Analysis of a Complex Kind

Week 3

Lecture 4: Complex Trigonometric Functions

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Motivation

Now that we have extended the exponential function to the complex plane, can we do the same thing for trigonometric functions?

- Recall: $e^{i\theta} = \cos\theta + i\sin\theta$.
- Therefore, $e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$.
- Hence $e^{i\theta} + e^{-i\theta} = 2\cos\theta$ and $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$.
- Thus $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$.

Definition

The *complex cosine and sine functions* are defined via

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \text{ and } \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Properties of Sine and Cosine

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \text{ and } \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

- $\sin z$ and $\cos z$ are analytic functions (in fact, entire).
- For real-valued z (i.e. $z = x + i \cdot 0$) the complex sine and cosine agree with the real-valued sine and cosine functions.
- $\cos(-z) = \dfrac{e^{-iz} + e^{iz}}{2} = \cos z.$
- $\sin(-z) = \dfrac{e^{-iz} - e^{iz}}{2i} = -\sin z.$
- $\cos(z + w) = \cos z \cos w - \sin z \sin w,$
  $\sin(z + w) = \sin z \cos w + \cos z \sin w.$

The Proofs of the Addition Formulae

Example: How do you prove that $\cos(z+w) = \cos z \cos w - \sin z \sin w$? Plug in the definitions and simplify!

$$
\begin{aligned}
& \cos z \cos w - \sin z \sin w \\
= \;& \left(\frac{e^{iz}+e^{-iz}}{2}\right)\left(\frac{e^{iw}+e^{-iw}}{2}\right) - \left(\frac{e^{iz}-e^{-iz}}{2i}\right)\left(\frac{e^{iw}-e^{-iw}}{2i}\right) \\
= \;& \frac{(e^{iz}+e^{-iz})(e^{iw}+e^{-iw}) + (e^{iz}-e^{-iz})(e^{iw}-e^{-iw})}{4} \\
= \;& \frac{e^{iz}e^{iw} + e^{iz}e^{-iw} + e^{-iz}e^{iw} + e^{-iz}e^{-iw} + e^{iz}e^{iw} - e^{iz}e^{-iw} - e^{-iz}e^{iw} + e^{-iz}e^{-iw}}{4} \\
= \;& \frac{2e^{iz}e^{iw} + 2e^{-iz}e^{-iw}}{4} \\
= \;& \frac{e^{i(z+w)} + e^{-i(z+w)}}{2} \\
= \;& \cos(z+w).
\end{aligned}
$$

Further Properties

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \text{ and } \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

- $\cos(z + 2\pi) = \dfrac{e^{i(z+2\pi)} + e^{-i(z+2\pi)}}{2} = \cos z$
  $\sin(z + 2\pi) = \sin z.$
- $\sin^2 z + \cos^2 z = 1$. Proof: Let $w = -z$ in the addition formula for cosine!
- $\sin(z + \frac{\pi}{2}) = \cos z$. Proof:

$$\begin{aligned} \sin\left(z + \frac{\pi}{2}\right) &= \frac{e^{i(z+\frac{\pi}{2})} - e^{-i(z+\frac{\pi}{2})}}{2i} \\ &= \frac{ie^{iz} - (-i)e^{-iz}}{2i} \\ &= \frac{e^{iz} + e^{-iz}}{2} = \cos z. \end{aligned}$$

The Zeros of Sine and Cosine

When is $\sin z = 0$? Remember: $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$.

$$\begin{aligned}
\sin z = 0 \quad &\Longleftrightarrow \quad e^{iz} = e^{-iz} \\
&\Longleftrightarrow \quad (iz) - (-iz) = 2k\pi i,\ k \in \mathbb{Z} \\
&\Longleftrightarrow \quad 2iz = 2k\pi i,\ k \in \mathbb{Z} \\
&\Longleftrightarrow \quad z = k\pi,\ k \in \mathbb{Z}.
\end{aligned}$$

Similarly: $\cos z = 0 \iff z = \dfrac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$.

Derivatives

What are the derivatives of sine and cosine?

$$
\begin{aligned}
\frac{d}{dz}\sin z &= \frac{d}{dz}\frac{e^{iz}-e^{-iz}}{2i} \\
&= \frac{ie^{iz}-(-i)e^{-iz}}{2i} \\
&= \frac{e^{iz}+e^{-iz}}{2} = \cos z.
\end{aligned}
$$

Similarly: $\dfrac{d}{dz}\cos z = -\sin z$.

Relation to Hyperbolic Functions

It is possible to express the complex sine and cosine solely in terms of the real sine and cosine as well as the real hyperbolic sine and cosine:

$$\begin{aligned}
\sin z &= \sin(x+iy) \\
&= \sin x \cos(iy) + \cos x \sin(iy) \\
&= \sin x \frac{e^{i(iy)} + e^{-i(iy)}}{2} + \cos x \frac{e^{i(iy)} - e^{-i(iy)}}{2i} \\
&= \sin x \frac{e^{-y} + e^{y}}{2} + \cos x \frac{e^{-y} - e^{y}}{2i} \\
&= \sin x \frac{e^{y} + e^{-y}}{2} + i \cos x \frac{e^{y} - e^{-y}}{2} \\
&= \sin x \cosh y + i \cos x \sinh y.
\end{aligned}$$

Similarly:

$$\cos z = \cos x \cosh y - i \sin x \sinh y.$$

Next Up...

Some amazing properties of analytic functions.

Example: If $f = u + iv$ is analytic in a domain and if u is constant throughout the domain, then f itself must be constant!