Subcritical and supercritical regimes in epidemic models of earthquake aftershocks

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[1] We present an analytical solution and numerical tests of the epidemic-type aftershock (ETAS) model for aftershocks, which describes foreshocks, aftershocks, and main shocks on the same footing. In this model, each earthquake of magnitude m triggers aftershocks with a rate proportional to $10^{\alpha m}$. The occurrence rate of direct aftershocks triggered by a single main shock decreases with the time from the main shock according to the "local" modified Omori law $K/(t + c)^p$ with $p = 1 + \theta$. Contrary to the usual definition, the ETAS model does not impose an aftershock to have a magnitude smaller than the main shock. Starting with a main shock at time $t = 0$ that triggers aftershocks according to the local Omori law, which in turn trigger their own aftershocks and so on, we study the seismicity rate of the global aftershock sequence composed of all the secondary and subsequent aftershock sequences. The effective branching parameter n, defined as the mean aftershock number triggered per event, controls the transition between a subcritical regime $n < 1$ and a supercritical regime $n > 1$. A characteristic time t^*, function of all the ETAS parameters, marks the transition from the early time behavior to the large time behavior. In the subcritical regime, we recover and document the crossover from an Omori exponent $1 - \theta$ for $t < t^*$ to $1 + \theta$ for $t > t^*$ found previously in the work of Sornette and Sornette for a special case of the ETAS model. In the supercritical regime $n > 1$ and $\theta > 0$, we find a novel transition from an Omori decay law with exponent $1 - \theta$ for $t < t^*$ to an explosive exponential increase of the seismicity rate for $t > t^*$. The case $\theta < 0$ yields an infinite n-value. In this case, we find another characteristic time τ controlling the crossover from an Omori law with exponent $1 - |\theta|$ for $t < \tau$, similar to the local law, to an exponential increase at large times. These results can rationalize many of the stylized facts reported for aftershock and foreshock sequences, such as (1) the suggestion that a small p-value may be a precursor of a large earthquake, (2) the relative seismic quiescence sometimes observed before large aftershocks, (3) the positive correlation between b and p values, (4) the observation that great earthquakes are sometimes preceded by a decrease of b-value, and (5) the acceleration of the seismicity preceding great earthquakes. *INDEX TERMS:* 3210 Mathematical Geophysics: Modeling; 7209 Seismology: Earthquake dynamics and mechanics; 7223 Seismology: Seismic hazard assessment and prediction; 7230 Seismology: Seismicity and seismotectonics; 7260 Seismology: Theory and modeling; *KEYWORDS:* ETAS, aftershock, foreshock, Omori law, triggering

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1. Introduction

[2] It is well known that the seismicity rate increases after a large earthquake, for time period up to 100 years [*Utsu et al.*, 1995], and distances up to several hundred kilometers [*Tajima and Kanamori*, 1985; *Steeples and Steeples*, 1996; *Kagan and Jackson*, 1998; *Meltzner and Wald*, 1999; *Dreger and Savage*, 1999]. The rate of the triggered events usually decays in time as the modified Omori law $n(t) = K/(t + c)^p$, where the exponent p is found to vary between 0.3 and 2 [*Davis and Frohlich*, 1991; *Kisslinger and Jones*, 1991; *Guo and Ogata*, 1995; *Utsu et al.*, 1995] and is often close to 1 (see, however, the works of *Kisslinger* [1993] and

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Gross and Kisslinger [1994] for alternative decay laws such as the stretched exponential).

[3] These triggered events are called aftershocks if their magnitude is smaller than the first event. However, the definition of an aftershock contains unavoidably a degree of arbitrariness because the qualification of an earthquake as an aftershock requires the specification of time and space windows. In this spirit, several alternative algorithms for the definition of aftershocks have been proposed [*Gardner and Knopoff*, 1974; *Molchan and Dmitrieva*, 1992] and there is no consensus.

[4] Aftershocks may result from several and not necessarily exclusive mechanisms [see *Harris*, 2001, and references therein]: pore pressure changes due to pore fluid flows coupled with stress variations, slow redistribution of stress by aseismic creep, rate-and-state dependent friction within faults, coupling between the viscoelastic lower crust and the brittle upper crust, stress-assisted microcrack corrosion [*Yamashita and Knopoff*, 1987; *Lee and Sornette*, 2000], slow tectonic driving of a hierarchical geometry with avalanche relaxation dynamics [*Huang et al.*, 1998], dynamical hierarchical models with heterogeneity, feedbacks, and healing [*Blanter et al.*, 1997], etc.

[5] Since the underlying physical processes are not fully understood, the qualifying time and space windows are more based on common sense than on hard science. Particularly, there is no agreement about the duration of the aftershock sequence and the maximum distance between aftershock and main shock. If one event occurs with a magnitude larger than the first event, it becomes the new main shock and all preceding events are retrospectively called foreshocks. Thus, there is no way to identify foreshocks from usual aftershocks in real time. There is also no way to distinguish aftershocks from individual earthquakes [*Hough and Jones*, 1997]. The aftershock magnitude distribution follows the Gutenberg–Richter distribution with similar b-value as other earthquakes [*Ranalli*, 1969; *Knopoff et al.*, 1982]. They have also similar rupture process. Moreover, an event can be both an aftershock of a preceding large event, and a main shock of a following earthquake. For example, the M = 6.5 Big Bear event is usually considered as an aftershock of the M = 7.3 Landers event, and has clearly triggered its own aftershock sequence. One can trace the difficulty of the problem from the long-range nature of the interactions between faults in space and time resulting in a complex self-organized crust.

[6] In view of the difficulties in classifying sometimes an earthquake as a foreshock, a main shock, or an aftershock, it is natural to investigate models in which this distinction is removed and to study their possible observable consequences. In this spirit, the epidemic-type aftershock (ETAS) model introduced by *Kagan and Knopoff* [1981, 1987] and *Ogata* [1988] provides a tool for understanding the temporal clustering of the seismic activity without distinguishing between aftershocks, foreshocks and main shock events. The ETAS model is a generalization of the modified Omori law, which takes into account the secondary aftershocks sequences triggered by all events. In this model, all earthquakes are simultaneously main shocks, aftershocks and possibly foreshocks. An observed ''aftershock'' sequence is in the ETAS model the result of the activity of all events triggering events triggering themselves other events, and so on, taken together. The ETAS model aims at modeling complex aftershock sequences and global seismic activity. The seismicity rate is given by the superposition of aftershock sequences of all events. Each earthquake of magnitude m triggers aftershock with a rate proportional to $10^{\alpha m}$ with the same coefficient α for all earthquakes. The occurrence rate of aftershocks decreases with the time from the main shock according to the modified Omori law $K/(t + c)^p$. The background seismicity rate is modeled by a stationary Poisson process with a constant occurrence rate μ. Contrary to the usual definition, the ETAS model does not impose an aftershock to have a magnitude smaller than the main shock. This way, the same law describes foreshocks, aftershocks, and main shocks. This model has been used to give short-term probabilistic forecast of seismic activity [*Kagan and Knopoff*, 1987; *Kagan and Jackson*, 2000; *Console and Murru*, 2001], and to describe the temporal and spatial clustering of seismic activity [*Ogata*, 1988, 1989, 1992, 1999, 2001; *Kagan*, 1991; *Felzer et al.*, 2002]. Although the elementary results on the stability of the process have been known for many years [*Kagan*, 1991], no attempt has been made to study this model analytically in order to characterize its different regimes and obtain a deeper understanding of the combined interplay between the model parameters (b, α, p, K, c, and μ) on the seismic activity. We stress below the contrast between previous works in the mathematical statistical literature and our results.

[7] It should be noted that the ETAS model suffers from an important defect: it is fundamentally a ''branching'' model [*Harris*, 1963; *Vere-Jones*, 1977], with no ''loops.'' What this means is that an event has a unique ''mother main shock'' and not several. In the real case, we can expect that some events may be triggered by the combined loading and action at distance in time and space of several previous earthquakes. Hence, events should have several ''mothers'' in general. This neglecting of ''loops'' is known in statistical physics as a ''mean-field'' approximation and allows us to simplify the analysis while still keeping the essential physics in a qualitative way, even if the details may not be precisely recovered quantitatively.

[8] *Sornette and Sornette* [1999] studied analytically a particular case of the ETAS model, in which the aftershock number does not depend on the main shock magnitude, i.e., for $\alpha = 0$. Starting with one event at time $t = 0$ and considering that each earthquake generates an aftershock sequence with a ''local'' Omori exponent $p = 1 + \theta$, where θ is a positive constant, they studied the decay law of the ''global'' aftershock sequence, composed of all secondary aftershock sequences. They found that the global aftershock rate decays according to an Omori law with an exponent $p = 1 - \theta$, smaller than the local one, up to a characteristic time t^*, and then recovers the local Omori exponent $p = 1 + \theta$ for time larger than t^*.

[9] Here, we generalize their analysis in the more general case $\alpha > 0$ of the ETAS model, which includes a realistic magnitude distribution. We study the decay law of the global aftershock sequence as a function of the model parameters (local Omori law parameters and magnitude distribution). In addition to giving more complete analytical results, we present numerical simulations that test these predictions. We also generalize the investigation and

analysis into the "supercritical" regime. Indeed, depending on the branching ratio n, defined as the mean aftershock number triggered per event, and on the sign of θ, three different regimes for the seismic rate $N(t)$ are found:

1. For $n < 1$ (subcritical regime), we recover the results of *Sornette and Sornette* [1999], i.e., we find a crossover from an Omori exponent $p = 1 - \theta$ for $t < t^*$ to $p = 1 + \theta$ for $t > t^*$.

2. For $n > 1$ and $\theta > 0$ (supercritical regime), we find a transition from an Omori decay law with exponent $p = 1 - \theta$ to an explosive exponential increase of the seismicity rate.

3. In the case $\theta < 0$, we find a transition from an Omori law with exponent $1 - |\theta|$ similar to the local law, to an exponential increase at large times, with a crossover time τ different from the characteristic time t^* found in the case $\theta > 0$.

[10] As we show below, these results can rationalize many properties of aftershock and foreshock sequences.

2. The Model

[11] We assume that a given event (the "mother") of magnitude $m_i \geq m_0$ occurring at time t_i gives birth to other events ("daughters") in the time interval between t and $t + dt$ at the rate

$$\phi_{m_i}(t - t_i) = \frac{K\ 10^{\alpha(m_i - m_0)}}{(t - t_i + c)^{1+\theta}} H(t - t_i) H(m_i - m_0), \qquad (1)$$

where H is the Heaviside function: $H(t - t_i) = 0$ for $t < t_i$ and 1 otherwise, m_0 is a lower bound magnitude below which no daughter is triggered.

[12] This temporal power law decay follows the same mathematical law as Omori's law for the rate of aftershocks following a main shock, albeit with the modification that we do not specify that aftershocks (daughter earthquakes) have to be smaller than the triggering event (mother earthquake). The exponential term $10^{\alpha(m - m_0)}$ describes the fact that the larger the magnitude m of the mother event, the larger is the number of daughters. The exponent $p = 1 + \theta$ of the "local" Omori's law has no reason a priori to be the same as the one measured macroscopically which is usually found between 0.8 and 1.2 with an often quoted median value 1. This is in fact the question we address: assuming the form (1) for the "local" Omori's law, is the global Omori's law still a power law and, if yes, how does its exponent depend on p? What are the possible regimes of aftershocks as a function of the parameters of the model?

[13] This model can be extended to describe the spatio-temporal distribution of seismic activity. Following the work of *Kagan and Knopoff* [1981], we can introduce a spatial dependence in (1) of the form

$$\phi_{m_i}(t - t_i, \vec{r} - \vec{r}_i) = \frac{K\ 10^{\alpha(m_i - m_0)}}{(t - t_i + c)^{1+\theta}} \rho(\vec{r} - \vec{r}_i) H(t - t_i) H(m_i - m_0), \qquad (2)$$

where $\rho(\vec{r} - \vec{r}_i)$ describes the probability distribution for an earthquake occurring at position $\vec{r}_i$ to trigger an event at position $\vec{r}$. This term takes into account the spatial dependence of the stress induced by an earthquake and enables us to model the spatial distribution of aftershocks clustered close to the main shock. In this paper, we restrict our analysis to the temporal ETAS model without spatial dependence because we are mainly interested in describing the temporal evolution of seismic activity. The complete model with both spatial and temporal dependence (2) has been studied by *Helmstetter and Sornette* [2002] to derive the joint probability distribution of the times and locations of aftershocks including the whole cascade of secondary aftershocks. When integrating the rate of aftershocks calculated for the spatiotemporal ETAS model over the whole space, we recover the results given in this paper for the temporal ETAS model. Therefore, the results given here for the temporal ETAS model can be compared with real aftershock sequences when using all aftershocks whatever their distance from the main shock.

[14] The model (1) is a branching process because each daughter has only one mother and not several, as shown in Figure 1. As we said in the introduction, this "mean-field" assumption simplifies considerably the complexity of the process and allows for an analytical solution that we shall derive in the sequel. The key parameter is the average number n of daughter earthquakes created per mother event. Assuming that the distribution $P(m)$ of earthquake sizes expressed in magnitudes m follows the Gutenberg–Richter distribution $P(m) = b\ \ln(10)\ 10^{-b(m - m_0)}$, the integral of $\phi_m(t)$ over time and over all magnitudes $m \geq m_0$ gives

$$n \equiv \int_0^{+\infty} dt \int_{m_0}^{+\infty} dm\ P(m) \phi_m(t) = n_0 \int_0^{\infty} \frac{dt}{(t + 1)^{1+\theta}}, \qquad (3)$$

where

$$n_0 \equiv \frac{K}{c^\theta} \frac{b}{b - \alpha}, \qquad (4)$$

which is finite for $b > \alpha$. Three cases are analyzed below: $n < 1$, $n = 1$ and $n > 1$. The case $n = 1$ corresponds to an average conservation of the number of events and can be associated with a brittle elastic crust without dissipation. The "dissipative" case $n < 1$ can be interpreted as corresponding to a crust possessing a viscoelastic component and/or a partial coupling with a lower ductile layer, such that a part of the energy is released aseismically. The case $n > 1$ corresponds to a process in which an earthquake sequence triggers an inflow of energy from surrounding regions that may lead to a local self-exciting amplification. It can also correspond to a coupling with other nonmechanical modes of energy storage, such as proposed by *Sornette* [2000b] and *Viljoen et al.* [2002] which can be triggered by an event and feed the ensuing earthquake sequence for a while. Of course, the supercritical process can only be transient and has to crossover to another regime.

