Analysis of a Complex Kind

Week 4

Lecture 5: The Riemann Mapping Theorem

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Conformal Mappings

So far we have seen the following:

- The conformal mappings from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$ are of the form $z \mapsto \dfrac{az+b}{cz+d}$.
- The conformal mappings from $\mathbb{C}$ to $\mathbb{C}$ are of the form $z \mapsto az+b$.

More is true:

- There is no conformal mapping $f : \mathbb{C} \to D$, where $D \subset \mathbb{C}$, $D \neq \mathbb{C}$.
- There is no conformal mapping $f : \hat{\mathbb{C}} \to D$, where $D \subset \mathbb{C}$.

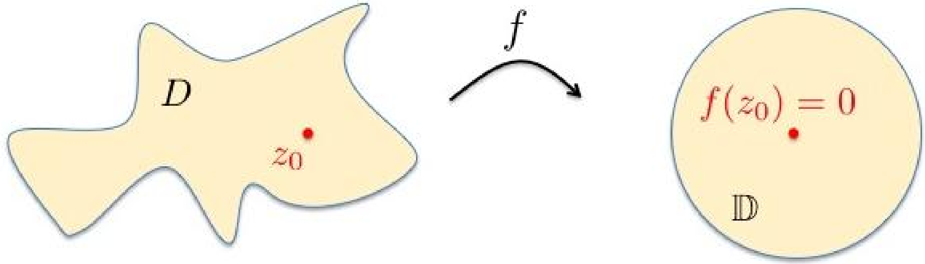
Question

What conformal mappings are there of the form $f : \mathbb{D} \to D$, where $\mathbb{D} = B_1(0)$ is the unit disk and $D \subset \mathbb{C}$?

The Riemann Mapping Theorem

Theorem

If D is a simply connected domain (= open, connected, no holes) in the complex plane, but not the entire complex plane, then there is a conformal map (= analytic, one-to-one, onto) of D onto the open unit disk $\mathbb{D}$.



We say that “D is conformally equivalent to $\mathbb{D}$.”

The Riemann Map

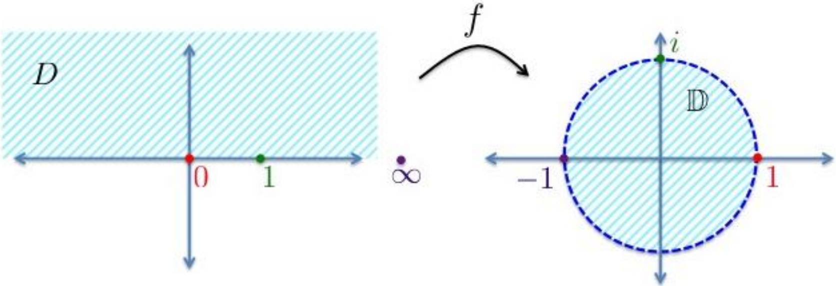
Let D be a simply connected domain. In order to find a *unique* conformal mapping f from D onto $\mathbb{D}$, we need to specify “3 real parameters”. For example, specify

- a point $z_0 \in D$ that is to be mapped to 0 under f (= 2 real parameters x_0, y_0)
- the argument of $f'(z_0)$ (= 1 real parameter), for example by requiring that $f'(z_0) > 0$.

The proof of the Riemann mapping theorem is beyond the scope of this course.

Instead, we'll look at some examples and applications.

1. Let D be the upper half plane, i.e. $D = \{z : \operatorname{Im} z > 0\}$. Then D can be mapped to $\mathbb{D}$ via (the restriction of) a Möbius transformation: Let f be the Möbius transformation that maps $0, 1, \infty$ to $1, i, -1$:



Then the line through $0, 1, \infty$ (the real axis!) must be mapped to the circle through $1, i, -1$ (the unit circle). Further, the domain to the left of the real axis (D) is then mapped to the domain to the left of the unit circle ($\mathbb{D}$), oriented by the ordering of the given points.

The Upper Half Plane - Finding the Riemann Map

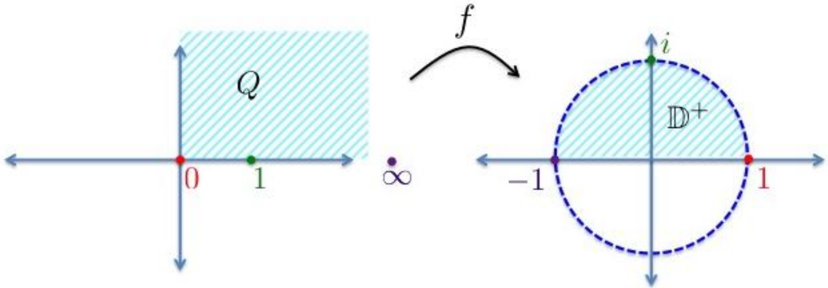
The restriction of the Möbius transformation f to the upper half plane D thus maps D onto $\mathbb{D}$. Can we find a formula for f? f maps $0, 1, \infty$ to $1, i, -1$.

