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Self-organized criticality in a crack-propagation model of earthquakes

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The distribution of seismic moment or energy of earthquakes is well described by the universal Gutenberg-Richter power law, $N(s) \approx s^{-1-b}$, where $b \approx 0.5$–0.6. We have constructed a simple dynamical model of crack propagation; when driven by slowly increasing shear stress, the model evolves into a self-organized critical state. A power-law distribution for earthquakes with $b \approx 0.4$ in two dimensions and $b \approx 0.6$ in three dimensions is found. The critical state is "at the edge of chaos," with algebraic growth in time of a small initial perturbation.

In 1956 Gutenberg and Richter[1] noted that the frequency of earthquakes where an energy E (or seismic moment s) is released follows the remarkably simple power law $N(s) \propto s^{-1-b}$ (in this paper the Gutenberg-Richter law is formulated in terms of seismic moment s, instead of earthquake magnitude[2]). The exponent b seems to be universal in the sense that it does not depend on the particular geographical area. In a recent analysis, Kagan[3] finds $b \approx 0.55$ using data from the Harvard earthquake catalog. The frequency of large earthquakes can thus be smoothly extrapolated from the frequency of small ones indicating a common mechanism.

It was realized already in 1976 by Vere-Jones[4] that the Gutenberg-Richter law, with a b value of 0.5, can be understood formally as the consequence of a chain reaction or branching process. Suppose that some activity is initiated by an instability somewhere in the crust of the earth—for instance, in a fault region. After each time step the activity may die, continue unchanged, or branch into two (or more) active sites. Eventually, a tree is generated where the total number of branches is a measure of the energy release during an earthquake. Only if the probability of branching exactly cancels the probability of death can the distribution of earthquakes follow the Gutenberg-Richter power-law distribution. Thus, the chain reaction has to be precisely critical. Until recently, however, there was no *dynamical* model, not to speak of a general theory, to support this picture.

In 1987 Bak, Tang, and Wiesenfeld[5,6] discovered that a large class of extended dissipative dynamical systems naturally organize themselves into a stationary critical state. In the critical state, there are avalanches of all sizes propagating throughout the system. Model calculations showed that the distribution of avalanches was indeed given by a power law. The most remarkable feature of the self-organized critical state is that it is *robust* with respect to essentially any modification of the system; this is crucial for the application to any realistic complex phenomenon in nature. The power law was confirmed by measurements on sandpile dynamics.[7]

Shortly thereafter, Bak *et al.*[8–10] suggested that the concept of self-organized criticality applies rather directly to earthquakes, and it was demonstrated that simple stick-slip models, similar to models that had already previously been proposed for earthquakes,[11,12] indeed evolve to the self-organized critical state. Random cellular automata models[8,9] and models with continuous deterministic loading[10] were studied with essentially identical results. The idea was further developed by several groups, extending it to account formally for Omori's law for aftershock distribution,[13] account for temporal correlations for large earthquakes,[14] and include inertia effects.[15]

While clearly establishing a general mechanism for the origin of the Gutenberg-Richter law in earthquakes, these early models have a couple of shortcomings. First, the b values do not agree well with the value $b \approx 0.5$–0.6 estimated from earthquake catalogs, indicating that the models were not in the correct universality class.[16] Second, the models were block-spring models with a stick-slip mechanism, where a block slides once the force exceeds a local critical friction force. This causes the force to be transferred to neighbor blocks. In real earthquakes, the instability is more likely to be caused by breakdown of elastic forces when the crust ruptures as the shear stress from tectonic plates grinding against each other builds up. Following the rupture, the local stress is released, but a *long-range* redistribution of elastic forces occurs. This dynamics is significantly different from the dynamics of the short-range stick-slip models, and the corresponding exponents can be different. In this paper we present a very simple crack-propagation model of earthquakes, in which the local breakdown of elastic force and long-range redistribution of stress are explicitly introduced. With a slow increase of shear stress, the model self-organizes into a critical state with b values in agreement with observations. Our model may also directly apply to the transformation from the elastic to the plastic regime of metals subjected to external stress. In fact, experiments by Bobrov and Lebyodkin[17] on stressed aluminum and niobium rods have revealed earthquakelike ruptures with a Gutenberg-Richter power-law distribution in the plastic regime.