[15] The case $b < \alpha$ requires a special attention. In absence of truncation or cutoff, it leads to a finite-time singularity due to the interplay between long-memory and extreme fluctuations [*Sornette and Helmstetter*, 2002]. However, it is more common to introduce a truncation or roll-off of the Gutenberg–Richter law at an upper magnitude. We can for example use a Gamma distribution of

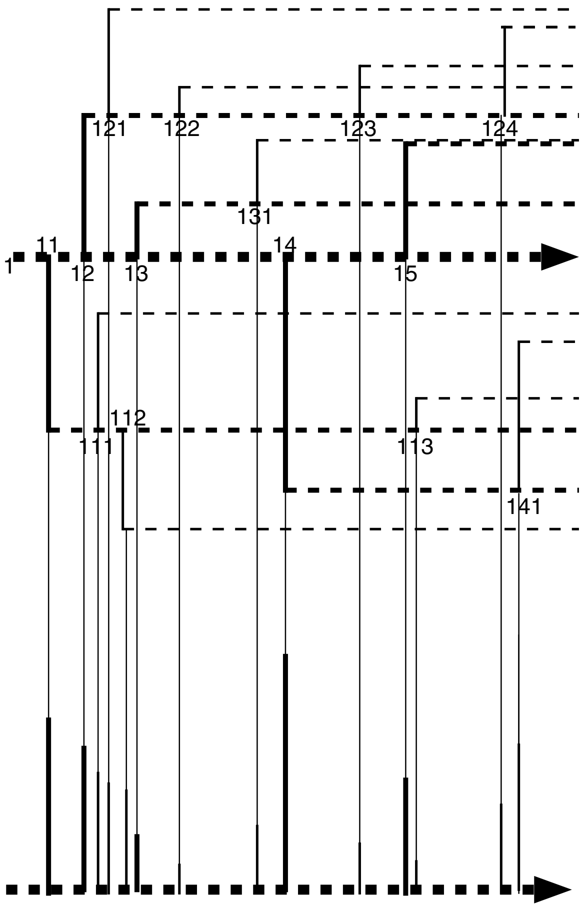

Figure 1. Schematic representation of the branching process associated with the ETAS model defined by (1) and (3). In this example, the thickest dashed line is the time arrow associated with the main shock indicated as "1." This main shock triggered five direct aftershocks (of first generation) denoted "11," "12," "13," "14," and "15" whose magnitudes are proportional to the length of their vertical lines (their position above or below the thickest dashed line is arbitrary and chosen to ensure a better visibility of the diagram). The aftershock "11" triggered three (secondary) aftershocks denoted "111," "112," and "113." The aftershock "12" triggered four aftershocks denoted "121," "122," "123," and "124." The aftershock "13" triggered a single aftershock denoted "131." The aftershock "14" also triggered a single aftershock denoted "141." The aftershock "15" did not trigger any aftershock. The observable catalog is the superposition of all these events, which are projected on the thick dashed line at the bottom of the figure, keeping the thickness as a code for the generation number of each event.

energies, which is a power law distribution tapered by an exponential tail. In this case, the branching ratio has been calculated by *Kagan* [1991] and is given by the approximate analytical expression valid for a corner magnitude m_c significantly larger than m_0,

$$n_0 = \frac{K}{c^\theta}\,\frac{b}{b-\alpha}\,\frac{10^{b(m_c-m_0)} - 10^{\alpha(m_c-m_0)}}{10^{b(m_c-m_0)} - 1}. \tag{5}$$

For a corner magnitude $m_c \gg m_0$, and for $\alpha < b$, we recover the expression (4) for n_0 obtained for the Gutenberg–Richter distribution without roll-off.

[16] Note that n is defined as the average over all main shock magnitudes of the mean number of events triggered by a main shock. It is thus grossly misleading to think of the branching ratio as giving the number of daughters to a given earthquake, because this number is extremely sensitive to the specific value of its magnitude. Indeed, the number of aftershocks to a given main shock increases exponentially with the main shock magnitude as given by (1), so that large earthquakes will have many more aftershocks than small earthquakes. From (1) and (3), we can calculate the mean number of aftershocks $N(M)$ triggered directly by a main shock of magnitude M

$$N(M) = n\frac{(b-\alpha)}{b}10^{\alpha(M-m_0)}. \tag{6}$$

As an example, take $\alpha = 0.8$, $b = 1$, $m_0 = 0$ and $n = 1$. Then, a main shock of magnitude $M = 7$ will have on average 80000 direct aftershocks, compared to only 2000 direct aftershocks for an earthquake of magnitude $M = 5$ and less than 0.2 aftershocks for an earthquake of magnitude $M = 0$.

[17] When $\theta > 0$, $\int_0^\infty \frac{dt}{(t+1)^{1+\theta}} = 1/\theta$ and the branching ratio $n = n_0/\theta$ is finite. In this regime, n is an increasing function of the rate K and a decreasing function of θ, c and $b - \alpha$.

[18] Even for $b > \alpha$ and $\theta > 0$, the average number of daughters per mother can be larger than one: $n > 1$. This regime corresponds to the supercritical regime of branching processes [*Harris*, 1963; *Sornette*, 2000a] in which the total number of events grows on average exponentially with time. If $n < 1$, there is less than one earthquake triggered per earthquake on average. This is the subcritical regime in which the number of events following the first main shock decays eventually to zero. The critical case $n = 1$ is at the borderline between the two regimes. In this case, there is exactly one earthquake on average triggered per earthquake and the process is exactly at the critical point between death on the long run and exponential proliferation.

[19] There is another scenario, occurring for $\theta \le 0$, in which the seismicity blows up exponentially with time. In this case, the integral $\int_0^\infty \frac{dt}{(t+1)^{1+\theta}}$ becomes unbounded. In principle, n becomes infinite: this does not invalidate the ETAS model per se. It only reflects the fact that the calculation of an average number of daughters per mother has become meaningless because of the anomalously slow decay of the kernel $\phi(t)$. This mechanism is reminiscent of that leading to anomalous diffusion and to aging in quenched random systems and spin glasses (see the work of *Sornette* [2000a] for an introduction). As in these systems, any estimation of the averages depend on the timescale of study: due to the extremely slow decay of $\phi(t)$, the number of daughters created beyond any time t far exceeds the number of daughters created up to time t. Notwithstanding the decay, its cumulative effect creates this dominance of the far future. This regime is the opposite of the situation where $\theta > 0$ where most of the daughters are created at relatively early times. Since the number of daughters born up to time t is an unbounded increasing function of t, it is intuitively appealing, as we show in Appendix A, that this regime should be

similar to the supercritical regime $n > 1$ discussed above in the case $\theta > 0$.

[20] Until now, we have discussed three issues related to the convergence of the ETAS sequences: (1) the condition $\theta > 0$ ensures convergence at large times, (2) the convergence at short times is obtained by the introduction of the regularization constant c in the generalized Omori's law, and (3) the condition $\alpha < b$ is a necessary condition for the finiteness of the number of daughters. Finally, we should stress the role of the "ultraviolet" cutoff m_0 on the magnitudes. In the ETAS model, only earthquakes of magnitude $m \geq m_0$ are allowed to give birth to aftershocks, while events of smaller magnitudes are lost for the epidemic dynamics. If such a cutoff is not introduced and no cutoff is put on the Gutenberg–Richter toward small magnitudes, the dynamics becomes completely dominated by the swarms of very tiny earthquakes, which individually has very low probability to generate aftershocks but become so numerous that their collective effect becomes overwhelming in the dynamics. We would thus have the unphysical situation in which a magnitude 7 or 8 earthquake may be triggered by tiny earthquakes of magnitudes -2 or less. We stress that the introduction of such a cutoff m_0 is a simple way to prevent such a situation to occur, but it does not mean that small earthquakes of magnitude below m_0 do not have their own aftershocks. It only means that such small earthquakes create aftershocks that can not participate in the epidemic process leading to significantly larger earthquakes; these small earthquakes live their separate life. This is why they are not registered by the ETAS model. This formulation is of course only an end-member of many possible regularization procedures, which are well known to be an ubiquitous requisite in mechanical models of rupture. An improvement of the ETAS model would be for instance to replace this abrupt cutoff m_0 by introducing a roll-off in the Gutenberg–Richter law for the aftershocks with a characteristic corner magnitude decreasing with the magnitude of the mother earthquake. This and other schemes will not be explored here, as we want to analyze the simplest version possible.

[21] We now describe briefly the connection with previous works in the mathematical statistics literature. As we said above, the model (1) belongs to the general class of branching models [*Moyal*, 1962; *Harris*, 1963]. The elementary results on the stability of the process, such as the condition $n < 1$, have been known for many years, and go back to the origin of the ETAS model as a special case (for discrete magnitudes) or extension (for continuous magnitudes) of the class of "mutually exciting point processes" introduced by *Hawkes* [1971, 1972] and *Hawkes and Adamapoulos* [1973]. A convenient mathematical overview is in Chapter 5 of the work of *Daley and Vere-Jones* [1988], especially Example 5.5(a) and associated exercises 5.5.2–5.5.6. For the ETAS model, the equations governing the probability generating functional, the probability of extinction within a given number of generations, the expectation measure for the total population, the second factorial moment (related to the covariance of the population) and their Fourier transform can be derived as special cases of results summarized there. In particular, the process initiated with a single event at the origin corresponds to the total progeny process for a general branching process model with time-magnitude state-space and a single ancestor at time $t = 0$; Exercise 5.5.6 gives the equations of the above cited variables for the case of fixed magnitudes (i.e., $\alpha = 0$). This direct probabilistic analysis in terms of generating functions effectively replaces the Wiener–Hopf theory in the present paper and mentioned also by *Hawkes* [1971, 1972] and *Hawkes and Oakes* [1974]. However, there is not explicit solutions given to these equations and there is no discussion of the change of regime from an effective Omori's law $1/t^{1-\theta}$ at early times to $1/t^{1+\theta}$ at long times, nor mention of the interesting supercritical case, as done in the present work.

[22] *Hawkes* [1971, 1972] and *Hawkes and Adamapoulos* [1973] use what is in effect an ETAS model with an exponential "bare" Omori's law rather than the power law $1/(t + c)^{1+\theta}$ defined in (1). *Hawkes and Adamapoulos* [1973] use it in an early study of earthquake data. The introduction of magnitudes is similar to the introduction of a marked process associated with a single point process [*Hawkes*, 1972]; however, the impact of magnitudes on the seismicity rate is assumed to be linear by *Hawkes* [1972] while it is multiplicative in the ETAS model. Our derivation presented in Appendix A of the solution of the ETAS model for the mean rate of earthquakes in terms of its Laplace transform recovers previous results. For instance, equation (17) in the work of *Hawkes and Oakes* [1974] is the same as our (A6) in Appendix A (up to a factor β stemming from taking the cumulative number in the work of *Hawkes and Oakes* [1974]). The key factor $Q(\beta)$ in (A7) corresponds to the quantity $G_1(0)$ in equation (5) in the work of *Hawkes* [1972]. The link between Hawkes' "mutually exciting point processes" and branching processes was made explicit by *Hawkes and Oakes* [1974].

[23] Some average properties of the ETAS model have been derived in the Master thesis of *Ramselaar* [1990]. Specifically, using the theory of Markov processes applied to branching processes, *Ramselaar* [1990] proves that, in the supercritical regime $n > 1$ (where n is the average branching ratio defined in (5)), the average number of aftershocks stemming from a common ancestor grows exponentially as $\sim e^{t/t^*}$ where t^* is the solution of $n\, R(c/t^*) = 1$ and the function R is defined in (A9). The solution of this equation $n\, R(c/t^*) = 1$ for t^* is the same as our t^* given by (12) and the exponential growth of Ramselaar is therefore the same as our result (17). We add on this asymptotic result, which is valid only at large times, by exhibiting the solution for the aftershock decay at early times. In addition, contrary to the incorrect claim of *Ramselaar* [1990] that "the Ogata earthquake process is critical or supercritical but is never subcritical," we demonstrate that the subcritical regime exhibits a rich phenomenology.

3. Analytical Solution

[24] We analyze the case where there is an origin of time $t = 0$ at which we start recording the rate of earthquakes, assuming that the largest earthquake of all has just occurred at $t = 0$ and somehow reset the clock. In the following calculation, we will forget about the effect of events preceding the one at $t = 0$ and count aftershocks that are created only by this main shock.

[25] Let us call $N_m(t)$ the rate of seismicity at time t and at magnitude m, that is, $N_m(t)\, dtdm$ is the number of events in

the time/magnitude interval $dt \times dm$. We define its expectation $\lambda_m(t)dtdm \equiv \mathrm{E}[N_m(t)\ dt\ dm]$, as the mean number of earthquakes occurring between t and $t + dt$ of magnitude between m and $m + dm$. $\lambda_m(t)$ is the solution of a self-consistency equation that formalizes mathematically the following process: an earthquake may trigger aftershocks; these aftershocks may trigger their own aftershocks, and so on. The rate of seismicity at a given time t is the result of this cascade process. The self-consistency equation that sums up this cascade reads

$$\lambda_m(t) \equiv \mathrm{E}[N_m(t)] = E\left[\int_{m_0}^{\infty} dm' \int_{-\infty}^{t} d\tau\ \phi_{m'}(t-\tau)P(m)N_{m'}(\tau)\right] \tag{7}$$

$$= P(m)\int_{m_0}^{\infty} dm' \int_{-\infty}^{t} d\tau\ \phi_{m'}(t-\tau)\mathrm{E}[N_{m'}(\tau)] \tag{8}$$

$$= P(m)\int_{m_0}^{\infty} dm' \int_{-\infty}^{t} d\tau\ \phi_{m'}(t-\tau)\lambda_{m'}(\tau). \tag{9}$$

If there is an external source $S(t,m)$, it should be added to the right-hand side of (9).

[26] The mean instantaneous rate $\lambda_m(t)$ at time t is the sum over all induced rates from all earthquakes of all possible magnitudes that occurred at all previous times. The rate of events at time t induced per earthquake that occurred at an earlier time τ with magnitude m' is equal to $\phi_{m'}(t-\tau)$. The term $P(m)$ is the probability that an event triggered by an earthquake of magnitude m' is of magnitude m. We assume that this probability is independent of the magnitude of the mother earthquake and is nothing but the Gutenberg–Richter law. This hypothesis can be easily relaxed if needed and $P(m)$ can be generalized into $P(m|m')$ giving the probability that a daughter earthquake is of magnitude m conditioned on the value m' of the magnitude of the mother earthquake. However, we do not pursue here this possibility as this hypothesis seems well founded empirically [*Ranalli*, 1969; *Knopoff et al.*, 1982]. The term $S(t, m)$ is an external source, which is determined by the physical process. We consider the case where a great earthquake occurs at the origin of time $t = 0$ with magnitude M. In this case, the external source term is

$$S(t,m) = \delta(t)\ \delta(m - M), \tag{10}$$

where δ is the Dirac distribution. Other arbitrary source functions can be chosen.

