Analysis of a Complex Kind

Week 2

Lecture 5: The Mandelbrot Set

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Finding the Mandelbrot Set

Recall: The Mandelbrot set is

$$M = \{c \in \mathbb{C} \,:\, J(z^2 + c) \text{ is connected}\}.$$

How could a computer check such a condition??

Theorem

Let $f(z) = z^2 + c$. Then $J(f)$ is connected if and only if 0 does not belong to $A(\infty)$, that is if and only if the orbit $\{f^n(0)\}$ remains bounded under iteration.

In fact, it is possible to show the following:

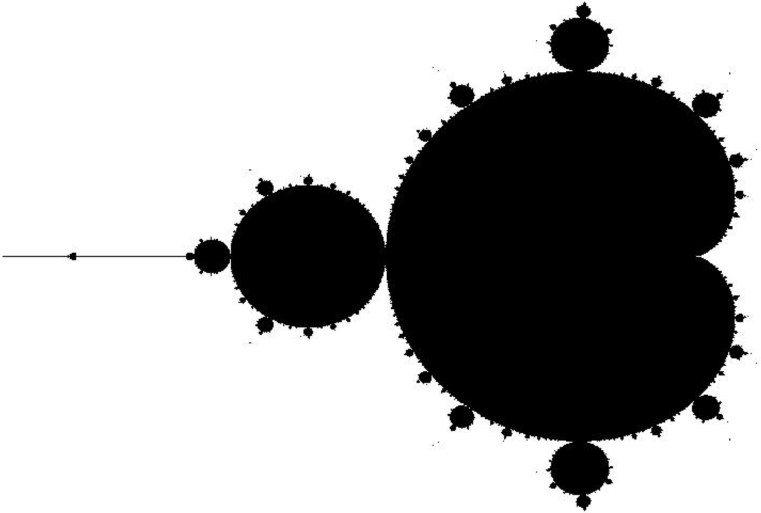
Theorem

A complex number c belongs to M if and only if $|f^n(0)| \leq 2$ for all $n \geq 1$ (where $f(z) = z^2 + c$).

Computer Algorithm to Plot the Mandelbrot Set

1. Choose a window $W = \{c_x + ic_y : c_{xmin} \leq c_x \leq c_{xmax}, c_{ymin} \leq c_y \leq c_{ymax}\}$ to display. If you want to see the entire Mandelbrot set, you'd want something like $W = \{c : -2 \leq c_x \leq 0.75, -1.5 \leq c_y \leq 1.5\}$.
2. As before, pick a largest number of iterations, *maxiter*. The larger this number, the more accurate your picture will get, but the slower the calculation will be.
3. For each pixel on your screen, choose a parameter c in your window W, corresponding to that pixel. We'll look at the corresponding polynomial $f(z) = z^2 + c$.
4. Calculate the iterates of 0 under this polynomial: $f(0) = c$, $f(f(0)) = c^2 + c$, $f(f(f(0))) = (c^2 + c)^2 + c, \ldots$ If one of these iterates satisfies that $|f^n(0)| > 2$, color the initial pixel white.
5. If you reach the maximum number of iterations, *maxiter*, without having left $\overline{B_2(0)}$, there's a good chance that the parameter c belongs to the Mandelbrot set M. Color this pixel black.

A First Look At M



A Prettier Picture

- Again, you can use different colors for those parameters $c \in \mathbb{C}$ for which 0 escapes to infinity under iteration, depending on how quickly the escape happens:
  - If $|f(0)| = |c| > 2$, color the corresponding pixel in color zero.
  - Otherwise, if $|f(f(0))| = |c^2 + c| > 2$, color the corresponding pixel in color one.
  - Otherwise, if $|f(f(f(0)))| = |(c^2 + c)^2 + c| > 2$, color the corresponding pixel in color two.

    $\vdots$
  - If $|f^n(0)| \leq 2$ for all $n \leq \mathit{maxiter}$, color the pixel corresponding to c black (or whatever other color you choose for your M).
- Zooming into the Mandelbrot set and coloring parameters by escape time yields beautiful pictures.

Pictures

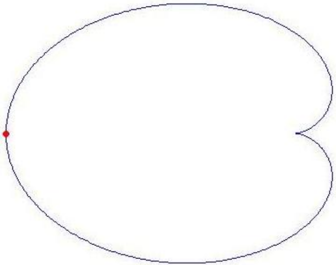
Let's look at some of those beautiful pictures!

