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Magnitude-Frequency Relations for Earthquakes Using a Statistical Mechanical Approach

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At very small magnitudes, observations indicate that the frequency of occurrence of earthquakes is significantly smaller than the frequency predicted by simple Gutenberg-Richter statistics. Previously, it has been suggested that the dearth of small events is related to a rapid rise in scattering and attenuation at high frequencies (i.e., the controversial "f_{max}" problem) and the consequent inability to detect these events with standard arrays of seismometers. However, several recent studies have suggested that instrumentation cannot account for the entire effect and that the decline in frequency may be real. Working from this hypothesis, we derive a magnitude-frequency relation for very small earthquakes that is based upon the postulate that the system of moving plates can be treated as a system not too far removed from equilibrium. As a result, it is assumed that in the steady state, the probability $P[E]$ that a segment of fault has a free energy E is proportional to the exponential of the free energy $P \propto \exp[-E / E_N]$. In equilibrium statistical mechanics this distribution is called the Boltzmann distribution. The probability weight E_N is the space-time steady state average of the free energy of the segment. Earthquakes are then treated as fluctuations in the free energy of the segments. With these assumptions, it is shown that magnitude-frequency relations can be obtained. For example, previous results obtained by the author can be recovered under the same assumptions as before, for intermediate and large events, the distinction being whether the event is of a linear dimension sufficient to extend the entire width of the brittle zone. Additionally, a magnitude-frequency relation is obtained that is in satisfactory agreement with the data at very small magnitudes. At these magnitudes, departures from frequencies predicted by Gutenberg-Richter statistics are found using a model that accounts for the finite thickness of the inelastic part of the fault zone. The inelastic thickness of the fault zone that is obtained is in general agreement with thicknesses found from field observations following earthquakes. Thus, departures from simple Gutenberg-Richter scaling are apparently due at very large magnitudes to the finite width of the brittle layer, and at very small magnitudes to the finite thickness of the inelastic fault zone.

1. Introduction and Background

A recent series of observations indicates that earth-quakes at very small magnitudes may not follow the statistical distribution originally proposed by *Gutenberg and Richter* [1942]; (hereafter referred to as GR). They showed that the cumulative frequency dn_o /dt of events with magnitudes larger than m can be represented by

$$dn_0 / dt = 10^{A_0} \ 10^{-bm} . \tag{1}$$

Here, A_0 characterizes the level of seismicity of the fault system, and the value of b determines the frequency of occurrence of large events relative to small events. Until recently, it had been thought that $b \approx 1$ always. However, the "brittle" seismogenic part of Earth is for the most part the uppermost 10-50 km of Earth's lithosphere. Earthquakes of intermediate size with magnitudes up to $m \sim 6$ have source dimensions smaller than $\sim$ 10 km and are thus relatively uninfluenced by the finite width of the brittle lithosphere. Large events with $m \sim 7$ and above have a length L that can greatly exceed the width of the brittle lithosphere. As a result, the way in which slip scales with event size changes. Using these ideas, *Rundle* [1989a] showed that the expected values for b for intermediate size events is b_I and for large events is $b_L = 1.5$. These predictions were recently confirmed by *Pacheco et al.* [1992] using world wide data, who found that $b_L/b_I = 1.5 \pm 5\%$ and that, in general, $b_I \approx 1$. Thus the departure from strict Gutenberg-Richter scaling at large magnitudes is apparently related to the finite depth of the brittle layer. In a study of large strike-slip earthquakes using high-quality data, *Romanowicz* [1992] also found that the frequency of earthquakes falls off at the expected rate and showed that the slip in these events reached saturation when the depth of the earthquake roughly equaled the depth extent of the brittle seismogenic zone.

Within the past 5 years, several observations indicate that there may also be a departure from GR scaling at very small magnitudes, and that this departure is not due to anomalous attenuation at high frequencies, nor to instrumental effects. *Aki* [1987], *Malin et al.* [1989], and *Rydelek and Sacks* [1989] all report observing a cumulative magnitude-frequency relation for small events that falls significantly below the rate given by (1) with $b \approx 1$ as m decreases below some cutoff magnitude m_c , where typically m_c is in the range of 1 to 3. Because the observations are controversial, in all cases, considerable effort has been expended to ensure that the decrease of small events is real and is not due to anomalous attenuation at high frequencies or to instrumental effects. Figure 1 shows data from *Aki* [1987]. Aki's data were obtained from a borehole seismograph station operated by the University of Southern California at Baldwin Hills in the middle of the Newport Inglewood fault zone. Circumstances clearly indicate that the smaller events must have originated within the fault zone. By contrast, Malin et al. obtained data from a highly sensitive array of borehole seismometers located near Parkfield, California. The events were carefully located on the San Andreas fault near the hypocenter of the 1966 earthquake, beneath Middle Mountain. Aki used moment magnitudes determined from the measurement of coda amplitude as a function of lapse time. Malin et al. used