[27] By construction of the kernel (1), it is natural to search the solution for $\lambda_m(t)$ as

$$\lambda_m(t) = P(m)\lambda(t), \tag{11}$$

which makes explicit in the solution the hypothesis of a separation of the variables magnitude and time. A. Helmstetter et al. (Mainshock are aftershocks of conditional foreshocks: How do foreshock statistical properties emerge from aftershock laws?, submitted to *Journal of Geophysical Research*, 2002) have shown that (11) is a correct ansatz for $\alpha \leq b/2$, which is the regime considered here. For $\alpha \geq b/2$, large fluctuations prevent the decoupling between time and magnitude to hold and lead to corrections to the predictions presented here, which, due to their complexity, will be described elsewhere. The ETAS model assumes that the time response and the magnitude response are independent at each generation. In reality and more generally, we can envision that the rate of activation of new earthquakes will depend on (1) the magnitude of the "mother" (which the ETAS model takes into account multiplicatively in (1)), (2) on the magnitude of the daughter (which is neglected in the ETAS model), and (3) on the time since the mother was born. Rather than having a very general kernel combining these three parameters nonlinearly, equation (1) is based on a hypothesis of independence between these different factors. In addition, assuming that the cascade of secondary aftershocks does not spoil this independence, this allows us to factorize them, leading to (11).

[28] The problem is then to determine the functional form of $\lambda(t)$, assuming that ϕ is given by (1). The integral equation (9) is a Wiener–Hopf integral equation [*Feller*, 1971]. It is well known [*Feller*, 1971; *Morse and Feshbach*, 1953] that, if $\phi(\tau)$ decays no slower than an exponential, then $\lambda(t)$ has an exponential tail $\lambda(t) \sim \exp[-rt]$ for large t with r solution of $\int \phi(x) \exp[rx]\, dx = 1$. This result implies that a global Omori's law cannot be obtained by the epidemic ETAS branching model with, for instance, local exponential relaxation rates. In the present case, $\phi(\tau)$ decays much slower than an exponential and a different analysis is called for that we now present. The solution of (9) is derived in Appendix A and is summarized in the following sections. For the sequel, it is useful to define the characteristic time

$$t^* \equiv c\left(\frac{n\ \Gamma(1-\theta)}{|1-n|}\right)^{\frac{1}{\theta}}, \tag{12}$$

where $\Gamma(x)$ is the Gamma function: $\Gamma(z) = \int_0^\infty du\ u^{z-1}e^{-u}$ which is nothing but $(z-1)!$ for positive integers z.

3.1. The Subcritical Regime n < 1 and θ > 0

[29] An approximation is made in the analytical solution so that the results presented below are only valid for $t \gg c$.

[30] We define the parameter S_0 that describes the external source term

$$S_0 = \frac{(b-\alpha)}{b} 10^{\alpha(M-m_0)}. \tag{13}$$

[31] Two cases must be distinguished.

1. For $c \ll t \ll t^*$, we get

$$\lambda_{t<t^*}(t) \sim \frac{S_0}{\Gamma(\theta)|1-n|}\ \frac{t^{*-\theta}}{t^{1-\theta}} \qquad \text{for } c \ll t \ll t^*. \tag{14}$$

2. For $t \gg t^*$, we obtain

$$\lambda_{t>t^*}(t) \sim \frac{S_0}{\Gamma(\theta)(1-n)}\ \frac{t^{*\theta}}{t^{1+\theta}} \qquad \text{for } t \gg t^*. \tag{15}$$

[32] We verify the self-consistency of the two solutions $\lambda_{t>t^*}(t)$ and $\lambda_{t<t^*}(t)$ by checking that $\lambda_{t>t^*}(t^*) = \lambda_{t<t^*}(t^*)$. In other words, t^* is indeed the transition time at which the "short-time" regime $\lambda_{t<t^*}(t)$ crosses over to the "long-time" regime $\lambda_{t>t^*}(t)$.

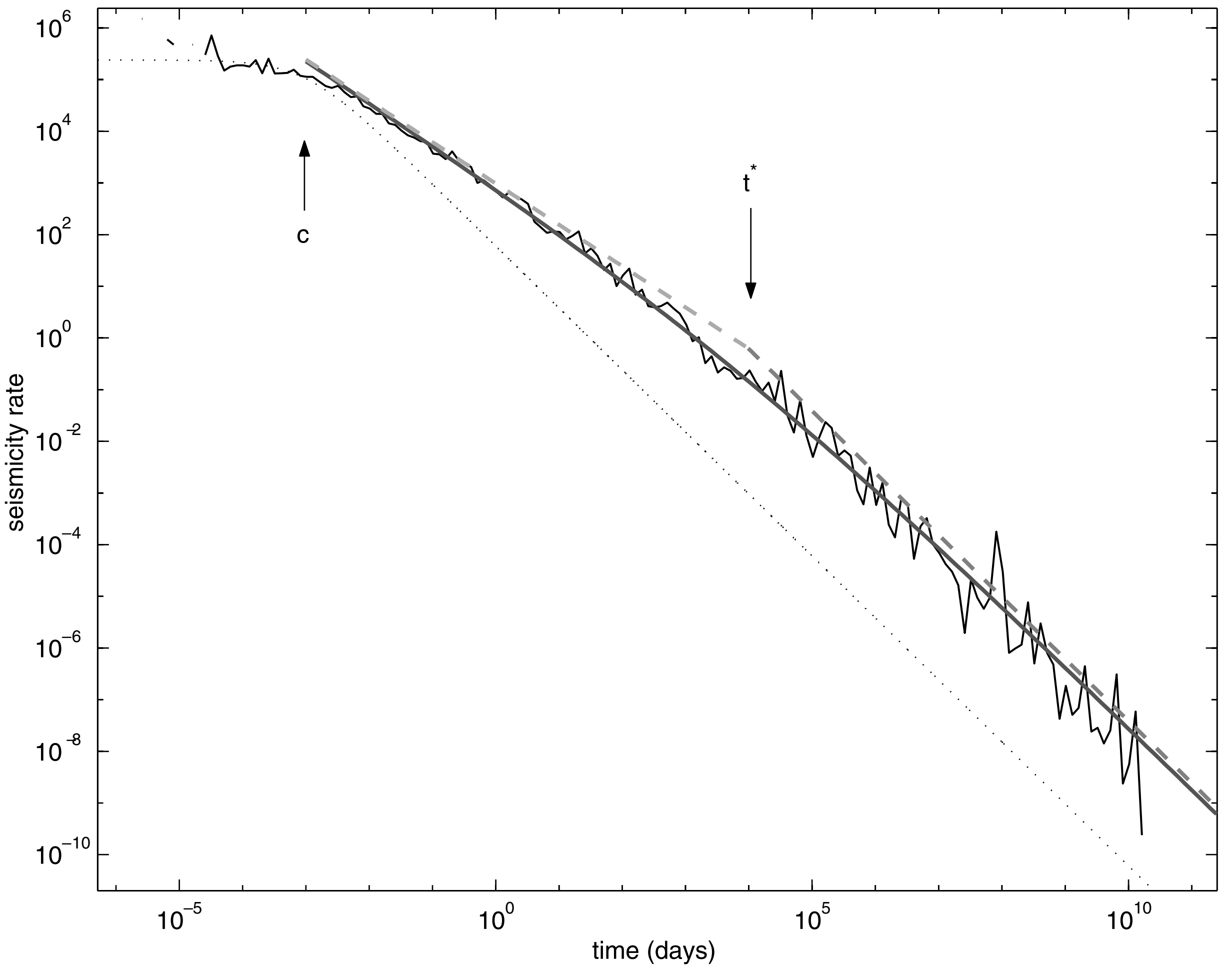

Figure 2. Seismicity rate $N(t)$ in the subcritical regime with $n = 0.95$. The noisy black line represents the seismicity rate obtained for a synthetic catalog generated using $K = 0.024$, $M = 6.8$, $m_0 = 0$, $c = 0.001$ day, $\alpha = 0.5$, $b = 1.0$, and $\theta = 0.2$, giving the characteristic time is $t^* = 4500$ days. The local Omori law with exponent $p = 1 + \theta = 1.2$ is shown for reference (dotted line). The analytical solution (16) is shown as the thick line. The two dashed lines represents the asymptotic solutions (14) for $t < t^*$ and (15) for $t > t^*$.

[33] The full expression of $\lambda(t)$ valid at all times $t \gg c$ is given by

$$\lambda(t) = \frac{S_0}{1-n}\,\frac{t^{*-\theta}}{t^{1-\theta}} \sum_{k=0}^{\infty} (-1)^k \frac{(t/t^*)^{k\theta}}{\Gamma((k+1)\theta)} \qquad (16)$$

Expression (16) provides the solution that describes the crossover from the $1/t^{1-\theta}$ Omori's law (14) at early times to the $1/t^{1+\theta}$ Omori's law (15) at large times. The series $\sum_{k=0}^{\infty}(-1)^k \frac{(t/t^*)^{k\theta}}{\Gamma((k+1)\theta)}$ is a series representation of a special Fox function [*Glöckle and Nonnenmacher*, 1993] (see Appendix A for details).

[34] The ETAS model has been simulated numerically using the algorithm described by *Ogata* [1998, 1999]. Starting with a large event of magnitude M at time $t = 0$, events are then simulated sequentially. After each event, we calculate the conditional intensity $\lambda(t)$ defined by

$$\lambda(t) = \sum_{t_i \le t} \frac{K\ 10^{\alpha(m_i - m_0)}}{(t - t_i + c)^{1+\theta}}$$

where t is the time of the last event and t_i and m_i are the times and magnitudes of all preceding events that occurred at time $t_i \le t$. The time of the following event is then determined according to the nonstationary Poisson process of conditional intensity $\lambda(t)$, and its magnitude is chosen in a Gutenberg–Richter distribution with parameter b. These simulations are compared to the theoretical predictions in Figure 2, which shows the aftershock seismic rate $\lambda(t)$ in the subcritical regime triggered by a main event of $M = 6.8$, for the parameters $K = 0.024$ (constant in (1)), the threshold $m_0 = 0$ for aftershock triggering, $c = 0.001$ day, $\alpha = 0.5$, a b-value $b = 1.0$ and $\theta = 0.2$ (corresponding to a local Omori's exponent $p = 1.2$). These parameters lead to a branching ratio $n = 0.95$ (equation (3)) and a characteristic crossover time $t^* = 4500$ days (equation (12)). The noisy black line represents the seismicity rate obtained for the synthetic catalog. The local Omori law with exponent $p = 1 + \theta = 1.2$ is shown for reference as the dotted line. The analytical solution (16) is shown as the thick line. The two dashed lines represent the approximation solutions (14) for $t < t^*$ and (15) for $t > t^*$.

3.2. The Supercritical Regime $n > 1$ and $\theta > 0$

[35] From the definition of the branching ETAS model for $n > 1$, it is clear that the number of events $\lambda(t)$ blows up exponentially for large times as $n - 1$ to a power proportional to the number t of generations. We shall show below that the rate of the exponential growth can be calculated explicitly, which yields $\lambda(t) \sim e^{t/t^*}$, where t^* has been defined in (12). However, there is an interesting early and intermediate time regime in the situation where a great event of magnitude M has just occurred at $t = 0$. In this case, the total seismicity is the result of two competing effects: (1) the total seismicity tends to decay according to the Omori's law governing the rate of daughter earthquakes triggered by the great event; (2) since each daughter may in turn trigger granddaughters, granddaughters may trigger grand-grand-daughters and so on with a number $n > 1$ of children per parent, the induced seismicity will eventually blow up exponentially. However, before blowing up, one can expect that seismicity will first decay because it is mainly controlled by the large rate $\sim 10^{\alpha(M-m_0)}$ directly induced by the great earthquake which decays according to its "local" Omori's law. This decay will be progressively perturbed by the proliferation of daughters of daughters of . . . and will crossover to the explosive exponential regime.

[36] At early times $c \ll t \ll t^*$, the early decay rate of aftershocks is the same $\approx (S_0/\Gamma(\theta)\,(n-1))\,(t^{*-\theta}/t^{1-\theta})$ as for the subcritical regime (14) (see Appendix A). However, as time increases, Appendix A shows that the decay of aftershock activity can be represented as a power law with an effective apparent exponent $\theta_{app} > \theta$ increasing progressively with time. The seismic rate will thus decay approximately as $\sim 1/t^{1-\theta_{app}(t)}$. Quantitatively, the large time behavior is (see Appendix A)

$$\lambda(t) \sim \frac{S_0}{(n-1)t^*\theta}\, e^{t/t^*} \tag{17}$$

exhibiting an exponential growth at large times. Expression (12) shows that $1/t^* \sim |1 - n|^{\frac{1}{\theta}}$. Thus, as expected, the exponential growth disappears as $n \rightarrow 1^+$.

[37] The full expression of $\lambda(t)$ valid at times $t \gg c$ is

$$\lambda(t) = \frac{S_0}{(n-1)}\,\frac{t^{*-\theta}}{t^{1-\theta}} \sum_{k=0}^{\infty} \frac{(t/t^*)^{k\theta}}{\Gamma((k+1)\theta)} \tag{18}$$

Expression (18) provides the solution that describes the crossover from the $1/t^{1-\theta}$ Omori's law at early times (14) to the exponential growth (17) at large times.

[38] Figure 3 tests these predictions by comparing them with direct numerical simulation of the ETAS model, in the case of a main shock of magnitude $M = 6$. The parameters of the synthetic catalog are $K = 0.024$ (constant in (1)), the threshold $m_0 = 0$ for aftershock triggering, $c = 0.001$ day, $\alpha = 0.5$, a b-value $b = 0.75$ and $\theta = 0.2$ (corresponding to a local Omori's exponent $p = 1.2$). These parameters lead to a branching ratio $n = 1.43$ (3) and a characteristic crossover time $t^* = 0.85$ day (12). The noisy black line represents the seismicity rate obtained for the synthetic catalog. The local Omori law with exponent $p = 1 + \theta = 1.2$ is shown for reference as the dotted line. The analytical solution (18) is shown as the thick line. The two dashed lines correspond to the approximative analytical solutions (14) and (17). At early times $c < t < t^*$, the decay of $N(t)$ is initially close to the prediction (14). For $t > t^*$, we observe that the analytical equation (16) is very close to the exponential solution (17), so as to be almost indistinguishable from it.

3.3. Case $\theta < 0$ Corresponding to a Local Omori's Law Exponent $p < 1$

[39] We have already remarked that, in this case, the integral $\int_0^\infty \frac{dt}{(t+1)^{1+\theta}}$ in the definition (3) of the branching ratio n becomes unbounded: the number of daughters created beyond any time t far exceeds the number of daughters created up to time t.

[40] Appendix A shows that the general equation (9) still holds and the general derivation starting with (A1)–(A5) still applies.

