

Entropy, energy, and proximity to criticality in global earthquake populations

Ian G. Main and Fahad H. Al-Kindy

Department of Geology and Geophysics, University of Edinburgh, United Kingdom

Received 15 September 2001; accepted 6 December 2001; published 11 April 2002.

[1] We examine the question of proximity of the global earthquake population to the critical point characterised by the energy E and entropy S based on annual frequency data from the Harvard CMT catalogue. The results are compared with a theoretical model corresponding to a Boltzmann probability density distribution of the form $p(E) \propto E^{-B-1} e^{-E/\theta}$. The data are consistent with the model predictions for fluctuations in the characteristic energy θ at constant B value, of the form $S \sim B \langle \ln E \rangle$. This approximation is valid for large θ , relative to the maximum possible event size, confirming that the Earth is perpetually in a near-critical state, reminiscent of self-organized criticality. However, the results also show fluctuations of $\pm 10\%$ in entropy that may be more consistent with the notion of intermittent criticality. A more precise definition of the two paradigms, and a similar analysis of numerical models, are both needed to distinguish between these competing models. **INDEX TERMS:** 3220 Mathematical Geophysics: Nonlinear dynamics; 7223 Seismology: Seismic hazard assessment and prediction; 7209 Seismology: Earthquake dynamics and mechanics; 7260 Seismology: Theory and modeling

1. Introduction

[2] It has been suggested that the Earth's brittle crust is maintained in a state of self-organized criticality, implying a system perpetually near global failure [Bak and Tang, 1989]. This implies large stress correlations would be maintained in the system, hence reducing the degree of predictability of individual earthquakes [Main, 1997; Geller et al., 1997]. However, many observations have been interpreted as implying instead a system maintained predominantly in a sub-critical state, with fluctuations during the largest earthquakes representing states of intermittent criticality [Jaume and Sykes, 1999], implying a degree of predictability in the population dynamics [Main, 1999a, Sykes et al., 1999; Sornette, 2000]. These include the spatial and temporal scaling relations of earthquake sequences [Utsu et al., 1995]; accelerating cumulative Benioff 'strain' [Bufe and Varnes, 1993]; finite correlation lengths inferred from the tail in the frequency-moment distribution [Kagan, 1999; Leonard et al., 2001]; and anomalous stress diffusion in the crust during earthquake triggering [Marsan et al., 2000]. However, the methods that have been used to establish intermittent criticality have been criticised on statistical [Gross and Rundle, 1998; Vere-Jones et al., 2001] and physical [Main, 1999b] grounds. Of particular concern is the inherent retrospective selection bias if an optimisation procedure is used to tune the data to the desired form [Mulargia, 2001]. Here we examine this problem in the most direct way possible using the tools of statistical mechanics, using global earthquake catalogue data to determine temporal fluctuations in the seismic energy release E and entropy S . The use of global data minimises the potential for selection bias. The form of the entropy-energy relationship is identical to that predicted for a

system that is in a near-critical state, but with short-term fluctuations in correlation length and entropy that may be significant. If confirmed, this would imply a finite but perhaps low degree of statistical predictability in the population dynamics.

2. Theory

[3] In classical statistical physics, macroscopic thermodynamic variables can be calculated from a basic knowledge of the micro-states available to the system with probability

$$p(E)dE = \frac{g_E \exp(-E/\theta)}{Z} dE, \quad (1)$$

where in thermodynamics $\theta = kT$, k is Boltzmann's constant, T the temperature, g_E is the degeneracy of states, i.e. the number of ways a particular energy level can be filled, and the partition function Z is the total number of micro-states — a normalising constant to ensure unit total probability [Mandl, 1988]. The degeneracy in earthquake populations occurs because energy is related to the surface rupture area A by $E \sim A^{3/2}$, so there are fewer ways of fitting a larger earthquake of a given rupture area into the total fault surface area of a fault population [Main and Burton, 1984; Rundle, 1993]. For earthquakes the density distribution of the degeneracy has the power-law form

$$g_E = E^{-B-1}/E_0^{-B}, \quad (2)$$

where E_0 is a scaling constant with the units of energy. Equation (2) corresponds to the Gutenberg-Richter law for the equivalent magnitude where typically the exponent $B \approx 2/3$. For a two-dimensional fault model, the correlation length is related directly to $\theta^{1/3}$, and tends to infinity precisely at the critical point [Stauffer and Aharony, 1994; Bruce and Wallace, 1989].

