Dynamics of a Traveling Density Wave Model for Earthquakes

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We propose a unifying new framework for friction and earthquakes. Predictions from theory include (1) scaling exponents, and (2) metastable lifetimes for nucleating slip droplets. Earthquake simulations produce Gutenberg-Richter distributions of events with b values similar to observed declustered values, and populations of characteristic earthquakes, with associated aftershocks. Time is a relevant scaling field, justifying statistical time-to-failure analyses. When applied to integrate-and-fire neurons, the theory yields an explicit equation for the cell potential. [S0031-9007(96)00206-2]

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A central problem in understanding earthquakes lies in clarifying the physics of the frictional sliding processes [1]. Current modeling approaches [2–8] rely on friction laws that are of either the cellular automaton (CA) jump-rule type, the simple Burridge-Knopoff model with inverse velocity dependence, or the experimentally determined Dieterich-Ruina (DR) parametrized state-variable model. Each of these laws has its difficulties.

Recent work [3, 11] has shown that classes of mean field CA models can be described by a spatially and temporally coarse grained field theoretic formulation, and that these models can often be treated as equilibrium models. We are therefore motivated to examine a broad class of models that contain the coarse grained CA models as a special case [12–17].

To summarize our main result: We have found that the large body of laboratory data is consistent with a field theoretic formulation involving a traveling density wave (TDW) Lyapunov functional, similar to the density wave Hamiltonians [18, 19] used to describe charge and spin density waves.

We begin by considering two elastic media in contact at a surface upon which slip $s(\mathbf{x}, t)$ can occur in response to applied shear stress. Sliding is driven [20] by the imposition of a spatially constant displacement s_B and stress σ_B on boundaries, or at distances far removed from the slip surface. For such a system, the elastic stress $\sigma^e(\mathbf{x}, t)$ on the slip surface can be written in terms of (1) a stress Green's function $T(\mathbf{x} - \mathbf{x}')$; (2) the stress σ_B, and (3) the slip deficit $\phi = s(\mathbf{x}, t) - s_B$:

$$\sigma^e(\mathbf{x}, t) = \int T(\mathbf{x} - \mathbf{x}')\phi(\mathbf{x}', t)\, d^2\mathbf{x}' + \sigma_B\,. \quad (1)$$

Note that the elastic stress is a functional of ϕ, which we denote as $\sigma^e[\phi]$, and that we consider one side of the surface "fixed," with a coordinate system $\mathbf{x}$ attached to the "moving" side.

There is also a frictional stress σ^f on the surface resisting slip. In formulating the general characteristics of the friction, we refer to the extensive literature [12–17] on friction and wear of solids, particularly the compilation of data described in Refs. [12, 13]. We note that σ^f arises from the cohesion between the sliding surfaces, and that the contact between irregular surfaces will vary as sliding progresses. Following Ref. [13], we therefore take σ^f at each location to be a functional:

$$\sigma^f = \sigma^f[s; c; r]\,. \quad (2)$$

In (2), $s(\mathbf{x}, t)$ is the slip, c represents a set of parameters describing the cohesion between the surfaces, and r represents a set of random parameters. Both of the parameter sets represented by c and r are in general site (location) and time dependent. Time dependence describes wearing of the surface, i.e., random irreversible alteration of surface properties as a result of microscopic surface deformation during sliding. Note that σ^f is a function of space and time through the space and time dependence of s, c, and r. Also, c should depend explicitly on the normal stress σ_N pushing the surfaces into contact.

