Analysis of a Complex Kind

Week 4

Lecture 4: Möbius Transformations, Part 2

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Review

Recall: $f(z) = \dfrac{az+b}{cz+d}$, with $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$ is called a Möbius transformation.

- f maps $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to $\hat{\mathbb{C}}$.
- f is a conformal map from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$.
- If $c = 0$, $d = 1$: $f(z) = az + b$ is a conformal map from $\mathbb{C}$ to $\mathbb{C}$.
- Möbis transformations map circles and lines to circles or lines.
- For distinct z_1, z_2, z_3, the Möbius transformation $f(z) = \dfrac{z - z_1}{z - z_3} \cdot \dfrac{z_2 - z_3}{z_2 - z_1}$ maps z_1, z_2, z_3 to $0, 1, \infty$, respectively.

Further Facts

- The composition of two Möbius transformations is a Möbius transformation, and so is the inverse.

- Given three distinct points z_1, z_2, z_3 and three distinct points w_1, w_2, w_3, there exists a unique Möbius transformation $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that maps z_j to w_j, $j = 1, 2, 3$.

  Proof:
  Let f_1 be the Möbius transformation that maps z_1, z_2, z_3 to $0, 1, \infty$.
  Let f_2 be the Möbius transformation that maps w_1, w_2, w_3 to $0, 1, \infty$.
  Then $f_2^{-1} \circ f_1$ maps z_1, z_2, z_3 to w_1, w_2, w_3, respectively.

Examples

Find the Möbius transformation f that maps 0 to -1, i to 0, and ∞ to 1.

1. $$f_1(z) = \frac{z-0}{\text{“}z-\infty\text{”}} \cdot \frac{\text{“}i-\infty\text{”}}{i-0} = \frac{z}{i}$$

   maps $0, i, \infty$ to $0, 1, \infty$.

   $$f_2(z) = \frac{z+1}{z-1} \cdot \frac{0-1}{0+1} = \frac{z+1}{-z+1}$$

   maps $-1, 0, 1$ to $0, 1, \infty$. To find f_2^{-1}, solve for z:

   $$\begin{aligned} w = \frac{z+1}{-z+1} \quad &\Longleftrightarrow \quad w(-z+1) = z+1 \\ &\Longleftrightarrow \quad -wz + w = z+1 \\ &\Longleftrightarrow \quad w-1 = z(1+w) \\ &\Longleftrightarrow \quad z = \frac{w-1}{w+1} \end{aligned}$$

Möbius transformation f that maps 0 to -1, i to 0, and ∞ to 1

1. So far: $f_1(z) = \frac{z}{i}$ maps $0, i, \infty$ to $0, 1, \infty$,

   and $f_2^{-1}(w) = \frac{w-1}{w+1}$ maps $0, 1, \infty$ to $-1, 0, 1$.

   Thus $f = f_2^{-1} \circ f_1$ is the desired map:

$$(f_2^{-1} \circ f_1)(z) = \frac{f_1(z) - 1}{f_1(z) + 1} = \frac{\frac{z}{i} - 1}{\frac{z}{i} + 1} = \frac{z - i}{z + i}.$$

Let's look at another approach:

Möbius transformation f that maps 0 to -1, i to 0, and ∞ to 1

2. f is of the form $f(z) = \dfrac{az+b}{cz+d}$.
   - Since $f(i) = 0$, we have $a \neq 0$, we can thus assume that $a = 1$.
   - Thus $f(z) = \dfrac{z+b}{cz+d}$. Since $f(i) = 0$, we have that $b = -i$.
   - Thus $f(z) = \dfrac{z-i}{cz+d}$. Since $f(\infty) = 1$, we have $c = 1$, and since $f(0) = -1$, we have $d = i$.
   - Thus $f(z) = \dfrac{z-i}{z+i}$.

Compositions of Möbius transformations

Fact

Every Möbius transformation is the composition of maps of the type

$$\begin{aligned} z &\mapsto az \quad (\textit{rotation\&dilation}) \\ z &\mapsto z+b \quad (\textit{translation}) \\ z &\mapsto \frac{1}{z} \quad (\textit{inversion}) \end{aligned}$$

Proof: Let f be a Möbius transformation.

1. Suppose first that $f(\infty) = \infty$. Then $f(z) = az + b$. This corresponds to the following composition:

$$z \overset{\text{rot. \& dil.}}{\longmapsto} az \overset{\text{translation}}{\longmapsto} az + b.$$

Compositions of Möbius transformations

2. Suppose next that $f(\infty) \neq \infty$. Then

$$\begin{aligned} f(z) &= \frac{az+b}{cz+d} \quad \text{with } c \neq 0 \\ &= \frac{\frac{a}{c}z + \frac{b}{c}}{z + \frac{d}{c}}, \end{aligned}$$

and we can thus assume that $c = 1$. So

$$f(z) = \frac{az+b}{z+d} = \frac{a(z+d)+b-ad}{z+d} = a + \frac{b-ad}{z+d}.$$

This corresponds to the following composition:

$$z \overset{\text{trans.}}{\longmapsto} z+d \overset{\text{inv.}}{\longmapsto} \frac{1}{z+d} \overset{\text{dil. \& rot.}}{\longmapsto} \frac{b-ad}{z+d} \overset{\text{trans.}}{\longmapsto} a + \frac{b-ad}{z+d}.$$

Images of Circles and Lines

We mentioned earlier the fact, that Möbius transformations map circles & lines to circles & lines. How would you prove this?

By the previous composition result it suffices to prove this fact for the three standard types (translation, rotation & dilation, inversion).

Clearly, dilations, rotations and translations preserve circles and lines as circles and lines, and so all that is left to show is that $f(z) = \frac{1}{z}$ maps circles & lines to circles & lines.

We saw the main ideas on how to do so during the last lecture.

Next up: The Riemann mapping theorem!