[4] The expectation value of the system energy is the first moment

$$\langle E \rangle = \int_{E_{\min}}^{E_{\max}} E p(E) dE, \quad (3)$$

and the entropy is defined by

$$S = - \int_{E_{\min}}^{E_{\max}} \ln p(E) p(E) dE. \quad (4)$$

The maximum energy E_{\max} here is an imposed physical upper bound, determined either by the size of the Earth in the most extreme case, or more likely by characteristic lengths in the structure of plate boundaries. The Gutenberg-Richter law is recovered as $\theta \rightarrow \infty$, preserving a finite energy through finite E_{\max} . From (1)–(4), it can be shown, for a system in a near-critical

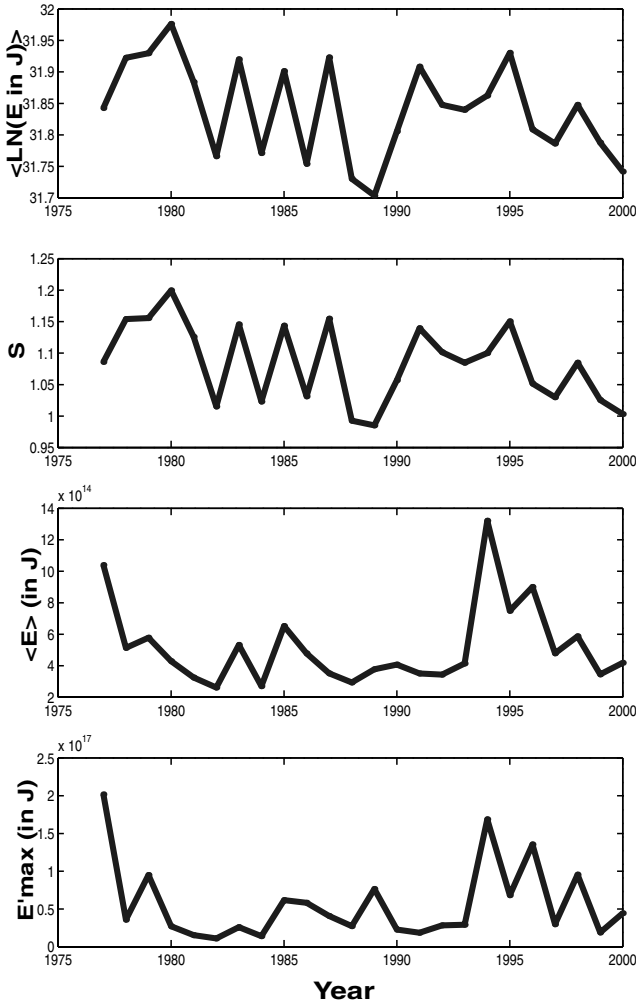


Figure 1. Temporal variation in $\langle \ln E \rangle$, entropy S , mean annual energy $\langle E \rangle$ and maximum energy E'_{\max} from the global occurrence of earthquakes from the CMT catalogue for the time period 1977–2000. Energy units are Joules.

state, using conventional logarithmic increments $\delta \log E$ to define the relevant frequencies, that

$$S = S_0 + B \langle \ln E \rangle, \quad (5)$$

where S_0 is a constant (see Appendix). In a sub-critical state with finite positive θ we would expect the local slope to be greater than B , and in a supercritical state (negative θ) we would expect a local slope less than B . Corollaries of (5) for finite, positive θ are $\langle E \rangle \propto \theta^{1-B}$, and $\partial S / \partial \langle E \rangle = \epsilon$, with ϵ a small positive number (see Appendix). Thus, for a system in a near-critical state, with significant fluctuations in θ at constant B -value, we would expect (i) a strong positive correlation between S and $\langle \ln E \rangle$ of slope B , (ii) a strong positive correlation between $\langle E \rangle$ and θ , and (iii) a weak positive correlation between S and $\langle E \rangle$.

