cap P$

Answer for exercise 2.4:

$$\begin{aligned}
 x \in (A \cup C) \cap (B \setminus C) &\Leftrightarrow (x \in A \vee x \in C) \wedge (x \in B \wedge x \notin C) \\
 &\Leftrightarrow ((x \in A \vee x \in C) \wedge (x \notin C)) \wedge x \in B \\
 &\Leftrightarrow ((x \in A \wedge x \notin C) \vee (x \in C \wedge x \notin C)) \wedge x \in B \\
 &\Leftrightarrow x \in A \wedge x \notin C \wedge x \in B \\
 &\Leftrightarrow x \in A \wedge x \in B \wedge x \notin C \\
 &\Leftrightarrow x \in (A \cap B) \setminus C.
 \end{aligned}$$

Answer for exercise 2.5: Use direct proof.

Answer for exercise 2.6: Prove separately each direction.

Answer for exercise 2.7:

$$\begin{aligned}(x, y) \in A \times (B \cap C) &\Leftrightarrow x \in A \wedge y \in B \cap C \\&\Leftrightarrow x \in A \wedge y \in B \wedge y \in C \\&\Leftrightarrow x \in A \wedge x \in A \wedge y \in B \wedge y \in C \\&\Leftrightarrow x \in A \wedge y \in B \wedge x \in A \wedge y \in C \\&\Leftrightarrow (x, y) \in A \times B \wedge (x, y) \in A \times C \\&\Leftrightarrow (x, y) \in (A \times B) \cap (A \times C).\end{aligned}$$

Answer for exercise 2.8:

$$\begin{aligned}x \in (A \cap B) \setminus B &\Leftrightarrow x \in (A \cap B) \wedge x \notin B \\&\Leftrightarrow (x \in A \wedge x \in B) \wedge x \notin B \\&\Leftrightarrow x \in A \wedge (x \in B \wedge x \notin B) \\&\Leftrightarrow x \in \emptyset.\end{aligned}$$

Answer for exercise 2.10: (In slovene.)

1. Napačna. Vzemi $A = \emptyset$, $B = \{\emptyset\}$, $B = \{\{\emptyset\}\}$.
2. Napačna. Vzemi isti primer kot v (a).
3. Pravilna. Dokaz s protislovjem. Recimo, da trditev ni pravilna. Naj bo $A \cap B \subseteq \overline{C}$, $A \cup C \subseteq B$ in naj obstaja $x \in A \cap C$. Torej je $x \in A$ in $x \in C$. Ker je po drugi predpostavki $A \cup C \subseteq B$, je $x \in B$. Sledi $x \in A \cap B$. Ker je po prvi predpostavki $A \cap B \subseteq \overline{C}$, je $x \in \overline{C}$. Protislovje, saj $x \in C$.
4. Napačna. Vzemi $A = C \neq B$.
5. Napačna. Vzemi tri paroma disjunktne neprazne množice.

Answer for exercise 2.21:

→ Let $A \subseteq B$. We will show that $A \cap \overline{B} = \emptyset$ holds. By assumption, we have

$$(\forall x)(x \in A \Rightarrow x \in B). \quad (1)$$

Suppose that there exists $x \in A \cap \overline{B}$. Then

$$\begin{aligned} x \in A \cap \overline{B} &\Rightarrow x \in A \wedge x \in \overline{B} \\ &\Rightarrow x \in A \wedge x \notin B \\ &\Rightarrow x \in B \wedge x \notin B \quad (\text{due to (1)}), \end{aligned}$$

a contradiction. Therefore, $A \cap \overline{B} = \emptyset$.

← Let $A \cap \overline{B} = \emptyset$. We will show that $A \subseteq B$. Take any $x \in A$. Then $x \notin \overline{B}$, since $A \cap \overline{B} = \emptyset$. Thus, $x \in B$. Since x was arbitrary, it follows that $A \subseteq B$.

1.

$$\begin{aligned} A \setminus B &= \{x; x \in A \wedge x \notin B\} \\ &= \{x; x \notin \overline{A} \wedge x \in \overline{B}\} \\ &= \{x; x \in \overline{B} \wedge x \notin \overline{A}\} \\ &= \overline{B} \setminus \overline{A}. \end{aligned}$$

Answer for exercise 3.4:

1. Domain: $\{1, 2, 3\}$, Range: $\{2, 3, 4\}$
2. Domain: $\{1, 2, 3, 4\}$, Range: $\{5, 3, 9\}$
3. Domain: $\{1, 3, 4\}$, Range: $\{2, 5\}$
4. Domain: $\{-1, 2, 3\}$, Range: $\{-1, 2, 3\}$
5. Domain: $\{2, 9\}$, Range: $\{0\}$

Answer for exercise 3.6: For $R_1 = \{(1,2), (2,3), (3,4)\}$ and $R_2 = \{(2,4), (3,5), (4,6)\}$:

$$R_2 \circ R_1 = \{(1,4), (2,5), (3,6)\}, \quad R_1 \circ R_2 = \{(2,4), (3,5)\}.$$

They are **not equal** because composition is not commutative.

Answer for exercise 3.7:

1. $R_2 \circ R_1 = \{(1,6), (2,4)\}$
2. $R_2 \circ R_1 = \{(1,4), (2,6), (3,4)\}$
3. $R_2 \circ R_1 = \{(4,5), (5,6), (6,4)\}$
4. $R_1 \circ R_1 = \{(4,4), (5,5), (6,6)\}$

Answer for exercise 3.8: $R_1 = \{(x,y) \mid y = x + 1\}$, $R_2 = \{(y,z) \mid z = y + 2\}$:

$$R_2 \circ R_1 = \{(x,z) \mid z = x + 3, x, z \in \mathbb{Z}\}.$$

Answer for exercise 3.9: For $R = \{(1,2), (2,3), (3,4)\}$:

$$R \circ R = \{(1,3), (2,4)\}, \quad R \circ R \circ R = \{(1,4)\},$$

while $R \circ R \circ R \circ R = \emptyset$

Answer for exercise 3.14: To check transitivity: If $(x,y) \in R$ and $(y,z) \in R$, then $(x,z) \in R$. Here, $(1,2) \in R$ and $(2,3) \in R$, but $(1,3) \notin R$. Thus, R is **not transitive**.