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seismic moments and corner frequencies to calculate equivalent local duration magnitudes and Brune Stress drops. *Rydelek and Sacks* [1989] do not discuss their method for estimating magnitude. Instead, they used comparisons between noisy (daytime) and less noisy (nighttime) periods, accounting for tidal effects, to establish the limits of detectability for their data. They analyzed data from several locations, including Hawaii, the southeast corner of Hokkaido, Japan, and Phlegrean Fields near Pozzuoli, Italy. Of these three data sets, only that taken in Hokkaido had a noise threshold low enough to display the small magnitude decline in frequency. However, this decline was similar in most respects to that shown in Figure 1.

In this paper, we argue that this departure of the observed frequency of events from that expected on the basis of the GR scaling line is due to the appearance of inelastic effects associated with the finite thickness of a fault zone. It is known, for example, that a fault zone is not a planar surface but instead consists of a fractal collection of faults and subfaults [*King*, 1983; *Okubo and Aki*, 1987; *Aviles et al.*, 1987]. Moreover, surface observations obtained following major earthquakes [e.g., *Lawson*, 1908] clearly indicate that rupture takes place throughout a zone normal to the general fault trace, whose thickness λ_c is of the order of several tens to perhaps several hundreds of meters. The basic idea used in the present paper is that very small events have a fixed source size λ_c and that the moment is determined only by the slip, which is thus independent of source size. This is in contrast to intermediate size events, in which slip scales linearly with source radius, and large events, in which slip is proportional to the depth of the brittle layer and thus saturates [*Rundle*, 1989a; *Romanowicz*, 1992; *Romanowicz and Rundle*, 1993]. Thus, in both the small and large events, slip is independent of source size, and the resulting frequency of events is less than predicted by (1).

This paper is organized according to the following logic. In the second section, the assumptions used in *Rundle* [1989a] are briefly recapitulated, the qualitative picture of a fault zone is given, and the basic relations needed for what follows are collected. The third section lays the basis for the statistical mechanical approach developed here, and in particular, we derive the fundamental probability distribution $P(E)$ for the free energy associated with a fault zone using the well known method of most probable distributions [e.g., *Huang*, 1987]. The fourth section uses $P(E)$ to recover the magnitude-frequency relations derived by *Rundle* [1989a] as a check on the method. The fifth section constructs a magnitude-frequency relation for small events, and we close with a discussion. An appendix provides several empirical relations used in the text.

2. Basic Picture and Assumptions

The qualitative picture, basic assumptions, and empirical relations used by *Rundle* [1989a] will be summarized here. The reader is referred to that paper for details. The basic philosophy is to start by focusing on a single spatial scale, then to extend the picture to all spatial scales. The approach is intimately tied into the idea of a cascade of spatial scales. Invariance of the physics (self-similarity or scale invariance) across spatial scales leads to a GR relation with a constant b value. A change in the b value at a given scale is associated with a change in the underlying physics.

We start with a simple qualitative picture of a fault zone. Consider a sliding surface (fault) with total area S_T embedded in an otherwise elastic medium. Sliding on the surface at the long-term average velocity V is driven by shear motion of the farfield boundaries of the elastic medium. Now divide S_T into N equal size areas, or patches, $S_N = S_T / N$. Denoting the state of slip at time t on each patch i by $u_i(t)$, one can write the deficit in slip $\phi_i(t)$ relative to the expected long-term offset on each patch as $\phi_i(t) = Vt - u_i(t)$. As described in other places [e.g., *Rundle*, 1988], sticking friction leads to an increase in $\phi_i(t)$, and thus to an increase in stress and stored mechanical energy on each patch. Eventually, a threshold in stress is attained, and the patch slips in an earthquake, reducing stress $\Delta\sigma_i$ and decreasing the amount of stored energy by $|\Delta E_i|$. *Kanamori and Anderson* [1975] have shown that $\Delta\sigma_i$ is roughly constant, $\Delta\sigma_i \equiv \Delta\sigma_T$, independent of the size of the event. However, it should be noted that the stress decrease due to slip on patch i averaged over all of S_T will be somewhat less than $\Delta\sigma_T$ because slip on patch i can increase the stress on its neighbors, and vice versa. Therefore, each patch needs to slip some number of times (possibly a large number) before the stress decrease averaged over all of S_T is $\Delta\sigma_T$.