3. Method

[5] In order to avoid problems of catalogue heterogeneity, we used seismic moment data from the Harvard Centroid Moment Tensor catalogue for the time period 1977–2000 inclusive, i.e. the era of widely-available digital seismic data. For this period the catalogue was found to be complete above moment magnitude 5.8. The bandwidth of seismic moment in the catalogues used corre-

sponded to 4 orders of magnitude. If the smallest moment corresponded to an elemental source area A_{\min} , then the total number of elemental source areas required to describe the system is $N = A_{\max} / A_{\min} = 10^{2.7} \approx 500$. We require N to be large for statistical stability in any thermodynamic treatment, although here N is much less than, say, Avogadro's number. The scalar moment for both catalogues was then converted to seismic energy using standard relationships [Kanamori, 1977]. Data for non-overlapping annual time windows was used to construct incremental frequency-energy data using $\delta \log E$ equivalent to 0.25 magnitude units. This was the highest resolution that could be achieved while avoiding empty bins in the determination of the frequency-energy relation, a requirement for a stable estimate of S . We calculated $\langle E \rangle$ from the individual summed energies, and S using a discrete version of (4) after appropriate normalisation to ensure unit total probability. The method is non-parametric - no curve fits were used to smooth the data.

4. Results

[6] The temporal evolution of S , $\langle E \rangle$, $\langle \ln E \rangle$, and E'_{\max} is shown on Figure 1. The annual maximum energy E'_{\max} is a proxy for θ [Bruce and Wallace, 1989]. Note the relatively large fluctuations in $\langle E \rangle$ (between 2 and 14×10^{14} J), associated with smaller fluctuations in S (between 1 and 1.2 in normalised units). There is a strong positive correlation between fluctuations in $\langle \ln E \rangle$ and S , and between $\langle E \rangle$ and E'_{\max} , but only a weak correlation if any between $\langle E \rangle$ and S . The primary data then qualitatively confirm criteria (i)–(iii).

[7] We test equation (5) quantitatively against the data in Figure 2. The fit is very good ($r^2 = 0.941$) with $B = 0.627 \pm 0.033$, confirming criterion (i) quantitatively. This value of B is indistinguishable from that obtained from a maximum likelihood analysis of CMT frequency-moment data [$B = 0.636 \pm 0.013$, Kagan, 1999; $B = 0.625 \pm 0.011$, Leonard et al., 2001]. The assumption of relatively constant value of B is validated, verifying that temporal fluctuations are dominated by changes in the correlation length via θ , similar to that found for spatial fluctuations [Kagan, 1999]. In ideal self-organized criticality, with only small fluctuations about the critical state allowed [Bak and Tang, 1989], the data would plot as a small cluster of points of slope B , around a central value of S and $\langle \ln E \rangle$. The fluctuations in the macroscopic thermodynamic variable S (on the order of $\pm 10\%$) may be more consistent with intermittent criticality.

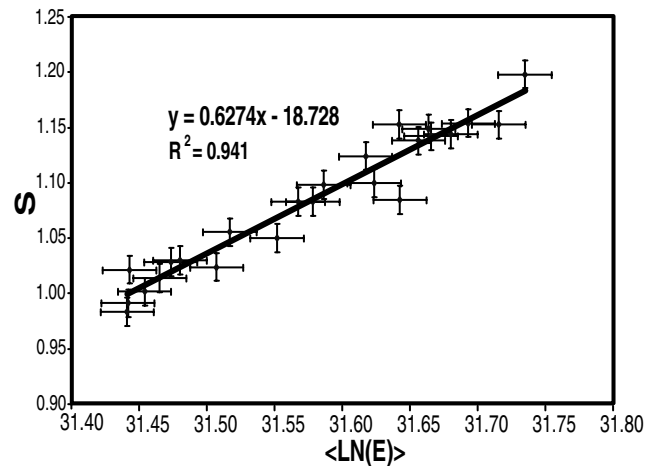


Figure 2. Correlation between S and $\langle \ln E \rangle$ from the data of Figure 1. The straight line corresponds to the best-fit regression line for equation (5). Error bars are shown at one standard deviation.