[41] Similarly to the supercritical case $n > 1$ of the regime $\theta > 0$, we find a crossover from a power law decay at early times to an exponential increase of the seismicity rate at large times. The characteristic time τ that marks the transition between these two regimes is given by

$$\tau = c\left(\frac{n_0\Gamma(|\theta|)}{1+\frac{n_0}{|\theta|}}\right)^{-\frac{1}{|\theta|}}. \tag{19}$$

[42] In contrast with the case $\theta > 0$, the early time behavior (i.e., $c \ll t \ll \tau$) of the global decay law in the case $\theta < 0$ is similar to the local Omori law:

$$\lambda(t) = \frac{S_0}{\left(1+\frac{n_0}{|\theta|}\right)\Gamma(|\theta|)}\,\frac{\tau^{-|\theta|}}{t^{1-|\theta|}} \tag{20}$$

[43] Similarly to the supercritical case $n > 1$ of the regime $\theta > 0$, the long time dependence of the regime $\theta < 0$ is controlled by a simple pole $1/\tau$ leading to a long-time seismicity growing exponentially

$$\lambda(t) = \frac{S_0}{\left(1+\frac{n_0}{|\theta|}\right)\tau|\theta|}\, e^{t/\tau} \tag{21}$$

This result is in agreement with the fact that the number of daughters born up to time t is an unbounded increasing function of t, and we should thus recover a regime similar to the supercritical case of $\theta > 0$.

[44] The full expression of $\lambda(t)$ valid at times $t > c$ is

$$\lambda(t) = \frac{S_0}{\left(1+\frac{n_0}{|\theta|}\right)}\,\frac{1}{t} \sum_{k=1}^{\infty} \frac{(t/\tau)^{k|\theta|}}{\Gamma(k|\theta|)} \tag{22}$$

Expression (22) provides the solution that describes the crossover from the local Omori law $1/t^{1-|\theta|}$ at early times to the exponential growth at large times.

[45] Figure 4 compares these predictions to a direct numerical simulation of the ETAS model, in the case of a main shock of magnitude $M = 7$. The parameters of the synthetic catalog are $K = 0.02$, $m_0 = 0$, $c = 0.01$ day, $\alpha = 0.5$, $b = 1$ and $\theta = -0.1$ (corresponding to a local Omori's

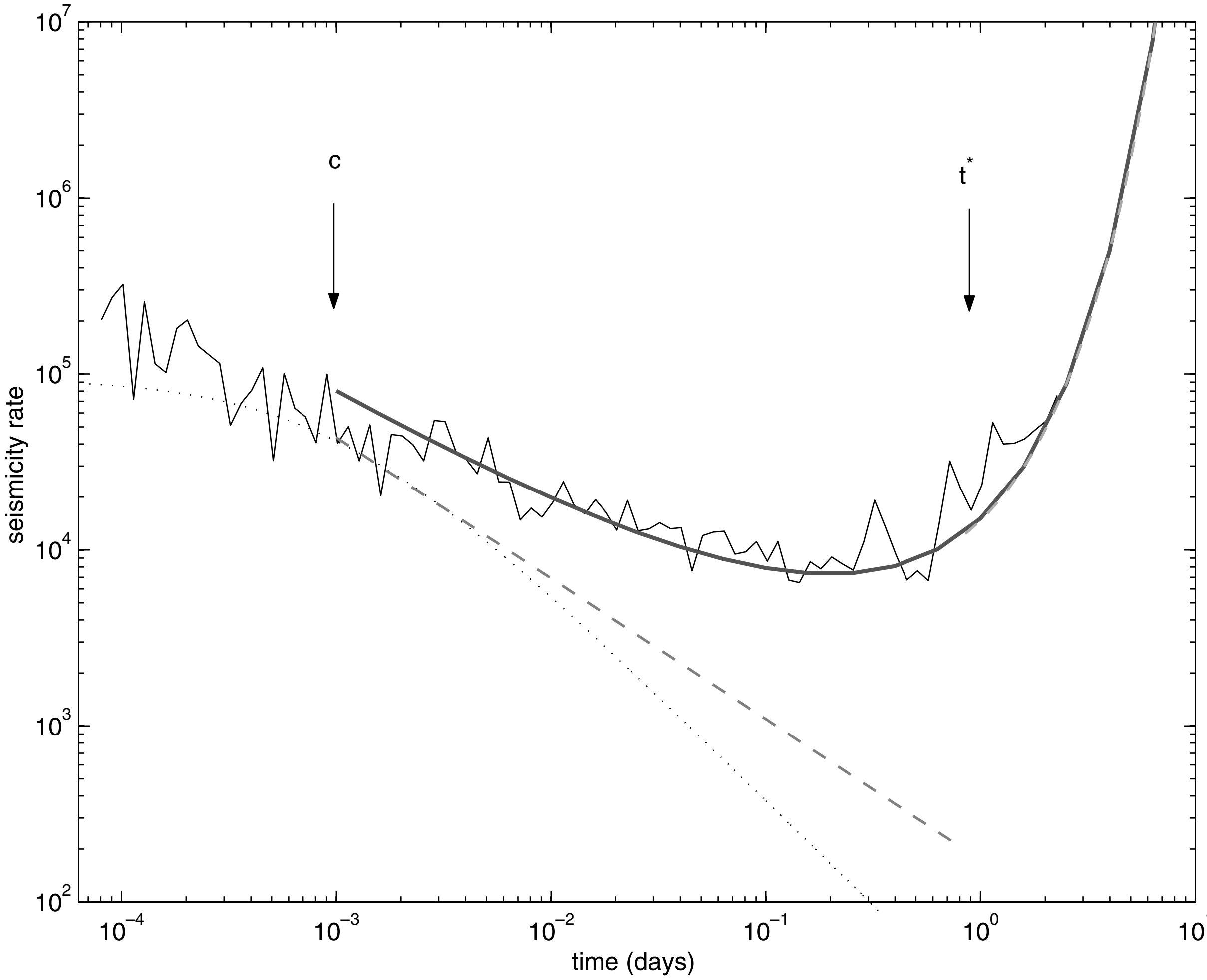

Figure 3. Seismicity rate $N(t)$ in the supercritical regime. Same legend as in Figure 2. The synthetic catalog was generated using the same parameters as for Figure 2, except for a lowest b-value of $b = 0.75$ and a smallest main shock magnitude $M = 6$, leading to a branching number $n = 1.43$ and a characteristic time $t^* = 0.85$ day. The analytical solution (thick line) is calculated from (18). The two dashed lines correspond to the approximative analytical solutions (14) and (17).

exponent $p = 0.9$). These parameters lead to a characteristic crossover time $\tau = 10^5$ day (equation (19)). The noisy black line represents the seismicity rate obtained for the synthetic catalog. The local Omori law with exponent $p = 1 + \theta = 0.9$ is shown for reference as the dotted line. The analytical solution (22) is shown as the thick line. The two dashed lines correspond to the approximative analytical solutions (20) and (21). At early times $c < t < \tau$, the decay of $\lambda(t)$ is initially close to the prediction (20). For $t > \tau$, we observe that the analytical equation (22) is very close to the exponential solution (21), so as to be almost indistinguishable from it.

4. Discussion

[46] Assuming that each event triggers aftershock sequences according to the local Omori law with exponent $1 + \theta$, we have shown that the decay law of the global aftershock sequence is different from the local one. Depending on the branching ratio n, which is a function of all ETAS parameters, we find two different regimes, the subcritical regime for $n < 1$ and the supercritical regime for $n > 1$ and $\theta > 0$. For the two regimes in the case $\theta > 0$, a characteristic time t^*, function of c, n and θ, appears in the global decay law $\lambda(t)$ and marks the transition between the early time behavior and the large time behavior. In the subcritical regime ($n < 1$), the global decay law is composed of two power laws. At early times ($t < t^*$), $\lambda(t)$ decays like $t^{-1+\theta}$. At large times ($t > t^*$) the global decay law recovers the local law $N(t) \sim t^{-1-\theta}$. In the supercritical regime ($n > 1$ and $\theta > 0$), the early times decay law is similar to that of the subcritical regime, and the seismicity rate increases exponentially for large times. The case $\theta < 0$ leads to an infinite n-value, due to the slow decay with time of the local Omori law. In this case, we find a transition from an Omori law with exponent $1-|\theta|$ similar to the local law, to an exponential increase at large times, with a crossover time τ different from the characteristic time t^* found in the case $\theta > 0$. Thus, the Omori law is only an approximation of the global decay law valid for some time periods and parameter values. The value of the local Omori exponent $p = 1$ is the only one for which the local and the global decay rate are

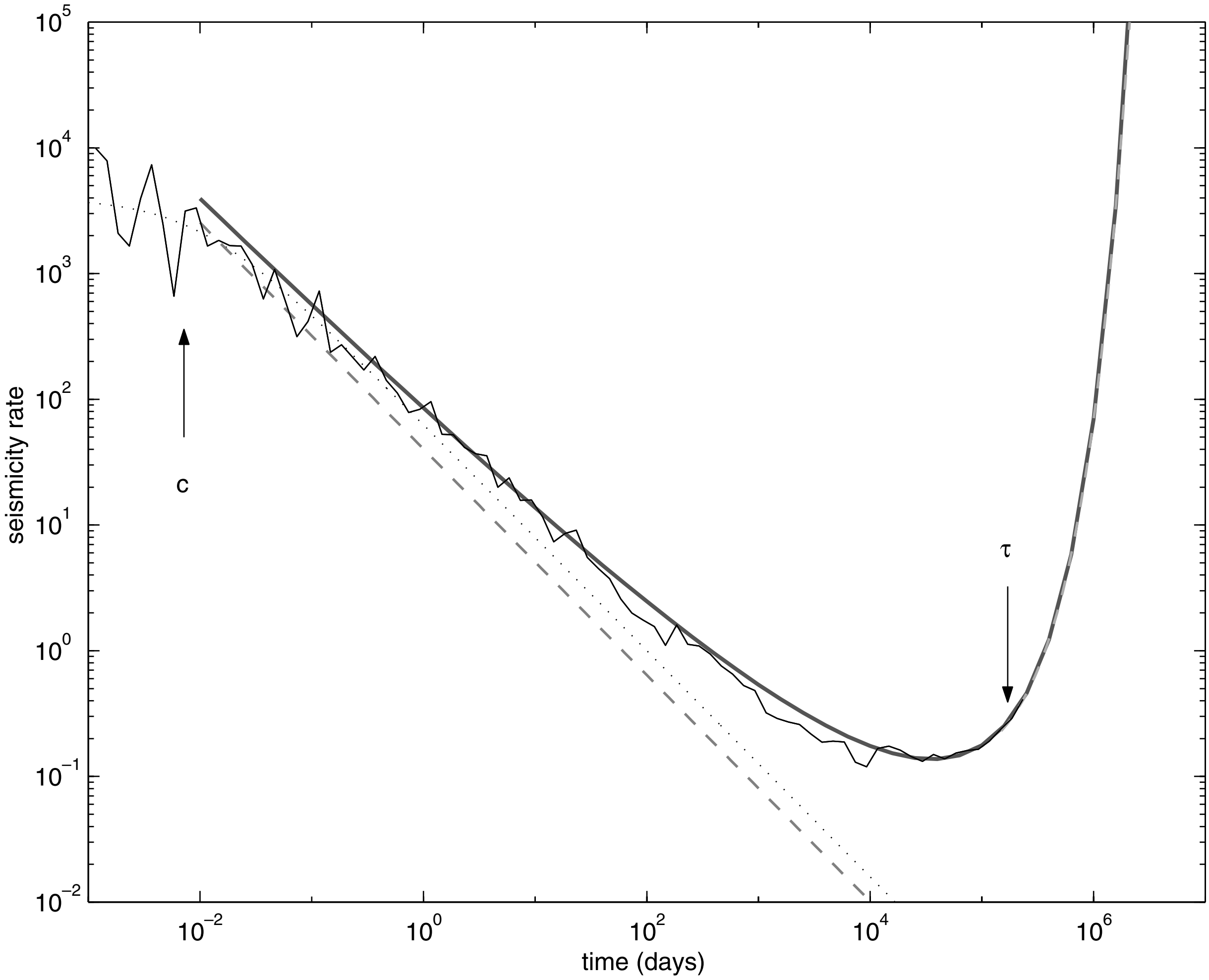

Figure 4. Seismicity rate $N(t)$ in the case $\theta < 0$ corresponding to a local Omori's law exponent $p < 1$. Same legend as in Figure 2. The synthetic catalog was generated using $K = 0.02$, $M = 7$, $m_0 = 0$, $c = 0.01$ day, $\alpha = 0.5$, $b = 1.0$, and $\theta = -0.1$, giving the characteristic time is $\tau = 10^5$ days. The analytical solution (thick line) is calculated from (22). The two dashed lines correspond to the approximative analytical solutions (20) and (21).

similar, and are both power laws without any characteristic time. For small n, t^* is very small so that in real data we should observe only the behavior $t > t^*$ characteristic of large times. The global decay law then appears similar to the local Omori law. On the contrary, for n close to 1, t^* is very large by comparison with the time period available in real data, and we should observe only the power law behavior $\lambda(t) \sim t^{-1+\theta}$ characteristic of early times, with a global p-value smaller than the local one. Changing n thus provides an important source of variability of the exponent p.

4.1. Estimation of n and t^* in Earthquake Data

[47] In real earthquake data, it is possible to evaluate the branching value n in order to determine if the seismic activity is either in the subcritical or the supercritical regime. The values of n and t^* can be evaluated from (3) and (12) as a function of the ETAS parameters b, $p = 1 + \theta$, c, K and α. The parameters of the ETAS model and their standard error can be inverted from seismicity data (time and magnitudes of each event) using a maximum likelihood method [*Ogata*, 1988]. We now discuss the range of the different parameters obtained from such inversion procedure.

1. The parameter α is found to vary between 0.35 to 1.7, and is often close to 0.5 [*Ogata*, 1989, 1992; *Guo and Ogata*, 1997]. An α-value of 0.5 means that a main shock of magnitude M will have on average 10 times more aftershocks than a main shock of magnitude $M - 2$, independently of M. Note that our definition of α is slightly different from that used by Ogata and we have divided his α-values by ln(10) to compare with our definition.

For some seismicity sequences, *Ogata* [1989, 1992] and *Guo and Ogata* [1997] found $\alpha > b$. According to (3), this leads to an infinite n-value if we use a Gutenberg–Richter magnitude distribution. As we said, a truncation of the magnitude distribution is needed to obtain a physically meaningful finite n-value because the seismicity rate is controlled by the largest events.

