Earthquake Cycles and Neural Reverberations: Collective Oscillations in Systems with Pulse-Coupled Threshold Elements

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(Received 6 October 1994)

Driven systems of interconnected blocks with stick-slip friction capture main features of earthquake processes. The microscopic dynamics closely resemble those of spiking nerve cells. We analyze the differences in the collective behavior and introduce a class of solvable models. We prove that the models exhibit rapid phase locking, a phenomenon of particular interest to both geophysics and neurobiology. We study the dependence upon initial conditions and system parameters, and discuss implications for earthquake modeling and neural computation.

PACS numbers: 87.10.+e, 05.20.–y, 64.60.Cn, 91.30.Px

Many natural phenomena involve interacting threshold elements. Slowly driven by some external force, single elements discharge rapidly when they reach a trigger threshold, then become quiescent again. This activity may in turn stimulate neighboring elements. Prominent examples include earthquakes, avalanches, forest fires, colonies of flashing fireflies, and assemblies of spiking nerve cells. The corresponding models, although closely related on a formal level, exhibit distinct collective properties that range from systemwide synchronization [1] to self-organized criticality [2,3]. Phase locking without global synchronization has been identified as an important borderline phenomenon between both cases [4,5] and has been observed under various conditions [2–12].

In this Letter, a class of solvable models is introduced. The dynamics resemble those of previous earthquake models [2,3,7–10] and integrate-and-fire neurons [1,6,11,12], and reduce to Abelian avalanches [8] in a limiting case. We prove that for nearest-neighbor couplings and periodic boundary conditions, the models rapidly relax to phase-locked solutions—the attractors are reached as soon as every element has discharged once. Next, we study the dependence upon initial conditions, coupling parameters, and boundary conditions. We then extend the class of interactions and discuss the connection with networks of spiking neurons. Finally, we outline implications for earthquake modeling and neural computation.

Geophysical description of the model.—Seismic activity occurs predominantly on faults located at the boundaries between tectonic plates. Relative plate movement leads to a slow accumulation of stress that is quickly released during earthquakes. We follow Refs. [2,3,7–10] and concentrate on the dynamics of a single fault, represented by a rectangular two-dimensional lattice. The stress at site i is modeled by a scalar variable F_i.

The duration of earthquakes, typically less than a minute, defines a first time scale of the fault dynamics. A second time scale, governed by the stress loading process, is given by the recurrence time between "characteristic events," the largest earthquakes on a fault. The shortest observed recurrence times are in the 10-yr range, more than 6 orders of magnitude longer than the duration of single events. Neglecting aftershocks, earthquakes may thus be approximated as *instantaneous* events, separated by silent episodes of uniform stress increase [2,3,7–10]. The remaining time scale is chosen such that $dF_i/dt = 1$.

Stick-slip friction is incorporated by a static fracture criterion. When one of the F_i reaches the threshold F_{th}, it is reset to zero. At the same time, the stresses F_{nn} of i's four nearest neighbors are increased by αF_{th}, where $\alpha < 1/4$. This rule provides an exact description of single slips in slider-block systems [13]; see [3].

Events with multiple slips occur if during and due to the relaxation of block i, a neighboring block j becomes unstable (when $F_j = F_{\mathrm{th}}$). In models that lack a short time scale and inertial terms, the resulting coupled motion can only be described approximately. For example, Olami, Feder, and Christensen [3] assume that block i has slipped to zero-stress position *before* block j starts to move. This implies that F_j usually exceeds F_{th} at the beginning of j's relaxation.

As an alternative, we consider the opposite extreme, where site j is reset as soon as F_{th} is reached. Accordingly, the fixed quantity αF_{th} is redistributed. To account for the previous excess stress $F_j - F_{\mathrm{th}}$, a fraction γ is accumulated by site j *after* the relaxation. If more than one site is unstable, a fixed update order is chosen as in [3].

Without loss of generality, F_{th} is set to unity [14]. The system is initialized with random values for the F_i, independently drawn from a uniform probability distribution with width $w \leq 1$ [15] so that $F_i \in [1 - w, 1]$. The dynamic can then be summarized as follows:

(i) Initialize the F_i randomly in $[1 - w, 1]$.