[8] Earthquake generation is neither ‘adiabatic’ (constant S) nor ‘isothermal’ (constant $\langle E \rangle$), but it is more consistent with the former. The regression coefficient between $\langle E \rangle$ and E'_{\max} is $r^2 = 0.805$, quantitatively verifying criterion (ii). Similarly the regression between S and $\langle E \rangle$ gives $r^2 = 0.046$, consistent with criterion (iii), but not sufficient to discriminate between models of self-organized criticality and intermittent criticality, because the slope is indistinguishable from zero within the errors of the regression. We conclude that globally the brittle Earth is in a near-critical state, where system fluctuations are dominated by fluctuations in θ rather than B . Large fluctuations in energy can be achieved with smaller fluctuations in entropy, implying a relatively unstable system. However, fluctuations in entropy may be quantitatively significant.

5. Discussion

[9] So far we have considered only the radiated seismic energy, and not the internal potential energy of the system. Although our knowledge of strain has increased remarkably recently due to the advent of global satellite data, our empirical knowledge of the energetic micro-states in the Earth is hampered by difficulties in measuring the local stress, and also by systematic spatial variations in seismic efficiency. However, numerical modelling has shown a strong positive correlation between the form of the distribution of radiated and internal potential energy in two-dimensional cellular automaton models where the Hamiltonian can be calculated explicitly [Rundle *et al.*, 1995]. This result also holds when the model is effectively tuned to variable seismic efficiency by altering the local degree of energy conservation [Main *et al.*, 2000]. If this holds also on the Earth, then large earthquakes would be more likely to occur within time periods when the internal potential energy is on average high and spatially correlated. This logical step holds for laboratory-scale fracture but is impossible to verify independently for the real Earth, where stresses can not be measured independently. It is also an explicit but unproven assumption in applying time-to-failure analysis to predicting earthquake occurrence within the framework of intermittent criticality [Bufe and Varnes, 1993; Jaume and Sykes, 1999]. The numerical models also confirm that, for a heterogeneous strength distribution, systematic changes in the radiated energy distribution result from variations in θ rather than B [Main *et al.*, 2000], as seen in the data here.

[10] The results demonstrate that statistical mechanical concepts from equilibrium thermodynamics can also apply in some circumstances to far-from equilibrium threshold systems such as earthquake populations. However, the results are unable to discriminate between a self-organized critical point system and one of intermittent criticality. If self-organized criticality is the null hypothesis, then intermittent criticality has not been established from the current analysis. The main caveat is that the observed $\pm 10\%$ fluctuations in a macroscopic state variable (entropy) may be significant. If this is so, then finite degree of predictability (in a general sense) would be expected in the system. However, the close proximity to the critical point implies that large variations in E can occur with only small changes in S , enabling large fluctuations around the phase space to occur rapidly, even if the system is intermittently critical. This study highlights (1) the need for calculations of S as well as E from the Hamiltonian in numerical models, to establish the significance of a $\pm 10\%$ fluctuation in S , and (2) a more precise definition of the difference between self-organized and intermittent criticality. The methods used here are based on the statistical mechanics of an earthquake population, and could be applied only to assessing the degree of time-dependent hazard rather than the prediction of individual events. Finally, there may be potential in applying the method outlined here to other critical or near-critical systems.

6. Conclusion

[11] We have examined the relationship between the macroscopic state variables of energy and entropy for global earthquakes as a function of changes in the energy distribution of energetic micro-states. The results confirm that global seismicity is in a near-critical state, with large fluctuations in mean energy occurring for small changes in entropy. The results are consistent with temporally fluctuating correlation length at constant B -value. The inferred B -value from the entropy-energy relationship of 0.627 ± 0.033 agrees well with independent maximum likelihood determinations from the same catalogue. The fluctuations in entropy are on the order of $\pm 10\%$. Whether or not this is large enough to discriminate between competing models of self-organized criticality and intermittent criticality depends on a more precise definition of the two than available at present, specifically the magnitude of entropy fluctuations that are allowed in a self-organized critical point system. The proximity to the critical point indicates that the predictability of the system may be finite, but low.

