Analysis of a Complex Kind

Week 3

Lecture 1: The Complex Derivative

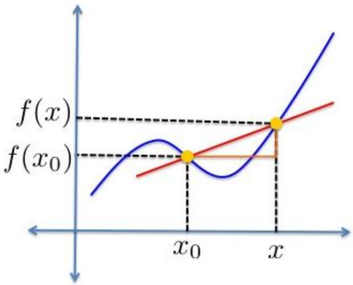
Petra Bonfert-Taylor

Quick Review from Calculus

Let $f : (a, b) \to \mathbb{R}$ be a real-valued function of a real variable, and let $x_0 \in (a, b)$. The function f is *differentiable at* x_0 if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

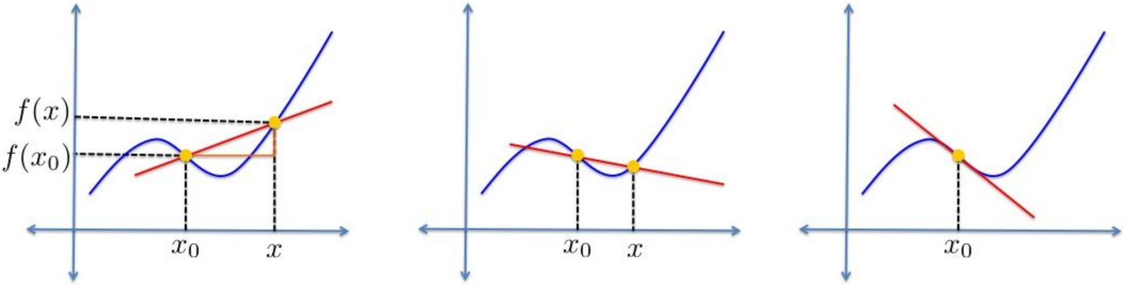
exists. If so, we call this limit the *derivative of f at x_0* and denote it by $f'(x_0)$.



$\dfrac{f(x) - f(x_0)}{x - x_0}$ is the slope of the secant line through the points $(x_0, f(x_0))$ and $(x, f(x))$.

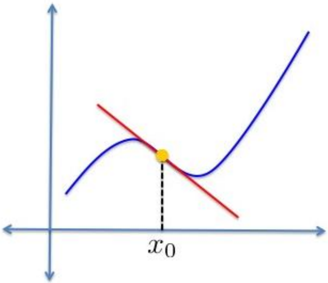
Secant Becomes Tangent

As x approaches x_0, the slope of the secant line changes:

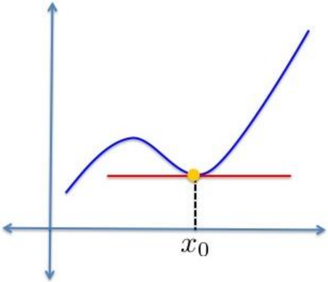


In the limit, the slopes approach the slope of the tangent line to the graph of f at x_0.

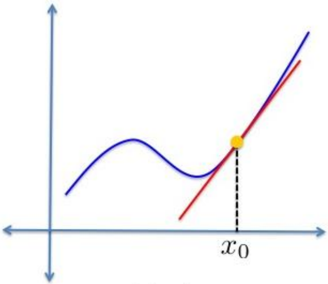
Examples



$f'(x_0) < 0$



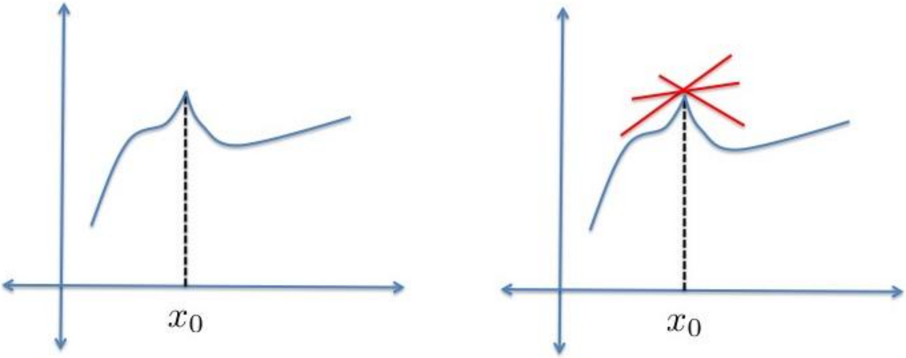
$f'(x_0) = 0$



$f'(x_0) > 0$

No Derivative

The derivative does not always exist! Here is an example:



The graph of f does not have a tangent line at x_0!

