Analysis of a Complex Kind

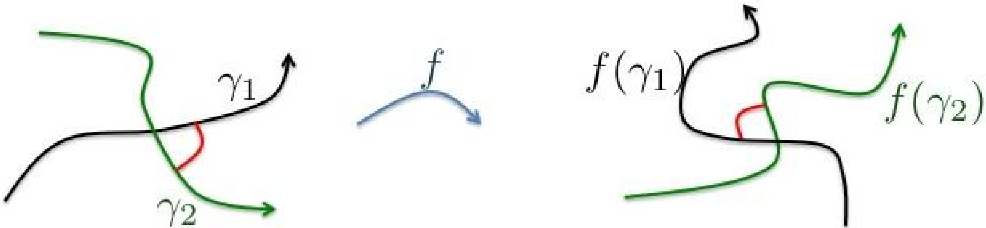
Week 4

Lecture 2: Conformal Mappings

Petra Bonfert-Taylor

What Is A Conformal Mapping?

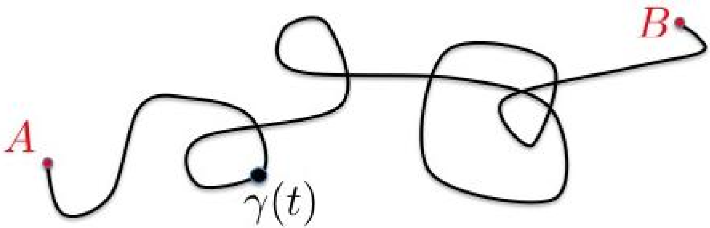
Intuitively, a conformal mapping is a “mapping that preserves angles between curves”.



To make this precise, we need to define curves as well as angles between curves.

Definition

A *path* in the complex plane from a point A to a point B is a continuous function $\gamma : [a, b] \to \mathbb{C}$ such that $\gamma(a) = A$ and $\gamma(b) = B$.



Examples:

1. $$\begin{aligned}\gamma(t) &= (2 + i) + e^{it},\ 0 \le t \le \pi \\ &= \underbrace{(2 + \cos t)}_{x(t)} + i\,\underbrace{(1 + \sin t)}_{y(t)}\end{aligned}$$

Examples

2. $$\begin{aligned}\gamma(t) &= (2+i) + t(-3-5i),\ 0 \le t \le 1 \\ &= (2-3t) + i(1-5t)\end{aligned}$$

3. $$\begin{aligned}\gamma(t) &= te^{it},\ 0 \le t \le 3\pi \\ &= (t\cos t) + i(t\sin t)\end{aligned}$$

4. $$\gamma(t) = \begin{cases} t(1+i), & 0 \le t \le 1 \\ t+i, & 1 < t \le 2 \\ 2+i(3-t), & 2 < t \le 3 \end{cases}$$

Curves

Definition

A path $\gamma : [a, b] \to \mathbb{C}$ is *smooth* if the functions $x(t)$ and $y(t)$ in the representation $\gamma(t) = x(t) + iy(t)$ are smooth, that is, have as many derivatives as desired.

In the above examples, (1), (2), and (3) are smooth, whereas (4) is *piecewise smooth*, i.e. put together ("concatenated") from finitely many smooth paths.

The term *curve* is typically used for a smooth or piecewise smooth path.

If $\gamma = x + iy : [a, b] \to \mathbb{C}$ is a smooth curve and $t_0 \in (a, b)$, then

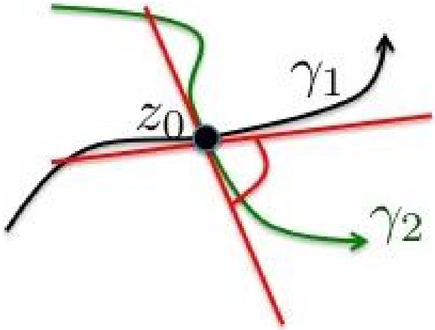
$$\gamma'(t_0) = x'(t_0) + iy'(t_0)$$

is a tangent vector to γ at $z_0 = \gamma(t_0)$.