A large α-value can be associated with seismic activity called "swarms," while a small α-value is observed for aftershock sequences with a single main shock and no

Table 1. ETAS Parameters, Branching Ratio n, and Characteristic Time t^* for the Sequences Studied by *Ogata* [1989, 1992]

Reference	Seismicity data	M_0	b	μ (day^{-1})	K	c (day)	p	α	n	t^* (day)
Ogata [1989]	Japan, 1895–1980	6.0	1.0	0.005	0.087	0.02	1.0	0.7	Inf	[a]
Ogata [1989]	Rat-Island 1963–1982	4.7	1.0	0.0	0.072	0.167	1.35	0.63	1.04	4600
Ogata [1989]	Nagano, 1978–1986	2.5	0.9	0.021	0.008	0.017	0.85	0.94	Inf	[b]
Ogata [1989]	Nagano aftershocks, 1986	2.9	1.2	0.0	0.032	0.038	1.14	0.73	0.92	4.10^6
Ogata [1992]	Worldwide shallow earthquakes	7.0	1.0	0.019	0.018	0.21	1.03	0.53	1.49	10^{17}
Ogata [1992]	Central Aleutian, 10 years	4.7	1.0	0.008	0.042	0.03	1.13	0.62	1.34	2200
Ogata [1992]	Tohoku, 95 years	6.0	1.0	0.0054	0.98	0.02	1.0	0.70	Inf	[a]
Ogata [1992]	Tokachi-Oki aftershocks, 1 year	4.8	1.0	0.14	0.015	0.23	1.28	0.98	4.03	1.5
Ogata [1992]	Niigata aftershocks, 150 days	4.0	1.0	0.075	0.0005	0.15	1.37	1.26	Inf	[b]
Ogata [1992]	Niigata aftershocks, 150 days	2.5	1.0	0.47	0.0002	1.10	1.72	1.34	Inf	[b]
Ogata [1992]	Izu Islands, 55 years	4.0	1.0	0.0038	0.062	0.012	1.143	0.155	0.96	10^8
Ogata [1992]	Izu Peninsula, 7 years	2.5	1.0	0.022	0.035	0.003	1.35	0.17	0.91	7.3
Ogata [1992]	Off east cost of Izu, 33 days	2.9	1.0	0.59	0.016	0.009	1.73	0.31	1.00	346
Ogata [1992]	Matsushiro swarm, 20 years	3.9	1.0	0.0006	0.092	0.13	1.14	0.27	1.21	2200
Ogata [1992]	Kanto, 1904–1916	5.4	1.0	0.028	0.010	0.010	1.00	0.62	Inf	[a]
Ogata [1992]	Kanto, 1916–1923	5.4	1.0	0.025	0.001	0.010	1.02	1.31	Inf	[b]
Ogata [1992]	Hachijo, 1938–1969	5.4	1.0	0.013	0.008	0.004	1.02	0.85	3.0	5.10^6
Ogata [1992]	Hachijo, 1969–1973	5.4	1.0	0.016	0.001	0.013	1.00	1.11	Inf	[a]
Ogata [1992]	Tonankai, 1933–1939	5.2	1.0	0.050	0.010	0.065	1.02	0.90	5.28	4.10^3
Ogata [1992]	Tonankai, 1939–1944	5.2	1.0	0.031	0.009	0.011	1.01	0.83	5.54	10^7
Ogata [1992]	Tokachi, 1926–1945	5.0	1.0	0.047	0.013	0.065	1.32	0.83	0.57	0.40
Ogata [1992]	Tokachi, 1945–1952	5.0	1.0	0.041	5.20	11.6	3.50	1.37	Inf	[b]
Ogata [1992]	Tokachi, 1952–1961	5.0	1.0	0.032	0.021	0.059	1.10	0.72	0.99	10^{22}
Ogata [1992]	Tokachi, 1961–1968	5.0	1.0	0.014	0.014	0.005	0.86	0.43	Inf	7.10^5[c]

We have computed n and t^* using (3) and (12) from the ETAS parameters K, α, c, $p = 1 + \theta$, and μ calculated by , *Ogata* [1989, 1992] using a maximum likelihood method. For most sequences, we have assumed $b = 1$ to evaluate n and t^* because b-value is not given by *Ogata* [1989, 1992]. Thus, there is a large uncertainty in the n and t^* values in the case where α is close to 1.

[a] t^* cannot be evaluated because $p = 1$.

[b] t^* cannot be evaluated because $\alpha > b$.

[c] τ is given instead of t^* because $\theta < 0$.

significant secondary aftershock sequences [*Ogata*, 1992, 2001].

2. The parameter c is usually found to be of the order of 1 hour [*Utsu et al.*, 1995]. In practice, the evaluation of c is hindered by the incompleteness of earthquake catalogs just after the occurrence of the main shock, due to overlapping aftershocks on the seismograms. A large c is often an artifact of a change of the detection threshold. Notwithstanding these limitations, well-determined nonzero c-value have been obtained for some aftershocks sequences [*Utsu et al.*, 1995]. Note that a nonzero c is required for the aftershocks rate to be finite just at the time of the main shock.

3. The "local" p-value, equal to $1 + \theta$, describes the decay law of the aftershock sequence triggered by a single earthquake. The local Omori law is the law $\phi(t)$ obtained by inverting the ETAS model on the data. The "global" p-value describes the decay law of the whole aftershock sequence, composed of all secondary aftershocks triggered by each aftershock. We have shown that the Omori law is only an approximation of the global decay law, so that in the subcritical regime the global p-value will change from $1 - \theta$ at early times to $1 + \theta$ at large times. *Guo and Ogata* [1997] measured both the local and global p-values for 34 aftershocks sequences in Japan, and found that the local p-value is usually slightly larger than the global p-value [*Guo and Ogata*, 1997]. This is in agreement with our prediction when identifying the local p-value with $1 + \theta$ (recovered at large times) and the global p-value with $1 - \theta$ found at early times. *Guo and Ogata* [1997] and *Ogata* [1992, 1998, 2001] found a local p-value smaller than one for some aftershocks sequences in Japan. Within the confine of the ETAS model, this corresponds to the case $\theta < 0$ discussed above and in Appendix A.

4. The parameter K measures the rate of aftershocks triggered by each earthquake, independently of its magnitude. Recall that the branching ratio n is proportional to K. It is usually found of the order of $K \approx 0.02$ [*Ogata*, 1989, 1992; *Guo and Ogata*, 1997], but large variations of K-value from 0.001 to 5 are reported by *Ogata* [1992].

5. The parameter μ measures the background seismicity rate that is supposed to arise from the tectonic loading. $\mu \simeq 0$ for an aftershock sequence triggered by a single main shock. This parameter has no influence on the branching ratio n. In real catalogs, the background seismicity only accounts for a small part of the seismic activity.

[48] We have computed the branching ratio n and the crossover time t^* from the ETAS parameters measured by *Ogata* [1989, 1992] for several seismicity sequences in Japan and elsewhere. The ETAS parameters and the n and t^* values are given in Table 1. When the b-value is not given in the text, we have computed n and t^* assuming a b-value equal to 1. We find that the n-value is either smaller or larger than 1. This means that the seismicity can be interpreted to be either in the subcritical or in the supercritical regime. An infinite n-value is found if the local p-value is smaller than one ($\theta < 0$) or if the α-value is larger than the b-value. For the same area, the ETAS parameters and the n and t^* values are found to vary in time, sometimes changing from the subcritical to the supercritical regime. The characteristic time t^* shows large spatial and temporal variability, ranging from 0.4 to 10^{22} days. Large t^* values are related to a branching ratio n close to one, i.e., close to the critical point $n = 1$. The ETAS model thus provides a picture of seismicity in which

subcritical and supercritical regimes are alternating in an intermittent fashion. As we shall argue, the determination of the regime may provide important clues and quantitative tools for prediction.

4.2. Implications of the ETAS Model in the Subcritical Regime $n < 1$

[49] In the subcritical regime, the ETAS model can explain many of the departures of the global aftershock decay law from a pure Omori law.

[50] The ETAS model contains by definition (and thus "explains") the secondary aftershock sequences triggered by the largest aftershocks that are often observed [*Correig et al.*, 1997; *Guo and Ogata*, 1997; *Simeonova and Solakov*, 1999; *Ogata*, 2001]. In the ETAS model, the fact that secondary aftershock sequences of large aftershocks can stand out above the overall background aftershock seismicity results from the factor $10^{\alpha(m_i - m_0)}$ in (1).

[51] Our analytical results may rationalize why some alternative models of aftershock decay work better than the simple modified Omori law. In the subcritical regime, we predict an increase of the apparent global p-value from $1 - \theta$ at early times to $1 + \theta$ at large times. To our knowledge, this change of exponent has never been observed. This change of power law may be approximated by the stretched exponential function proposed by *Kisslinger* [1993] and *Gross and Kisslinger* [1994] to fit aftershocks sequences. In the stretched exponential model, the rate of aftershocks $\lambda(t)$ is defined by

$$\lambda(t) = K\, t^{q-1} e^{-(t/t_0)^q}, \qquad (23)$$

where q, K and t_0 are constants. At early times, this function decays as a power law $1/t^{1-q}$ with apparent Omori's exponent $1 - q$. For times larger than the relaxation time t_0, the seismicity rate decays exponentially in the argument $(t/t_0)^q$. For $q < 1$, this decay is much slower than exponential and can be accounted for by an apparent power law with larger exponent. Figure 5 compares the stretched exponential function with the analytical solution of the ETAS model (16) with parameters $t^* = t_0$ and $\theta = q$, and with the Omori law of exponent $p = 1 - q$. These three laws have the same power law behavior at early times, and then both the stretched exponential and the analytical solution (16) decay faster than the Omori law at large times. The fact that it is very difficult to distinguish the decay laws described by power laws and by stretched exponential has been illustrated by *Laherrère and Sornette* [1998] in many examples including earthquake size and fault length distributions. *Kisslinger* [1993] and *Gross and Kisslinger* [1994] compared this function to the modified Omori law $\lambda(t) = K\,(t + c)^{-p}$ for several aftershocks sequences in southern California. They found that the stretched exponential fit often works better for the sequences with a small p-value or a large q-value, indicative of a slow decay for small times. This is in agreement with our result that in the subcritical regime a slowly decaying aftershock sequence (global p-value smaller than one) will then crossover to a more rapid decay for time larger than t^*. The relaxation time t_0 ranges between 2 and 380 days for the sequences that are better fitted by the stretched exponential [*Kisslinger*, 1993]. This parameter is analogous to t^* found in our model, because these two parameters define the transition from the early time power law decay to another faster decaying behavior for large times. To further validate our results, these aftershocks sequences should be fitted using (16) to compare our results with the stretched exponential function and determine if the transformation of the early time power law decay is better fitted by a stretched exponential falloff or an increase in the apparent Omori exponent from $1 - \theta$ to $1 + \theta$ as predicted by our results.

[52] The ETAS model can also rationalize some correlations found empirically between seismicity parameters. It may explain the rather large variability of the global empirical p-value. *Guo and Ogata* [1995] have reported a positive correlation between the Gutenberg–Richter b-value and the p-value (exponent of the global Omori law) for several aftershock sequences in Japan. A similar correlation has also been found by *Kisslinger and Jones* [1991] for several aftershock sequences in southern California, but this correlation was detectable only if the earthquake sequences were separated into thrust and strike slip events. This positive correlation between b and global p values is expected from our analysis. From (3), we see that a small b-value is associated with a large n-value. For $n \simeq 1$, the characteristic time t^* is very large, so that the global aftershock rate decays as a power law with exponent $1 - \theta$ over a large time interval. For $n > 1$ and $\theta > 0$, we see an apparent global p-value smaller than $1 - \theta$ which decreases with time. In contrast, for large b-values, the branching ratio n is small and the characteristic time t^* is very small. In this case, only the large time behavior is observed with a larger exponent $1 + \theta$. Consequently, in the subcritical regime, our results predict a change of the global p-value from $1 - \theta$ for small b-value and times $t \ll t^*$ to $1 + \theta$ for large b-values. There is also a positive correlation between p-value and b-value in the supercritical regime. For $n > 1$ or $\theta < 0$, the global aftershock sequence is characterized by an apparent exponent p smaller than $1 - |\theta|$ which decreases with time. Then, we expect the apparent exponent p to be all the smaller, the smaller is the b-value, because the characteristic times t^* for $\theta > 0$ or τ for $\theta < 0$ decreases with b. The variability of the global p exponent reported by *Guo and Ogata* [1995] and *Kisslinger and Jones* [1991] may thus be explained by a change of b-value and a constant local p exponent. However, the results of *Guo and Ogata* [1997] contradict this interpretation. *Guo and Ogata* [1997] studied the same aftershocks sequences than *Guo and Ogata* [1995] but they measured the local p-value of the ETAS model. They still found a large variability in the local p-value, and a positive correlation between this local p-value and the b-value.

4.3. Implications of the ETAS Model in the Supercritical Regime and in the Case $\theta < 0$

[53] In the regime where the mean number of aftershocks per main shock is larger than one (i.e., $n > 1$), the mean rate of aftershocks increases exponentially for large times. However, because of the statistical fluctuations, the aftershock sequence has a finite probability to die. This probability of extinction can be evaluated for the simple branching model without time dependence [*Harris*, 1963]. Therefore, a branching ratio larger than 1 does not imply necessarily that the number of aftershocks will be infinite. If n is not too large, and if the number of aftershocks is small, there is a