Appendix

[12] From (1) and (2) of the main text we have the incremental probability

$$p(E)dE = \alpha E^{-B-1} \exp(-E/\theta) dE, \quad (A1)$$

where $\alpha = E_0^B Z^{-1}$ is a scaling constant. If we use logarithmic increments of energy $p(E)\delta \ln E$

$$\frac{d \ln E}{dE} = \frac{1}{E} \quad \text{or} \quad dE = E d \ln E, \quad (A2)$$

then the partition function is

$$Z = \int_{E_{\min}}^{E_{\max}} \left(\frac{E}{E_0} \right)^{-B} \exp\left(-\frac{E}{\theta}\right) d \ln E, \quad (A3)$$

the energy is

$$\langle E \rangle = Z^{-1} \int_{E_{\min}}^{E_{\max}} E \left(\frac{E}{E_0} \right)^{-B} \exp\left(-\frac{E}{\theta}\right) d \ln E, \quad (A4)$$

and the entropy is

$$S = -Z^{-1} \int_{E_{\min}}^{E_{\max}} \left(\frac{E}{E_0} \right)^{-B} \exp\left(-\frac{E}{\theta}\right) \ln \left[\frac{(E/E_0)^{-B} \exp(-E/\theta)}{Z} \right] d \ln E. \quad (A5)$$

[13] If we expand the logarithmic term in the kernel of (A5), we find exactly that

$$S = \ln Z - B \ln E_0 + B \langle \ln E \rangle + \langle E \rangle / \theta. \quad (A6)$$

For a system of finite mean energy near the critical point $\theta \rightarrow \infty$, $\langle E \rangle / \theta$ tends to zero, and can be neglected. Near the critical point, we find

$$\frac{\partial \ln Z}{\partial \theta} = \frac{1}{Z} \frac{\partial Z}{\partial \theta} = \frac{\langle E \rangle}{\theta^2}. \quad (A7)$$

Again as $\theta \rightarrow \infty$ we find $\partial \ln Z / \partial \theta \sim 0$, and hence $Z \sim \text{constant}$. Thus from (A6) a cross-plot of S and $\langle \ln E \rangle$ should approximate to a straight line with slope B and intercept $S_0 \approx \ln Z - B \ln E_0$ [equation (5) and criterion (i) of the main text].

[14] From (A2) and (A4) we have after a change of variables to $x = E/E_{\max}$, with $E_{\max} \gg E_{\min}$,

$$\langle E \rangle = \theta^{1-B} E_{\min}^B B \int_0^{x_{\max}} x^{-B} e^{-x} dx. \quad (\text{A8})$$

The integral is the incomplete gamma function. For $B = 2/3$, it is near but less than the value of the complete gamma function of 1.786 [Kagan, 1993]. Thus $\langle E \rangle \approx \theta^{1-B} E_{\min}^B$, implying a positive correlation between $\langle E \rangle$ and θ [criterion ii].

[15] From (A6) it follows that

$$\frac{\partial S}{\partial \langle E \rangle} = \frac{\partial \ln Z}{\partial \langle E \rangle} + B \frac{\partial \langle \ln E \rangle}{\partial \langle E \rangle} + \frac{1}{\theta}. \quad (\text{A9})$$

For $Z \sim \text{constant}$, it can be shown from the definitions of $\langle E \rangle$ and $\langle \ln E \rangle$ that

$$\frac{\partial \langle \ln E \rangle}{\partial \langle E \rangle} = \frac{\partial \langle \ln E \rangle}{\partial \theta} \bigg/ \frac{\partial \langle E \rangle}{\partial \theta} \approx \frac{\langle \ln E \rangle}{\langle E \rangle}. \quad (\text{A10})$$

For the values reported here $1/\theta \approx 10^{-17} \text{ J}^{-1}$, and $\langle \ln E \rangle / \langle E \rangle \approx 31.85 / (8 \times 10^{14}) = 4 \times 10^{-14} \ln(\text{J})/\text{J}$. Hence, $\partial S / \partial \langle E \rangle \approx \varepsilon$ where $\varepsilon \ll 1$ but $\varepsilon > 1/\theta$ [criterion iii].

[16] **Acknowledgments.** We thank Yan Kagan for his insightful comments on a previous manuscript, and two anonymous reviewers for their thoughtful and constructive reviews. FAK's contribution was funded by Petrol Development of Oman (PDO).

