Analysis of a Complex Kind

Week 2

Lecture 4: How To Find Julia Sets

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Finding Julia Sets

- Recall: For $f(z) = z^2 + c$, where $c \in \mathbb{C}$, the Julia set of f is the boundary of $A(\infty)$, where $A(\infty)$ is the "basin of attraction to infinity", i.e. $A(\infty) = \{z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty\}$.
- Let's start by finding all those $z \in \mathbb{C}$ for which $\{f^n(z)\}$ stays bounded:

$$K(f) = \{z \in \mathbb{C} : \{f^n(z)\} \text{ is bounded}\}$$

  is the *filled-in Julia set* of f.
- Examples:
  - $f(z) = z^2$ (i.e. $c = 0$): Then $K(f) = \{z : |z| \leq 1\}$, and $J(f) = \{z : |z| = 1\}$.
  - $f(z) = z^2 - 2$ (i.e. $c = -2$): Then $K(f) = [-2, 2]$, and $J(f) = [-2, 2]$.
  - $f(z) = z^2 - 1$ (i.e. $c = -1$): $K(f) = ?$

The Julia Set of $f(z) = z^2 - 1$

$f(z) = z^2 - 1$, i.e. $c = -1$. What is $K(f)$? Let's check some orbits ...

- $z = 0$: $f(0) = -1$, $f^2(0) = f(-1) = 0$, $f^3(0) = f(0) = -1$, $f^4(0) = 0, \ldots$
  The orbit is thus $0, -1, 0, -1, 0, -1, \ldots$
  This is called a *periodic orbit*, and 0 is a *periodic point* of *period* 2. Clearly, $0 \in K(f)$.
- $z = 1$: $1, 0, -1, 0, -1, 0, -1, \ldots$ So 1 isn't itself a periodic point (the orbit never returns to 1), but it's the next best thing: it runs into a periodic orbit. We thus call 1 a *pre-periodic point*. Again, $1 \in K(f)$.
- $z = \dfrac{1+\sqrt{5}}{2}$: Then

$$f(z) = \left(\frac{1+\sqrt{5}}{2}\right)^2 - 1 = \frac{1+2\sqrt{5}+5}{4} - 1 = \frac{2+2\sqrt{5}}{4} = \frac{1+\sqrt{5}}{2} = z.$$

The point $\dfrac{1+\sqrt{5}}{2}$ is a *fixed point* of f and thus belongs to $K(f)$ as well.

The Julia Set of $f(z) = z^2 - 1$, cont.

So far everything we've tried belongs to $K(f)$. Let's find some more orbits.

- $z = -2$: $-2, 3, 8, 63, \ldots \to \infty$. Thus $-2 \in A(\infty)$.
- $z = i$: $i, -2, 3, 8, 63, \ldots \to \infty$. Thus $i \in A(\infty)$.
- How do we know for sure that the orbit of -2 goes off to ∞?

$f(z) = z^2 - 1$. What Orbits Go Off To ∞?

- One can show that if x lies on the real axis and is smaller than $-\frac{1+\sqrt{5}}{2}$, or larger than $\frac{1+\sqrt{5}}{2}$, then $x \in A(\infty)$.
- More generally, if $z \in \mathbb{C}$ with $|z| > \frac{1+\sqrt{5}}{2}$, then $z \in A(\infty)$.
- Even more generally, if $z \in \mathbb{C}$ such that $|f^n(z)| > \frac{1+\sqrt{5}}{2}$, for some n, then $z \in A(\infty)$.
- A similar condition holds for general quadratic polynomials $f(z) = z^2 + c$:

Theorem

Let $f(z) = z^2 + c$, and let $R = \dfrac{1+\sqrt{1+4|c|}}{2}$. Let $z_0 \in \mathbb{C}$. If for some $n > 0$ we have that $|f^n(z_0)| > R$, then $f^n(z_0) \to \infty$ as $n \to \infty$, i.e. $z_0 \in A(\infty)$, so $z_0 \notin K(f)$.

This sounds like a condition a computer could check!

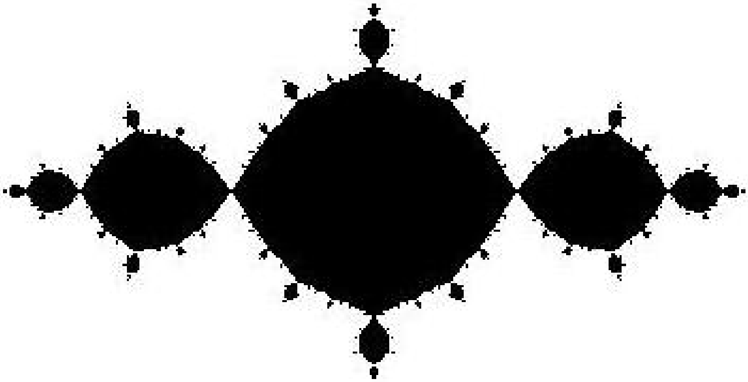
A Computer Algorithm to Find the Filled-In Julia Set