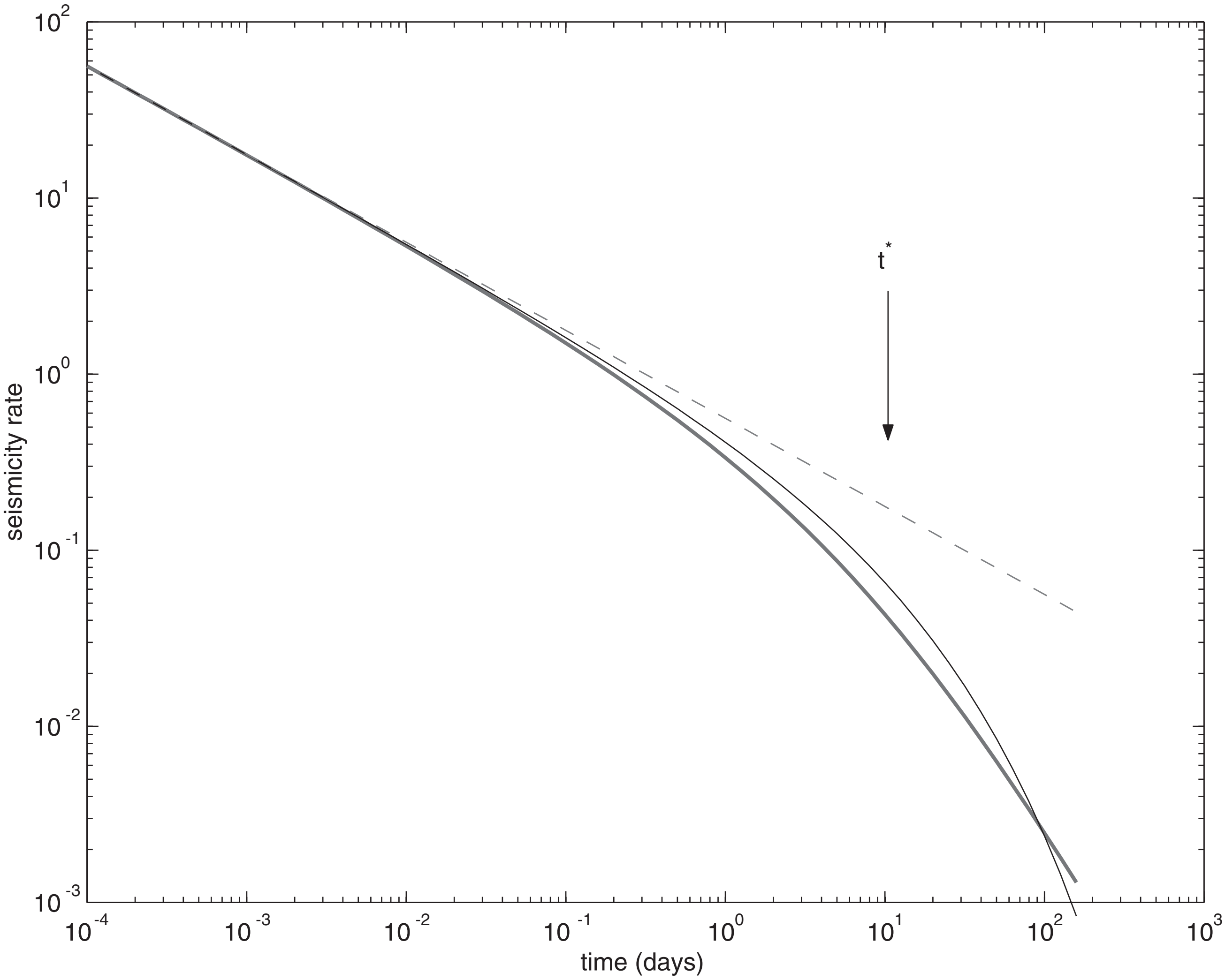

Figure 5. Comparison between the three decay laws of aftershock sequences: Omori law with $p = 0.7$ (dashed line), stretched exponential with $q = 0.3$, and $t_0 = 10$ days (thin black line) and our analytical solution in the subcritical regime (16) for $\theta = q = 1 - p = 0.3$ and $t^* = t_0 = 10$ days (solid gray line). At early times $t \ll t^*$, the three functions are similar and decay as $t^{-0.7}$. At large times, the stretched exponential function and the analytical solution of the ETAS model decay more rapidly than the Omori law. For times up to $t = 10\ t^*$, the stretched exponential function is a good approximation of the ETAS model solution and describes the transition from a power law decay at early times to a faster decay law.

significant probability that the aftershock sequences will die, as observed in numerical simulations of the ETAS model. If the characteristic time t^* is very large, the aftershock sequence may not remain supercritical long enough for the exponential increase to be observed. Even if the large time exponential acceleration is rarely observed in real seismicity, it may explain the acceleration of the deformation before material failure. The early times behavior of the seismic activity preceding the exponential increase has also important possible implications for earthquake prediction, and can rationalize some empirically proposed seismic precursors, such as the low p-value [*Liu*, 1984; *Bowman*, 1997], or the relative seismic quiescence preceding large aftershocks [*Matsu'ura*, 1986; *Drakatos*, 2000].

[54] It is widely accepted that about a third to a half of strong earthquakes are preceded by foreshocks [e.g., *Jones and Molnar*, 1979; *Bowman and Kisslinger*, 1984; *Reasenberg*, 1985, 1999; *Reasenberg and Jones*, 1989; *Abercrombie and Mori*, 1996], i.e., are preceded by an unusual high seismicity rate for time periods of the order of days to years, and distance up to hundreds kilometers. However, there is no reliable method for distinguishing foreshocks from aftershocks. Indeed, the ETAS model makes no arbitrary distinctions between foreshocks, main shocks, and aftershocks and describes all earthquakes with the same laws. While this seems a priori paradoxical, our analysis of the ETAS model provides a useful tool for identifying foreshocks, i.e., earthquakes that are likely to be followed by a larger event, from usual aftershocks that are seldom followed by a larger earthquake. The characterization of foreshocks will be performed in statistical terms rather than on a single-event basis. In other words, we will not be able to say whether any specific event is a precursor. It is the ensemble statistics that may betray a foreshock structure.

[55] The crux of the method is that, when seismicity falls in the regime with a branching ratio $n > 1$, the corresponding earthquake sequences can be identified as foreshocks. This is because the supercritical regime corresponds to an exponentially accelerating seismicity for times larger than t^*: by a pure statistical effect, the larger number of earthquakes of any size will sample more and more the branch of the Gutenberg–Richter law toward large events. Thus by the sheer weight of numbers, larger and larger earthquakes will occur as time increases. Of course, we are not implying any precise deterministic growth law, but statistically, the largest events should indeed grow significantly, the more so, the more within the supercritical regime, the larger the branching ratio $n > 1$. Conversely, this argument implies that, in the subcritical regime, the triggered events are usual aftershocks, because a main shock is unlikely to be followed by a larger triggered event. Foreshock sequences can thus be identified by evaluating the branching ratio n from the inversion of seismic data (times and magnitudes of an earthquake sequence) for the ETAS parameters. There is however a finite probability that a triggered event in the subcritical regime be larger than the triggering event, and thus the triggering event will be a foreshock of the triggered event. Therefore, foreshocks can be observed even in the subcritical regime, but they are less frequent than aftershocks.

[56] A note of caution is in order: the direct estimation of n and t^* or τ may be quite imprecise if the number of events is small. Based on our analysis and our results, the foreshock regime can be nevertheless identified with relatively good confidence if one assumes an upper bound for the local exponent p. Let us assume for instance that the local p-value is smaller than 1.3 (i.e., $\theta < 0.3$); according to our results, the global exponent p cannot become smaller than $1 - \theta = 0.7$ in the subcritical regime. In contrast, in the supercritical regime, we have shown that the apparent exponent is smaller than or at most equal to $1 - \theta$. Therefore, a measure of the global p-value yielding a value smaller than 0.7, is always associated with the supercritical regime. As we said above, *Guo and Ogata* [1997] and *Ogata* [1992, 1998, 2001] found a local p-value smaller than one for some aftershocks sequences in Japan corresponding to the case $\theta < 0$. A small global p-value can thus also result from a small local p-value. In sum, a small global p-value results either from a larger than one local p-value in the supercritical regime $n > 1$ or from a small (smaller than 1) local p-value before the exponential growth regime.

[57] Such a small p-value precursor was first proposed empirically by *Liu* [1984], who studied several aftershock sequences of moderate earthquakes that have been followed by a large earthquake. He proposed that a p-value smaller than 1 is a signature of a foreshock sequence, whereas $p > 1$ is associated with normal aftershock sequences with a single main shock in the past. He suggested that p-values close to one characterize double main shock sequences. These empirical rules are part of the earthquake prediction method used in China [*Liu*, 1984; *Zhang et al.*, 1999]. The small precursory p-value has been used with other precursors to predict the occurrence of a $M = 6.4$ earthquake in China following another $M = 6.4$ earthquake 3 months later [*Zhang et al.*, 1999]. A precursor associated with a small global p-value has also been observed by *Bowman* [1997] for a sequence in Australia. In 1987, several $M = 4$–5 earthquakes occurred in a region that was not seismically active before, and triggered a large number of aftershocks characterized by an abnormally low p-value of 0.3. A sequence of three $M \geq 6$ occurred 1 year later, followed by an aftershock sequence with a more standard p-value of 1.1. *Simeonova and Solakov* [1999] have also reported a very low p-value of 0.5, for one sequence of aftershocks in Bulgaria, which was followed 1 year latter by a larger earthquake. The first part of the aftershock sequence was well fitted by a modified Omori law, and then a significant deviation occurred with an abnormally high aftershock rate by comparison with the prior trend. This departure from an Omori law is expected from our results for an aftershock sequence in the supercritical regime and the very low value of the exponent p can be interpreted as the apparent exponent within the crossover from the $1/t^{1-\theta}$ decay (14) at early times to the exponential explosion (17) at times $t > t^*$ (see Figure 3).

[58] In addition to the small precursory p-value predicted in the regime $n > 1$, we have shown that this regime is also characterized by a decrease of the apparent global p-value with time. Such a decrease of p-value has also been identified as a precursor by *Liu* [1984].

[59] Other patterns may be a signature of the supercritical regime. The relative precursory quiescence suggested by *Drakatos* [2000] may also be explained by our results. In contrast to the "absolute" quiescence which detects changes in the background seismicity after removing the aftershocks from the catalog [e.g., *Wyss and Habermann*, 1988], the "relative" quiescence [*Matsu'ura*, 1986; *Drakatos*, 2000] takes into account the aftershocks and detects changes in seismic activity after a large main shock by comparison with the usual Omori law decay of aftershocks. *Drakatos* [2000] studied several aftershock sequences in Greece, which contains large aftershocks, i.e., aftershock with magnitude no smaller than $M - 1.2$, where M is the main shock magnitude. For each sequence, he fitted the aftershock sequence by a modified Omori law up to the time of the large aftershock using a maximum likelihood method. He found that large aftershocks were often preceded by a relative quiescence by comparison with an Omori law, with an increase of the seismicity rate just before the large main shock occurrence. Such a departure from an Omori law is predicted by our results in the supercritical regime. Indeed, in the supercritical regime, large aftershocks are likely to occur when the earthquake rate $N(t)$ changes from an Omori law to the exponential explosion for times close to t^*.

[60] To illustrate this concept, we have performed a simulation of the ETAS model in the supercritical regime and have applied the same procedure as used by *Drakatos* [2000] to fit the synthetic aftershock sequence by an Omori law up to the time of the first large aftershock. The parameters of the synthetic catalog are $K = 0.024$, $m_0 = 0$, $c = 0.001$ day, $\alpha = 0.5$, $b = 0.8$ and $\theta = 0.2$, yielding $n = 1.27$ and $t^* = 4.6$ day. Figure 6 represents the cumulative aftershock number as a function of time for the synthetic catalog and the fit with a modified Omori law. From this figure, we see a clear relative seismic quiescence, as defined by a cumulative aftershock number smaller than that predicted by the fit. The aftershock activity recovers the level predicted by the fit at the time of the large aftershock. All

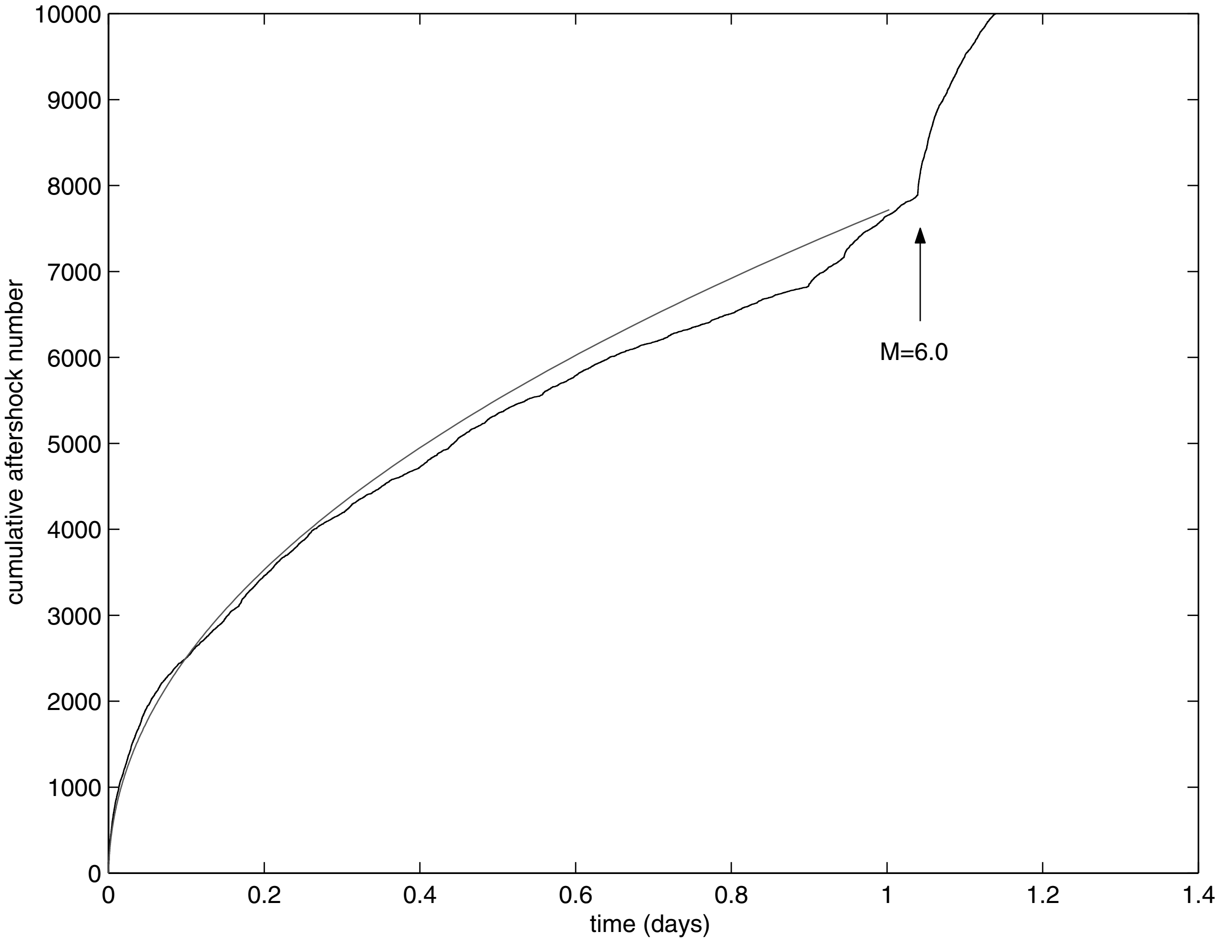

Figure 6. Cumulative aftershock number in the supercritical regime from a synthetic catalog generated using a branching ratio $n = 1.27$, $\theta = 0.2$, and $t^* = 4.6$ days. The main shock magnitude is $M = 7.0$. The thin line is a fit by an Omori law evaluated for time before the occurrence of the first $M \geq 6.0$ aftershock. This fit gives an apparent global p-value of 0.58. Relative seismic quiescence (by comparison with an Omori law) is observed before the occurrence of the $M = 6.0$ aftershock, due to the transition from an Omori law decay with exponent $p = 1 - \theta = 0.8$ for time $t \ll t^*$ to an exponential increase of the seismicity rate for time $t \gg t^*$.

theses results are similar to those obtained by *Drakatos* [2000].