Now consider a state of deformation of the system (solid + surface) induced by a far field displacement s_B. The balance of forces [20, 21] on the surface requires that

$$\sigma^e[\phi] = \sigma^f[\phi + s_B; c; r] \quad (3)$$

using the definition of ϕ. We now use the fact that (3) can be obtained as the Euler-Lagrange equation corresponding to a functional potential [20, 21]. We define $U[\phi] = E[\phi] - S[\phi + s_B, \phi]$, where U is a Lyapunov functional [22] that plays a role similar to an equilibrium "free energy functional," and $\sigma^f = -\delta S/\delta\phi$. The functional $S[\phi + s_B, \phi]$ defines the energy associated

with the cohesive forces acting on the surface [23]. It is straightforward [3, 21, 23] to show that the "elastic energy" $E[\phi]$ exists and has the form

$$E[\phi] = \int\int \{ -(1/2)[T(\mathbf{x} - \mathbf{x}')\phi(\mathbf{x},t)\phi(\mathbf{x}',t)\, d^2\mathbf{x}'] + \sigma_B \phi(\mathbf{x},t)\}\, d^2\mathbf{x}\,. \quad (4)$$

Similarly, σ^f can be written [11, 12] as a Fourier cosine series in $\phi + s_B$ using expansion parameters A_n, κ_n, and $h = A_0$, from which $S[\phi + s_B, \phi]$ can then be obtained by functional integration

$$S[\phi + s_B, \phi] = \int \Big[\sum_n A_n \cos\{\kappa_n(\phi + s_B + \varepsilon_n)\} - h\phi \Big] d^2\mathbf{x}\,. \quad (5)$$

The parameters A_n, κ_n, and h describe the cohesive ("c") properties of the frictional force, and the random phases ε_n describe general time dependent changes ("r") associated with wearing of the surface. Note that in general A_n, κ_n, and h also have a random part (r) associated with processes of wear on the surface.

We are interested here in models with $\sigma_B = 0$, and in which the boundaries are translating uniformly, so that $s_B = Vt$, where V is the velocity of the moving side with respect to the fixed side. Therefore $\sigma^e[\phi] = \sigma^f[\phi + Vt]$, and the instantaneous balance of forces at a site $\mathbf{x}$ can be viewed in the (σ, ϕ) plane as the intersection (see Fig. 1) of a fixed curve $\sigma^e[\phi]$ with a leftward moving wave $\sigma^f[\phi + Vt]$. The intersection point ϕ^* is therefore time dependent: ϕ^* slowly becomes more negative (strain accumulation) until a point is reached at which ϕ^* increases rapidly in a positive direction (slip event). If the cohesion varies sufficiently in space or in amplitude, unstable stick slip occurs at fixed far field displacement; otherwise, the surface slips in a stable manner in response to increasing far field displacement.

We make the assumption that the system is instantaneously evolving toward states of greater stability. In that case, it can be shown [22] that the dynamics are obtained in terms of the associated Ito-Langevin equation, a constant mobility Γ, and a noise η:

$$\frac{\partial \phi}{\partial t} = -\Gamma \frac{\delta U}{\delta \phi} + \eta(\mathbf{x}, t)\,. \quad (6)$$

The lattice average $\langle \eta(\mathbf{x},t)\eta(\mathbf{x}',t')\rangle = \beta^{-1}\delta(t - t') \times \delta(\mathbf{x} - \mathbf{x}')$, and β is a constant.

To explore the consequences of these ideas, suppose that σ^f is dominated by only one of the Fourier terms for some wave number κ:

$$U[\phi] = \int\int \big(-(1/2)[T(\mathbf{x} - \mathbf{x}')\phi(\mathbf{x},t)\phi(\mathbf{x}',t)\, d^2\mathbf{x}'] - 2\gamma\cos[\kappa\{\phi(\mathbf{x},t) + Vt + \varepsilon\}] + h\phi(\mathbf{x},t)\big)\, d^2\mathbf{x}\,. \quad (7)$$

Both 2γ and h depend in general on the normal stress σ_N, which in turn depends [24] on V. Using (6), the