THE MODEL AND NUMERICAL SIMULATIONS

The situation that we want to describe is illustrated in Fig. 1. We are focusing on the entire (potential) earthquake region rather than individual preexisting faults. A d-dimensional medium is subjected to a slowly increasing external stress field. When the stress somewhere exceeds a critical value (which is must be eventually since the stress is ever increasing), the shear stress is released while the medium undergoes a local shear deformation (rupture). This causes a very anisotropic redistribution of elastic forces, falling off roughly as $1/r^d$ with the distance from the instability:[18] Somewhere the shear force increases; somewhere it decreases. This redistribution takes place essentially with the speed of sound, i.e., much faster than the geological time scale involved in the buildup of stress. At the positions with increased stress, the force may now exceed the critical force causing further rupture, and so on; the local instability may cause a chain reaction.

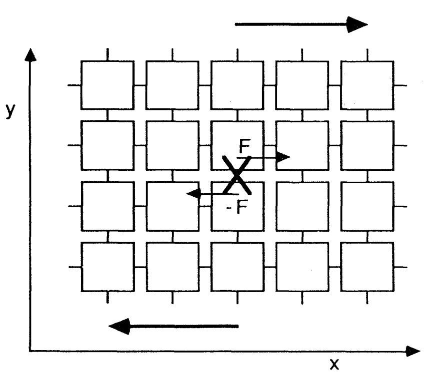

FIG. 1. Schematic illustration of fault region subjected to external shear stress. The vertical springs represent shear stress; the horizontal ones represent axial stress. The model is driven by slowly increasing the shear forces. When the local stress is larger than the threshold stress, the spring breaks, causing a redistribution of elastic forces. The broken spring will be frozen immediately, i.e., the bond will be replaced with a new spring of zero stress.

The model is defined on a d-dimensional hypercube, with mass elements on lattice sites; the nearest neighbors are connected by springs. Let $\sigma_x(\mathbf{r})$ be the stress in the bond between blocks at $\mathbf{r}$ and $\mathbf{r}+\mathbf{e}_x$ ($\mathbf{e}_x$ is the unit vector at x direction); this represents a force $\sigma_x(\mathbf{r})$ on the block at $\mathbf{r}$ and $-\sigma_x(\mathbf{r})$ on the block at $\mathbf{r}+\mathbf{e}_x$. Similarly we can define the stress in other directions. We only consider one component of the forces. Our model does not describe the details of the dynamical process of ruptures, but only the stress distribution before and after each event. Since the earth is at rest between earthquakes, the total force on each mass element from its neighbors must be zero before and after. Therefore, the stress σ_x and σ_y must satisfy the following equation (we only consider two-dimensional lattice here, generalization to three dimensional situations is straightforward):

$$\sigma_x(\mathbf{r})+\sigma_y(\mathbf{r})-\sigma_x(\mathbf{r}-\mathbf{e}_x)-\sigma_y(\mathbf{r}-\mathbf{e}_y)=0 \; . \tag{1}$$

In the beginning all deformations are elastic, and we can assume Hooke's law for the stress (i.e., $\sigma \propto \Delta u$, where Δu is the difference between displacements at two neighbor sites). Thus the sum of stress along a closed loop must be zero. Let us take the border of a unit square as the closed loop; then we have

$$\sigma_x(\mathbf{r})+\sigma_y(\mathbf{r}+\mathbf{e}_x)-\sigma_x(\mathbf{r}+\mathbf{e}_y)-\sigma_y(\mathbf{r}_y)=0 \; . \tag{2}$$

The slow increase of the external shear stress is represented by adding a small shear stress to all vertical bonds (say, in the y direction). As the stress is increased, somewhere in the system the stress becomes larger than the threshold stress which we assigned to be random initially; the spring will then break, causing a rupture. Let us consider the effect of breaking, say, a vertical spring connecting the blocks at $\mathbf{r}_0$ and $\mathbf{r}_0+\mathbf{e}_y$. After the break, the original stress (let it be σ_0) is reduced to zero; this will cause force unbalance and subsequent stress redistribution. The additional stress caused by the local rupture can be viewed as the effect of applying a force F at $\mathbf{r}_0$ and $-F$ at $\mathbf{r}_0+\mathbf{e}_y$ (F should have the opposite sign of σ_0; see Fig. 1). The new stress distribution can then be calculated as $\sigma_{\text{new}}=\sigma_{\text{old}}+\sigma'$, where σ' is induced by the "dipole" force. We assume elastic redistribution of the stress, so σ' satisfies Eqs. (1) and (2) with additional forces F and $-F$ added to Eq. (1) at positions $\mathbf{r}_0$ and $\mathbf{r}_0+\mathbf{e}_y$, respectively. The equations above can be solved easily to obtain σ':[19]

