Analysis Portfolio

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Dear Dr. Shannon,

This portfolio is written with the intention to summarize key concepts learned over the semester. There is a broad range of proofs, ranging from chapter 1 to chapter 4. I felt that these proofs key in demonstrating my understanding of the material to employers in the mathematical field. Though I am not exactly sure where my mathematical and computational skills will take me, these proofs demonstrate my ability to problem solve and persevere until I reach the desired result.

Recently, I have been studying the words of prominent figures in the world, such as Dwayne Johnson, Will Smith, and Arnold Schwarzenegger to help build my understanding of the word 'success'. This whole semester I have struggled with time management and the complexity of my course work. I have been afraid of failure in terms of hindering my grade. However, I have learned that this fear of failure is even more so damaging to my life and academic career than succeeding in my coursework 100 percent of the time. Failing opens the doors to understanding and prospering. If I live in constant fear of getting something wrong, I will never take the opportunity to learn and build the foundation required to succeed. Denzel Washington helped me understand that, not failing means that I didn't even try.

This portfolio not only represents the coursework for the semester in Analysis, but it also represents my struggle to let myself fail and improve, fail and improve, fail and improve. This ultimately reaches out to employers who are looking to hire me for my skills, but are also looking for someone to grow with them, someone who is willing to be pushed to the ground and stand back up, ready to figure out what went wrong and how I can fix it. Most employers glance at your credentials, but are more concerned with your ability to problem solve. Thus, this portfolio is a statement of my ability to persevere, fail, and refine. This is a statement of my ability to never stop trying and reaching to be the best I can be. Good isn't good enough if it can be better, and better isn't good enough if it can be best.

Thank you for your help this semester. Not only have you provided me important knowledge of analysis and calculus, but you have also pushed me to learn more about myself and my journey to success. And hopefully, this portfolio is a demonstration of my journey to both you and my future employers.

1 Introduction

This portfolio combines a large range of proofs from the Analysis (Math 451) curriculum. The set includes proofs that summarize the main concepts of Analysis, such as (1) Set Theory, (2) Functions, (3) The positive numbers, (4) Limits, (5) Continuity, and (6) Differentiation. The proofs are presented in a chronological manner in reference to the chapters of the textbook, "An Introduction to Analysis" by Gerald G. Bilodeau, Paul Thie, and G. Keough.

Proofs Included: (1) Proof that $f(C \cap D) = f(C) \cap f(D)$, (2) Proof that f is one-to-one if and only if $f(C \cap D) = f(C) \cap f(D)$, (3) Proof that $2n^3 + 3n^2 + n$ is always divisible by 6, (4) Proof of the Product Rule, (5) Proof of Leibniz's formula

2 Proofs

2.1 Proof that $f(C \cap D) = f(C) \cap f(D)$

Let $f : A \to B$ be a one-to-one function and C and D subsets of A. Prove that $f(C \cap D) = f(C) \cap f(D)$.

Proof:

Part 1: Show $f(C \cap D) \subseteq f(C) \subseteq f(D)$

Let $y \in f(C \cap D)$.
Then $y = f(x)$ for some $x \in (C \cap D)$ by the definition of image.
Then $x \in C$ and $x \in D$ by the definition of intersection.
Since $x \in C, f(x) \in f(C)$.
Since $x \in D, f(x) \in f(D)$.
Since $y = f(x) \in f(C)$ and $y = f(x) \in f(D), y = f(x) \in f(C) \cap f(D)$ by the definition of intersection.
Thus, $f(C \cap D) \subseteq f(C) \cap f(D)$.

Part 2: Show $f(C) \cap f(D) \subseteq f(C \cap D)$.

Assume $f : A \to B$ is injective.
Let $y \in f(C) \cap f(D)$.
Then $y \in f(c)$ and thus $y = f(a)$ for some $a \in C$.
Also, $y = f(b)$ for some $b \in D$.
Since f is injective, $f(a) = f(b)$ implies $a = b$.
Thus, $a = b \in (C \cap D)$, and so $y = f(a) = f(b) \in f(C \cap D)$.
Thus, $f(C) \cap f(D) \subseteq f(C \in D)$.

Since in both parts the Left Hand Side and Right Hand Side are subsets of each other, $f(C \cap D) = f(C) \cap f(D)$.

Narrative: This proof demonstrates definitions associated with sets and functions. In order to show that the $f(C \cap D) = f(C) \cap f(D)$, we need to show that both sides are subsets of each other. We then use the definition of intersection and image to show how elements in sets and functions are related. The result is important for the next proof.