The Angle Between Curves

Definition

Let γ_1 and γ_2 be two smooth curves, intersecting at a point z_0. The *angle* between the two curves at z_0 is defined as the angle between the two tangent vectors at z_0.



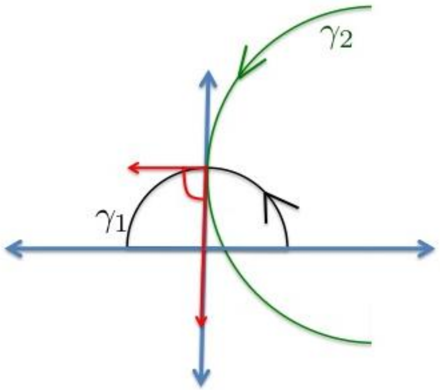
Example

Let $\gamma_1 : [0, \pi] \to \mathbb{C}$, $\gamma_1(t) = e^{it}$ and

$\gamma_2 : \left[\frac{\pi}{2}, \frac{3\pi}{2}\right) \to \mathbb{C}$, $\gamma_2(t) = 2 + i + 2e^{it}$.

Then $\gamma_1\left(\frac{\pi}{2}\right) = \gamma_2(\pi) = i$. Furthermore,

$$\begin{aligned}
\gamma_1'(t) = ie^{it}, \quad \gamma_1'\left(\frac{\pi}{2}\right) &= ie^{i\frac{\pi}{2}} = i^2 = -1, \\
\gamma_2'(t) = 2ie^{it}, \quad \gamma_2'(\pi) &= 2ie^{i\pi} = 2i(-1) \\
&= -2i.
\end{aligned}$$



The angle between these curves at i is thus $\frac{\pi}{2}$.

Conformality

Definition

A function is conformal if it preserves angles between curves. More precisely, a smooth complex-valued function g is *conformal at z_0* if whenever γ_1 and γ_2 are two curves that intersect at z_0 with non-zero tangents, then $g \circ \gamma_1$ and $g \circ \gamma_2$ have non-zero tangents at $g(z_0)$ that intersect at the same angle.

A *conformal mapping* of a domain D onto V is a continuously differentiable mapping that is conformal at each point in D and maps D one-to-one onto V.

Theorem

If $f : U \to \mathbb{C}$ is analytic and if $z_0 \in U$ such that $f'(z_0) \neq 0$, then f is conformal at z_0.

Reason: If $\gamma : [a, b] \to U$ is a curve in U with $\gamma(t_0) = z_0$ for some $t_0 \in (a, b)$, then

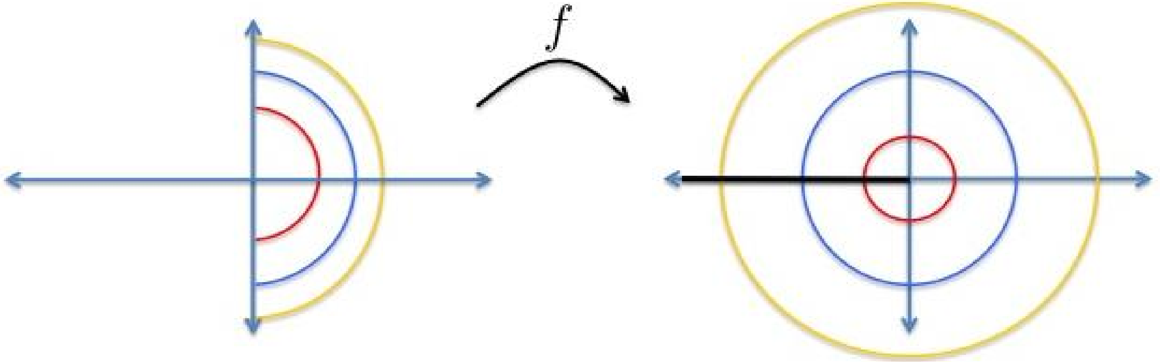
$$(f \circ \gamma)'(t_0) = f'(\gamma(t_0)) \cdot \gamma'(t_0) = \underbrace{f'(z_0)}_{\in \mathbb{C} \setminus \{0\}} \cdot \gamma'(t_0).$$

Thus $(f \circ \gamma)'(t_0)$ is obtained from $\gamma'(t_0)$ via multiplication by $f'(z_0)$ (= rotation & stretching).