(ii) If $F_i \geq 1$ and if i is next in the update scheme then

$$F_i \rightarrow F_i' = \gamma(F_i - 1)\,, \tag{1}$$

$$F_{\mathrm{nn}} \rightarrow F_{\mathrm{nn}}' = F_{\mathrm{nn}} + \alpha\,. \tag{2}$$

(iii) Repeat step (ii) until $F_i < 1$ for all i.

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(iv) If the condition of step (ii) does not apply then

$$dF_i/dt = 1 \quad \text{for all } i\,. \tag{3}$$

Two cases, $\gamma = 0$ and $\gamma = 1$, are of particular interest. For $\gamma = 1$, the update order in (iii) does not influence the evolution of the system in terms of its stable configurations (all $F_i < 1$). In addition, fast, (1) and (2), and slow dynamics, (3), commute. The system is Abelian and equivalent to an avalanche model proposed by Gabrielov [8]. For $\gamma = 0$, the stick-slip model of Feder and Feder [10] is recovered in the limit $\alpha = 1/4$. The model of Ref. [3] is obtained if (2) is replaced by

$$F_{\rm nn} \rightarrow F'_{\rm nn} = F_{\rm nn} + \alpha F_i\,. \tag{4}$$

If only a single site is relaxed at a time, (2) and (4) are identical. Events with multiple slips differ, in general, and give rise to distinct collective properties.

Phase locking.—For periodic boundary conditions, the Abelian model ($\gamma = 1$) approaches cyclic oscillations with period $P = 1 - 4\alpha$ [8,12]. In what follows, we prove that an attractor is reached as soon as every site has slipped once, and that this result holds for all $0 \leq \gamma \leq 1$.

Any site i slips at most once during one event. To cause a second failure, at least one neighbor would have to discharge twice before the ith element does so. By induction, this is impossible because $\alpha < 1/4$.

The stress increase due to slipped neighbors does not depend on whether they relax in a single or several events. This implies that during any time interval of length P no site i relaxes more than once: F_i increases by at most $1 - 4\alpha$ due to (3) and by up to 4α if all neighbors slip. At least the same total amount is lost in a single slip (1).

It is next shown that transients have a finite duration. Let $t_{\max}$ denote the first time where all sites have failed at least once, t_i the last instance where site i slides before $t_{\max}$, $t_{\min}$ the minimum of all t_i, and j a site that fails at $t_{\min}$ without being triggered by other sites at that time. By definition, all sites discharge at least once in $[t_{\min}, t_{\max}]$. This means, in particular, that every neighbor of j slips at least once between $t_{\min}$ and $t_{\max}$. Each event adds α to F_j. The total increase ΔF_j is thus at least $4\alpha + t_{\max} - t_{\min}$. By assumption, site j fails once in $[t_{\min}, t_{\max}]$ so that $\Delta F_j < 1$ or $t_{\max} - t_{\min} < P$.

The result implies that every site fails exactly once in $[t_{\min}, t_{\max}]$ and no site fails in $(t_{\max} - P, t_{\min})$. Since $t_{\max} \leq w$, this proves that in finite time $t_{\max}$, a limit cycle is approached in the sense that $F_i(t) = F_i(t - P)$ for $t > t_{\max}$ (Fig. 1) [17]. The argument also shows that the attractor is reached as soon as every element has slipped once. This finishes the proof [18].

For $\gamma < 1$, P-periodic oscillations with one slip per cycle cannot occur if a site is driven above threshold because a single stress drop would exceed unity, the total stress increase over one cycle. This means that the system relaxes to fine-tuned states, where if site i is triggered by n of its neighbors at time t, $F_i(t^-) = 1 - n\alpha$ [19]. It follows that although every *toppling sequence* of the Abelian model can be realized for $\gamma < 1$, the volume of all attractors is greatly reduced in terms of the F_i.