If the average frictional properties remain roughly the same through time, then one can speak of an aver age slip $\langle u_i \rangle$ on the patch, and an average recurrence interval $T_i = \langle u_i \rangle / V$ for the patch. One can also define the time τ_T it takes for events of size S_N to reduce the stress over S_T by the amount $\Delta\sigma_T$. Patch i must on average slip τ_T / T_i times during the time interval τ_T. A reasonable observational estimate for τ_T is the glob-ally averaged recurrence interval for great earthquakes, 100-300 years. For the sake of clarity and simplicity it is henceforth assumed that all of the N patches have the same average properties. Thus each patch of size S_N is assigned the average recurrence interval $T_i \rightarrow T_N \equiv 1/N)\Sigma_i T_i$. The average slip $\langle u_i \rangle \equiv u_N$ at scale N is then $u_N = VT_N$, so that T_N is a measure of the average slip on patches at scale N.

Observations indicate that the area ruptured in a year by events in any small magnitude band of width Δm is constant [*Rundle*, 1989a]. This leads to the conclusion that the area available for events at any given spatial scale is also constant, equal to S_T. The scale independence of the area is equivalent to the assumption that the rate of occurrence of events in a magnitude band Δm can be freely integrated over all bands with integration weight $\rho(m) = 1$ to obtain the cumulative frequency of events. The following are other fundamental postulates upon which the results of *Rundle* [1989a] are based:

1. Distinct events at any spatial scale N are treated as weakly interacting during slip. For example, this will clearly be the case if in fact the area S_T at scale N is actually disjoint, so that the various patches with area S_N are spatially separate from each other.

2. The average rate of occurrence for events of size S_N is N^{α}/τ_T, i.e., the average rate scales as a power law. α is a number that may be a function of time, but $\alpha = 1$ generally for strict self-similarity in space and time.

3. Probability Distribution

One can derive the probability that a given fault patch is associated with free energy E_i. To proceed, one uses the method of most probable distributions that is described in most elementary texts on statistical mechanics. For example, details of the derivation omitted here are given by *Huang* [1987, pp. 80-82].

Consider patch i at scale N. As a result of motion of the far field, the deficit in slip $\phi_i(t) = Vt - u_i(t)$ increases. As a result, both the stress and mechanical energy in crease on the patch. Now define the stored free energy associated with patch i, $E_i > 0$, as the mechanical energy resulting from elastic, gravitational, cohesive, and other forces [e.g., *Huang*, 1987; *Rundle*, 1989b; *Rundle and Klein*, 1989]. The free energy is stored in the medium as a direct result of the increase in the slip deficit. For example, if the sliding surface were frictionless, all $\phi_i(t) = 0$, all stress would be zero, and there would be no stored free energy. Thus, sliding on the fault serves to reduce the free energy and, in turn, to generate heat directly on the sliding surface, as well as elastic wave radiation that is eventually dissipated by inelastic effects. The physics of sliding can therefore be understood applying the same physical principles as are used in the study of nucleation in other physical systems, i.e., that the system is seeking a minimum free energy state [e.g., *Rundle*, 1989b; *Rundle and Klein*, 1989].

On physical grounds, it is expected that only a few patches on S_T slide during a given time interval. While the free energy E_p of some patches may decline due to sliding, the free energy of other patches increases due to farfield shear motion or due to transfer of stress from sliding patches to neighboring patches. Thus, the total free energy E associated with the deficit in slip on all the patches should fluctuate around some value E_0. Now imagine that N is a very large number, and define the number of patches that have free energy between E_p and $E_p + \Delta E_p$ as n_p. In accordance with common terminology, n_p are called occupation numbers. If the system described here were in equilibrium instead of in steady state, the distribution function obtained would be called a microcanonical ensemble, because the number of patches ("particles") is fixed at N, while the energy is required to fluctuate near E. As a consequence, one has the two constraints:

$$\Sigma_p n_p = N$$
$$\Sigma_p n_p E_p = E \ . \tag{2}$$

To derive the result, one writes the number of ways ("complexions") $\Omega(n)$ in which the N patches can be distributed into the occupation numbers $\{n_p : p = 1,...,k\}$:

