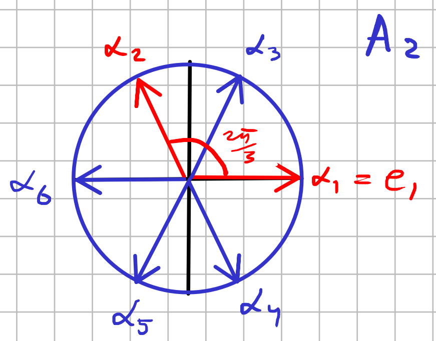
Алгебри Лі. Д/З N3. Корешков.

N1 A_2. $\circ-\circ$.



$\langle A_2 \rangle = E^2$, $|B| = 2$.

$B = \{\alpha_1, \alpha_2\}$

$$\begin{cases} \angle(\alpha_1, \alpha_2) = \frac{2\pi}{3} \\ |\alpha_1| = |\alpha_2| \end{cases}$$

Покладемо $\alpha_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_e$ $\alpha_2 = \begin{pmatrix} \cos\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} \end{pmatrix}_e = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}_e$

$C = (c_{ij}) = (\langle \alpha_i, \alpha_j \rangle)_{i,j=\overline{1,2}}$

$C_{11} = C_{22} = 2$

$C_{12} = \langle \alpha_1, \alpha_2 \rangle = 2\frac{(\alpha_1, \alpha_2)}{\cancel{(\alpha_2, \alpha_2)}_1} = 2\cancel{|\alpha_1|}^1\cancel{|\alpha_2|}^1\cancel{\cos\frac{2\pi}{3}}^{-\frac{1}{2}} = -1$

$C_{21} = \langle \alpha_2, \alpha_1 \rangle = 2\frac{(\alpha_1, \alpha_2)}{\cancel{(\alpha_1, \alpha_1)}_1} = 2\cancel{|\alpha_1|}^1\cancel{|\alpha_2|}^1\cancel{\cos\frac{2\pi}{3}}^{-\frac{1}{2}} = -1$

$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ – матриця Картана A_2.

N2

$B = \{\alpha_1, \alpha_2\}$ – база кор. сис. A_2.

Двоїста база $\check{B} = \{\check{\alpha} \mid \alpha \in B\}$, $\check{\alpha} = \frac{2\alpha}{(\alpha, \alpha)}$

$\check{\alpha}_1 = \frac{2\alpha_1}{\cancel{(\alpha_1, \alpha_1)}_1} = 2\alpha_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$\check{\alpha}_2 = \frac{2\alpha_2}{\cancel{(\alpha_2, \alpha_2)}_1} = 2\alpha_2 = \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}$

$|\check{\alpha}_1| = |\check{\alpha}_2|$

$(\check{\alpha}_1, \check{\alpha}_2) = -2$

$\cos(\check{\alpha}_1, \check{\alpha}_2) = -2/|\check{\alpha}_1||\check{\alpha}_2| = -\frac{1}{2}$ $\Rightarrow$ $\angle(\check{\alpha}_1, \check{\alpha}_2) = \frac{2\pi}{3}$

Таким чином $\check{A}_2 \sim A_2$, як і очікувалось.

Вагова система для A_2 задається фундаментальними вагами.
Позначу множину фундаментальних ваг як $\Omega = \{\omega_1, \omega_2\}$.
$|\Omega| = |B|$.

Умова на Ω: $\langle \omega_k, \alpha_m \rangle = \delta_{km}$

Це можна записати як

$$\Omega_B = \begin{pmatrix} | & | & & | \\ \omega_1^B & \omega_2^B & \cdots & \omega_n^B \\ | & | & & | \end{pmatrix}, \quad \Omega_{kl} = \text{коефіцієнт розкладу } \omega_l \text{ біля } \alpha_k$$

$$A = \begin{pmatrix} | & | & & | \\ \alpha_1^e & \alpha_2^e & \cdots & \alpha_n^e \\ | & | & & | \end{pmatrix}, \quad A_{kl} = \alpha_l^k = (\alpha_l, e_k).$$

$$\langle x, \alpha_m \rangle = \sum_{i=1}^n x_i^B \langle \alpha_i, \alpha_m \rangle = \sum_{i=1}^n x_i^B C_{im} = C_m^T \vec{x}^B$$

$\forall k, m$: $\langle \omega_k, \alpha_m \rangle = \delta_{km}$. Тобто $C^T \Omega = I \Rightarrow \Omega = C^{-T}$

Ω дає координати в базисі B.

