Båth's law derived from the Gutenberg-Richter law and from aftershock properties

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[1] The empirical Båth's law states that the average difference in magnitude between a mainshock and its largest aftershock is 1.2, regardless of the mainshock magnitude. Following *Vere-Jones*' [1969] and *Console et al.* [2003], we show that the origin of Båth's law is to be found in the selection procedure used to define mainshocks and aftershocks rather than in any difference in the mechanisms controlling the magnitude of the mainshock and of the aftershocks. We use the ETAS model of seismicity, which provides a more realistic model of aftershocks, based on (i) a universal Gutenberg-Richter (GR) law for all earthquakes, and on (ii) the increase of the number of aftershocks with the mainshock magnitude. Using numerical simulations of the ETAS model, we show that this model is in good agreement with Båth's law in a certain range of the model parameters. *INDEX TERMS:* 7209 Seismology: Earthquake dynamics and mechanics; 7223 Seismology: Seismic hazard assessment and prediction; 7260 Seismology: Theory and modeling. **Citation:** Helmstetter, A., and D. Sornette, Båth's law derived from the Gutenberg-Richter law and from aftershock properties, *Geophys. Res. Lett.*, *30*(20), 2069, doi:10.1029/2003GL018186, 2003.

1. Introduction

[2] Båth's law [*Båth*, 1965] predicts that the average magnitude difference Δm between a mainshock and its largest aftershock is 1.2, independently of the mainshock magnitude. Many studies have validated Båth's law, with however large fluctuations of Δm between 0 and 3 from one sequence to another one [e.g., *Felzer et al.*, 2002; *Console et al.*, 2003]. In addition to providing useful information for understanding earthquake processes, Båth's law is also important from a societal view point as it gives a prediction of the expected size of the potentially most destructive aftershock that follows a mainshock.

2. Vere-Jones' Interpretation of Båth's Law

[3] Båth's law is often interpreted as an evidence that mainshocks are physically different from other earthquakes and have a different magnitude distribution [e.g., *Utsu*, 1969]. In contrast, *Vere-Jones* [1969] offered a statistical interpretation, elegant in its simplicity, which consisted in viewing the magnitudes of the mainshock and largest aftershock as the first and second largest values of a set of independent identically distributed (iid) random variables distributed according to the same GR distribution $P(m) \sim 10^{-bm}$. If the same minimum threshold m_0 applies for both aftershocks and mainshocks, this model predicts that Δm has the same density distribution $P_{\Delta m}(\Delta m) \sim 10^{-b\Delta m}$ as the GR distribution of the sample [*Vere-Jones*, 1969] with a mean $\langle \Delta m \rangle$ equal to $1/(b \ln 10) \approx 0.43$ for $b \approx 1$. Thus, rather than a distribution peaked at $\Delta m \approx 1.2$, *Vere-Jones*' interpretation predicts an exponential distribution with an average significantly smaller than Båth's law value. Such discrepancies have been ascribed to different magnitude thresholds chosen for the definition of mainshocks and largest aftershocks and to finite catalog size effects [*Vere-Jones*, 1969; *Console et al.*, 2003]. Improved implementation of *Vere-Jones'* model by *Console et al.* [2003], taking into account the fact that the minimum threshold for aftershock magnitudes is smaller than for mainshocks, has shown that this model provides a much better fit to the data, but that there is still a minor discrepancy between this model and the observations. The results of [*Console et al.*, 2003] for a worldwide catalog and for a catalog of seismicity of New Zealand are not completely explained by this model, the observed value of $\langle \Delta m \rangle$ being still a little larger than predicted. *Console et al.* [2003] interpret this result as possibly due to "a change in the physical environment before and after large earthquakes" but they do not rule out the existence of a possible bias that may explain the discrepancy between their model and the observations. We propose in section 3 a simple statistical interpretation of Båth's law, which can explain this discrepancy without invoking any difference in the mechanisms controlling the magnitude of the mainshock and of the aftershocks.