$$\Omega\{n_p\} = \frac{N!}{n_1!\, n_2!\, n_3! \ldots n_k!} \ . \tag{3}$$

The entropy Ξ associated with the set (n_p) is $\Xi = k_B \log\Omega$, where k_B is Boltzmann's constant. We wish to find the most probable distribution for the occupation numbers corresponding to the maximum entropy state. Thus, we maximize (3) subject to the conditions (2) by using the method of Lagrange multipliers. In fact, the result is most easily obtained by maximizing the logarithm of (3). Assuming that N is very large, and thus that Stirling's formula is applicable, one obtains for the most probable distribution of occupation numbers $\langle n_i \rangle$ the simple result:

$$\langle n_p \rangle \equiv N\,P[E_p] = N\,Z^{-1} \exp[-E_p / E_N] \ , \tag{4}$$

which is the exponential (Boltzmann) distribution. The constant of proportionality Z^{-1} is obtained from the normalization condition:

$$N = \int_0^\infty N\, P[E_p]\, dE_p \ . \tag{5}$$

Thus

$$Z = E_N \ . \tag{6}$$

If all patches are identical in properties, then the probability that the ith patch has an energy near E_i is $E_i\,P[E_i]$, as is implied by (2). The distribution (4) has been seen in numerical simulations of slider-block models (J.B. Rundle et al., manuscript in preparation, 1993), lending support to the general approach.

The factor E_N in the exponential is the average energy about which patch i fluctuates. In the kinetic theory of gases, the canonical ensemble involves a system at equilibrium that is held in close contact with a large (effectively infinite) heat bath that maintains the system at a constant temperature T [e.g., *Huang*, 1987]. A classical system in the canonical ensemble also has a Boltzmann distribution in energy. By calculating the average energy of system particles in the kinetic theory of gases, it is trivial to show that the factor equivalent to E_N is just the quantity $k_B T$.

In our nonequilibrium, steady state system, the motion of the far field acts to produce mechanical energy available to induce sliding on the fault zone. Sporadic but ongoing sliding of the patches reduces the amount of mechanical energy, turning it into heat that then escapes from the system. It has been assumed that the system of fault patches is so large that the energy fluctuates with small amplitude fluctuations about some average value $E_o = N\,E_N$. Clearly, that energy should depend both on the size S_N and the average slip $u_N \propto T_N$ in an event. Thus we take

$$E_N \equiv \chi\, S_N\, T_N \ . \tag{7}$$

where the probability weight χ is the energy density, per unit of area, per unit of recurrence time interval. While it is difficult to obtain a direct measurement of the value of χ, it turns out fortuitously that χ cancels out of all relevant expressions that we derive. We address the issue of the value of χ in a later section. Finally, it can be seen that E_N is proportional to the average seismic moment of events of size S_N, since $S_N\,T_N \propto S_N\,u_N$.

4. MAGNITUDE-FREQUENCY RELATIONS FOR INTERMEDIATE AND LARGE EVENTS

It is instructive to use the distribution function $P[E_i]$ to retrieve the results obtained by *Rundle* [1989a] for intermediate and large events. As in work by *Rundle* [1989a], we assume for the moment that the fault plane is of negligible thickness, but this assumption will be reconsidered below. It will be recalled that the basic picture of Earth is an upper part that consists of a cold, brittle elastic layer in which earthquake stresses build up and are released, overlying a hotter, more ductile lower regime in which stresses cannot build up significantly due to the tendency of the material to flow under applied stresses. Moreover, let the fault plane be of width W in the downdip direction and of length $L >> W$ in the horizontal (strike) direction. This picture is consistent with that summarized by *Scholz* [1982].

Earthquakes whose maximum linear sizes are less than W will in general not be affected by the finite thickness of the brittle layer. Events of this type will be called intermediate sized earthquakes. They behave as if the fault plane were unbounded, and they can grow freely in any direction within the plane of the fault. On the other hand, large sized earthquakes are events that are big enough that the finite width W of the brittle seismogenic fault zone bounds their extent in the depth direction. To grow to a significantly large size, events must therefore extend horizontally along the fault zone. For these large events, it will be assumed that the slip u_i scales with the minimum dimension of the event size, in agreement with well-known predictions of elastic dislocation theory and recent reassessment of data on strike-slip fault zones [*Romanowicz*, 1992]. The slip therefore saturates if the event extends over the entire width W of the fault zone, so that u_i is then proportional to W. There has been controversy about this assumption, and a full discussion of this issue is given by *Romanowicz and Rundle* [1993].

