Analysis of a Complex Kind

Week 3

Lecture 2: The Cauchy-Riemann Equations

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Recall: A complex function f can be written as

$$f(z) = u(x, y) + iv(x, y),$$

where $z = x + iy$ and u, v are real-valued functions that depend on the two real variables x and y.

Example: $f(z) = z^2 = (x + iy)^2 = \underbrace{x^2 - y^2}_{u(x,y)} + i \cdot \underbrace{2xy}_{v(x,y)}$.

Thus $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$.

For example, if $z = 2 + i$, then clearly $f(z) = (2 + i)^2 = 4 + 4i + i^2 = 3 + 4i$.

Alternatively, since $z = x + iy$, we have $x = 2$ and $y = 1$, thus

$$u(x, y) = u(2, 1) = 2^2 - 1^2 = 3 \quad \text{and} \quad v(x, y) = v(2, 1) = 2 \cdot 2 \cdot 1 = 4,$$

thus

$$f(2 + i) = u(2, 1) + iv(2, 1) = 3 + 4i.$$

$f(z) = z^2 = u(x, y) + iv(x, y), u(x, y) = x^2 - y^2, v(x, y) = 2xy.$

We already know:

- f is differentiable everywhere in $\mathbb{C}$
- $f'(z) = 2z$ for all $z \in \mathbb{C}$.

Let's look at the function $u(x, y)$ more carefully.

- If we fix the variable y at a certain value, then u only depends on x. For example, if we only consider $y = 3$, then $u(x, y) = u(x, 3) = x^2 - 9$.
- We can now differentiate this function with respect to x according to the rules of calculus and find that the derivative is $2x$.
- We write $\frac{\partial u}{\partial x}(x, 3) = u_x(x, 3) = 2x$, and, more generally, for arbitrary (fixed) y,

$$\frac{\partial u}{\partial x}(x, y) = u_x(x, y) = 2x.$$

- This is called the *partial derivative of u with respect to x*.

Similarly, we can look at $v(x, y)$.

- For example, $v(x, 3) = 2 \cdot x \cdot 3 = 6x$, and the derivative of this function with respect to x is 6.
- Thus $\frac{\partial v}{\partial x}(x, 3) = v_x(x, 3) = 6$.
- More generally, for arbitrary (fixed) y, we have

$$\frac{\partial v}{\partial x}(x, y) = v_x(x, y) = 2y.$$

- This is called the *partial derivative of v with respect to x.*

$f(z) = z^2 = u(x,y) + iv(x,y), u(x,y) = x^2 - y^2, v(x,y) = 2xy.$

Obviously, we can do the same thing by fixing x and differentiating with respect to y!

- Example: Let $x = 2$. Then $u(x,y) = u(2,y) = 4 - y^2$, and $\frac{\partial u}{\partial y}(x,y) = u_y(2,y) = -2y$.
- More generally, $\frac{\partial u}{\partial y}(x,y) = u_y(x,y) = -2y$.
- This is called the *partial derivative of u with respect to y.*
- Similarly, $v(2,y) = 4y$, and $\frac{\partial v}{\partial y}(2,y) = v_y(2,y) = 4$, More generally, $\frac{\partial v}{\partial y}(x,y) = v_y(x,y) = 2x$.
- This is called the *partial derivative of v with respect to y.*

Putting It All Together

Here is what we have found:

$$f(z) = z^2 = u(x,y) + iv(x,y), u(x,y) = x^2 - y^2, v(x,y) = 2xy.$$

Derivatives:

$$f'(z) = 2z, u_x(x,y) = 2x, u_y(x,y) = -2y, v_x(x,y) = 2y, v_y(x,y) = 2x.$$

We notice:

- $u_x = v_y$
- $u_y = -v_x$
- $f' = \underbrace{u_x + iv_x}_{=f_x} = \underbrace{-i(u_y + iv_y)}_{=-if_y}$.

Coincidence???