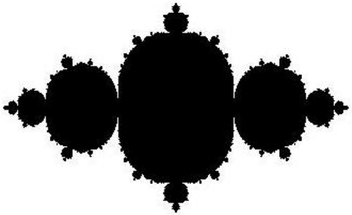
Properties of the Mandelbrot Set

- *M* is a connected set (Douady, Hubbard, 1982).
- *M* is contained in the disk of radius 2, centered at zero.
- The boundary of *M* is very intricate- this is where you'll find the most beautiful zooms.
- Moreover, for *c*-values near the boundary of *M*, their Julia sets have many different patterns. Here are some examples:
  - The boundary of the main cardioid is given by $c = \frac{1}{2}e^{i\theta} - \frac{1}{4}e^{2i\theta}$, $0 \leq \theta < 2\pi$. Writing $\theta = 2\pi\alpha$, $0 \leq \alpha < 1$, we can distinguish whether α is a rational or an irrational number.

Let $c = \frac{1}{2}e^{2\pi i\alpha} - \frac{1}{4}e^{4\pi i\alpha}$, where $0 \leq \alpha < 1$ is a rational number.

- Then α is of the form $\frac{p}{q}$.
- The parameter c is an attachment point of another "bud" to the Mandelbrot set, and the Julia set for $f(z) = z^2 + c$ looks similar to the Julia sets for parameter values within the bud.
- Example: $\alpha = \frac{1}{2}$. Then $c = \frac{1}{2}e^{\pi i} - \frac{1}{4}e^{2\pi i} = -\frac{1}{2} - \frac{1}{4} = -0.75$. Here is a picture for $J(z^2 - 0.75)$:

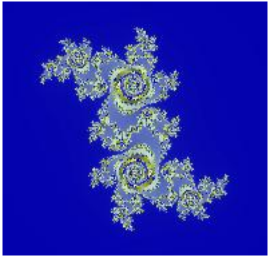




The Case of an Irrational α

Let $c = \frac{1}{2}e^{2\pi i\alpha} - \frac{1}{4}e^{4\pi i\alpha}$, where $0 \leq \alpha < 1$ is a irrational number.

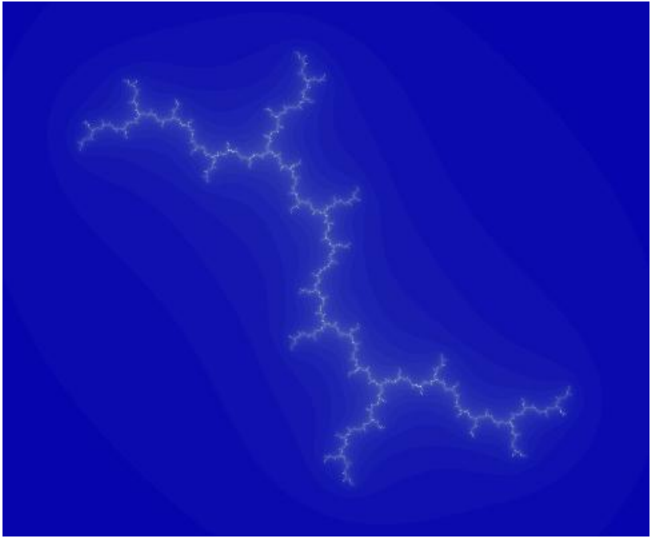
- Thus there are no values p and q such that $\alpha = \frac{p}{q}$.
- Julia sets for such values look more intricate and come in several “flavors”!
- Here is an example: $\alpha = \dfrac{1+\sqrt{5}}{2}$. Then $c \equiv -0.390540870218401\ldots - 0.586787907346969\ldots i$. For $f(z) = z^2 + c$, the interior of $K(f)$ has a so-called “Siegel disk”, in which iteration looks like a rotation by angle α.



"Many" (we'll clarify this later) points in the boundary of M are co-called *Misiurewicz points*:

- A point $c \in \mathbb{C}$ is called a Misiurewicz point if the orbit of 0 under $f(z) = z^2 + c$ is pre-periodic, but not periodic.
- Example: $c = i$: Then $f(z) = z^2 + i$, and the orbit of 0 under f is $0,\ i,\ -1+i,\ -i,\ -1+i,\ -i,\ \ldots$
- Clearly, Misiurewicz points c belong to M since the orbit of 0 under $f(z) = z^2 + c$ is bounded.
- Properties:
  - Let c be a Misiurewicz point. Then $J(f) = K(f)$, i.e. $K(f)$ has no interior.
  - Misiurewicz points are dense in ∂M.
  - The Mandelbrot set is self-similar under magnification near Misiurewicz points. Note: The Mandelbrot is "quasi-self-similar" everywhere: small, slightly different versions of itself can be found at arbitrary small scales.

The Julia Set For $c = i$



A Zoom Into The Mandelbrot Set At $z = i$

Let's look at the Mandelbrot set near $z = i$.

Big Open Conjecture

Here is one of the big outstanding conjectures in the field of complex dynamics:

Conjecture

The Mandelbrot set is locally connected, that is, for every $c \in M$ and every open set V with $c \in V$, there exists an open set U such that $c \in U \subset V$ and $U \cap M$ is connected.