To begin, we calculate the mean $<E>$ and standard deviation $<\sigma_E>$ of the energy of the ith patch using the probability distribution $P[E]$:

$$<E> = \int_0^\infty E\, P[E]\; dE = \chi\, S_N\, T_N \tag{8}$$

$$<E^2> = \int_0^\infty E^2\; P[E]\; dE = 2\,(\chi\, S_N T_N)^2 \tag{9}$$

$$\sigma_E = \sqrt{(<E^2> - <E>^2)} = \chi\, S_N\, T_N\ . \tag{10}$$

Using the definition of seismic moment (A1) and the stress drop-slip relation (A3) for a square fault patch, one can write

$$\frac{\chi\, T_N}{\sigma_E} = \left(\frac{\Delta\sigma_T}{2.2\, M_o}\right)^{2/3} . \tag{11}$$

Multiplying both sides by S_T/τ_T, and using the moment magnitude relation (A2), one obtains

$$\frac{\chi\, S_T\, T_N}{\sigma_E \tau_T} = \left[\left(\frac{S_T}{\tau_T}\right)\left(\frac{\Delta\sigma_T}{2.2}\right)^{2/3} 10^{-10.7}\right] 10^{-m} \tag{12}$$

The left-hand side of (12) is just the total amount of energy per year available to produce events of magnitude m, ($\chi\, S_T\, T_N/\tau_T$), divided by the root mean square fluctuation in energy, σ_E, of one such event. The left-hand side is therefore the average number of such fluctuations, or events, per year. To see this more explicitly, one can substitute for σ_E from (10) in (12):

$$\frac{\chi\, S_T\, T_N}{\sigma_E \tau_T} = \frac{1}{\chi\, S_N\, T_N} \cdot \frac{\chi\, S_T\, T_N}{\tau_T} = \frac{[S_T/S_N]}{\tau_T}$$

$$= \frac{N}{\tau_T} \equiv \omega. \tag{13}$$

The expression $N/\tau_T \equiv \omega$ was obtained by *Rundle* [1989a] as the result of a rescaling argument often used in the study of critical phenomena [e.g., *Ma*, 1976]. ω is then the rate of occurrence of events of size S_N. It was also pointed out by *Rundle* [1989a] that an apparent rate of occurrence $\omega' = N^{\alpha}/\tau_T$ might be observed if sampling or statistics are incomplete. The parameter α may depend upon time, but over long periods of time, self-similarity requires that $\alpha = 1$. In the remainder of this paper, we consider only examples in which $\alpha = 1$. Generalization to cases in which $\alpha \neq 1$ is obvious [e.g., *Rundle*, 1989a].

Note explicitly that the form in which (13) is written indicates that the quantity given by $\chi\, S_T\, T_N/\tau_T$ is indeed the average yearly energy available to produce events of a size which recur on a given patch of area S_N at intervals T_N. Breaking the expression down a bit further, the quantity χS_T is the amount of energy available per year for events of any given size, and so T_N/τ_T must be the fraction of this energy that is expended in producing events of that size. The rms fluctuation in energy σ_E for events associated with intervals T_N is what determines how many events occur. These points will become important in the derivation of the small magnitude limit.

The probability density dn_I'/dt for the rate of occurrence of intermediate events of magnitude m is thus given by (12). The cumulative rate of occurrence dn_I/dt of intermediate events with magnitude greater than m is then given by the integral of (12);

$$dn_I/dt = \int_m^\infty (dn_I'/\, dt\,)\; \rho(m)\; dm\,. \tag{14}$$

Recall that the quantity $\rho(m)$, the integration weight giving the relative fault areas in different magnitude bands, has been assumed equal to 1 on the basis of self similarity. Thus:

$$dn_I/dt = \left[\left(\frac{S_T}{\tau_T}\right)\left(\frac{\Delta\sigma_T}{2.2}\right)^{2/3} 10^{-11.1}\right] 10^{-m}$$

$$= 10^{A_I}\; 10^{-m}. \tag{15}$$

Thus the expected value is $b_I = 1$, although $b_I = \alpha$ in the more general case of incomplete sampling. More-over, the prefactor to the exponential depends on α in the general case. Expression (15) is identical to that obtained by *Rundle* [1989a] for $\alpha = 1$.