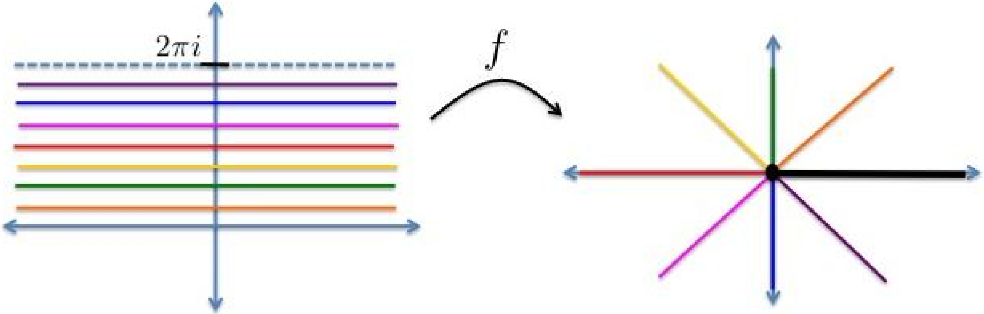
If γ_1, γ_2 are two curves in U through z_0 with tangent vectors $\gamma_1'(t_1)$ and $\gamma_2'(t_2)$, then $(f \circ \gamma_1)'(t_1)$ and $(f \circ \gamma_2)'(t_2)$ are both obtained from $\gamma_1'(t_1)$ and $\gamma_2'(t_2)$, respectively, via multiplication by $f'(z_0)$. The angle between them is thus preserved.

Examples

- $f(z) = z^2$ maps $U = \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$ conformally onto $\mathbb{C} \setminus (-\infty, 0]$.

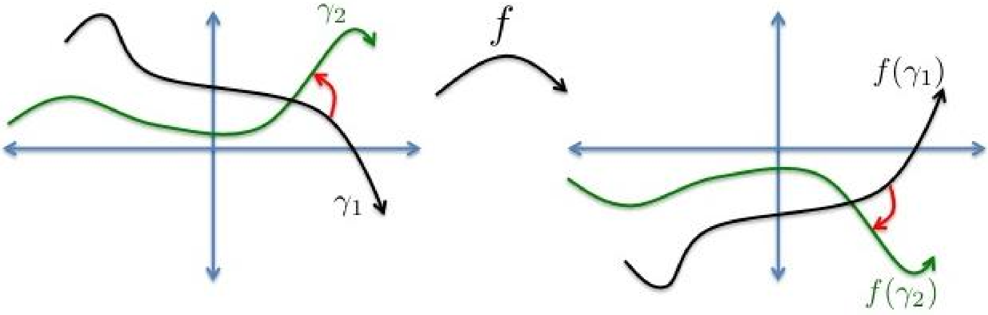


- $f(z) = e^z$ is conformal at each point in $\mathbb{C}$ (f is analytic in $\mathbb{C}$ and $f'(z) \neq 0$ in $\mathbb{C}$). Since f is not one-to-one in $\mathbb{C}$, it is not a conformal mapping from $\mathbb{C}$ onto $\mathbb{C} \setminus \{0\}$.
  However, if you choose $D = \{z \,|\, 0 < \operatorname{Im} z < 2\pi\}$, then f maps D conformally onto $f(D) = \mathbb{C} \setminus [0, \infty)$.



More Examples

- $f(z) = \overline{z}$ is one-to-one and onto from $\mathbb{C}$ to $\mathbb{C}$, however, angles between curves are reversed in orientation:



f is thus not conformal anywhere.