[4] Notwithstanding the appealing simplicity of *Vere-Jones*' interpretation and its success to fit the data, this model does not provide a realistic model of aftershocks, and misses some important properties of seismicity. In particular, it does not take into account the fact that aftershocks represent only a subset of the whole seismicity, which are selected as events that occurred within a space-time window around and after a larger event, called the mainshock, which is supposed to have triggered these earthquakes. We first consider as a mainshock only the largest earthquake of a catalog of N events which have independent magnitudes drawn according to the GR law $10^{-b\,(m-m_0)}$ with a minimum magnitude m_0. Only a small subset of size $N_{\rm aft}$ of the whole catalog occur in the specified space-time window used for aftershock selection. The largest event in the whole catalog has an average magnitude given by $\langle m_M \rangle \approx m_0 + (1/b) \log_{10} N$. Let us sort the magnitudes of all events in the

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catalog by descending order: $m_1 > m_2 > \ldots > m_N$. The largest aftershock, within the subset of aftershocks of size $N_{\rm aft}$, has an expected overall rank equal to $\approx N/N_{\rm aft}$. Using the distribution of the magnitude difference $m_1 - m_j$ between the largest earthquake (with rank equal to 1) and the event of rank j given by [*Vere-Jones*, 1969] and assuming $N \gg N_{\rm aft} \gg 1$, the average magnitude difference between the mainshock and its largest aftershock is thus given by

$$\langle \Delta m \rangle = \langle m_M - m_A \rangle \approx \frac{1}{b} \log_{10} (N/N_{\rm aft}) \quad (1)$$

This expression (1) shows that if the mainshock is taken to be the largest event in the catalog, then the magnitude difference $\langle \Delta m \rangle$ is likely to be substantially larger than that predicted by *Vere-Jones*' initial formulation. That is because the subset formed by the aftershocks is significantly smaller than the original catalog of size N, thus the rank of the largest aftershock is not 2 in general. In this sense, the mainshock in a sequence is not a member of the set of aftershocks [*Utsu*, 1969; *Evison and Rhoades*, 2001]. The mainshock appears to be an outlier when we compare the mainshock magnitude with the aftershock magnitude distribution, even if all events in the initial catalog have the same magnitude distribution. The fact that the mainshock does not belong to the subset of aftershocks does not however imply that mainshocks are physically different from other earthquakes, in contradiction with previous claims of *Utsu* [1969], but simply results from the rules of aftershock selection. Expression (1) retrieves Båth's law only for a specific value of the number of aftershocks $N_{\rm aft} = 10^{-b\langle \Delta m \rangle}\, 10^{b(m_M - m_0)}$ with $\langle \Delta m \rangle = 1.2$.

[5] We use in section 3 the ETAS model of seismicity, in order to take into account the rules of aftershock selection in time, space and magnitude, and to take into account the increase of aftershock productivity with the maisnhock magnitude. We will generalize the result (1) in the case where the mainshock is not the largest event of the whole catalog.

[6] Using an approach similar to expression (1) and taking Båth's law as given led *Michael and Jones* [1998] and *Felzer et al.* [2002] to deduce that the number of earthquakes triggered by an earthquake of magnitude m is proportional to $\sim 10^{\alpha m}$, with $\alpha = b$. We shall see below using numerical simulations that the ETAS model is also consistent with $\alpha < b$.

3. Båth's Law and the ETAS Model

[7] In order to shed light on the explanation of Båth's law, and to investigate the effects of the selection procedure for aftershocks, we need a complete model of seismicity, which describes the distribution of earthquakes in time, space and magnitude, and which incorporates realistic aftershock properties. We thus study the Epidemic Type Aftershock Sequence model (ETAS) of seismicity, introduced by [*Kagan and Knopoff*, 1981; *Ogata*, 1988]. The ETAS model assumes that each earthquake triggers aftershocks with a rate (productivity law) increasing as $\rho(m) = K 10^{\alpha(m-m_0)}$ with its magnitude m. A crucial assumption of the ETAS model is that all earthquakes have the same magnitude distribution, given by the GR law, independently of the past seismicity. The seismicity results from the sum of an external constant average loading rate and from earthquakes triggered by these sources in direct lineage or through a cascade of generations.

[8] It can be shown that the average total number of aftershocks $\langle N_{\rm aft}(m_M) \rangle$ (including the cascade of indirect aftershocks) has the same dependence with the mainshock magnitude

$$\langle N_{\rm aft}(m_M) \rangle = \frac{K}{1-n} 10^{\alpha(m_M - m_0)} \quad (2)$$

