Analysis of a Complex Kind

Week 2

Lecture 3: Iteration of Quadratic Polynomials, Julia Sets

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Quadratic Polynomials

We'll be looking at polynomials of the form $f(z) = z^2 + c$, where $c \in \mathbb{C}$ is a constant. We'll study how the behavior of the iterates of f depends on c.

- What about other quadratic polynomials? Shouldn't we be looking, more generally, at $p(z) = az^2 + bz + d$, for constants $a, b, d \in \mathbb{C}$?
- It turns out, for each triple of constants (a, b, d) there is exactly one constant c such that $p(z) = az^2 + bz + d$ and $f(z) = z^2 + c$ "behave the same under iteration".
- Why? Given a, b and d, we define $c = ad + \frac{b}{2} - \left(\frac{b}{2}\right)^2$. Then letting $\varphi(z) = az + \frac{b}{2}$ one can check that $p(z) = \varphi^{-1}(f(\varphi(z)))$ for all z.

Quadratic Polynomials, cont.

- $p(z) = \varphi^{-1}(f(\varphi(z)))$ for all z.
- We write this as $p = \varphi^{-1} \circ f \circ \varphi$ (read: "phi inverse composed with f composed with phi"). Here is the miracle that happens under iteration:

$$\begin{aligned} p \circ p &= (\varphi^{-1} \circ f \circ \varphi) \circ (\varphi^{-1} \circ f \circ \varphi) = \varphi^{-1} \circ f \circ f \circ \varphi, \quad \text{so} \\ p^2 &= \varphi^{-1} \circ f^2 \circ \varphi \\ p^3 &= \varphi^{-1} \circ f^3 \circ \varphi \\ &\vdots \\ p^n &= \varphi^{-1} \circ f^n \circ \varphi \end{aligned}$$

- It thus suffices to study the iteration of quadratic polynomials of the form $f(z) = z^2 + c$.

The Julia Set

- The *Julia set* (named after the French mathematician Gaston Julia, 1893-1978) of $f(z) = z^2 + c$ is the set of all $z \in \mathbb{C}$ for which the behavior of the iterates is “chaotic” in a neighborhood.
- The *Fatou set* (named after the French mathematician Pierre Fatou, 1878-1929) is the set of all $z \in \mathbb{C}$ for which the iterates behave “normally” in a neighborhood.
- What does this mean??
- The iterates of f behave normally near z if nearby points remain nearby under iteration.
- The iterates of f behave chaotically at z if in any small neighborhood of z the behavior of the iterates depends sensitively on the initial point. We’ll clarify this in examples!

First Example

Let's look at $c = 0$, that is $f(z) = z^2$. Then $f^n(z) = z^{(2^n)}$.

Writing $z = re^{i\theta}$, we see that $f^n(z) = r^{(2^n)} \cdot e^{i \cdot 2^n \theta}$. Thus:

- If $|z| < 1$, then $|f^n(z)| = |z|^{(2^n)} \to 0$ as $n \to \infty$, so $f^n(z) \to 0$ as $n \to \infty$.
- If $|z| > 1$, then $|f^n(z)| \to \infty$ as $n \to \infty$, so we say that $f^n(z) \to \infty$ as $n \to \infty$.
- If $|z| = 1$ then $z = e^{i\theta}$, so $f^n(z) = e^{i2^n\theta}$, thus $|f^n(z)| = 1$ for all n.

The Julia set of $f(z) = z^2$

We notice: In any little disk around a point z with $|z| = 1$, there are points w with $|w| > 1$ (and for which thus $f^n(w) \to \infty$), and other points w with $|w| < 1$ (and for which thus $f^n(w) \to 0$).

The unit circle $\{z : |z| = 1\}$ is thus the locus of chaotic behavior, whereas $\underbrace{\{z : |z| > 1\}}_{\text{iterates attracted to } \infty}$ and $\underbrace{\{z : |z| < 1\}}_{\text{iterates attracted to } 0}$ form the locus of normal behavior.

We write $J(f) = \{z : |z| = 1\}$ (Julia set) and $\mathcal{F}(f) = \{z : |z| > 1\} \cup \{z : |z| < 1\}$ (Fatou set).

The Basin of Attraction to ∞

More generally, let's look at $f(z) = z^2 + c$. Let

$$A(\infty) = \{z : f^n(z) \to \infty\} \quad \text{“basin of attraction to } \infty\text{”.}$$

Theorem

The set $A(\infty)$ is open, connected and unbounded. It is contained in the Fatou set of f. The Julia set of f coincides with the boundary of $A(\infty)$, which is a closed and bounded subset of $\mathbb{C}$.

Recap:

- The Julia set is a closed and bounded set.
- The Fatou set is open and unbounded and contains $A(\infty)$.
- Also: $J(f) \cap \mathcal{F}(f) = \emptyset$ and both sets are “completely invariant” under f, meaning that $f(J) = J$ and $f(\mathcal{F}) = \mathcal{F}$.

Another Example

Let's look at another example: $f(z) = z^2 - 2$.
It is hard to calculate and understand the iterates $f^n(z)$!
There is a trick! Conjugate f with

$$\varphi(w) = w + \frac{1}{w}, \varphi : \{w \,:\, |w| > 1\} \to \mathbb{C} \setminus [-2, 2].$$

f maps $[-2, 2]$ to $[-2, 2]$ and $\mathbb{C} \setminus [-2, 2]$ to $\mathbb{C} \setminus [-2, 2]$. We can thus look at $\varphi^{-1} \circ f \circ \varphi$.

Recall: $f(z) = z^2 - 2$, $\varphi(w) = w + \frac{1}{w}$. What is $\varphi^{-1}(f(\varphi(w)))$?

$$\begin{aligned}
f(\varphi(w)) &= (\varphi(w))^2 - 2 \\
&= \left(w + \frac{1}{w}\right)^2 - 2 \\
&= w^2 + \frac{1}{w^2} + 2w\frac{1}{w} - 2 \\
&= w^2 + \frac{1}{w^2} \\
&= \varphi(w^2), \quad \text{so} \\
\varphi^{-1}(f(\varphi(w))) &= w^2.
\end{aligned}$$

$\varphi^{-1} \circ f \circ \varphi$, cont.

- Recall: $f(z) = z^2 - 2$, $\varphi(w) = w + \dfrac{1}{w}$.

$$\varphi^{-1}(f(\varphi(w))) = w^2, \text{ or } f(z) = \varphi(g(\varphi^{-1}(z))), \text{ where } g(w) = w^2.$$

- Thus, on $\mathbb{C} \setminus [-2, 2]$, the function $f(z) = z^2 - 2$ behaves like $g(w) = w^2$ behaves on the exterior of the closed unit disk.
- Since the iterates $g^n(w)$ tend to ∞ for $|w| > 1$ we conclude that $f^n(z) \to \infty$ as $n \to \infty$ for all $z \in \mathbb{C} \setminus [-2, 2]$.
- Thus $A(\infty) = \mathbb{C} \setminus [-2, 2]$, and thus $J(f) = [-2, 2]$.

Wrap-up

We have looked at two examples so far and found their Julia sets:

- $f(z) = z^2$. We found that $J(f) = \{z : |z| = 1\}$, the unit circle.
- $f(z) = z^2 - 2$. We found that $J(f) = [-2, 2]$, the closed interval from -2 to 2 on the real axis.

These two examples are exceptional in that their Julia sets are "smooth". In fact, they are the only examples amongst all $f(z) = z^2 + c$ with smooth Julia sets!

Here are some pictures of other Julia sets. We'll learn how to create these during the next lecture.