References

- Bak, P., and C. Tang, Earthquakes as a self-organized critical phenomenon, *J. Geophys. Res.*, **94**, 15,635–15,637, 1989.
- Bruce, A., and D. Wallace, Critical point phenomena: universal physics at large length scales, in P. Davies (ed.), *The New Physics*, Cambridge University Press, Cambridge, UK, 1989.
- Bufe, C. G., and D. Varnes, Predictive modelling of the seismic cycle in the Greater San Francisco Bay region, *J. Geophys. Res.*, **98**, 9871–9883, 1993.
- Geller, R. J., D. D. Jackson, Y. Y. Kagan, and F. Mulargia, Geoscience—Earthquakes cannot be predicted, *Science*, **275**, 1616–1617, 1997.
- Gross, S., and J. Rundle, A systematic test of time-to-failure analysis, *Geophys. J. Int.*, **133**, 57–64, 1998.
- Jaume, S. C., and L. R. Sykes, Evolving towards a critical point: a review of

- accelerating seismic moment/energy release prior to large and great earthquakes, *Pure Appl. Geophys.*, **155**, 279–305, 1999.
- Kagan, Y. Y., Statistics of characteristic earthquakes, *Bull. Seismol. Soc. Am.*, **83**, 7–24, 1993.
- Kagan, Y. Y., Universality of the seismic moment-frequency relation, *Pure Appl. Geophys.*, **155**, 537–573, 1999.
- Kanamori, H., The energy release in great earthquakes, *J. Geophys. Res.*, **82**, 2981–2987, 1977.
- Leonard, T., O. Papasouliotis, and I. G. Main, A Poisson model for identifying characteristic size effects in frequency data: application to frequency-size distributions for global earthquakes, 'starquakes' and fault lengths, *J. Geophys. Res.*, **106**, 13,473–13,484, 2001.
- Main, I. G., Earthquakes — long odds on prediction, *Nature*, **385**, 19–20, 1997.
- Main, I. G., Nature debate on earthquake prediction: concluding remarks, http://helix.nature.com/debates/earthquake/equake_frameset.html, 1999a.
- Main, I. G., Applicability of time-to-failure analysis to accelerated strain before earthquakes and volcanic eruptions, *Geophys. J. Int.*, **139**, F1–F6, 1999b.
- Main, I. G., and P. W. Burton, Information theory and the earthquake frequency magnitude distribution, *Bull. Seismol. Soc. Am.*, **74**, 1409–1426, 1984.
- Main, I. G., G. O'Brien, and J. Henderson, Statistical physics of earthquakes: Comparison of distribution exponents for source area and potential energy and the dynamic emergence of log-periodic quanta, *J. Geophys. Res.*, **105**, 6105–6126, 2000.
- Mulargia, F., Retrospective selection bias (or the benefit of hindsight), *Geophys. J. Int.*, **146**, 489–496, 2001.
- Mandl, F., *Statistical Physics*, 2nd ed., Wiley, New York, 1988.
- Marsan, D., C. J. Bean, S. Steacy, and J. McCloskey, Observation of diffusion processes in earthquake populations and implications for the predictability of seismicity systems, *J. Geophys. Res.*, **105**, 28,081–28,094, 2000.
- Rundle, J. B., Magnitude-frequency relations for earthquakes using a statistical mechanical approach, *J. Geophys. Res.*, **98**, 21,943–21,949, 1993.
- Rundle, J. B., W. Klein, S. Gross, and D. L. Turcotte, Boltzmann fluctuations in numerical simulations of nonequilibrium lattice threshold systems, *Phys. Rev. Lett.*, **75**, 1658–1661, 1995.
- Sornette, D., *Critical point phenomena in the natural sciences*, in Chaos, Fractals, Self-Organisation and Disorder: Concepts and Tools, Springer Series in Synergetics, 2000.
- Stauffer, D. and A. Aharony, *Introduction to percolation theory*, Taylor and Francis, London, 1994.
- Sykes, L. R., B. E. Shaw, and C. H. Scholz, Rethinking earthquake prediction, *Pure Appl. Geophys.*, **155**, 207–232, 1999.
- Utsu, T., Y. Ogata, and R. S. Matsu'ura, The centenary of the Omori formula for a decay law of aftershock activity, *J. Phys. Earth*, **43**, 1–33, 1995.
- Vere-Jones, D., R. Robinson, and W. Z. Yang, Remarks on the accelerated moment release model: problems of model formulation, simulation and estimation, *Geophys. J. Int.*, **144**, 517–531, 2001.

I. G. Main and F. H. Al-Kindy, Department of Geology and Geophysics, West Mains Road, Edinburgh EH9 3JW, United Kingdom. (Ian.Main@glg.ed.ac.uk; fahad.chaos@ed.ac.uk)