Definition

A complex-valued function f of a complex variable is *(complex) differentiable* at $z_0 \in$ domain(f) if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists.}$$

If this limit exists, it is denoted $f'(z_0)$ or $\frac{df}{dz}(z_0)$ or $\left.\frac{d}{dz} f(z)\right|_{z=z_0}$.

Example:

- $f(z) = c$ (a constant function, $c \in \mathbb{C}$). Let $z_0 \in \mathbb{C}$ be arbitrary. Then

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{c - c}{z - z_0} = 0 \to 0 \text{ as } z \to z_0.$$

Thus $f'(z) = 0$ for all $z \in \mathbb{C}$.

Other Forms of the Difference Quotient

Instead of

$$\frac{f(z) - f(z_0)}{z - z_0}$$

we often write $z = z_0 + h$ (where $h \in \mathbb{C}$), and the difference quotient becomes

$$\frac{f(z_0 + h) - f(z_0)}{h} \quad \text{or simply} \quad \frac{f(z + h) - f(z)}{h},$$

where we'll take the limit as $h \to 0$.

Further examples:

- $f(z) = z$. Then

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{(z_0 + h) - z_0}{h} = \frac{h}{h} = 1 \to 1 \text{ as } h \to 0.$$

  So $f'(z) = 1$ for all $z \in \mathbb{C}$.

More Examples

- $f(z) = z^2$. Then

$$\frac{f(z_0+h) - f(z_0)}{h} = \frac{(z_0+h)^2 - z_0^2}{h} = \frac{2z_0 h + h^2}{h} = 2z_0 + h \to 2z_0 \text{ as } h \to 0.$$

Thus $f'(z) = 2z$ for all $z \in \mathbb{C}$.

- $f(z) = z^n$. Then

$$\begin{aligned} \frac{f(z_0+h) - f(z_0)}{h} &= \frac{(z_0+h)^n - z_0^n}{h} \\ &= \frac{(z_0^n + nhz_0^{n-1} + \frac{n(n-1)}{2}h^2 z_0^{n-2} + \cdots + h^n) - z_0^n}{h} \\ &= nz_0^{n-1} + h\left(\frac{n(n-1)}{2} z_0^{n-2} + \cdots\right) \to nz_0^{n-1} \text{ as } h \to 0. \end{aligned}$$

Thus $f'(z) = nz^{n-1}$ for all $z \in \mathbb{C}$.

Theorem

Suppose f and g are differentiable at z, and h is differentiable at $f(z)$. Let $c \in \mathbb{C}$. Then

1. $(cf)'(z) = cf'(z)$.
2. $(f+g)'(z) = f'(z) + g'(z)$.
3. $(f \cdot g)'(z) = f'(z)g(z) + f(z)g'(z)$.
4. $\left(\dfrac{f}{g}\right)'(z) = \dfrac{g(z)f'(z) - f(z)g'(z)}{(g(z))^2}$, *for* $g(z) \neq 0$.
5. $(h \circ f)'(z) = h'(f(z))f'(z)$.