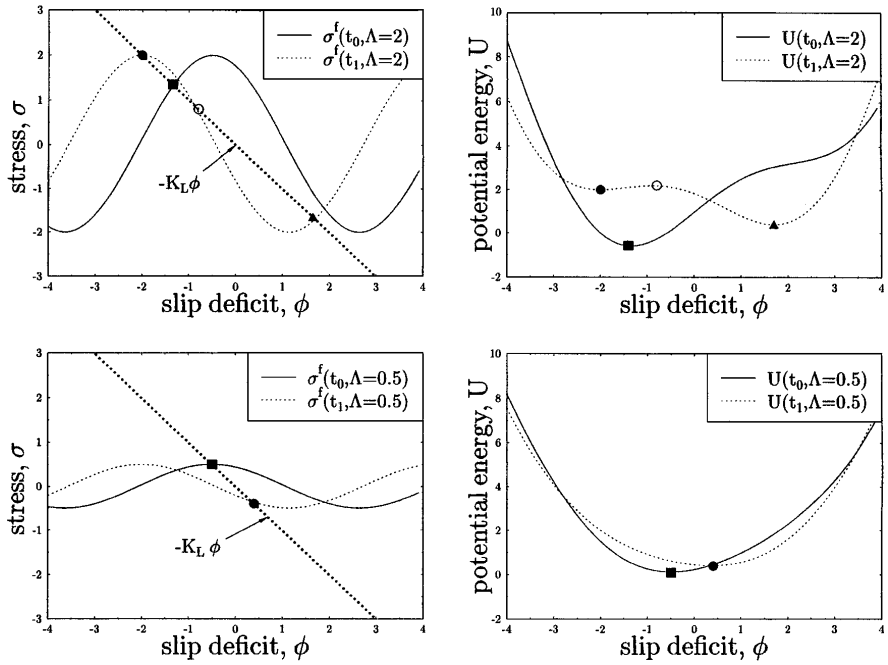

FIG. 1. Plot of stress $\sigma(t)$ and potential energy $U(t)$ against ϕ in mean field limit for $\Lambda = 2$ (top) and $\Lambda = 0.5$ (bottom) and for two distinct times $t_1 > t_0$. Top: At time t_0, only one globally stable state exists (square); at time t_1, metastable (filled circle), unstable (open circle), and globally stable (triangle) states now exist. Bottom: At both times t_0 and t_1, only one globally stable state exists (square at t_0 and circle at t_1).

corresponding Ito-Langevin equation [22] is

$$\frac{\partial \phi(\mathbf{x},t)}{\partial t} = \Gamma\Big(\int T(\mathbf{x}-\mathbf{x}')\phi(\mathbf{x}',t)\, d^2\mathbf{x}' - 2\gamma\kappa \sin[\kappa\{\phi(\mathbf{x},t) + Vt + \varepsilon(\mathbf{x},t)\}] - h\Big) + \eta(\mathbf{x},t)\,. \quad (8)$$

We search for spatially uniform solutions $\phi_0(t)$ to (7) and (8), with $\eta = 0$ and $\varepsilon = 0$. The functional density is $u_0[\phi_0] = K_L\phi_0^2 - 2\gamma\cos\{\kappa(\phi_0 + Vt)\} + h\phi_0$, where $K_L > 0$ is minus the integral of $T(\mathbf{r})$. Clearly if κ and γ are small, $u_0[\phi_0]$ is concave up, and has only a single (global) minimum. On the other hand, if γ or κ are sufficiently large, $u_0[\phi_0]$ may have more than one minimum, with all but the lowest energy state being metastable. More than one minimum is possible when $\partial^2 u_0/\partial\phi_0^2 = K_L + 2\gamma\kappa^2\cos\{\kappa(\phi_0 + Vt)\}$ has at least one zero. Metastability can occur only if the parameter $\Lambda > \Lambda_c = 1$, where $\Lambda = 2\gamma\kappa^2/K_L$. As will be clear from the discussion below, $\Lambda - \Lambda_c$ can be regarded as a scaling field for this potential. Qualitatively, Λ can be interpreted as indicating the number of local minima. An immediate prediction of our theory is that as the stiffness of the surrounding elastic medium K_L is increased, a transition should occur, from the appearance of sudden unstable jumps (decay from metastability), to stable sliding. Such a transition is commonly observed [7, 9, 10, 13, 15] in laboratory experiments.