Another Example

$$\begin{aligned} f(z) &= 2z^3 - 4z + 1, \quad z = x + iy \\ &= 2(x+iy)^3 - 4(x+iy) + 1 \\ &= 2(x^3 + 3x^2 iy + 3xi^2y^2 + i^3y^3) - 4x - 4iy + 1 \\ &= \underbrace{(2x^3 - 6xy^2 - 4x + 1)}_{=u(x,y)} + i\underbrace{(6x^2y - 2y^3 - 4y)}_{=v(x,y)}. \end{aligned}$$

Then

$$u_x(x,y) = 6x^2 - 6y^2 - 4 \qquad v_x(x,y) = 12xy$$
$$u_y(x,y) = -12xy \qquad v_y(x,y) = 6x^2 - 6y^2 - 4.$$

Again: $u_x = v_y$ and $u_y = -v_x$!

$f(z) = 2z^3 - 4z + 1$, $f'(z) = ???$

So far:

$$u_x(x,y) = 6x^2 - 6y^2 - 4 \qquad v_x(x,y) = 12xy$$
$$u_y(x,y) = -12xy \qquad v_y(x,y) = 6x^2 - 6y^2 - 4.$$

How about $f'(z)$?

$$\begin{aligned} f'(z) &= 6z^2 - 4, \quad z = x + iy \\ &= 6(x+iy)^2 - 4 \\ &= (6x^2 - 6y^2 - 4) + 12ixy \\ &= u_x(x,y) + iv_x(x,y) = -i(u_y(x,y) + iv_y(x,y)) \\ &= f_x(z) = -if_y(z). \end{aligned}$$

Theorem

Suppose that $f(z) = u(x,y) + iv(x,y)$ is differentiable at a point z_0. Then the partial derivatives u_x, u_y, v_x, v_y exist at z_0 and satisfy there:

$$u_x = v_y \quad \text{and} \quad u_y = -v_x.$$

These are called the Cauchy-Riemann Equations. Also,

$$\begin{aligned} f'(z_0) &= u_x(x_0,y_0) + iv_x(x_0,y_0) = f_x(z_0) \\ &= -i(u_y(x_0,y_0) + iv_y(x_0,y_0)) = -if_y(z_0). \end{aligned}$$

A Word About The Proof

To prove this theorem, you'd look at the difference quotient

$$\frac{f(z_0 + h) - f(z_0)}{h}$$

whose limit as $h \to 0$ must exist if f is differentiable at z_0.

- You'd let h approach 0 along the real axis only, then along the imaginary axis only. Both times the limit must exist and both limits must be the same.
- Equating these two limits with each other and recognizing the partial derivatives in the expressions yields the Cauchy-Riemann equations.

Another Example

Let $f(z) = \overline{z} = x - iy$.
Then $u(x, y) = x$ and $v(x, y) = -y$, so

$$\begin{aligned} u_x(x,y) &= 1 \qquad & v_x(x,y) &= 0 \\ u_y(x,y) &= 0 \qquad & v_y(x,y) &= -1. \end{aligned}$$

Since $u_x \neq v_y$ (while $u_y = -v_x$) for all z, the function f is not differentiable anywhere.

Recall: If f is differentiable at z_0 then the Cauchy-Riemann equations hold at z_0. Is the reverse statement also true? I.e.:

Question

Are the Cauchy-Riemann equations a guarantee for differentiability?

In other words, if f satisfies the Cauchy-Riemann equations at a point z_0 then does this imply that f is differentiable at z_0?

Sufficient Conditions for Differentiability

The answer: Almost...!

Theorem

Let $f = u + iv$ be defined on a domain $D \subset \mathbb{C}$. Then f is analytic in D if and only if $u(x, y)$ and $v(x, y)$ have continuous first partial derivatives on D that satisfy the Cauchy-Riemann equations.

Example: $f(z) = e^x \cos y + ie^x \sin y$. Then

$$\begin{aligned} u_x(x, y) &= e^x \cos y & \qquad v_x(x, y) &= e^x \sin y \\ u_y(x, y) &= -e^x \sin y & \qquad v_y(x, y) &= e^x \cos y. \end{aligned}$$

Thus the Cauchy-Riemann equations are satisfied, and in addition, the functions u_x, u_y, v_x, v_y are continuous in $\mathbb{C}$. Therefore, the function f is analytic in $\mathbb{C}$, thus entire.

The Complex Exponential Function

This function

$$f(z) = e^x \cos y + ie^x \sin y$$

is the complex exponential function.

We'll study it in the next lecture.