Але нам потрібні координати в базисі e. $\Omega_e = A\Omega$.

Отже, $\Omega_e = AC^{-T}$.

Повернемося до №2.

$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. $\det C = 4-(+1) = 3$. $C^T = C$. $C^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = C^{-T}$.

Дійсно: $CC^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \cdot \frac{1}{3} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \cdot \frac{1}{3} = I$.

$$\Omega = C^{-T} = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix} \Rightarrow \boxed{\begin{aligned} \omega_1 &= (2/3 \;\; 1/3)_B = 2/3\,\alpha_1 + 1/3\,\alpha_2 \\ \omega_2 &= (1/3 \;\; 2/3)_B = 1/3\,\alpha_1 + 2/3\,\alpha_2 \end{aligned}}$$

Перевіримо:

$$\left.\begin{aligned} \langle \omega_1, \alpha_1 \rangle &= 2/3 \langle \alpha_1, \alpha_1 \rangle^{2} + 1/3 \langle \alpha_2, \alpha_1 \rangle^{-1} = 4/3 - 1/3 = 1 \\ \langle \omega_1, \alpha_2 \rangle &= 2/3 \langle \alpha_1, \alpha_2 \rangle^{-1} + 1/3 \langle \alpha_2, \alpha_2 \rangle^{2} = -2/3 + 2/3 = 0 \\ \langle \omega_2, \alpha_1 \rangle &= 1/3 \langle \alpha_1, \alpha_1 \rangle + 2/3 \langle \alpha_2, \alpha_1 \rangle = 0 \\ \langle \omega_2, \alpha_2 \rangle &= 1/3 \langle \alpha_1, \alpha_2 \rangle + 2/3 \langle \alpha_2, \alpha_2 \rangle = -1/3 + 4/3 = 1. \end{aligned}\right\} \langle \omega_k, \alpha_m \rangle = \delta_{km}$$

В e-базисі:

$$\Omega_e = A\Omega = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix} = \begin{pmatrix} 2/3 - 1/6 & 1/3 - 2/6 \\ \frac{\sqrt{3}}{6} & \frac{2\sqrt{3}}{6} \end{pmatrix} =$$

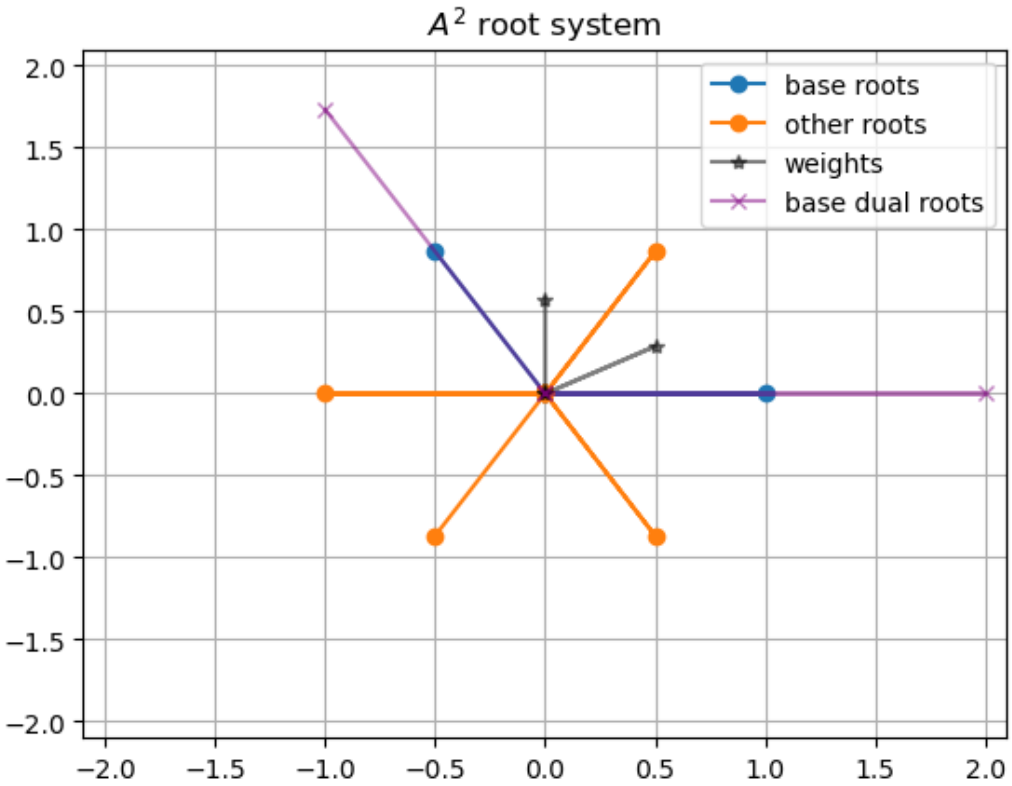
$$= \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{3} \end{pmatrix} \Rightarrow \boxed{\begin{aligned} \omega_1 &= \left(\tfrac{1}{2} \;\; \tfrac{\sqrt{3}}{6}\right)_e \\ \omega_2 &= \left(0 \;\; \tfrac{\sqrt{3}}{3}\right)_e \end{aligned}}$$