1. Given c, let $R = \frac{1+\sqrt{1+4|c|}}{2}$.
2. Choose a window $W = \{x + iy : a \le x \le b, c \le y \le d\}$ to display. If you want to see the entire Julia set, you'd want $B_R(0) \subset W$. But at some point it also may be interesting to zoom into the Julia set!
3. The computer can't keep checking all iterates $f^n(z)$, at some point it'll have to stop. So pick a largest number, *maxiter*, up to which it will check. The larger this number, the more accurate your picture will get, but the slower the calculation will be.
4. For each pixel on your screen, choose a point z in your window W, corresponding to that pixel.
5. Calculate the iterates of z: $f(z), f(f(z)), f(f(f(z))), \ldots$ If one of these iterates satisfies that $|f^n(z)| > R$, color the initial pixel white.
6. If you reach the maximum number of iterations, *maxiter*, without having left $B_R(0)$, there's a good chance that z belongs to $K(f)$. Color this pixel black.

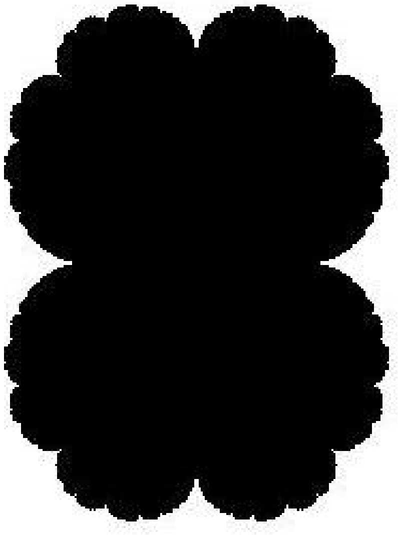
Examples

This gives you an image of the filled-in Julia set. Let's look at some examples..

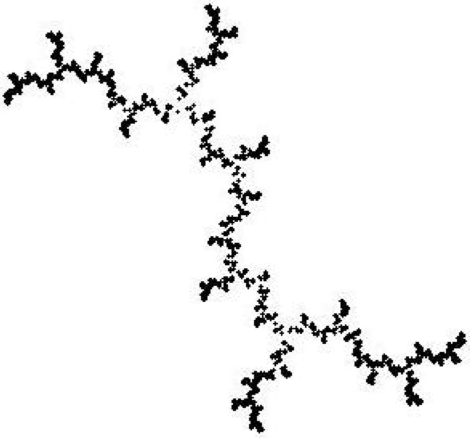
Julia Set of $f(z) = z^2 - 1$



Julia Set of $f(z) = z^2 + \frac{1}{4}$



Julia Set of $f(z) = z^2 + i$



Julia Set of $f(z) = z^2 + 1$

Prettier Pictures

- The pictures of the filled-in Julia sets we have produced so far are only black and white.
- For colorful pictures, choose colors $c_0, c_1, c_2, \ldots, c_{maxiter}$ for $A(\infty)$ as well as a color for the filled-in Julia set.
- If $|z| > R$, color the corresponding pixel with color c_0.
- Otherwise, if $|f(z)| > R$, color the corresponding pixel with color c_1.
- Otherwise, if $|f(f(z))| > R$, color the corresponding pixel with color c_2.

$$\vdots$$

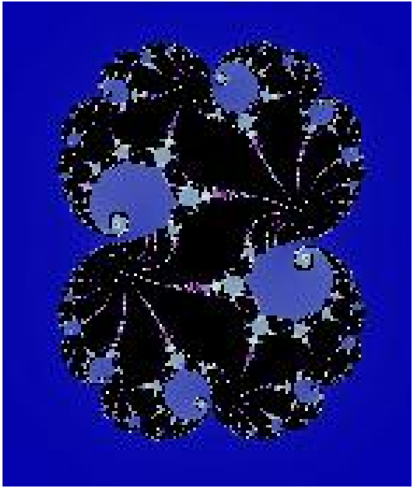
- Otherwise (i.e. if $|f^n(z)| \leq R$ for all $n \leq maxiter$), color the corresponding pixel in your chosen color for the filled-in Julia set.