as the number of direct aftershocks $\rho(m_M)$ given above [*Helmstetter and Sornette*, 2002]. n is the branching ratio, defined as the average number of directly triggered earthquakes per earthquake, averaged over all magnitudes. Using this model, *Felzer et al.* [2002] have argued that α must be equal to b in order to obtain an average difference in magnitude $\langle \Delta m \rangle$ that is independent of the mainshock magnitude. This result is in apparent disagreement with the empirical observation $\alpha \approx 0.8 < b \approx 1$ reported by *Helmstetter* [2003] using a catalog of seismicity for Southern California. The analysis of *Felzer et al.* [2002] neglects the fluctuations of the number of aftershocks from one sequence to another one. We have however shown recently (A. Saichev et al., Anomalous scaling of offspring and generation numbers in branching processes, submitted to *Europhys. Lett.*, 2003, hereinafter referred to as Saichev et al., submitted manuscript, 2003; (http://arXiv.org/abs/cond-mat/0305007)) that there are huge fluctuations of the number of aftershocks per sequence for the same mainshock magnitude. We show below, using numerical simulations of the ETAS model, that taking into account these fluctuations has important effects on the estimation of $\langle \Delta m \rangle$ and on its dependence with m_M.

[9] The average magnitude $\langle m_A \rangle$ of the largest event in a catalog of $N_{\rm aft}$ aftershocks with magnitudes larger than m_0 distributed according to the GR law is given by [*Feller*, 1971]

$$\langle m_A \rangle = m_0 - \int_0^1 \frac{N_{\rm aft}(1-x)^{N_{\rm aft}-1} \ln(x)}{b \ln(10)} dx \quad (3)$$

$$\approx m_0 + \log_{10}(N_{\rm aft})/b \qquad \text{for } N_{\rm aft} \gg 1. \quad (4)$$

We derive below an approximate expression for $\langle \Delta m \rangle$ in the ETAS model, which neglects the fluctuations of $N_{\rm aft}$, i.e., which replaces $N_{\rm aft}$ by its average value (2) in (4). Using this approximation, we obtain

$$\langle \Delta m \rangle \approx \frac{b-\alpha}{b}(m_M - m_0) - \frac{1}{b} \log_{10}\left(\frac{K}{1-n}\right). \quad (5)$$

This approximate relation thus predicts an increase of $\langle \Delta m \rangle$ with the mainshock magnitude if $\alpha < b$. Expression (5) thus predicts that Båth's law is only recovered for $\alpha = b$ [*Felzer et al.*, 2002]. Using numerical simulations of the ETAS model, we find however that the large fluctuations in aftershock numbers due to the cascades of triggered events modify significantly the prediction (5). Adding the constraint that aftershocks are usually chosen to be smaller than the mainshock further alters the prediction (5).

[10] We have generated synthetic catalogs with the ETAS model to measure $\langle \Delta m \rangle$ for different values of the mainshock magnitude. In this first test, we start the simulation

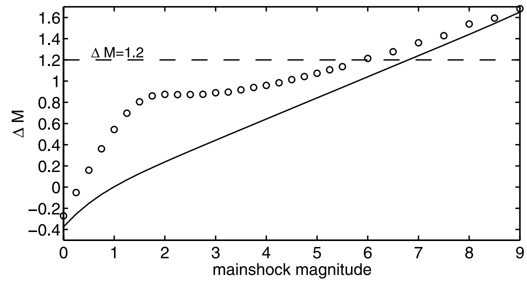

Figure 1. Average magnitude difference $\langle \Delta m \rangle$ between the mainshock and its largest aftershock as a function of the mainshock magnitude (open circle), for numerical simulations of the ETAS model with parameters $n = 0.8$, $\alpha = 0.8$, $b = 1$, $m_0 = 0$ and with an Omori exponent $p = 1.2$. The continuous line is the prediction using the approximate analytical solution (5) for $\langle \Delta m \rangle$.

with a mainshock of magnitude m_M, which generates a cascade of direct and indirect aftershocks. We select as "aftershocks" all earthquakes triggered directly or indirectly by the mainshock, without any constraint in the time, location, or magnitude of these events. For $\alpha = 0.8$ and $b = 1$, we find that $\langle \Delta m \rangle$ is much larger than predicted by (5), and increases slower with the mainshock magnitude, in better agreement with Båth's law than the analytical solution (5) (Figure 1). We have checked that the average number of aftershocks $\langle N_{\rm aft}(m_M) \rangle$ is in good agreement with the analytical solution (2), and thus a discrepancy with (2) is not an explanation for the difference between the results of the numerical simulations and the analytical prediction (5).