To recover the magnitude frequency relation found by *Rundle* [1989a] for large events, we proceed as above with small changes. The picture of an earthquake now is an event whose downdip depth is a constant, equal to the downdip width W of the fault zone within the brittle seismogenic layer. The length of the event L along strike is assumed to be considerably greater than the depth, $L >> W$. Thus we use (10) and (A3) as before to write

$$\frac{\chi\, T_N}{\sigma_E} = \left(\frac{\Delta\sigma_T}{f\, M_o}\right)^{2/3}. \tag{16}$$

Multiplying both sides by $S_T/\tau_T = LW/\tau_T$, and using the moment-magnitude relation (A2) yields

$$\frac{\chi\, S_T\, T_N}{\sigma_E \tau_T} = \left[\left(\frac{S_T\, W}{f\, \tau_T}\right) \Delta\sigma_T\;\; 10^{-16.1}\right] 10^{-1.5m} = \gamma \tag{17}$$

for the probability density of the long term rate of occurrence $\gamma = dn_L'/dt$. Equation (17) is identical to equations (17) and (18) of Rundle [1989a] again with $\alpha = 1$. The cumulative frequency of large events with $\alpha = 1$ is again found by integrating (17) over $[m, \infty)$:

$$dn_L/dt = \left[\left(\frac{S_T\, W}{f\, \tau_T}\right) \Delta\sigma_T\;\; 10^{-16.6}\right] 10^{-1.5m}$$

$$= 10^{A_L}\; 10^{-1.5m}. \tag{18}$$

The more general expression with $\alpha \neq 1$ is given by *Rundle* [1989a]. In this case, one finds that $b_L = 1.5\alpha$, with again an expected value $\alpha = 1$ for complete sampling and good statistics.

5. MAGNITUDE FREQUENCY RELATION FOR SMALL EVENTS

Taken at face value, the seismicity data in Figure 1 indicate that something unusual and unexpected is occurring at small magnitudes, namely, that the frequency of small earthquakes is considerably less than would be predicted using Gutenberg-Richter statistics. Other data obtained by *Malin et al.* [1989] and *Rydelek and Sacks* [1989] support this contention. All of these authors have taken considerable care to account for the limits of instrumental detectability in their observations. The remainder of this paper is primarily devoted to an analysis of these observations, and to the construction of a magnitude-frequency relation to explain them. However, it should be clear that the extent to which the analysis represents reality rests almost entirely upon the assumption that the trend in the data arises from a source, and not a path, effect. In addition, since the frequency of small events must smoothly join on to the curve representing the frequency of larger events, the correct b values for intermediate and large events are needed to ensure that small event frequencies are accurately calculated.

The change in scaling at $m \sim 7$ is associated with a change in the physics of the faulting process, namely, the change from effectively unbounded events to events that are affected by the finite width W of the fault zone. The scaling of slip changes, and in turn the scaling in moment changes as well. All of these effects are associated with the appearance of the length scale W in the physics. In searching for a rationale for the departure of the observed frequency from the Gutenberg-Richter prediction shown in Figure 1, it is therefore appealing to ascribe the effect to the appearance of another length scale.

The length scale that most naturally suggests itself is the finite thickness λ_c of the fault zone. Earthquakes are a result of elastic stresses building up across a fault zone as the result of relative motion in the far field. These stresses are spatially correlated over macroscopic distances of the order of the maximum linear size of the earthquake. However, within a typical fault zone, deformation is often characterized by inelastic processes, including enhanced or retarded pore fluid flow, "plastic" deformation of fault gouge, and so forth. Therefore events that have a maximum dimension less than λ_c should be rare, since a physical picture of correlated elastic stress across such a fault zone is questionable.

We therefore hypothesize that small events involve a change in the physical process, and a consequent change in scaling. Specifically, we propose a physical picture in which small events have a fixed linear size equal to λ_c, the minimum length over which elastic stress can be correlated. In this idea, fault diameter can be no less than the thickness of the inelastic region, so that the fault area for these events is always λ_c^2. The moment is then determined entirely by the slip, quantities which are given by

$$T_N \propto u_N = (\sqrt{S_N}\ \Delta\sigma_T / f\mu)$$

and

$$M_0 = \mu \lambda_c^2 (\sqrt{S_N}\ \Delta\sigma_T / f\mu) \ . \tag{19}$$

The definition of u_N is implied by continuity as $S_N \to 0$. To account for a smooth crossover to this new physical process at small size scales, we therefore propose that an extra term be added to the average energy E_N given by (7):