- f is of the form $f(z) = \dfrac{az+b}{cz+d}$. Since $f(\infty) \neq \infty$, we have $c \neq 0$ and can thus assume $c = 1$.
- Thus $f(z) = \dfrac{az+b}{z+d}$. Since $f(\infty) = -1$ we conclude that $a = -1$.
- Then $f(z) = \dfrac{-z+b}{z+d}$. Since $f(0) = 1$ we have $\dfrac{b}{d} = 1$, so $b = d$.
- So $f(z) = \dfrac{-z+b}{z+b}$. Since $f(1) = i$ we have $\dfrac{-1+b}{1+b} = i$, so $b = i$.

Thus $f(z) = \dfrac{-z+i}{z+i}$ maps the upper half plane D conformally onto the unit disk $\mathbb{D}$.

The First Quadrant to the Upper Half of the Unit Disk

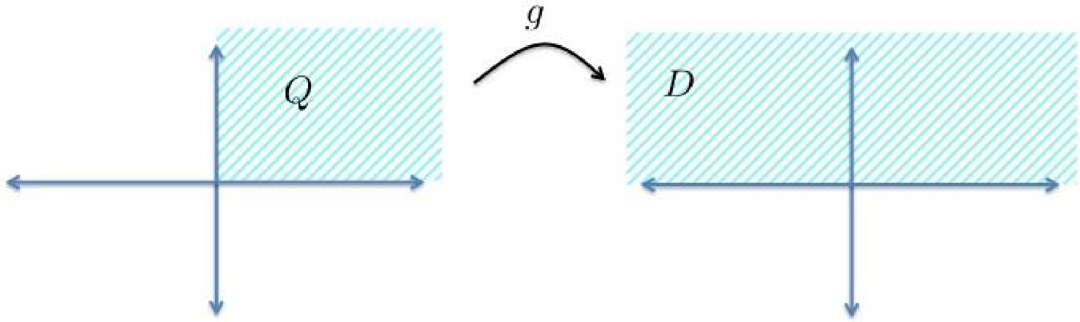
2. Let Q be the first quadrant, i.e. the domain in the complex plane, bounded by the positive real axis and the positive imaginary axis. Since the map f from the previous example maps 0 to 1, i to 0, and ∞ to -1, it maps the line through $0, i, \infty$ (i.e. the imaginary axis) to the line through $1, 0, -1$ (i.e. the real



axis).
Hence the the restriction of f to Q maps Q conformally onto the upper half of the unit disk, $\mathbb{D}^+$.

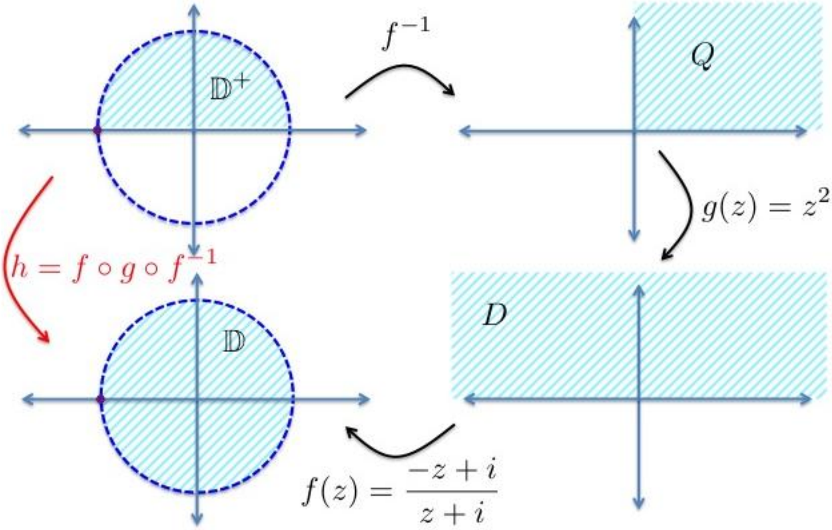
The First Quadrant to the Upper Half Plane

3. The map $g(z) = z^2$ is injective and analytic in the first quadrant Q.



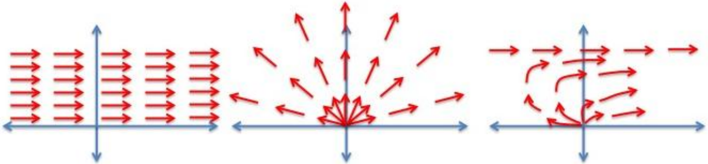
g maps Q conformally onto its image, namely the upper half plane D.

The previous three examples help us construct the Riemann map from $\mathbb{D}^+$ to $\mathbb{D}$:



Applications

- Many problems are easier to solve in the unit disk (or some other "nice" standard region) than in the region they are formulated in.
- Solutions can be found in the standard region, then transported back to the original region via a Riemann map.
- Example: Fluid flow can be modeled nicely in the upper half plane:



a flow in x-direction | a flow from a source at the origin | a combination of the two

- To understand a similar fluid flow in another region, map this flow from the upper half plane to the desired region using the Riemann map.
- Other examples: electrostatics, heat conduction, aerodynamics, etc.