Перевіримо ще раз:

$$\begin{pmatrix} \check{\alpha}_1 = (2, 0) \\ \check{\alpha}_2 = (-1, \sqrt{3}) \end{pmatrix}$$

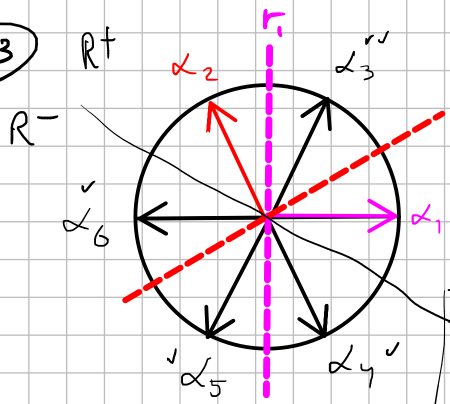
$$\left.\begin{aligned} (\omega_1, \check{\alpha}_1) &= \left(\tfrac{1}{2} \;\; \tfrac{\sqrt{3}}{6}\right) \cdot (2 \;\; 0) = 1 \\ (\omega_1, \check{\alpha}_2) &= \left(\tfrac{1}{2} \;\; \tfrac{\sqrt{3}}{6}\right) \cdot (-1 \;\; \sqrt{3}) = -\tfrac{1}{2} + \tfrac{3}{6} = 0 \\ (\omega_2, \check{\alpha}_1) &= \left(0 \;\; \tfrac{\sqrt{3}}{3}\right) \cdot (2 \;\; 0) = 0 \\ (\omega_2, \check{\alpha}_2) &= \left(0 \;\; \tfrac{\sqrt{3}}{3}\right) \cdot (-1 \;\; \sqrt{3}) = \tfrac{\sqrt{3}\sqrt{3}}{3} = 1. \end{aligned}\right\} (\omega_k, \check{\alpha}_m) = \delta_{km}$$

Рисуночек:



A^2 root system
base roots
other roots
weights
base dual roots
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0

N3



A_2

$$\left[\begin{array}{l}\alpha_1 = (1,0)_e \\ \alpha_2 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)_e\end{array}\right]$$

- r_1 діє як $(x_1, x_2)_e \mapsto (-x_1, x_2)_e$
- $r_2(x) = x - 2\frac{(x,\alpha_2)}{\cancel{(\alpha_2,\alpha_2)}_{1}}\alpha_2 =$

$$= x - 2\left[\begin{pmatrix}x_1\\x_2\end{pmatrix}_e \cdot \begin{pmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}_e\right]\cdot\begin{pmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}_e =$$

$$= x - 2\left(-\frac{x_1}{2} + \frac{\sqrt{3}x_2}{2}\right)\begin{pmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}_e =$$

$$= \begin{pmatrix} x_1 + \frac{1}{\cancel{2}}\cdot\cancel{2}\left(-\frac{x_1}{2} + \frac{\sqrt{3}x_2}{2}\right) \\ x_2 - \frac{\sqrt{3}}{\cancel{2}}\cdot\cancel{2}\left(-\frac{x_1}{2} + \frac{\sqrt{3}x_2}{2}\right)\end{pmatrix} =$$