The potential $u_0[\phi_0]$ is shown for a variety of times in Fig. 1 for an example in which $\Lambda > 1$. In addition to Λ, time "t" can also be regarded as a kind of scaling field for the line of ("first order") critical points defining the spinodal. It can be observed from Fig. 1 that when the system is in a metastable energy state higher than the global minimum, the passage of time drives down the height of the energy barrier toward the spinodal, and the system nucleates into the lower energy state. This process recurs with period $P = 2\pi/\kappa V$ so that the spinodal occurs at times $t = t_{\rm sp} + nP$ ($(n =$ integer). The equation obtained to describe the nucleation process is precisely that obtained in studies of spinodal nucleation [25], but where the scaling field is proportional to $\delta t = (t_{\rm sp} - t)\mathrm{mod}(P)$. We state the results here and present the details elsewhere [26].

For a model with long wavelength fluctuations are present, the field $\phi = \phi(\mathbf{x},t)$ and the Lyapunov functional density u_0 is modified by the presence of a term $K_C|\nabla\phi|^2$:

$$U'[\phi] = \int\int \{(1/2)[K_L\phi^2 + K_C|\nabla\phi|^2] - 2\gamma\cos[\kappa(\phi + Vt)] + h\phi\}, \quad (9)$$

where the constant K_C is proportional to the second moment of the stress Green's function $T(\mathbf{r})$. Writing $\phi(\mathbf{x},t) = \Phi(t_{\rm sp}) + \psi(\mathbf{x},t)$, where $\Phi(t_{\rm sp})$ is a spatial constant, and with $|\psi(\mathbf{x},t)| \ll |\Phi|$, we expand about the spinodal. To first order in δt, the second scaling field for this problem, the equation for $\psi(\mathbf{x},t)$ is found to be

$$-K_C\nabla^2\psi + K_L V\delta t - \alpha\psi^2 = 0\,, \quad (10)$$

where $\alpha = |\gamma\kappa^3\sin\{\kappa(\Phi + Vt_{\rm sp})\}|$. Note that the coefficient of the third term does not vanish at the spinodal $\delta t = 0$ and that $\partial\psi/\partial t = 0$. Equation (10) is identical to the Euler-Lagrange equation obtained for nucleation near the spinodal in magnetic systems [25], binary fluids, alloys [27], and for tensile fracture nucleation [23]. The solution to Eq. (10) is known [25] to be given in the form $\psi(\mathbf{x},t) \sim (K_L V\delta t)^{1/2}\tilde{\psi}(|\mathbf{x}|/\xi)$, where the correlation length $\xi \sim (K_L V\delta t)^{-1/4}$ and $\tilde{\psi}$ is nonzero and bounded at $t = t_{\rm sp}$. This is precisely the kind of low-amplitude, large spatial-extent solution that has been observed in recent field data [28]. Observations indicate that both foreshock and aftershock frequency are proportional to an inverse power of δt [29, 30].

Several predictions are immediately evident [25]. For small V, which is the case of interest for earthquake faults, the lifetime Γ in the metastable state is $\Gamma \sim \exp\{K_C(K_L V\delta t)^{3/2-d/4}\}$, where d is the dimension of space ($d = 2$ for a planar fault). Moreover, one expects that nucleation will occur near the Becker-Döring limit $K_C(K_L V\delta t)^{3/2-d/4} \propto \beta^{-1}$, where the constant of proportionality is in the range of 2 to 4. The scaling properties of the seismicity near the spinodal imply exponents for a mean field spinodal, specifically $\tau = 5/2$. Finally, a scaling regime with critical exponents for the Ginzburg-Landau model in $d = 2$ should be seen as Λ approaches 1 from below. Using simple scaling arguments [26], we predict (1) the corresponding Gutenberg-Richter b value for a single fault to be $b = 3(\zeta - 1)/2c$, where c is the exponent in the scaling relation between seismic moment M_0 and area A, $M_0 \propto A^c$, and ζ is the exponent of the frequency-size distribution in the scaling region, either τ or $\tau - 1$, as appropriate; and (2) an Omori law for foreshocks can be derived [26] with an Omori exponent [30] $p = 1/\sigma_m = 1$, where σ_m is the (magnetic field) surface exponent.