$$E_N = \chi S_N T_N + \kappa \lambda_c^3 T_N \ . \tag{20}$$

The new quantity κ is the amount of energy dissipated, per unit of volume per year, by inelastic deformation within the fault zone. Therefore, as magnitude de-creases, the dissipation can no longer be pictured as occurring on a fault plane of zero thickness. Instead, bulk dissipation within the nonzero-thickness fault zone must now be taken into account. In fact, this new picture implies the relation

$$\chi \approx \lambda_c \kappa \ . \tag{21}$$

To obtain the magnitude frequency relation corresponding to (20), start by defining the quantity

$$S_c = \frac{\kappa}{\chi} \lambda_c^3 = \lambda_c^2 \tag{22}$$

using (21). Proceeding as before, using (A1)-(A3), then multiplying by S_T / τ_T, one obtains for the probability density for the frequency of occurrence of small events dn_S'/dt:

$$dn_S'/dt = \frac{\chi S_T T_N}{\sigma_E \tau_T} = \left[\frac{S_T}{S_c \tau_T}\right] \frac{1}{z^{-1} 10^m + 1} \tag{23}$$

where

$$z^{-1} = 10^{-m_c} = \left[\frac{f 10^{16.1}}{\Delta\sigma_T}\right]^{2/3} \frac{1}{S_c} \ . \tag{24}$$

It can be seen that (23) is an example of a Fermi-Dirac distribution in magnitude. The quantity z is analogous to the fugacity in equilibrium statistical mechanics. However, the relevant physically observable quantity is the critical magnitude, m_c, which is related to the event size S_c and therefore to the fault thickness λ_c. Using (22) and (24),

$$\lambda_c = \left[10^{.5 m_c + 5.5}\right] \Delta\sigma_T^{1/3} \ . \tag{25}$$

To obtain the cumulative frequency of events dn_S/dt greater than magnitude m, (23) is integrated over $[m, \infty]$:

$$\begin{aligned} dn_S / dt &= \frac{S_T}{S_c \tau_T \log 10} \log\left[1 + 10^{m_c - m}\right] \\ &= 10^{A_I - m_c} \log\left[1 + 10^{m_c - m}\right] \ . \end{aligned} \tag{26}$$

The quantity A_I is the value of the coefficient found for intermediate earthquakes and can be obtained directly from expression (15). For events with magnitudes m significantly larger than m_c, relation (26) goes asymptotically to (15) as expected, since, for x small, $\log(1 + x) \to x$.

6. DISCUSSION AND SUMMARY

Equation (26) is plotted as the dashed line in Figure 1 and provides a reasonable fit to *Aki's* [1987] small magnitude and intermediate magnitude data. In fitting the data, we have used $A_I = \log_{10}(143) = 2.16$, and $m_c = 1.45$ as values. Note because of the specific numerical values used in the moment-magnitude relation (A2), units used for calculations in the above equations are cgs units. From the value obtained for m_c, relation (25) can be used to obtain a value λ_c. Assuming that $\Delta\sigma_T = 1$ *MPa* (= 10^7 dyn/cm^2 = 10 bars), one finds that $\lambda_c = 75$ m, whereas $\Delta\sigma_T = 10$ *MPa*, one finds $\lambda_c = 35$ m. By way of comparison, in the San Francisco earthquake of 1906 [e.g., *Lawson*, 1908; pp. 64, 96-100], fences offset by the fault movement were generally deformed over a fault zone of tens to hundreds of meters thick. Assuming that the surface offsets are roughly indicative of fault thicknesses at depth, the value of 35-75 m for λ_c is reasonable.

As a matter of interest, values for the parameters χ and κ can be estimated, although the accuracy of the estimates is unknown. The probability weight χ is the average rate of energy dissipated by slip over S_T per year, divided by S_T. For Earth as a whole, and assuming that S_T is one Earth circumference multiplied by 100 km depth (= 4×10^{12} m^2; see *Rundle*, 1989a], and a seismic efficiency $\eta \approx 0.1$ [H. Kanamori, personal communication, 1987], one obtains a value for $\chi \approx 5 \times 10^{17}$ j / (0.1 x 4 x 10^{12}) = 1.25×10^6 j /m^{2}yr. Assuming a value $\lambda_c = 50$ m, then $\kappa \approx 2.5 \times 10^4$ j /m^{3}yr.