$$\sigma'_{x,y}(\mathbf{r})=-FG_{x,y}(\mathbf{r}-\mathbf{r}_0)+FG_{x,y}(\mathbf{r}-(\mathbf{r}_0+\mathbf{e}_y)) \; , \tag{3}$$

where $G_{x,y}$ are the lattice Green functions for the equations

$$G_x(\mathbf{r})=\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}dk_x dk_y \frac{(1-e^{ik_x})e^{i\mathbf{k}\cdot\mathbf{r}}}{4-2(\cos k_x+\cos k_y)} \; , \tag{4}$$

$$G_y(\mathbf{r})=\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}dk_x dk_y \frac{(1-e^{ik_y})e^{i\mathbf{k}\cdot\mathbf{r}}}{4-2(\cos k_x+\cos k_y)} . \tag{5}$$

After the rupture, the stress on the bond is reduced to zero. In order that the forces at position $\mathbf{r}_0$ and $\mathbf{r}_0+\mathbf{e}_y$ be balanced, F should be related to σ_0 by the condition $F+\sigma_0+\sigma'_y(\mathbf{r}_0)=0$. It is easy to show that $\sigma'_y(\mathbf{r}_0)=-F/d$; hence $F=-[d/(d-1)]\sigma_0$. We assume that the bond which has broken will be frozen immediately, i.e., we replace the old broken bond with a new spring of zero stress; any additional deformation of the bond will again be elastic. Also, we assign a new random threshold stress to the new bond.[20] Note that the stress is not conserved; after a rupture both the local stress and the total stress are reduced.

The dynamics of the model can be summarized as follows.

(i) Increase of shear stress: $\sigma_y(\mathbf{r})\rightarrow\sigma_y(\mathbf{r})+p$ at each time step.

(ii) Rupture: When $|\sigma_i(\mathbf{r}_0)\equiv\sigma_0|>\sigma_i^{\text{thr}}$, then at next time step (a) on the broken bond,

$$\sigma_i(\mathbf{r}_0)\rightarrow\sigma_i(\mathbf{r}_0)-\sigma_0 ;$$

(b) on other bonds,

$$\sigma_i(\mathbf{r})\rightarrow\sigma_i(\mathbf{r})+\frac{d}{d-1}\sigma_0(G_i(\mathbf{r}-\mathbf{r}_0)-G_i(\mathbf{r}-(\mathbf{r}_0+\mathbf{e}_i))) ;$$

(c) new threshold stress: $\sigma_i^{\text{thr}}(\mathbf{r}_0)\rightarrow$random number $\in[0,1]$.

In the beginning we set all the stress to be zero and assign the threshold stress to be a random number between 0 and 1; thus all deformations are elastic initially. As the stress builds up, local isolated ruptures start occurring, which gradually lead to chain reactions. As the process continues, bigger and bigger chain reactions, representing bigger and bigger earthquakes, take place. Eventually the system enters a statistically stationary state where the local release of stress during ruptures balances the global increase of stress. When comparing with real earthquake data, we assume that the crust of the earth has had sufficient time to reach this highly excited nonequilibrium state during the geological evolution of the earth so that the earthquakes that we observe are those of the stationary state. The state of the crust as we observe it, with fault areas, etc., should be seen as snapshots of a continuously evolving dynamical process. The details, including the positions of active faults, will change but the statistical properties will remain the same.

Figure 2 shows the number of local ruptures (springs breaking) versus time in a simulation of a 40×40 system, where the shear is increased by $p=0.0001$ per unit time. Note that earthquakes occur in different sizes. Figure 3 shows the structure of a single large earthquake. The line segments represent the displacement (proportional to the release of stress) at the bonds which have ruptured during the earthquake. Note: (i) The ruptures occur along fault lines (or planes in the 3D model) parallel to the forces, which have developed dynamically. (ii) The fracture sites

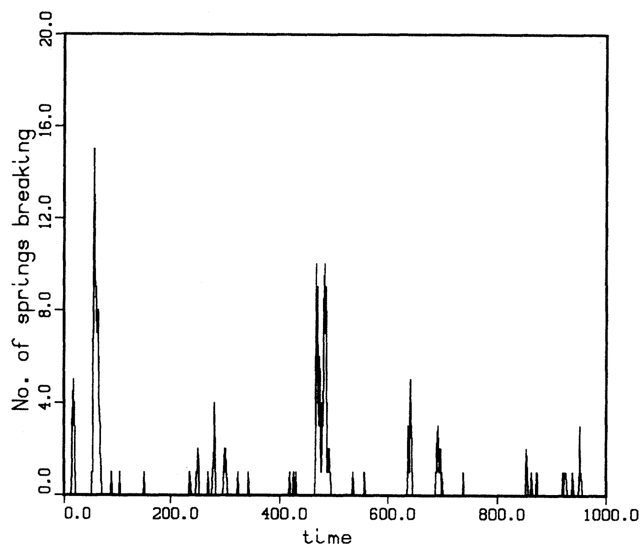

FIG. 2. Evolution of activity of a sequence of earthquakes for a 40×40 system.