2.2 Proof that f is one-to-one if and only if $f(C \cap D) = f(C) \cap f(D)$

Let $f : A \to B$. Show that f is one-to-one if and only if $f(C \cap D) = f(C) \cap f(D)$ for every pair of sets C and D in A.

Part 1: If f is injective, $f(C \cap D) = f(C) \cap f(D)$.

By Part 2 of the previous proof, we have already shown that if f is injective, $f(C \cap D) = f(C) \cap f(D)$.

Part 2: If $f(C \cap D) = f(C) \cap f(D)$, then f is injective.

We first assume $f(C \cap D) \subseteq f(C) \cap f(D)$ for all subsets of $C, D \in A$.
Let $x_1, x_2 \in A$.
Let $C = x_1 and D = x_2$.
If $x_1 \neq x_2$, then $(C \cap D) = \emptyset$, thus $f(C \cap D) \subseteq f(C) \cap f(D)$, which also equals $f(\emptyset) = \emptyset$.
Thus, $f(C) \cap f(D) = \emptyset$, which implies $f(x_1) \neq f(x_2)$.
Thus, f is injective.

Narrative: In this proof, we use the result of the first proof as well as the definition of subsets and injective functions to show that the statement holds both directions. This proof was chosen because it demonstrates a difficult concept for me understand. Being able to use a given piece of information, such as the injectivity of a function, takes understanding of the concept.

2.3 Proof that $2n^3 + 3n^2 + n$ is always divisible by 6

Show by induction that $2n^3 + 3n^2 + n$ is always divisible by 6.

Proof:

Base Case: First, we want to test the statement for $n = 1$. Plugging in 1, we get $2(1)^3 + 3(1)^2 + 1 = 6$, and thus, since 6 is obviously divisible by 6, the statement is true for $n = 1$.

Induction Hypothesis: Next, we want to assume that the statement will hold for $n = k$:

$$2k^3 + 3k^2 + k = 6m$$

for some $m, k \in \mathbf{N}$.

Induction Step: Now, we want to show that the statement will hold for all values, or $n = k + 1$.

$$\begin{aligned}
2(k+1)^3 + 3(k+1)^2 + (k+1) &= 2k^3 + 9k^2 + 13k + 6 \\
&= (2k^3 + 3k^2 + k) + 6k^2 + 12k + 6 \\
&= 6m + 6k^2 + 12k + 6 \text{ by the Induction Hypothesis} \\
&= 6(m + k^2 + 2k + 1), \text{which is divisible by 6}
\end{aligned}$$

Thus by the Principle of Mathematical Induction, the statement is true for $n = k + 1$ thus true for all $n \in \mathbf{N}$.

Narrative: This proof is a classic example of induction on the positive integers. I find the results of these proofs to be very interesting and mostly easy to follow. I also tend to like very algebraic proofs that fizzle out into something completely satisfying and understandable. This proof was also selected because induction is a fundamental proof method when dealing with the positive integers.

2.4 Proof of the Product Rule

Prove that $(f \cdot g)'(c) = f'(c)g(c) + g'(c)f(c)$.

Proof:

By the definition of the derivative,

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Thus,

$$
\begin{aligned}
(f \cdot g)'(c) &= \lim_{x \to c} \frac{(f \cdot g)(x) - (f \cdot g)(c)}{x - c} \\
&= \lim_{x \to c} \frac{f(x)g(x) - f(c)g(c)}{x - c} \\
&= \lim_{x \to c} \frac{f(x)g(x) - f(c)g(x) + f(c)g(x) - f(c)g(c)}{x - c} \\
&= \lim_{x \to c} \frac{g(x)(f(x) - f(c)) + f(c)(g(x) - g(c))}{x - c} \\
&= \lim_{x \to c} \frac{g(x)(f(x) - f(c))}{x - c} + \lim_{x \to c} \frac{f(c)(g(x) - g(c))}{x - c} \\
&= f'(c)g(c) + f(c)g'(c)
\end{aligned}
$$

Thus, by the definition of the derivative, $(f \cdot g)'(c) = f'(c)g(c) + g'(c)f(c)$.

Narrative: This proof is one of my all time favorite proofs. The algebra of this proof boils down nicely and quickly the results show what you want to see. The product rule itself is a fundamental property of derivatives and is used commonly, and proving it gives me a sense of understanding. In the proof of the product rule, Theorem 4.1.8 is used to assume that at the point c, since $f(x)$ is differentiable, it is also continuous (Bilodeau et al., 2010). Thus, when the limit of $g(x)$ and $f(x)$ were taken, they were equal to the function at point c. My enjoyment of this proof also extends into similar proofs, such as the quotient rule, the sum rule and difference rule. Also, this proof was included because it is necessary to have for the next proof.