In a general sense, it might seem strange to some readers that statistical mechanics could have anything to do with earthquakes. However, statistical mechanical methods, which include scaling (fractals) and the renormalization group [e.g., *Turcotte*, 1992], percolation theory [e.g., *Stauffer*, 1985], nucleation [e.g., *Rundle*, 1989b], chaos [e.g., *Turcotte*, 1992; see also *Schuster*, 1989], and self-organized criticality [e.g., *Bak et al.*, 1988], are rapidly being applied to many problems in the earth sciences. Statistical mechanics can describe not only the world of the very small and complex but also the very large and complex. It seems likely that applications of statistical mechanical methods to problems in the earth sciences will only continue to increase in the future.

APPENDIX

We collect several empirical relations that are used in the text [see *Rundle*, 1989a]. The definition of seismic moment M_o for the ith patch is

$$M_0 = \mu S_i u_i \tag{A1}$$

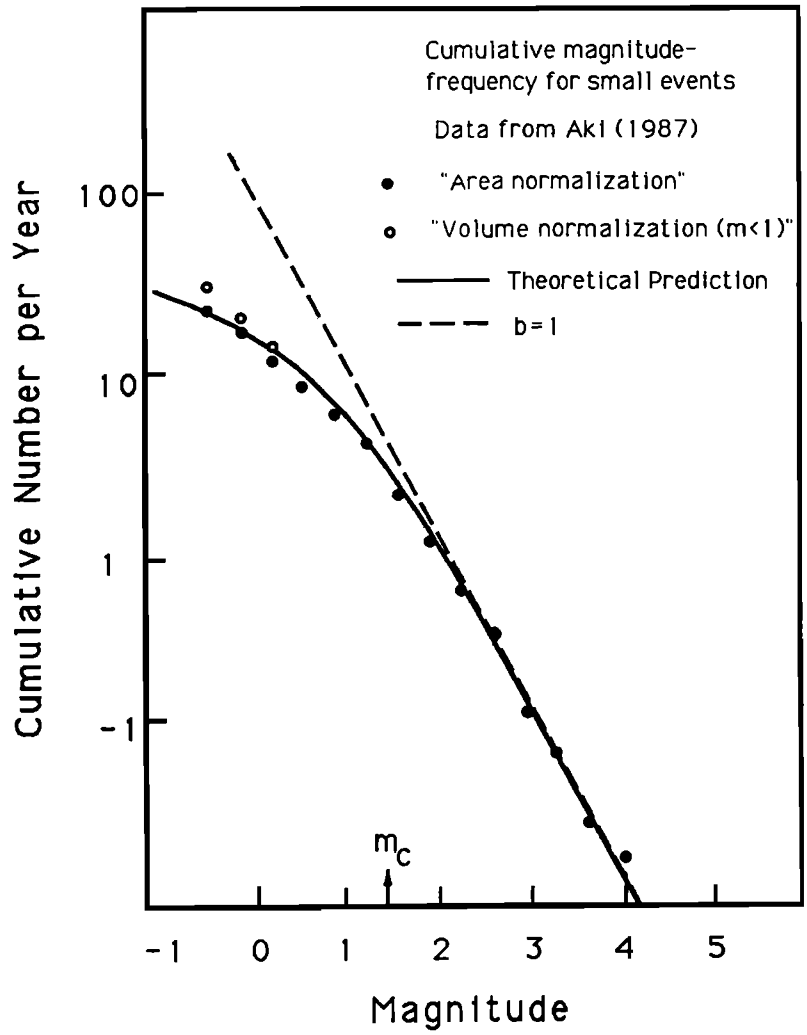

Fig. 1. Data obtained by *Aki* [1987] in a study of small earthquakes in the Baldwin Hills section of the Newport-Inglewood fault zone. Dashed line is a Gutenberg-Richter frequency obtained by fitting equation (1) with $A_o = \log_{10}(143) = 2.16$, $b = 1$. Solid line is a fit using the theoretical prediction (equation (26)) with $A_I = 2.16$, $m_c = 1.45$ for the critical magnitude event.

where μ is the shear modulus. The general magnitude-moment relation is [*Hanks and Kanamori*, 1979]

$$1.5\, m = \log_{10} M_o - 16.1 \tag{A2}$$

and the stress drop ($\Delta\sigma_T$) slip relation is

$$\Delta\sigma_T = (f\, \mu\, u_i)/R \tag{A3}$$

where the factor $f = 2.2$ for a square fault of area S_i [e.g., *Rundle and Kanamori*, 1987]; $f = 1.3$ for a long strike slip fault; $f = 1.7$ for a long dip slip fault. Also, $R = \sqrt{S_i}$ for a square fault, and R = fault width = W for a long strike slip or dip slip fault [see *Rundle*, 1989a; *Romanowicz and Rundle*, 1993].

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