of the single event may be disconnected, although the activity is clearly clustered. The earthquake appears to have a fractal structure.

We can measure the size of an earthquake by the "seismic moment" s, or total energy release, which is proportional to the total number of sites which have ruptured following the initial instability. Figures 4(a) and 4(b) show the distribution of seismic moments from a total of 10 000 events. The linearity of the log-log plot indicates that the distribution is a power law, i.e., the crust of the earth is at criticality. As usual, the interruption of the power law at large sizes is interpreted as a finite-size effect; there is no intrinsic typical size of earthquake.

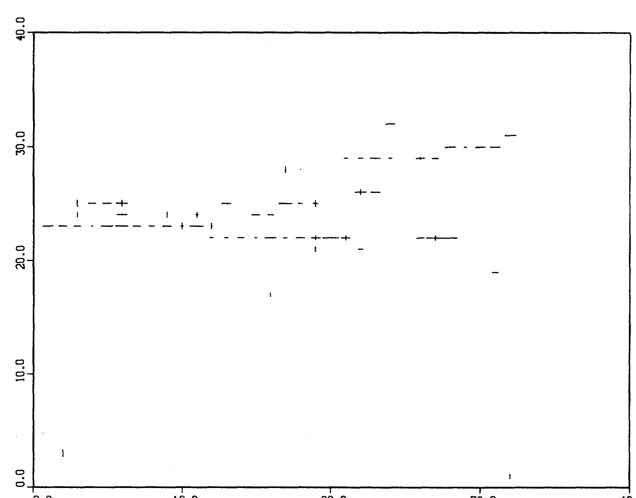

FIG. 3. The structure of a single earthquake. The line segments represent the displacement (proportional to the release of stress) at the sites that have ruptured.

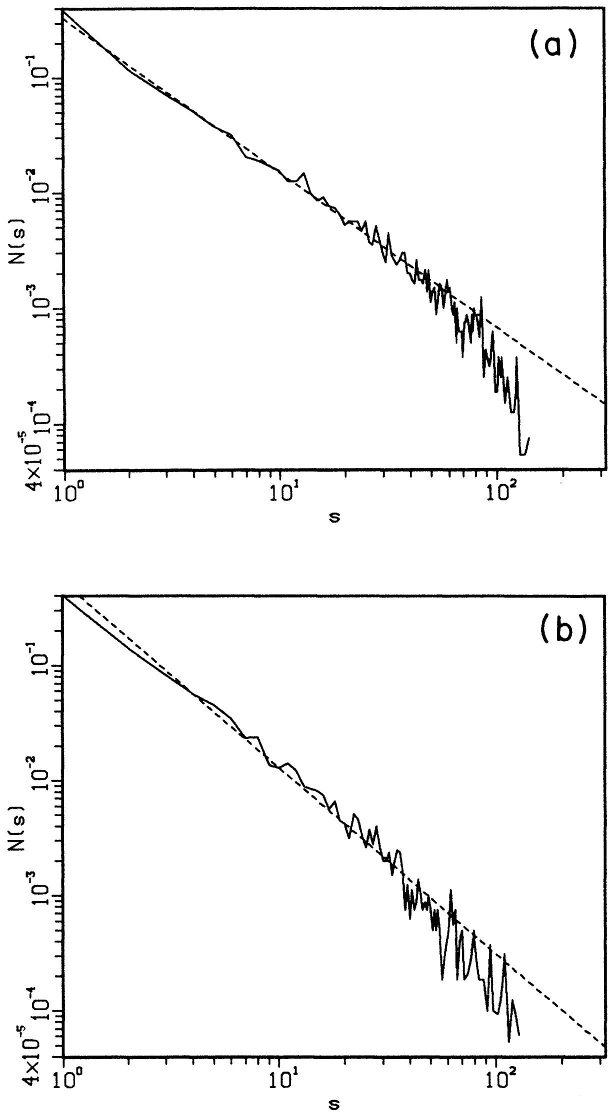

FIG. 4. Distribution of earthquakes generated in the stationary state of the model for (a) a 40×40 lattice, (b) a $20\times20\times20$ lattice. 10 000 earthquakes were included to obtain the statistics. The linear behavior of the log-log plot indicates a Gutenberg-Richter law with a b value of approximately 0.4 in two dimensions and 0.6 in three dimensions. The latter value compares well with the value 0.55 derived from an analysis of the Harvard earthquake catalog. Some coarse graining of the data has been done in obtaining the curves.