[11] The large fluctuations of the total number of aftershocks are at the origin of the discrepancy between the observed $\langle \Delta m \rangle$ and the prediction (5), which neglects the fluctuations of the number of aftershocks. *Saichev et al.* [submitted manuscript, 2003] have recently demonstrated that the total number of aftershocks in the ETAS model in the regime $\alpha > b/2$ has an asymptotic power-law distribution in the tail with an exponent of the cumulative distribution smaller than 1, even in the subcritical regime (defined by a branching ratio $n < 1$). These huge fluctuations arise from the cascades of triggering and from the power-law distribution of the number of triggered earthquakes per triggering earthquake appearing as a combination of the GR law and of the productivity law $\rho(m)$. Practically, this means that the aftershock number fluctuates widely from realization to realization and the average will be controlled by a few sequences that happen to have an unusually large number of aftershocks. Numerical simulations show that $\langle \Delta m \rangle$ is not controlled by the average number of aftershocks, but by its "typical" value, which is much smaller than the average value. Therefore, the expression (5) of $\langle \Delta m \rangle$ obtained by replacing $N_{\rm aft}$ by its average value (2) in (3) is a very bad approximation. Using the exact distribution of the number of aftershocks given in [*Saichev et al.*, submitted manuscript, 2003], we can obtain the asymptotic expression for large m_M, which recovers the dependence of $\langle \Delta m \rangle$ with m_M predicted by (5).

[12] For large mainshock magnitudes, the relative fluctuations of the total number of aftershocks per mainshock are weaker. Therefore, the obtained average magnitude difference $\langle \Delta m \rangle$ tends to recover the linear dependence (5) with the mainshock magnitude, represented by the continuous line in Figure 1. Our numerical simulations show that a constant value of $\langle \Delta m \rangle$ in a wide range of magnitudes can be reproduced using the ETAS model if $\alpha < b$. Our results also predict that Båth's law should fail for large mainshock magnitudes according to (5) if α is smaller than b. It is however doubtful that this deviation from Båth's law can be observed in real data as the number of large mainshocks is small.

[13] While the impact of fluctuations in the number of aftershocks produces a value of $\langle \Delta m \rangle$ larger than predicted by (5) and in better agreement with Båth's law, the average magnitude difference $\langle \Delta m \rangle \approx 0.9$ for $m_0 + 2 < m_m < m_0 + 5$ remains smaller than the empirical value $\langle \Delta m \rangle \approx 1.2$. However, we have not yet taken into account the constraints of aftershock selection, which will further modify $\langle \Delta m \rangle$. In the simulations shown in Figure 1, all earthquakes triggered (directly or indirectly) by the mainshock have been considered as aftershocks even if they were larger than the mainshock. In real data, the difficulty of identifying aftershocks and the usual constraint that aftershocks are smaller than the mainshock can be expected to affect the relation between $\langle \Delta m \rangle$ and the mainshock magnitude. The selection of aftershocks requires the choice of a space-time window to distinguish aftershocks from background events. A significant fraction of aftershocks can thus be missed. As a consequence, the value of Δm will increase.

[14] In order to quantify the impact of these constraints, we have generated synthetic catalogs using the ETAS model, which include a realistic spatio-temporal distribution of aftershocks. Specifically, according to the ETAS model, the number of aftershocks triggered directly by an event of magnitude m, at a time t after the mainshock and at a distance r is given by

$$\phi_m(t,r) = n\frac{(b-\alpha)}{b}10^{\alpha(m-m_0)}\frac{\theta c^{\theta}}{(t+c)^p}\frac{\mu d_m{}^{\mu}}{(r+d_m)^{1+\mu}}. \qquad (6)$$

where n is the branching ratio, p is the exponent of the local Omori's law (which is generally larger than the observed Omori exponent) and d_m is the characteristic size of the aftershock cluster of a magnitude m earthquake given by $d_m = 0.01 \times 10^{0.5m}$ km.

[15] We have then applied standard rules for the selection of aftershocks. We consider as a potential mainshock each earthquake that has not been preceded by a larger earthquake in a space-time window $R_C \times T_c$. This rule allows us to remove the influence of previous earthquakes and to obtain an estimate of the rate of seismicity triggered by this mainshock. The constant R_c is fixed equal to the size $\approx$100 km of the largest cluster in the catalog and $T_c = 100$ days. We then define aftershocks as all events occurring in a space time window $R(m_M) \times T(m_M)$ after a mainshock of magnitude m_M, where both $R(m_M) = 2.5 \times 10^{(1.2m_M-4)/3}$ km and $T(m_M) = 10/3 \times 10^{(2/3)(m_M-5)}$ days increase with the mainshock magnitude m_M [*Kagan*, 1996; *Console et al.*, 2003].