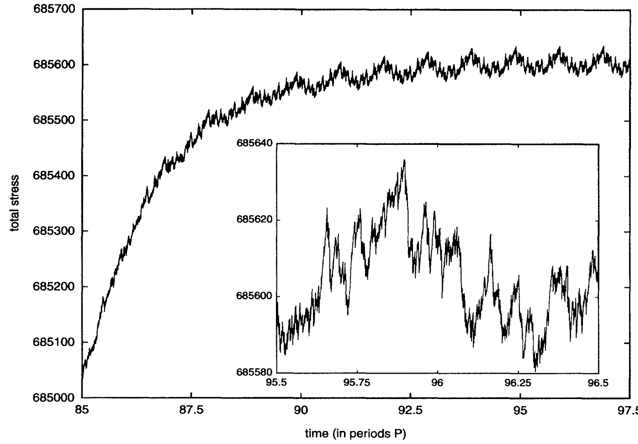

FIG. 1. Transient and periodic limit cycle. Time evolution of the total stress $F_{\rm total} = \sum_i F_i$ in a system with 1024^2 sites and periodic boundary conditions; $\gamma = 1$, $w = 1$, and $\alpha = 0.2495$. Vertical drops of size Δ correspond to events with $P^{-1}\Delta = 500\Delta$ slipped sites [16].

In the limit $\alpha \rightarrow 0$, (2) and (4) become identical. This may explain why the model of Ref. [3] quickly freezes in P-periodic states for small to intermediate α [4].

Parameter dependence.—For small α and large enough w, events are expected to be localized with exponential size-frequency relation [20] as observed in numerical experiments (Fig. 2). If, on the other hand, $w \leq \alpha$, the very first rupture has to lead to a systemwide avalanche, followed by a globally synchronized oscillation. If $w \leq 2\alpha$, a slipped row triggers the entire adjacent row as soon as a single element of the second row fails. The example illustrates domino effects due to stress accumulation.

If $w + 4\alpha < 1$, a slipped site is reset to a value that is smaller than the minimal stress of blocks that have not yet failed. Since the system is driven uniformly, slipped sites cannot catch up with the other elements. Thus every cell slips exactly once during the transient. This implies that the time evolution depends only on the ratio $r = w/\alpha$ and that toppling sequences do *not* depend on γ.

Slips of yet unruptured areas have to occur if their corners reach threshold. At time t, the stress of a corner site is at least $1 - w + 2\alpha + t$, the stress of the block that failed first is at most $4\alpha + t$. This means that for $w + 2\alpha < 1$ all sites slip once before the first site slips twice. The previous bound $w + 4\alpha < 1$ can thus be improved to $w + 2\alpha < 1$.

Numerical experiments were performed on lattices with up to 1024^2 sites. Data were taken from single limit cycles and multiple runs, and indicate that the size-frequency relation is self-averaging. Below the line $w + 2\alpha = 1$, it follows a power law, frequency (event size $>n$) $\propto n^{-B}$, at $r = 2.44 \pm 0.02$. A numerical value of $B = 0.79 \pm 0.05$ was obtained. Sample averages from the first event after reaching the attractors give $B_{\rm first} = 0.05 \pm 0.03$.

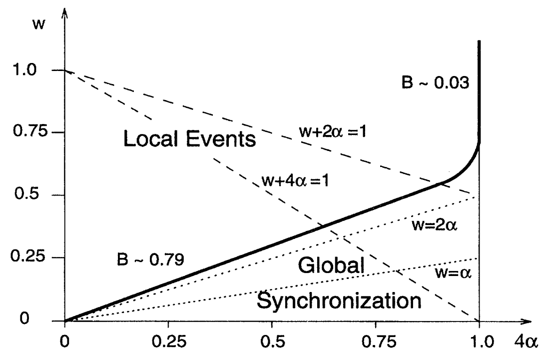

FIG. 2. Dependence upon internal dynamics (α) and initial conditions (w) for periodic boundary conditions. Local events and global synchronization are separated by a line where the size-frequency relation obeys a power law. The solid line represents numerical results for $\gamma = 1$. The line is valid for arbitrary γ if $w + 2\alpha < 1$. The other auxiliary lines refer to theoretical bounds discussed in the text.