To investigate these predictions, we carried out simulations on a 100×100 lattice of sites in $d = 2$. We solved [31] the Ito-Langevin equation (8) with $\Lambda = 2$, and $\kappa = 1$, $\eta = 0.01\rho$, $\varepsilon = 5.6\rho$, where ρ is a uniformly distributed random number between $[0, 1]$. Results for the scaling properties of the frequency-area relation are shown in Fig. 2. In this plot, sites interact with $q = 13^2 - 1 = 168$ neighbors, via an interaction $T(r) \sim K_C/r^3$, where $r = |\mathbf{x} - \mathbf{x}'|$. Both nucleation events ("characteristic earthquakes"), and a scaling regime for cluster numbers [32] n_s of size s of smaller events, with $\tau \approx 5/2$, can be seen. The location of the peak is controlled [31] by the amplitude of η. The corresponding b value for the spinodal fluctuations is ~ 2.23, which arises because $c \approx 1$ for this model [31] and $\zeta = \tau = 5/2$. This b value is roughly twice as large

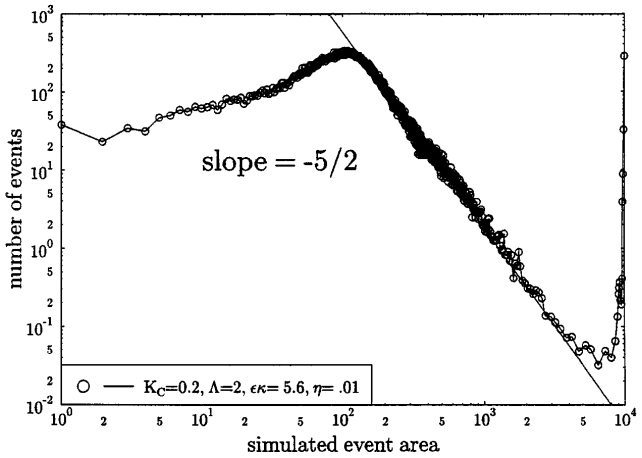

FIG. 2. Number of simulated events against area of simulated event on a log-log plot.

as commonly observed values for regional fault systems [33]. However, our preliminary results [31] with large η indicate that $\zeta = \tau - 1 = 3/2$, and $b \approx 0.75$, closer to the observed $b_{\rm obs} \approx 0.8$ characteristic of declustered earthquakes in southern California [33]. Simulations with these b values have been observed [31]. Finally, aftershock (relaxation) events have been seen [31] following larger events in the simulations.

Three final remarks are in order. (1) With $h \propto V + O(V^2)$, the Ito-Langevin equation (8) is suggestive of the state variable relaxation equation discussed in Ref. [6], with the state variable θ identified with the slip deficit ϕ. (2) Under the Ito-Langevin dynamics (8), the Lyapunov functional $U[\phi(t)]$, Eq. (7), with $\varepsilon = \eta = 0$, and with spatially uniform physical constants, will undergo a "downhill march" on its energy landscape. Because U is bounded from below, it can be shown [26, 34, 35] that $U[\phi(t + P)] \leq U[\phi(t)]$, and the system approaches a periodical limit cycle. This will also be true for systems with quenched disorder. (3) If ϕ is identified with cell potential, and V with injected current divided by cell capacitance [35] for an integrate-and-fire neuron, (8) can be regarded as an explicit, rather than the usual implicit [34] equation for the evolution of the cellular potential. Decay of a neuron from metastability is then identified with the transmission of an action potential. The network learning problem might be made easier for models of this type by the use of the explicit evolution equation.

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