The slope of the line is $1+b$. In two dimensions, the b value is approximately 0.4; in three dimensions the value is approximately 0.6. We cannot entirely rule out that $b=0.5$, the value corresponding to a simple uncorrelated chain reaction.[4] We would expect that all but the very largest earthquakes should be represented by the 3D values since the spatial extension of those quakes is less than the thickness of the crust; for larger earthquakes there might be a crossover to the smaller two-dimensional b value.

In any case, the b value is in excellent agreement with the value $b \approx 0.55$ obtained by Kagan from the Harvard catalog.[3] We interpret the power law as a strong indication that the crust of the earth is indeed at the self-organized critical state, and we interpret the agreement of b values as an indication that we have identified the relevant mechanism.

What can we learn about the prospects of earthquake forecasting from this? Our ability to make predictions depends on the way a small initial uncertainty evolves in time. In chaotic systems the deviation grows exponentially, by definition, as $\Delta \approx e^{\lambda t}$ so that after a characteristic time of order $1/\lambda$ we lose track of the system. In self-organized critical systems, the deviation only grows as a power law,[10] i.e., the dynamics is weakly chaotic. To check this, we add a small random (spatially uncorrelated) perturbation to configurations of the critical state.[21] Figure 5 shows the growth of this small perturbation in time, averaged over 2000 perturbations. Not surprisingly, we find the growth to be a power law $\Delta \approx t^{c}$ with exponent $c \approx 1.1$. The system is exactly at the edge of chaos, where the uncertainty grows algebraically rather than exponentially. We gradually lose our ability to predict, but much more slowly than for chaotic systems. There is not a well-defined critical time beyond which we cannot predict, just as there is no characteristic size or duration of the individual earthquake. The self-organized critical state is scale invariant in both time and space.

It has been suggested that self-organized criticality might provide a general mechanism for the formation of fractal structures.[22] Indeed, earthquakes seem too clustered on fractal sets.[23–25] It would be interesting to check if our model, or modifications of the model, for instance, as suggested by Ito and Matsuzaki,[13] can account for this observation.

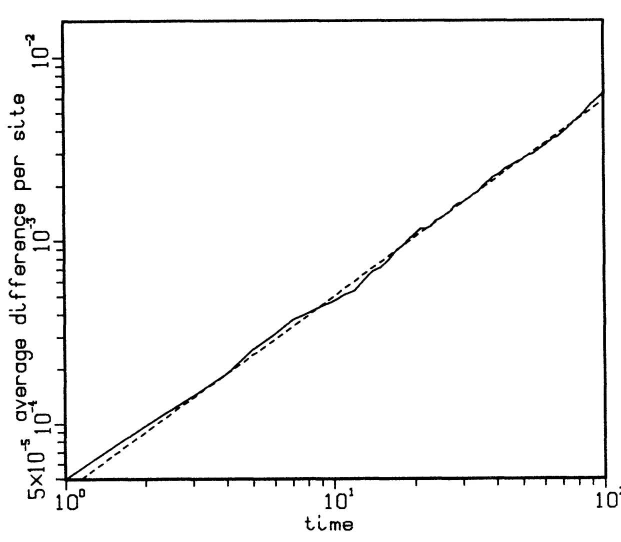

FIG. 5. Growth of small initial random perturbation with time. The power-law behavior indicated that the system is at the edge of chaos. The slope yields an exponent $c \approx 1.1$. The simulation is done on a 40×40 system with $p=0.0003$. The data are averaged over 20 000 perturbations.

SOME ANALYTICAL CONSIDERATIONS

The numerical simulations indicate that the b value is close to the value $\frac{1}{2}$, corresponding to a chain reaction with independent branching events.[4] In fact, this value is the "mean-field" value for self-organized critical phenomena. From renormalization-group theory for critical phenomena it is well known that mean-field exponents are correct in dimensions above the so-called upper critical dimension d_{cr}. One of the authors[26,27] found the upper critical dimension for the short-range models of self-organized criticality to be 4. We now argue that in the present model the critical dimension is lowered to 3, in which case the mean-field result $b=0.5$ (which is within the numerical accuracy) should be exact in three dimensions.

