Analysis of a Complex Kind

Week 4

Lecture 1: Inverse Functions of Analytic Functions

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The Logarithm Function

Motivation: Given $z \in \mathbb{C} \setminus \{0\}$, find $w \in \mathbb{C}$ such that $e^w = z$.
How? Write $z = |z|e^{i\theta}$, then $e^w = |z|e^{i\theta}$.
Next, write $w = u + iv$. Then $e^u e^{iv} = |z|e^{i\theta}$.
Thus $e^u = |z|$ and $e^{iv} = e^{i\theta}$, so $u = \ln|z|$ and $v = \theta + 2k\pi = \arg z$.

Definition

For $z \neq 0$ we define

$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z, \quad \text{the principal branch of logarithm,}$$

and

$$\begin{aligned} \log z &= \ln|z| + i \arg z, \quad \text{a multi-valued function} \\ &= \operatorname{Log} z + 2k\pi i, \quad k \in \mathbb{Z}. \end{aligned}$$

Examples

$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z.$$

- $\operatorname{Log} 1 = \ln|1| + i \operatorname{Arg} 1 = 0.$
- $\operatorname{Log} i = \ln|i| + i\dfrac{\pi}{2} = i\dfrac{\pi}{2}.$
- $\operatorname{Log}(-1) = \ln|-1| + i\pi = i\pi.$
- $\operatorname{Log}(1+i) = \ln\sqrt{2} + i\dfrac{\pi}{4}.$

Continuity of the Logarithm Function

$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z.$$

We notice:

- $z \mapsto |z|$ is continuous in $\mathbb{C}$.
- $z \mapsto \ln|z|$ is continuous in $\mathbb{C} \setminus \{0\}$.
- $z \mapsto \operatorname{Arg} z$ is continous in $\mathbb{C} \setminus (-\infty, 0]$.
- Thus, $\operatorname{Log} z$ is continuous in $\mathbb{C} \setminus (-\infty, 0]$.
- However,
  - as $z \to -x \in (-\infty, 0)$ from above, $\operatorname{Log} z \to \ln x + i\pi$, and
  - as $z \to -x$ from below, $\operatorname{Log} z \to \ln x - i\pi$,

  so $\operatorname{Log} z$ is not continuous on $(-\infty, 0)$ (and not defined at 0).

Is the Logarithm Function Analytic?

Fact

The principal branch of logarithm, Log z*, is analytic in* $\mathbb{C} \setminus (-\infty, 0]$.

What is its derivative?

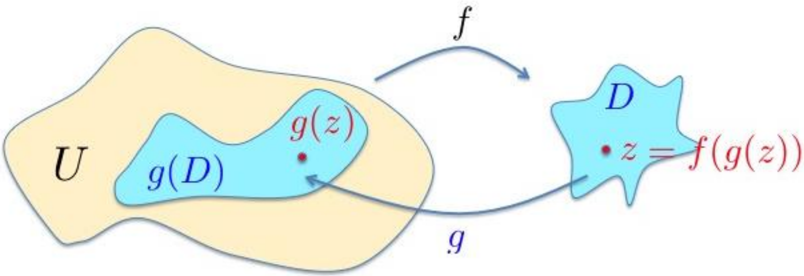
$$\begin{aligned} \text{Since } e^{\operatorname{Log} z} &= z, \quad \text{we find} \\ e^{\operatorname{Log} z} \cdot \frac{d}{dz} \operatorname{Log} z &= 1 \quad \text{so} \\ \frac{d}{dz} \operatorname{Log} z &= \frac{1}{z}. \end{aligned}$$

More General Theorem

Theorem

Suppose that $f : U \to \mathbb{C}$ is an analytic function and there exists a continuous function $g : D \to U$ from some domain $D \subset \mathbb{C}$ into U such that $f(g(z)) = z$ for all $z \in D$. Then g is analytic in D, and

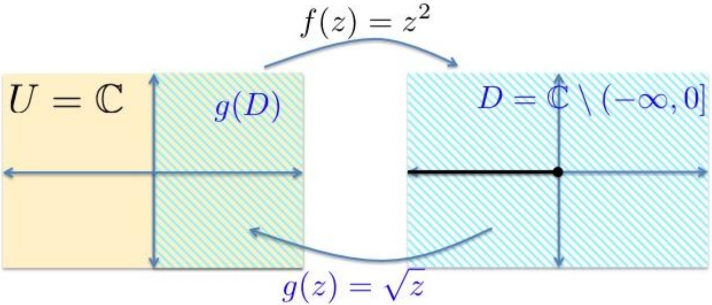
$$g'(z) = \frac{1}{f'(g(z))} \quad \text{for } z \in D.$$



Let $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z^2$. Then $f'(z) = 2z$.
Let $g : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$, $g(z) = \sqrt{z}$ be the principal branch of the square root.
Then

- $f(g(z)) = z$ for all $z \in D = \mathbb{C} \setminus (-\infty, 0]$
- g is continuous in D, thus
- g is analytic in D, and

$$\begin{aligned} g'(z) &= \frac{1}{f'(g(z))} \\ &= \frac{1}{2g(z)} \\ &= \frac{1}{2\sqrt{z}}. \end{aligned}$$



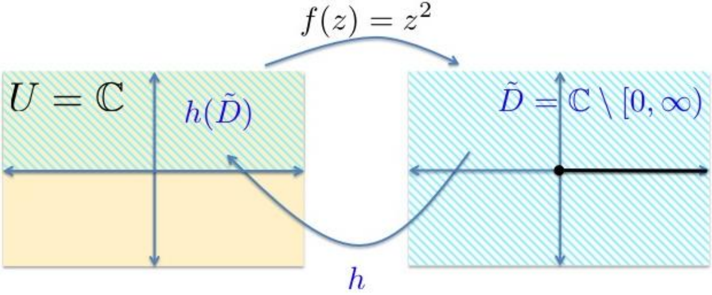
Again, let $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z^2$. Then $f'(z) = 2z$.

This time, let $h : \mathbb{C} \setminus [0, \infty) \to \mathbb{C}$, $h(z) = \begin{cases} \sqrt{z}, & \operatorname{Im} z \geq 0, \\ -\sqrt{z}, & \operatorname{Im} z < 0 \end{cases}$.

Then

- $f(h(z)) = z$ for all $z \in \tilde{D} = \mathbb{C} \setminus [0, \infty)$.
- h is continuous in $\tilde{D}$, thus
- h is analytic in $\tilde{D}$, and

$$h'(z) = \frac{1}{f'(h(z))} = \frac{1}{2h(z)}.$$



Some Terminology

Let's finish up by recalling some terminology: Let $f : U \to V$ be a function.

- f is *injective* (also called 1-1) provided that $f(a) \neq f(b)$ whenever $a, b \in U$ with $a \neq b$.
- f is *surjective* (also called onto) provided that for every $y \in V$ there exists an $x \in U$ such that $f(x) = y$.
- f is a *bijection* (also called 1-1 and onto) it f is both injective and surjective.

Examples:

- $f : \{z \in \mathbb{C} \,|\, \mathrm{Re}\, z > 0\} \to \mathbb{C} \setminus (-\infty, 0]$, $f(z) = z^2$ is a bijection.
- $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z^2$ is not injective but is surjective.
- $f : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$, $f(z) = \sqrt{z}$ is injective but not surjective.