2.5 Proof of Leibniz's formula

Show by induction that

$$D^n(f \cdot g) = \sum_{k=0}^{n} \binom{n}{k} (D^{n-k} f)(D^k g)$$

Proof:

Base Case: We have already shown for the first derivative of the product of two functions, we can use the product rule, but by using Leibniz's formula:

$$\begin{aligned} D^1(f \cdot g) &= \binom{1}{0}(D^{1-0}f)(D^0g) + \binom{1}{1}(D^0f)(D^1g) \\ &= f'g + fg' \end{aligned}$$

Thus the statement is true for $n = 1$.

Induction Hypothesis: Next, we want to assume the statement will hold for $n = m$.

$$D^m(f \cdot g) = \sum_{k=0}^{m} \binom{m}{k}(D^{m-k}f)(D^kg)$$

for some $m \in N$.

Induction Step: Now we want to show that the statement will hold for all values, or $n = m + 1$.

$$\begin{aligned}
D^{m+1}(f \cdot g) &= D(D^m(f \cdot g)) \\
&= D\left(\sum_{k=0}^{m} \binom{m}{k}(D^{m-k}f)(D^k g)\right) \text{ by the Induction Hypothesis} \\
&= \sum_{k=0}^{m} \binom{m}{k} D((D^{m-k}f)(D^k g)) \\
&= \sum_{k=0}^{m} \binom{m}{k}((D^{m-k+1}f)(D^k g) + (D^{m-k}f)(D^{k+1}g)) \\
&= \sum_{k=0}^{m} \binom{m}{k}(D^{m-k+1}f)(D^k g) + \sum_{k=0}^{m} \binom{m}{k}(D^{m-k}f)(D^{k+1}g) \\
&= \sum_{k=0}^{m} \binom{m}{k}(D^{m-k+1}f)(D^k g) + \sum_{k=1}^{m+1} \binom{m}{k-1}(D^{m-k+1}f)(D^k g) \\
&= \binom{m}{0}(D^{m+1}f)(D^0 g) + \sum_{k=1}^{m} \binom{m}{k}(D^{m-k+1}f)(D^k g) \\
&+ \sum_{k=1}^{m+1} \binom{m}{k-1}(D^{m-k+1}f)(D^k g) \\
&= \binom{m}{0}(D^{m+1}f)(D^0 g) + \binom{m}{m}(D^0 f)(D^{m+1}g) \\
&+ \sum_{k=1}^{m} \binom{m}{k}(D^{m-k+1}f)(D^k g) + \sum_{k=1}^{m} \binom{m}{k-1}(D^{m-k+1}f)(D^k g) \\
&= \binom{m}{0}(D^{m+1}f)(g) + \binom{m}{m} f(D^{m+1}g) + \sum_{k=1}^{m}\left(\binom{m}{k} + \binom{m}{k-1}\right)(D^{m-k+1}f)(D^k g) \\
&= \binom{m}{0}(D^{m+1}f)(g) + \binom{m}{m} f(D^{m+1}g) + \sum_{k=1}^{m} \binom{m+1}{k}(D^{m-k+1}f)(D^k g) \\
&= \sum_{k=1}^{m+1}(D^{(m+1)-k}f)(D^k g)
\end{aligned}$$

Thus by the Principle of Mathematical Induction, the statement is true for $n = m + 1$ thus true for all $n \in \mathbf{N}$.

Narrative: This proof uses extra properties to get to the desired result, such as Pascals Triangle Identity which states that $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ and the property that $\binom{n}{0} = \binom{n}{n} = 1$ for all n. This proof is extremely similar to the proof of the binomial theorem and can be tedious to work out. This proof was chosen because of it's complexity compared to the first induction proof in this set. I also chose this proof because the result is similar to what the binomial theorem states, but for functions.

3 Conclusion

This set of proofs is only a small sample of Analysis. There are many more definitive proofs on very fundamental aspects of calculus, but this set highlights interesting and important results. I chose these proofs because the algebra was interesting and rather simple to understand. They were also selected to cover pieces of several chapters to try and summarize my understanding of some important concepts of analysis.

Work Cited

Bilodeau, G. G., Thie, P. R., & Keough, G. E. (2010). An Introduction to Analysis (2nd ed.). Sudbury, MA: Jones and Bartlett.