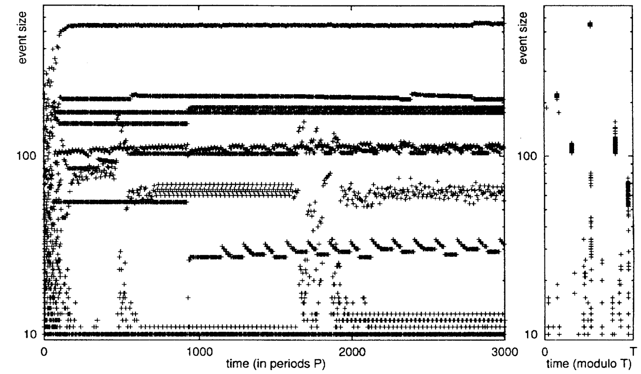

FIG. 3. Intermittent behavior. Each cross marks the size (≥ 10) and time of an event in a system with 40^2 sites and open boundary conditions; $\gamma = 0$, $w = 1$, and $\alpha = 0.24$. The system exhibits almost periodic behavior where events "repeat" after a time $T = 1 - 3\alpha = 7P$, as is clearly visible when event times are shown modulo T (here for $1000P < t < 3000P$).

The difference is due to nonuniform stress accumulation during the lattice filling process and is also reflected in temporal inhomogeneities of the limit cycles. For $w \gg 1$ and $\alpha \to 1/4$, transient times diverge when measured in terms of P, and $B = B_{\text{first}} = 0.03 \pm 0.03$ for $\gamma = 1$ [20].

Open boundary conditions.—Because of the reduced number of neighbors, sites located at the edges (and corners) of the lattice receive less pull and cannot sustain the maximum failure rate P^{-1}. The resulting dynamical defects propagate into the bulk and prevent complete phase locking. For $\gamma = 1$, synchronized clusters gradually change due to the loss and new recruitment of phase-locked cells, and one observes (quasi-)periodic behavior [8] with exponential size-frequency relations for $\alpha < 1/4$. For $\gamma = 0$, the fine tuning described earlier leads to a different intermittent behavior, where clusters remain virtually unchanged over many cycles before they suddenly merge or break apart (Fig. 3). The main slip frequency shifts from $(1 - 4\alpha)^{-1}$ to $(1 - 3\alpha)^{-1}$, demonstrating that the boundary strongly effects the collective behavior, a situation characteristic for extended driven systems.

Long transients and intermittency complicate quantitative numerical investigations. Extensive simulations reveal that for small γ (at least) sample averages exhibit self-organized criticality. As in Ref. [3], the exponent B depends on α. For $\gamma = 0$, it is $B = 0.40 \pm 0.05$ at $\alpha = 0.1$ and $B = 0.52 \pm 0.05$ (see also [10]) for $\alpha \to 1/4$.

Model extensions and biological interpretation.—To describe more general interactions, (2) is replaced by

$$F_j \to F_j' = F_j + T_{ji}, \tag{5}$$

where T_{ji} denotes the size of a pulse from i to j. Possible extensions include systems of arbitrary dimension and aspect ratio, and inhomogeneous, nonisotropic, and long-range interactions to represent granularity or depth-dependent material properties. For non-negative coupling strengths that satisfy the condition

$$\sum_i T_{ji} = A \leq 1 \quad \text{for all } j, \tag{6}$$

the convergence proof remains valid, with 4α replaced by A. Notice that symmetric couplings, $T_{ij} = T_{ji}$, are *not* required in the proof.

To account for spatially varying loading I_i and nonseismic stress decrease due to slow creep, Eq. (3) could be replaced by a leaky-integrator equation

$$dF_i/dt = -F_i\tau^{-1} + I_i. \tag{7}$$

The equation points to a potentially rich connection between slider-block models and networks of integrate-and-fire neurons [21]. In the new context (7) describes the time evolution of cell potentials F_i due to leakage currents and external stimuli I_i. When cell i reaches the firing threshold F_{th}, it resets and emits a short electrochemical pulse. Depending on the sign of the synaptic efficacy T_{ji}, this pulse excites or inhibits other cells [22].