[61] In the case $n > 1$, our results predict an exponential increase of the seismicity rate at large times. Because we assume that the magnitude distribution is independent of time, the same exponential acceleration is expected for both the cumulative energy release and the cumulative number of earthquakes. *Sykes and Jaumé* [1990] found that several large earthquakes in the San Francisco Bay area where preceded by an acceleration of the cumulative energy release that can be fitted by an exponential function, as predicted by our results. In laboratory experiments of rupture, several studies have also observed an exponential acceleration of the seismic energy release before the macroscopic rupture [*Scholz*, 1968; *Meredith et al.*, 1990; *Main et al.*, 1992].

[62] More recently, many studies have reported an acceleration of seismicity prior to great events (see the works of *Sammis and Sornette* [2002] and *Vere-Jones et al.* [2001] for reviews) but they used a power law instead of an exponential law to fit the acceleration of seismicity. A power law increase of the seismicity before rupture is predicted by several statistical models of rupture in heterogeneous media, which consider the global rupture or the great earthquake as a critical point (see the work of *Sornette* [2000a] for a review). Note that it is often difficult to distinguish in real data an exponential increase from a power law increase, especially with a small number of points and for times far from the rupture time. No systematic study has been undertaken that compares these two laws to test if the acceleration of the seismicity is better fitted by a power law rather than by an exponential law (see, however, the work of *Johansen et al.* [1996]).

[63] We have stressed that the ETAS model is fundamentally a mean-field approximation (branching process) which neglects "loops," i.e., multiple interactions (see Figure 1). An important consequence of this approximation is that the supercritical regime cannot lead to a growth rate faster than exponential. Indeed, recall that an exponential growth is characterized by a time derivative of the number of events proportional to the number of events $dN/dt = N/t^*$, i.e., is fundamentally a *linear* process. In a sequel to the present

work [*Sornette and Helmstetter*, 2002], we show however that for $b < \alpha$, the impact of the largest earthquake induces an effective nonlinearity which leads to a faster-than-exponential growth rate, possibly leading to a finite-time singularity [*Sammis and Sornette*, 2002]. A faster-than-exponential growth rate may also be obtained by introducing multiple interactions between earthquakes and positive feedback: rather than the linear law $dN/dt = N/t^*$ expressing the condition that each "daughter" has only one "mother," we may expect an effective law $dN/dt \sim N^{\delta}$, with $\delta > 1$ providing a measure of the effective number of ancestors impacting directly on the birth of a daughter. We may thus expect that an improvement of the ETAS model beyond the "mean-field" approximation would lead to power law acceleration of seismicity in some regions of the parameter space.

[64] Other precursory patterns may also be related to the supercritical regime: they comprise the precursory earthquake swarm or burst of aftershocks [*Evison*, 1977; *Keilis-Borok et al.*, 1980a, 1980b; *Molchan et al.*, 1990; *Evison and Rhoades*, 1999]. Swarms are earthquake sequences characterized by high clustering in space and time and the occurrence of several large events with magnitude larger than $M - 1$, where M is the magnitude of the largest event. A burst of aftershocks is a sequence of one or more main shocks with abnormally large number of aftershocks at the beginning of their aftershock sequences [*Keilis-Borok et al.*, 1980a]. From our results, an abnormally high aftershock rate or a sequence with several large events are expected in the supercritical regime.

4.4. Temporal Change of *n*-Value and Transition from One Regime to the Other One

[65] It is often reported that the b and p values vary in space and time [e.g., *Smith*, 1981; *Guo and Ogata*, 1995, 1997; *Wiemer and Katsumata*, 1999]. We have documented that a part of the observed variation of the exponent p may not be genuine but result from an inadequate parameterization of a more complex reality. Because n and t^* are function of b, p and the other ETAS parameters, we expect the fundamental parameters of the ETAS model, namely n and t^*, to vary significantly in space and time. The branching ratio n plays the role of a "control" parameter quantifying the distance from the critical point $n = 1$ between the subcritical and the supercritical regime; t^* is a crossover time and is sensitive to details of the systems. As a consequence, it is very reasonable to expect that the Earth's crust will change from the subcritical to the supercritical regime and vice versa, as a function of time and location.

[66] Expression (3) shows that the branching ratio n is a decreasing function of b. Accordingly, this may rationalize the observation that large earthquakes are sometimes preceded by a decrease of the b-value [e.g., *Smith*, 1981]. A decrease of the b-value leads to an increase of the n-value that can move the seismicity from the subcritical to the supercritical regime, and thus increase the probability to observe a large earthquake. Other ETAS parameters (α, K, p, and c) may also change in time and move the seismicity from one regime to the other one. *Ogata* [1989] measured the ETAS model parameters before and after the 1984 Western Nagano Prefecture earthquake ($M = 6.8$). He found that the seismic activity preceding the main shock was characterized by lower b, c, and K parameters and local p values than the seismicity following the main shock. He also obtained a larger α-value for the seismicity preceding the main shock. All these changes of parameters, except the change in K, lead to a larger n-value before the main shock than after. Before the main shock, n is in principle infinite because the local p-value is smaller than one. As we already discussed, this corresponds to an explosive supercritical regime of growing seismicity. After the main shock, we find $n = 0.92$ and $t^* = 10^6$ days, using the determination of the ETAS parameters. The seismicity has thus changed from a supercritical regime before the main shock to a subcritical regime after the main shock.

5. Conclusion

[67] We have provided analytical solutions of the ETAS model, which describes foreshocks, aftershocks and main shocks on the same footing. Each event triggers an aftershock sequence with a rate that decays according to the local Omori law with an exponent $p = 1 + \theta$. The number of aftershocks per event increases with its magnitude. We suggest that the Earth's crust at a given time and location may be characterized by its branching ratio n, quantifying its regime. We propose that n is a fundamental parameter for understanding and characterizing the organization of the seismicity within the Earth's crust. In the subcritical regime ($n < 1$), the global rate of aftershocks (including secondary aftershocks) decays with the time from the main shock with a decay law different from the local Omori law. We find a crossover from an Omori exponent $1 - \theta$ for $t < t^*$ to $1 + \theta$ for $t > t^*$. The modified Omori law is thus only an approximation of the decay law of the global aftershock sequence. In the supercritical regime ($n > 1$ and $\theta > 0$), we find a novel transition from an Omori decay law with an exponent $1 - \theta$ at early times to an explosive exponential increase of the seismicity rate at large times. The case $\theta < 0$ leads to an infinite n-value, due to the slow decay with time of the local Omori law. In this case, we find a transition from an Omori law with exponent $1 - |\theta|$ similar to the local law, to an exponential increase at large times, with a crossover time τ different from the characteristic time t^* found in the case $\theta > 0$. These results can rationalize many of the stylized facts reported for foreshock and aftershock sequences, such as the suggestion that a small p-value may be a precursor of a large earthquake, the relative seismic quiescence preceding large aftershocks, the positive correlation between b and p-values, the observation that great earthquakes are sometimes preceded by a decrease of b-value and the acceleration of the seismicity preceding great earthquakes.

[68] Finally, we would like to mention that our analysis can be generalized to various other choices of the local Omori law and of the magnitude distribution. The ETAS model can also be extended to describe the spatial distribution of the seismicity [*Helmstetter and Sornette*, 2002].

Appendix A: Technical Derivation of the Analytical Solution

[69] In this appendix, we provide the technical derivation of the results used in the main text for the subcritical and supercritical regimes. We start from (9).

A.1. General Derivation for θ > 0

[70] The integral over τ is the convolution of $\lambda_{m'}$ with $\phi_{m'}$. Since there is an origin of time and we have a convolution operator, the natural tool is the Laplace transform $\hat{f}(\beta) \equiv \int_0^{+\infty} f(t) e^{-\beta t} dt$. Applying the Laplace transform to (9) yields

$$\hat{\lambda}_m(\beta) = \hat{S}(\beta, m) + P(m) \int_{m_0}^{\infty} dm' \ \hat{\phi}_{m'}(\beta) \ \hat{\lambda}_{m'}(\beta). \tag{A1}$$

where the r.h.s. has used the convolution theorem that the Laplace transform of a convolution of two functions is the product of the Laplace transform of the two functions. Let us now apply the integral operator $\int_{m0}^{\infty} dm \hat{\phi}_m(\beta)$ on both sides of (A1) and define

$$\lambda(\beta) \equiv \int_{m_0}^{\infty} dm \ \hat{\phi}_m(\beta) \ \hat{\lambda}_m(\beta), \tag{A2}$$

$$Q(\beta) \equiv \int_{m_0}^{\infty} dm \ \hat{\phi}_m(\beta) \ P(m), \tag{A3}$$

and

$$\mathcal{S}(\beta) \equiv \int_{m_0}^{\infty} dm \ \hat{\phi}_m(\beta) \ \hat{S}(\beta, m). \tag{A4}$$

Then, expression (A1) yields

$$\lambda(\beta) = \mathcal{S}(\beta) + Q(\beta)\lambda(\beta), \tag{A5}$$

whose solution is

$$\lambda(\beta) = \frac{\mathcal{S}(\beta)}{1 - Q(\beta)}. \tag{A6}$$

This expression gives $\lambda_m(t)$ after inversion of the integral operator $\int_{m0}^{\infty} dm\hat{\phi}_m(\beta)$ and of the Laplace transform.

[71] The key quantity controlling the dependence of $\lambda_m(t)$ is

$$Q(\beta) = \frac{K}{\theta c^{\theta}} \left(\int_{m_0}^{\infty} dm \ 10^{\alpha(m-m_0)} \ P(m) \right) \left(\theta \int_0^{\infty} dt \frac{e^{-\beta c t}}{(t+1)^{1+\theta}} \right), \tag{A7}$$

obtained by replacing the expression of $\phi_m(t)$ defined in (1) and normalizing $t/c \rightarrow t$. Using $P(m) = \ln(10)\ b 10^{-b(m-m_0)}$, we obtain

$$Q(\beta) = n \ R(\beta c), \tag{A8}$$

where we have used the expression (3) of n and defined

$$R(\beta) \equiv \theta \int_0^{\infty} dt \frac{e^{-\beta t}}{(t+1)^{1+\theta}} = \theta \ e^{\beta} \ \beta^{\theta} \ \Gamma(-\theta, \beta)$$
$$= 1 - e^{\beta} \ \beta^{\theta} \ \Gamma(1-\theta, \beta), \tag{A9}$$

where

$$\Gamma(a, x) = \int_x^{\infty} dt \ e^{-t} \ t^{a-1} \tag{A10}$$

is the (complementary) incomplete Gamma function [*Abramowitz and Stegun*, 1964] and we have used $\Gamma(1+a, x) = a\Gamma(a, x) + x^a \ e^{-x}$ obtained by integration by part. Using the expansion of the incomplete Gamma function [*Olver*, 1974]

$$\Gamma(a, x) = \Gamma(a) - \sum_{k=0}^{+\infty} \frac{(-1)^k \ x^{a+k}}{k!(a+k)}, \qquad \text{for } a > 0, \tag{A11}$$

we obtain

$$R(\beta) = 1 - \Gamma(1-\theta) \ \beta^{\theta} + \frac{1}{1-\theta} \ \beta + \mathcal{O}(\beta^{1+\theta}, \beta^2, \beta^{2+\theta}, \beta^3, \ldots). \tag{A12}$$

It is possible, using the full expansion of the incomplete Gamma function, to estimate the value of $\lambda(\beta)$ when the second term $\frac{1}{1-\theta}\ \beta$ of the expansion cannot be neglected anymore compared with the term proportional to β^{θ}. Thus, the expansion (A12) using the first two terms only $R(\beta) = 1 - \Gamma(1-\theta)\ \beta^{\theta}$ becomes invalid for $\beta > [\Gamma(1-\theta)\ (1-\theta)]^{1/(1-\theta)}$, i.e., for times smaller than $[\Gamma(2-\theta)]^{-1/(1-\theta)}$. For all practical purpose, this is a small value and we can use safely the expansion (A12) in the following calculations.

[72] Let us now make explicit $\lambda(\beta)$:

$$\lambda(\beta) = \frac{K}{\theta c^{\theta}} \ R(\beta c) \int_{m_0}^{\infty} dm \ 10^{\alpha(m-m_0)} \int_0^{\infty} dt \ \lambda_m(t) \ e^{-\beta t}. \tag{A13}$$

[73] Using the definition of $\lambda(t)$ given by (11) and the factorization of the times and magnitudes in (A13), we obtain

$$\lambda(\beta) = nR(\beta c)\hat{\lambda}(\beta), \tag{A14}$$

where

$$\hat{\lambda}(\beta) = \int_0^{\infty} dt \ \lambda(t) \ e^{-\beta t}. \tag{A15}$$

Replacing (A14) in (A6) gives

$$\hat{\lambda}(\beta) = \frac{\mathcal{S}(\beta)}{nR(\beta c)(1 - nR(\beta c))}. \tag{A16}$$

[74] When a great earthquake occurs at the origin of time $t = 0$ with magnitude M, $S(t,m) = \delta(t)\ \delta(m - M)$, expression (A4) gives

$$\mathcal{S}(\beta) = \frac{K}{\theta c^{\theta}} \ 10^{\alpha(M-m_0)} \ R(\beta c). \tag{A17}$$

Thus, expression (A16) becomes

$$\hat{\lambda}(\beta) = \frac{b-\alpha}{b2} \ \frac{10^{\alpha(M-m_0)}}{(1 - nR(\beta c))}. \tag{A18}$$

[75] The dependence of $\hat{\lambda}(\beta)$ on β is uniquely controlled by the denominator $1 - n\ R(\beta c)$.

A.2. The Subcritical Regime $n < 1$

[76] The analysis proceeds exactly as in [*Sornette and Sornette*, 1999]. For $0 < \theta < 1$, and for small β (large times), $\hat{\lambda}(\beta)$ given by (A18) is

$$\hat{\lambda}(\beta) = \frac{S_0}{1 - n\left[1 - d(\beta c)^\theta\right]} = \frac{S_0}{(1-n)}\left(\frac{1}{1+(\beta t^*)^\theta}\right), \quad \text{(A19)}$$

where t^* is defined by (12) and the external source term S_0 is defined by (13). We retrieve equation (13) in the work of *Sornette and Sornette* [1999] with the correspondence $t_0 \to c$.

[77] Two cases must be distinguished.