To begin with, let us consider the probability $P(\mathbf{r}-\mathbf{r}')$ [or its Fourier transform $P(\mathbf{k})$] that a rupture (on a vertical bond) at site $\mathbf{r}'$ (for simplicity let $\mathbf{r}'=0$ below) causes another bond to break at $\mathbf{r}$. This probability is proportional to additional stress field $\sigma'(\mathbf{r})$ applied to the vertical bond induced by the original broken bond, or any of the subsequent ruptures in a critical chain reaction. We shall consider only vertical shear stress.

From Eqs. (3) and (5), it is easy to see that the Fourier transform of the additional stress at vertical bond in leading order, is $\sigma'_k \propto k_y^2/k^2 = \cos^2(\alpha)$, where α is the angle between the horizontal plane and $\mathbf{k}$. To include the effect of higher-order terms and the contribution from the horizontal bonds, a term $-ak^2$ to the above expression should be added:

$$\Sigma(\mathbf{k}) = \cos^2(\alpha) - ak^2 \ . \tag{6}$$

The rupture of one bond can initiate a rupture [with probability proportional to $\Sigma(\mathbf{k})$] at another bond, which in turn can initiate a third rupture, etc. Making a summation over a geometric series in order to take into account this chain reaction of breaking the vertical bonds, we end up with the final probability propagator

$$P(\mathbf{k}) = \Sigma(\mathbf{k}) + \Sigma(\mathbf{k})^2 + \Sigma(\mathbf{k})^3 + \cdots \tag{7}$$

$$\approx 1/(ak^2 + 1 - \cos^2\alpha) \approx 1/(ak^2 + \alpha^2) \ . \tag{8}$$

There is no constant "mass" term in the denominator due to the criticality of the theory. Essentially, the propagator is only "massless" in the $(d-1)$-dimensional plane $\alpha=0$. This propagator has been studied by Larkin and Khmelnitskii[28] in their study of critical behavior of Ising dipolar magnets, and by Cowley[29] in a study of $k=0$ structural phase transitions. In the latter study the singular behavior was actually caused by long-range elastic forces. The effect is precisely to lower the upper critical dimension from 4 to 3.

Some cautionary remarks are in place at this point. The analytic theory for self-organized criticality is not advanced to the level that one can actually prove that the upper critical dimension is 4 for short-range forces and therefore 3 for long-range forces, since a good deal of intuitive phenomenological considerations were employed in the original analytical theory. Also, our numerical results seem to favor a slightly larger b value.

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[2]The Gutenberg-Richter law is often formulated in terms of the earthquake magnitude m: $\log_{10}Q(m)=c-\beta m$ (where Q is the number distribution of the earthquake greater than m). The relation between the seismic moment (or energy) s and the magnitude m has also been estimated as $\log_{10}s=c'+dm$. Combining these two relations we have the power-law distribution of seismic moment $N(s)=dQ/ds \propto s^{-1-b}$, with $b=\beta/d$; this is the formulation we use in the paper. Note that both b and β are referred to as the b value in the literature.

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[16]Based on our experience from criticality in equilibrium thermodynamic models, we might hope that the exponents depend only on the symmetry and dimensionality of the physical systems, but not on any physical details. For instance, the critical point of liquid helium, and structural and magnetic phase transitions in solid-state systems, are all described by the exponents of the Ising model. This concept of universality allows direct comparison of simple model calculations with measurements on much more complex real physical systems,

as long as the dimension and symmetry are correct.

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[18]When a local deformation occurs, it will induce a long-range redistribution of elastic forces. Because there is no external force introduced in this process, the effect on the medium far away from the rupture site is well represented by a "dipole" field in analogy with the problem of local rearragnement of charges in electrostatics.

[19]We assume redistribution of elastic force is over an infinite medium, and the equations are solved for an infinite system. For our finite system this corresponds to an open boundary condition; the activity outside the system is neglected.

[20]A more realistic situation is to assign a smaller threshold stress to broken bonds. In this case the earthquakes are most likely to happen in certain "fault" regions, which are also generated dynamically. Preliminary results show that the scaling behavior remains the same.

[21]When doing this simulation, we use a fixed threshold stress value σ_0^{thr} at all bonds; thus the dynamics is completely deterministic. We have checked that the power law and the b value are not affected by this change.

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