Simulations of leaky integrate-and-fire models with excitatory nearest-neighbor couplings and uniform inputs exhibit three salient features [12]: (a) within a very short time [23] the networks converge to locally synchronized solutions; (b) on a longer time scale, clusters of coherent neurons slowly reorganize [24]; and (c) the early cluster shape depends significantly on the initial conditions. With nonuniform "grey values" for the $F_i(0)$, areas with small "brightness" variation w are encoded by large populations of synchronized neurons, as already suggested by Fig. 2. The results demonstrate that coupled integrate-and-fire neurons are able to quickly bind objects—here defined as regions of similar grey value—by synchronized firing patterns that are held in a dynamic short-time memory through neural reverberations.

Discussion.—On some faults, large earthquakes have repeated with remarkable coherence over a few cycles [25]. According to a prevalent view, inhomogeneities are needed to generate these characteristic events [26]. Our results prove, however, that complex "seismic cycles" can emerge as *collective* phenomena in *homogeneous* single-fault models. At the same time, self-organized criticality (with exponents that depend on the amount of stress dissipation) may occur, at least on the level of ensemble averages.

The similarity between avalanche models and networks of spiking neurons has led to speculations about a neurobiological role of self-organized criticality [11]. However, whereas for earthquakes and sandpiles, the main interest is in the properties of the stationary state, for neural computation, it is the convergence process which does the computation and is thus of particular interest. Furthermore, computations must be taken rapidly, and in any event the assumption of constant external input implicit in all models breaks down at longer times.

As shown by the present model, coupled integrate-and-fire neurons are able to perform rapid computations. For $\gamma = 1$, the model dynamics can be described by a downhill motion on an "energy landscape" generated by a Lyapunov function [16]. This makes it possible to "program" specific tasks [12] and extend our understanding of collective computation in attractor networks from previous models based on a firing-rate description to biologically more realistic models with spiking neurons.

The comparison with Ref. [3] shows that minute details of the local dynamics can have a pronounced effect on the emergent behavior of nonequilibrium systems [27]. For open boundary conditions, replacing (4) by (2) leads from apparently chaotic trajectories to approximately phase-locked solutions. The oscillation frequency of these solutions is strongly influenced by the boundary conditions. Phase locking itself, however, appears to be a robust phenomenon in systems with fixed stress transfer. The results indicate that *quantized pulses* may play an important role for the rapid formation and long maintenance of metastable oscillations in pulse coupled systems.

We thank John Rundle for valuable discussions that stimulated this work. We acknowledge suggestions by the participants of the "Workshop on Natural Hazard Reduction" at the Santa Fe Institute where the main ideas of this paper were first presented to a wider audience.

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[14] For $F_{\text{th}} = 2\pi$, the F_i may be interpreted as phases.
[15] More generally one may consider systems with slips to a positive stress value. They can be mapped onto the model if $w > 1$. Thus we also discuss this case.
[16] For $\gamma = 1$, $-F_{\text{total}}$ is a Lyapunov function if evaluated at times $t_0 + kP$, where $t_0 > 0$ and $k \in \mathbb{N}$ [12].
[17] Mathematically speaking, a (continuous) limit set of stable P-periodic solutions is reached.
[18] The update order in (iii) does not enter the proof. One may also replace (1) by *any* rule with $F_i - 1 \geq F_i' \geq 0$.
[19] The same argument applies to P-periodic states in [3].
[20] Global synchronization occurs if stress variations can be smoothed out ($\gamma = 0$) during long transients ($w \gg 1$).
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[22] For uniform all-to-all couplings and $\gamma = 1$, one recovers the Kuramoto model [6], for $\gamma = 0$, a variant of[1]; (5) is not applied in [1] if a cell has already fired at that instant.
[23] For a square lattice with $L = 200^2$ sites, typical transient times are less than $4P$ at $A = 0.8$ and $8P$ at $A = 0.9$.
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