[16] The results for different values of α are represented in Figure 2. For intermediate mainshock magnitude, the average magnitude difference $\langle \Delta m \rangle$ for $\alpha = 0.8$ is significantly larger than found in Figure 1 without the selection procedure, because mainshocks which trigger a larger event

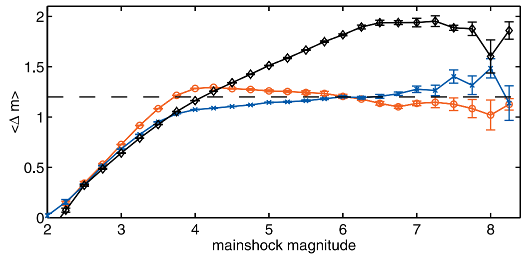

Figure 2. Average magnitude difference $\langle \Delta m \rangle$ between a mainshock and its largest aftershock, for numerical simulations of the ETAS model with parameters $b = 1$, $c = 0.001$ day, $p = 1.2$, a minimum magnitude $m_0 = 2$, a maximum magnitude $m_{max} = 8.5$ and a constant loading $\mu = 300$ events per day. Each curve corresponds to a different value of the ETAS parameters: $\alpha = 0.8$ and $n = 0.76$ (crosses), $\alpha = 0.5$ and $n = 0.8$ (diamonds) and $\alpha = 1$ and $n = 0.6$ (circles). The error bars gives the uncertainty of $\langle \Delta m \rangle$ (1 standard deviation). The horizontal dashed line is the empirical value $\langle \Delta m \rangle = 1.2$.

are rejected, and because the rules of selection (with a time-space window $R(m)$ and $T(m)$ increasing with m) reject a large number of aftershocks, especially for small mainshocks. For small magnitude m_M, $\langle \Delta m \rangle$ is small and then increases rapidly with m. This regime is not pertinent because most mainshocks do not trigger any aftershock and are thus rejected from the analysis. Most studies have considered only mainshocks with magnitude $m \geq m_0 + 2$, where m_0 is the minimum detection threshold. For $\alpha = 0.8$ or $\alpha = 1$, the magnitude difference is ≈ 1.2 in a large range of mainshock magnitudes, in agreement with Båth's law. For $\alpha = 1$, there is a slight decrease of $\langle \Delta m \rangle$ with m_M. For $\alpha = 0.5$, we observe a fast increase of $\langle \Delta m \rangle$ with m_M, which is not consistent with the observations of Båth's law [e.g., *Felzer et al.*, 2002; *Console et al.*, 2003]. The shape of the curves $\langle \Delta m \rangle$ is mostly controlled by α. The other parameters of the ETAS model and the rules of aftershock selection increase or decrease $\langle \Delta m \rangle$ but do not change the scaling of $\langle \Delta m \rangle$ with the mainshock magnitude.

4. Discussion and Conclusion

[17] We have first shown that the standard interpretation of Båth's law in terms of the two largest events of a self-similar set of independent events is incorrect. We have stressed the importance of the selection process of aftershocks, which represent only a subset of the whole seismicity catalog. Our point is that the average magnitude difference $\langle \Delta m \rangle$ is not only controlled by the magnitude distribution but also by the aftershock productivity. A large magnitude difference $\langle \Delta m \rangle$ can be explained by a low aftershock productivity.

[18] Using numerical simulations of the ETAS model, we have shown that this model is in good agreement with Båth's law in a certain range of the model parameters. We have pointed out the importance of the selection process of aftershocks, of the constraint that aftershocks are smaller than the mainshock and of the fluctuation of the number of aftershocks per sequence in the determination of the value of $\langle \Delta m \rangle$, and in its apparent independence as a function of the mainshock magnitude. In the ETAS model, the cascades of multiple triggering induce large fluctuations of the total number of aftershocks. These fluctuations in turn induce a modification of the scaling of $\langle \Delta m \rangle$ with the mainshock magnitude by comparison with the predictions neglecting these fluctuations. The constraints due to aftershock selection further affect the value of $\langle \Delta m \rangle$. Observations that $\langle \Delta m \rangle$ does not vary significantly with the mainshock magnitude requires that the exponent of the aftershock productivity law is in the range $0.8 < \alpha < 1$. Båth's law is thus consistent with the regime $\alpha < b$ in which earthquake triggering is dominated by the smallest earthquakes [*Helmstetter*, 2003].

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