1. $\beta t^* < 1$ corresponds to $t > t^*$ by identifying as usual the dual variable β to t in the Laplace transform with $1/t$. In this case, we can expand $\frac{1}{1+(\beta t^*)^\theta}$, which leads to

$$\hat{\lambda}_{t>t^*}(\beta) \sim \frac{S_0}{1-n}\left[1 - (\beta t^*)^\theta\right]. \quad \text{(A20)}$$

We recognize the Laplace transform of a power law of exponent θ, i.e.,

$$\lambda_{t>t^*}(t) \sim \frac{S_0}{\Gamma(\theta)(1-n)} \frac{t^{*\theta}}{t^{1+\theta}} \qquad \text{for t > t*}. \quad \text{(A21)}$$

2. For $t < t^*$, $\beta t^* > 1$ and (A19) can be written with a good approximation as

$$\hat{\lambda}_{t<t^*}(\beta) = \frac{S_0}{(1-n)(\beta t^*)^\theta} \sim \beta^{-\theta}. \quad \text{(A22)}$$

Denoting $\Gamma(z) \equiv \int_0^{+\infty} dt\, e^{-t} t^{z-1}$, we see that $\int_0^{+\infty} dt\, e^{-\beta t}\, t^{z-1} = \Gamma(z)\, \beta^{-z}$. Comparing with (A22), we thus get

$$\lambda_{t<t^*}(t) \sim \frac{S_0}{\Gamma(\theta)(1-n)} \frac{t^{*-\theta}}{t^{1-\theta}} \qquad \text{for t < t*}. \quad \text{(A23)}$$

[78] We verify the self-consistency of the two solutions $\lambda_{t>t^*}(t)$ and $\lambda_{t<t^*}(t)$ by checking that $\lambda_{t>t^*}(t^*) = \lambda_{t<t^*}(t^*)$. In other words, t^* is indeed the transition time at which the "short-time" regime $\lambda_{t<t^*}(t)$ crosses over to the "long-time regime $\lambda_{t>t^*}(t)$.

[79] We now calculate the full expression of $\lambda(t)$ valid at all times. We expand

$$\frac{1}{(\beta t^*)^\theta+1} = \frac{1}{(\beta t^*)^\theta} \frac{1}{(\beta t^*)^{-\theta}+1} = \frac{1}{(\beta t^*)^\theta} \sum_{k=0}^{\infty} (-1)^k (\beta t^*)^{-k\theta}, \quad \text{(A24)}$$

Thus, by taking the inverse Laplace transform

$$\lambda(t) = \frac{S_0}{1-n} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\beta\, e^{\beta t} \sum_{k=0}^{\infty} (-1)^k (\beta t^*)^{-(k+1)\theta}. \quad \text{(A25)}$$

The inverse Laplace transform of $\beta^{-(k+1)\theta}$ is $t^{(k+1)\theta-1}/\Gamma((k+1)\theta)$. This allows us to write

$$\lambda(t) = \frac{S_0}{1-n} \frac{t^{*-\theta}}{t^{1-\theta}} \sum_{k=0}^{\infty} (-1)^k \frac{(t/t^*)^{k\theta}}{\Gamma((k+1)\theta)} \quad \text{(A26)}$$

Expression (A26) provides the solution that describes the crossover from the $1/t^{1-\theta}$ Omori's law (A23) at early times to the $1/t^{1+\theta}$ Omori's law (A21) at large times. The series $\sum_{k=0}^{\infty}(-1)^k \frac{(t/t^*)^{k\theta}}{((k+1)\theta)}$ is a series representation of a special Fox function [*Glöckle and Nonnenmacher*, 1993] and it is also related to the generalized Mittag–Leffler function.

[80] For large times $t \gg t^*$, a direct numerical evaluation of $\lambda(t)$ from (A26) is impossible due to the very slow convergence of the series. The padé summation method [*Bender and Orzag*, 1978] can be used to improve the convergence of this series and to evaluate numerically (A26) for all times.

A.3. The Supercritical Regime $n > 1$

[81] We can analyze this regime by putting $n > 1$ in (A18) which can be written under a form similar to (A19):

$$\hat{\lambda}(\beta) = \frac{S_0}{(1 - nR(\beta c))} = \frac{S_0}{dn(\beta c)^\theta - (n-1)} = \frac{S_0}{(n-1)}\left(\frac{1}{(\beta t^*)^\theta - 1}\right), \quad \text{(A27)}$$

In the second and third equalities of (A27), we have used the small β-expansion (A12) of $R(\beta c)$ valid for $0 < \theta < 1$.

[82] At early times $c \ll t \ll t^*$, i.e., $\beta t^* \gg 1$, $\hat{\lambda}(\beta) \approx \frac{S_0}{(n-1)(\beta t^*)^\theta}$ which is the Laplace transform of (A23): thus, the early decay rate of aftershocks is the same $\sim 1/t^{1-\theta}$ as for the subcritical regime (A23). However, as time increases, the dual β of t decreases and $\hat{\lambda}(\beta)$ grows faster than $\sim(\beta c)^{-\theta}$ due to the presence of the negative term $-(n-1)$. This can be seen as an apparent exponent $\theta_{app} > \theta$ increasing progressively such that $dn(\beta t^*)^\theta - 1 \approx C(\beta t^*)^{\theta_{app}}$, where C is a constant. Note that $\theta_{app} > \theta$ for the pure power law $C(\beta c)^{\theta_{app}}$ to mimic the acceleration induced by the negative correction $-(n-1)$. The seismic rate will thus decay approximately as $\sim 1/t^{1-\theta_{app}(t)}$.

[83] The large time behavior is controlled by the pole at $\beta = 1/t^*$ of $\hat{\lambda}(\beta)$. Close to $1/t^*$,

$$\hat{\lambda}(\beta) \approx \frac{S_0}{(n-1)\theta} \frac{1}{\beta t^* - 1}. \quad \text{(A28)}$$

The inverse Laplace transform is thus

$$\lambda(t) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} d\beta\, e^{\beta t}\, \hat{\lambda}(\beta) \sim \frac{S_0}{(n-1)t^*\theta} e^{t/t^*} \quad \text{(A29)}$$

exhibiting the exponential growth at large times. Expression (12) shows that $1/t^* \sim |1-n|^{\frac{1}{\theta}}$. Thus, as expected, the exponential growth disappears as $n \to 1^+$.

[84] We now calculate the full expression of $\lambda(t)$ valid at all times. We expand

$$\frac{1}{(\beta t^*)^\theta - 1} = \frac{1}{(\beta t^*)^\theta} \frac{1}{1 - (\beta t^*)^{-\theta}} = \frac{1}{(\beta t^*)^\theta} \sum_{k=0}^{\infty} (\beta t^*)^{-k\theta}, \quad \text{(A30)}$$

[85] Thus

$$\lambda(t) = \frac{S_0}{(n-1)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\beta\, e^{\beta t} \sum_{k=0}^{\infty} (\beta t^*)^{-(k+1)\theta}. \quad \text{(A31)}$$

The inverse Laplace transform of $1/\beta^{(k+1)\theta}$ is $t^{(k+1)\theta-1}/\Gamma((k+1)\theta)$. This allows us to write

$$\lambda(t) = \frac{S_0}{(n-1)} \frac{t^{*-\theta}}{t^{1-\theta}} \sum_{k=0}^{\infty} \frac{(t/t^*)^{k\theta}}{\Gamma((k+1)\theta)} \tag{A32}$$

Expression (A32) provides the solution that describes the crossover from the $1/t^{1-\theta}$ Omori's law at early times to the exponential growth at large times. Note that the solution (A32) can be obtained directly from (A26) by removing the alternating sign $(-1)^k$ in the sum. The solution (A32) retrieves the two regimes discussed before.

1. For $t < t^*$, the sum in (A32) is close to $1/\Gamma(\theta)$, which leads to

$$\lambda(t) \approx \frac{S_0}{\Gamma(\theta)(n-1)} \frac{t^{*-\theta}}{t^{1-\theta}}. \tag{A33}$$

2. For $t \geq t^*$, the sum dominates. The sum is very similar to the series expansion of e^{t/t^*} and is actually proportional to e^{t/t^*} for large t. This result is obvious for $\theta = 1$ since the series expansion becomes identical to that of e^{t/t^*}. This can be justified for other values of θ as follows. For $\theta \to 0$, the discrete sum transforms into a continuous integral of the type

$$\int_0^{\infty} dx\; t^x/\Gamma(x). \tag{A34}$$

A saddle-node approximation, performed using the Stirling approximation (which already gives a very good precision for small z) $\Gamma(z) \approx \sqrt{2\pi}\, e^{-z}\, z^{z-\frac{1}{2}}$, shows that the saddle node of the integrant occurs for $x \approx t/t^*$, which then gives $\lambda(t) \sim e^{t/t^*}$. For arbitrary θ, we can use the Poisson's summation rule

$$\sum_{r=-\infty}^{+\infty} f(r) = \int_{-\infty}^{+\infty} du\, f(u) + \sum_{q=1}^{+\infty} \int_{-\infty}^{+\infty} du\, f(u) \cos[2\pi q u], \tag{A35}$$

on the function defined by

$$f(r) \equiv \frac{(t/t^*)^{r\theta}}{\Gamma(r\theta+\theta)}, \qquad \text{for } r \geq 0 \tag{A36}$$

and $f(r) = 0$ for $r < 0$. The left-hand side of (A35) is nothing but the semi-infinite sum in (A32). The first term in the right-hand side retrieves the integral (A34) encountered for the case $\theta \to 0$. This term thus contributes a term proportional to e^{t/t^*}. All the other terms contribute negative powers of t and are thus negligible compared to the exponential for $t > t^*$. This can be seen from the fact that each term with $q \geq 1$ is similar to the sum in (A26) for the subcritical case with alternating signs. The larger q is, the faster is the frequency of alternating signs, and the smaller is the integral. The leading dependence $\lambda(t) \sim e^{t/t^*}$ valid for any $0 \leq \theta \leq 1$ retrieves the limiting behavior already given in (A29) from a different approach for large times $t \gg t^*$. It has also been proved rigorously by *Ramselaar* [1990].

A.4. Case $\theta < 0$ Corresponding to a Local Omori's Law Exponent $p < 1$

[86] The general equation (9) still holds in this case and the general derivation starting with (A1)–(A6) still applies. The key quantity controlling the dependence of $\lambda_m(t)$ is still $Q(\beta)$ defined by (A7). Writing $\theta = -|\theta|$, we have

$$Q(\beta) = n_0\, R'(\beta c), \tag{A37}$$

where n_0 is defined by (4) and

$$R'(\beta) \equiv \int_0^{\infty} dt \frac{e^{-\beta t}}{(t+1)^{1-|\theta|}} = e^{\beta}\, \beta^{-|\theta|}\, \Gamma(|\theta|, \beta) \tag{A38}$$

where $\Gamma(a, x)$ is the (complementary) incomplete Gamma function defined by (A10). Using the exact expansion (A11), we obtain

$$Q(\beta) = n_0\, e^{\beta c}\, (\beta c)^{-|\theta|} \left(\Gamma(|\theta|) - \sum_{k=0}^{+\infty} \frac{(-1)^k\, (\beta c)^{|\theta|+k}}{k!\, (|\theta|+k)} \right). \tag{A39}$$

For small β's (i.e., large times), expression (A39) has the following leading behavior

$$Q(\beta) = n_0\, \Gamma(|\theta|)\, (\beta c)^{-|\theta|} - \frac{n_0}{|\theta|} + n_0 \Gamma(|\theta|)\, (\beta c)^{1-|\theta|} + h.o.t. \tag{A40}$$

where $h.o.t.$ stands for higher-order terms in the expansion in increasing powers of βc.

[87] The source term $S(\beta)$ in the denominator of $\hat{\lambda}(\beta)$ given by (A6) is now given by

$$S(\beta) = K\, c^{|\theta|}\, 10^{\alpha(M-m_0)}\, R'(\beta c). \tag{A41}$$

Expression (A6) for $\hat{\lambda}(\beta)$ then yields

$$\hat{\lambda}(\beta) = \frac{S_0}{1 - Q(\beta c)}, \tag{A42}$$

where $R'(\beta c)$ is given by (A38), n_0 is defined by (4) and S_0 is defined by (13). The dependence of $\hat{\lambda}(\beta)$ on β is uniquely controlled by the denominator $1 - Q(\beta c) = 1 - n_0 R'(\beta c)$.

[88] Using (A40), we get the leading behavior for small βc

$$\begin{aligned} \hat{\lambda}(\beta) &= \frac{S_0}{1 + \frac{n_0}{|\theta|} - n_0 \Gamma(|\theta|)(\beta c)^{-|\theta|}} \\ &= \frac{S_0}{\left(1 + \frac{n_0}{|\theta|}\right)} \frac{1}{\left(1 - (\beta\tau)^{-|\theta|}\right)} \end{aligned} \tag{A43}$$

where the characteristic time τ is given by (19).

[89] At early times $c < t < \tau$, $(\beta\, \tau)^{-|\theta|} < 1$ so that

$$\hat{\lambda}(\beta) \approx \frac{S_0}{\left(1 + \frac{n_0}{|\theta|}\right)} \left(1 + (\tau\beta)^{-|\theta|}\right). \tag{A44}$$

By applying the inverse Laplace transform, the constant term contributes a Dirac function $\delta(t)$ which is irrelevant as the calculation is valid only for $t > c$. The other term $(\tau\beta)^{-|\theta|}$ gives

$$\lambda(t) = \frac{S_0}{\left(1 + \frac{n_0}{|\theta|}\right)\Gamma(|\theta|)} \frac{\tau^{-|\theta|}}{t^{1-|\theta|}}. \tag{A45}$$

The early time behavior of $\lambda(t)$ is thus similar to the local Omori law $1/t^{1-|\theta|}$.

[90] Similarly to the supercritical case $n > 1$ of the regime $\theta > 0$, the long time dependence of the regime $\theta < 0$ is controlled by a simple pole $\beta^* = 1\tau$.

[91] Thus, the long-time seismicity is given by

$$\lambda(t) = \frac{S_0}{\left(1 + \frac{n_0}{|\theta|}\right)\tau|\theta|} e^{t/\tau} \tag{A46}$$

[92] We can also calculate the full expression of $\lambda(t)$ valid at all times $t > c$. We expand

$$\frac{1}{1 - (\beta\tau)^{-|\theta|}} = \sum_{k=0}^{\infty} (\beta\tau)^{-k|\theta|}, \tag{A47}$$

Removing the constant term, which by the inverse Laplace transform contributes a Dirac function $\delta(t)$ which is irrelevant as the calculation is valid only for $t \gg c$, we get

$$\lambda(t) = \frac{S_0}{\left(1 + \frac{n_0}{|\theta|}\right)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\beta\, e^{\beta t} \sum_{k=1}^{\infty} (\beta\tau)^{-k|\theta|}. \tag{A48}$$

The inverse Laplace transform of $1/\beta^{k|\theta|}$ is $t^{k|\theta|-1}/\Gamma(k|\theta|)$. This allows us to write

$$\lambda(t) = \frac{S_0}{\left(1 + \frac{n_0}{|\theta|}\right)} \frac{1}{t} \sum_{k=1}^{\infty} \frac{(t/\tau)^{k|\theta|}}{\Gamma(k|\theta|)} \tag{A49}$$

Expression (A49) provides the solution that describes the crossover from the local Omori law $1/t^{1-|\theta|}$ at early times to the exponential growth at large times.

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