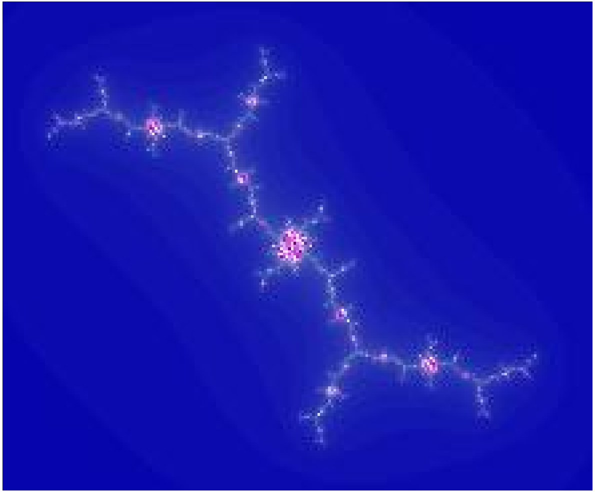
Remarks and Examples

- The larger your constant *maxiter*, the more detail you can see, but the longer the calculation will take.
- For small windows (i.e. zooms into the Julia set), a higher precision is required to avoid round-off errors.

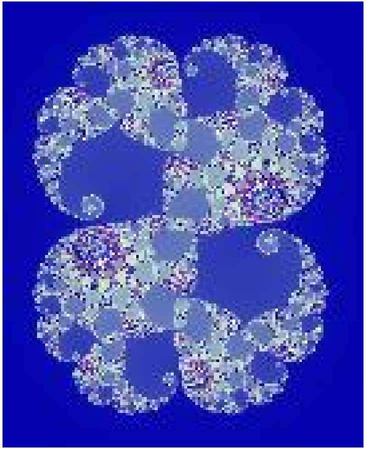
Let's look at some examples.

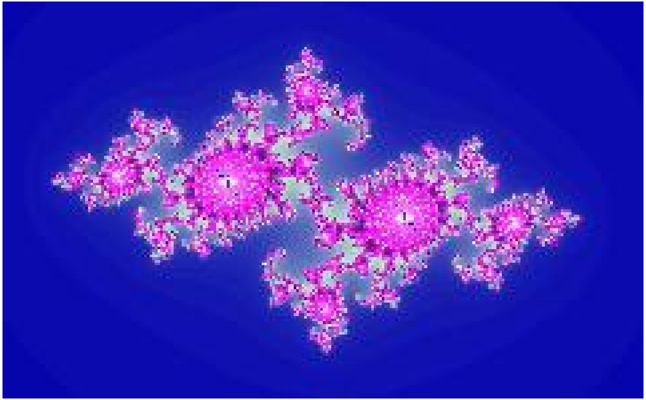
Colorful Julia Set Examples



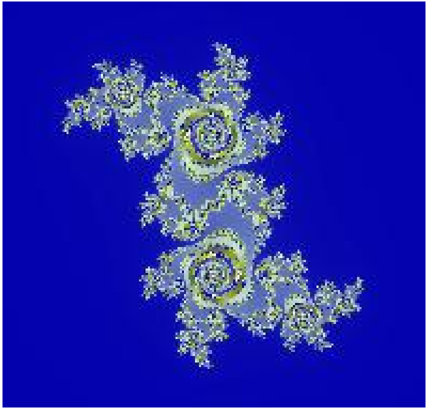


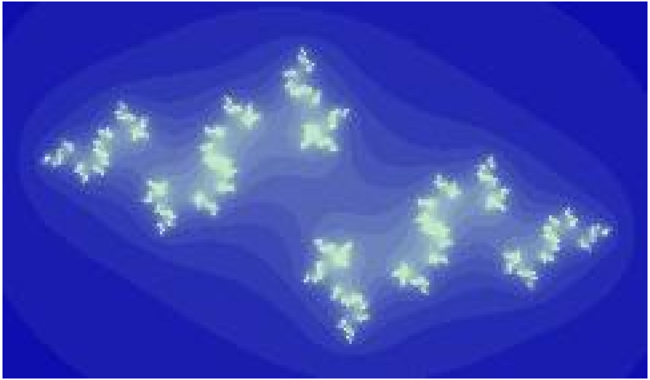
Colorful Julia Set Examples





Colorful Julia Set Examples





The Mandelbrot Set

We noticed the following in all of the pictures we looked at: The Julia set of $f(z) = z^2 + c$ is either "in one piece" or "totally dusty".
This leads us to the following definition:

Definition

The *Mandelbrot set* M is the set of all parameters $c \in \mathbb{C}$ for which the Julia set $J(f)$ of $f(z) = z^2 + c$ is connected.

Note: The Mandelbrot set is a subset of the *parameter space* (the space of all possible c-values), whereas Julia sets are sets of z-values.
Examples:

- $0 \in M$ since $J(z^2) = \{z : |z| = 1\}$.
- $-2 \in M$ since $J(z^2 - 2) = [-2, 2]$.
- $\frac{1}{4} \in M$ according to picture of $J(z^2 + \frac{1}{4})$.
- $1 \notin M$ according to picture of $J(z^2 + 1)$.

The Mandelbrot Set

How do you determine whether for a given $c \in \mathbb{C}$ the Julia set $J(z^2 + c)$ is connected?

Can a computer perform this task?