Examples

- $f(z) = 5z^3 + 2z^2 - z + 7$. Then
  $f'(z) = 5 \cdot 3z^2 + 2 \cdot 2z - 1 = 15z^2 + 4z - 1$.
- $f(z) = \dfrac{1}{z}$. Then
  $f'(z) = \dfrac{z \cdot 0 - 1 \cdot 1}{z^2} = \dfrac{-1}{z^2}$.
- $f(z) = (z^2 - 1)^n$. Then
  $f'(z) = n(z^2 - 1)^{n-1} \cdot 2z$.
- $f(z) = (z^2 - 1)(3z + 4)$. Then
  $f'(z) = (2z)(3z + 4) + (z^2 - 1) \cdot 3$.
- $f(z) = \dfrac{z}{z^2 + 1}$. Then
  $f'(z) = \dfrac{(z^2 + 1) - z \cdot 2z}{(z^2 + 1)^2} = \dfrac{1 - z^2}{(1 + z^2)^2}$.

A Non-Example

Let $f(z) = \text{Re}(z)$. Write $z = x + iy$ and $h = h_x + ih_y$. Then

$$\frac{f(z+h) - f(z)}{h} = \frac{(x + h_x) - x}{h} = \frac{h_x}{h} = \frac{\text{Re}\, h}{h}.$$

Does this have a limit as $h \to 0$?

- $h \to 0$ along real axis: Then $h = h_x + i \cdot 0$, so $\text{Re}\, h = h$, and thus the quotient evaluates to 1, and the limit equals 1.
- $h \to 0$ along imaginary axis: Then $h = 0 + i \cdot h_y$, so $\text{Re}\, h = 0$, and thus the quotient evaluates to 0, and the limit equals 0.
- $h_n = \frac{i^n}{n}$, then $\frac{\text{Re}\, h_n}{h_n} = \frac{\text{Re}\, i^n}{i^n} = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$ has no limit as $n \to \infty$.

f is not differentiable anywhere in $\mathbb{C}$.

Another Non-Example

Let $f(z) = \overline{z}$. Then

$$\frac{f(z+h) - f(z)}{h} = \frac{\overline{z} + \overline{h} - \overline{z}}{h} = \frac{\overline{h}}{h}.$$

- If $h \in \mathbb{R}$ then $\frac{\overline{h}}{h} = 1 \to 1$ as $h \to 0$.
- If $h \in i\mathbb{R}$ then $\frac{\overline{h}}{h} = -1 \to -1$ as $h \to 0$.

Thus $\frac{\overline{h}}{h}$ does not have a limit as $h \to 0$, and f is not differentiable anywhere in $\mathbb{C}$.

Differentiability Implies Continuity

Fact

If f is differentiable at z_0 then f is continuous at z_0.

Proof:

$$\lim_{z \to z_0} (f(z) - f(z_0)) = \lim_{z \to z_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \cdot (z - z_0) \right) = f'(z_0) \cdot 0 = 0.$$

Note however that a function can be continuous without being differentiable (we'll see an example shortly).

Analytic Functions

Definition

A function f is *analytic* in an open set $U \subset \mathbb{C}$ if f is (complex) differentiable at each point $z \in U$.
A function which is analytic in all of $\mathbb{C}$ is called an *entire* function.

Examples:

- polynomials are analytic in $\mathbb{C}$ (hence entire).
- rational functions $\dfrac{p(z)}{q(z)}$ are analytic wherever $q(z) \neq 0$.
- $f(z) = \overline{z}$ is not analytic.
- $f(z) = \operatorname{Re} z$ is not analytic.

Another Example

Let $f(z) = |z|^2$. Then

$$
\begin{aligned}
\frac{f(z+h) - f(z)}{h} &= \frac{|z+h|^2 - |z|^2}{h} \\
&= \frac{(z+h)(\overline{z} + \overline{h}) - |z|^2}{h} \\
&= \frac{|z|^2 + z\overline{h} + h\overline{z} + h\overline{h} - |z|^2}{h} \\
&= \overline{z} + \overline{h} + z \cdot \frac{\overline{h}}{h}.
\end{aligned}
$$

Thus...

- If $z \neq 0$ then the limit as $h \to 0$ does not exist.
- If $z = 0$ then the limit equals 0, thus f is differentiable at 0 with $f'(0) = 0$.
- f is not analytic anywhere.
- Note: f is continuous in $\mathbb{C}$!

The Cauchy Riemann Equations!