$$= \begin{pmatrix} \frac{x_1}{2} + \frac{\sqrt{3}x_2}{2} \\ \frac{\sqrt{3}x_1}{2} - \frac{x_2}{2}\end{pmatrix}_e$$

$$r_1 = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} \qquad r_2 = \begin{pmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}$$

$$W = W_0 = \langle r_1, r_2\rangle$$

- $r_1\alpha_1 = -\alpha_1 = \alpha_6 = (-1, 0)_e =$
  $= -1\cdot\alpha_1 \quad \underline{\in R^-}$
- $r_2\alpha_2 = -\alpha_2 = \alpha_4 = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)_e =$
  $= -1\cdot\alpha_2 \quad \underline{\in R^-}$
- $r_1\alpha_2 = \alpha_3 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)_e =$
  $= \alpha_2 - \cancel{\langle\alpha_2,\alpha_1\rangle}^{-1}\alpha_1 = \alpha_2 + \alpha_1 \quad \underline{\in R^+}$

$$\boxed{\begin{array}{l}\alpha_3 = \alpha_1 + \alpha_2\\ \alpha_3 - \alpha_1 = \alpha_2 \in R^+\\ \alpha_3 - \alpha_2 = \alpha_1 \in R^+\\ \alpha_3 \geqslant \alpha_1, \quad \alpha_3 \geqslant \alpha_2\end{array}}$$

- $r_2\alpha_1 = \alpha_1 - \langle\alpha_1,\alpha_2\rangle\alpha_2 = \alpha_1 + \alpha_2 = \alpha_3$ – було
- $r_1r_2\alpha_1 = r_1r_1\alpha_2 = \alpha_2$ – було
- $r_1r_2\alpha_2 = r_1(\alpha_4) = -r_1\alpha_2 = -\alpha_3 = \alpha_5 = -\alpha_1 - \alpha_2 \quad \underline{\in R^-}$
- $r_2r_1\alpha_1 = r_2\alpha_6 = -r_2\alpha_1 = -\alpha_3 = \alpha_5$ – було
- $r_2r_1\alpha_2 = r_2\alpha_3 = r_2\alpha_2 + r_2\alpha_1 = -\alpha_2 + \alpha_3 = -\alpha_2 + \alpha_1 + \alpha_2 = \alpha_1$ – було
- $r_1r_2r_1\alpha_1 = -r_1r_2\alpha_1 = -r_1\alpha_3 = -r_1(\alpha_1 + \alpha_2) = \alpha_1 - r_1\alpha_2 =$
  $= \alpha_1 - \alpha_1 - \alpha_2 = -\alpha_2 = \alpha_4$ – було
- $r_1r_2r_1\alpha_2 = r_1(\alpha_1) = -\alpha_1 = \alpha_6$ – було
- $r_2r_1r_2\alpha_1 = r_2(\alpha_2) = -\alpha_2 = \alpha_4$ – було
- $r_2r_1r_2\alpha_2 = r_2(\alpha_5) = \alpha_3$ – було.

Породили все A_2 як $W_0(B)$.

$R_1 = \{\alpha_1, \alpha_2, \alpha_3 = \alpha_1 + \alpha_2\}$ $\qquad$ $R_2 = \{\alpha_6 = -\alpha_1, \alpha_4 = -\alpha_2, \alpha_5 = -\alpha_1 - \alpha_2\}$

Порядок :

- α_1 та α_2 не порівнювані, бо $\alpha_1 - \alpha_2,\ \alpha_2 - \alpha_1 \notin R^+$.
- $\alpha_1 \leq \alpha_3$, $\alpha_2 \leq \alpha_3$ — задає порядок на R^+

$$\begin{array}{ccccc} — & & \alpha_3 & & — \ \deg = 2 \\ & \swarrow & & \searrow & \\ — & \alpha_1 & & \alpha_2 & — \ \deg = 1 \end{array}$$

$\deg(\alpha_3) = \deg(\alpha_1 + \alpha_2) = 1 + 1 = 2$

$\deg(\alpha_1) = \deg(\alpha_2) = 1$.

- Старший корінь — α_3.