Analysis of a Complex Kind

Week 1

Lecture 5: Topology in the Plane

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- Circles and disks: center $z_0 = x_0 + iy_0$, radius r.
  - $B_r(z_0) = \{z \in \mathbb{C} : z$ has distance less than r from $z_0\}$ *disk of radius r, centered at z_0.*
  - $K_r(z_0) = \{z \in \mathbb{C} : z$ has distance r from $z_0\}$ *circle of radius r, centered at z_0.*
- How do we measure distance?

- $$d = \sqrt{(x-x_0)^2 + (y-y_0)^2} \\ = |(x-x_0) + i(y-y_0)| \\ = |z - z_0|.$$
- So $B_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$ and $K_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}$.

Interior Points and Boundary Points

Definition

Let $E \subset \mathbb{C}$. A point z_0 is an *interior point of E* if there is some $r > 0$ such that $B_r(z_0) \subset E$.

Definition

Let $E \subset \mathbb{C}$. A point b is a boundary point of E if every disk around b contains a point in E and a point not in E.
The *boundary* of the set $E \subset \mathbb{C}$, ∂E, is the set of all boundary points of E.

Open and Closed Sets

Definition

A set $U \subset \mathbb{C}$ is *open* if every one of its points is an interior point.
A set $A \subset \mathbb{C}$ is *closed* if it contains all of its boundary points.

Examples:

- $\{z \in \mathbb{C} : |z - z_0| < r\}$ and $\{z \in \mathbb{C} : |z - z_0| > r\}$ are open.
- $\mathbb{C}$ and $\emptyset$ are open.
- $\{z \in \mathbb{C} : |z - z_0| \leq r\}$ and $\{z \in \mathbb{C} : |z - z_0| = r\}$ are closed.
- $\mathbb{C}$ and $\emptyset$ are closed.
- $\{z \in \mathbb{C} : |z - z_0| < r\} \cup \{z \in \mathbb{C} : |z - z_0| = r \text{ and } \operatorname{Im}(z - z_0) > 0\}$ is neither open nor closed.

Definition

Let E be a set in $\mathbb{C}$.
The *closure* of E is the set E together with all of its boundary points: $\overline{E} = E \cup \partial E$.
The *interior* of E, $\mathring{E}$ is the set of all interior points of E.

Examples:

- $\overline{B_r(z_0)} = B_r(z_0) \cup K_r(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq r\}$.
- $\overline{K_r(z_0)} = K_r(z_0)$.
- $\overline{B_r(z_0) \setminus \{z_0\}} = \{z \in \mathbb{C} : |z - z_0| \leq r\}$.
- With $E = \{z \in \mathbb{C} : |z - z_0| \leq r\}$, $\mathring{E} = B_r(z_0)$.
- With $E = K_r(z_0)$, $\mathring{E} = \emptyset$.

Connectedness

Intuitively: A set is connected if it is “in one piece”. How do we make this precise?

Definition

Two sets X, Y in $\mathbb{C}$ are *separated* if there are disjoint open set U, V so that $X \subset U$ and $Y \subset V$. A set W in $\mathbb{C}$ is *connected* if it is impossible to find two separated non-empty sets whose union equals W.

Example:

$$X = [0, 1) \quad \text{and} \quad Y = (1, 2]$$

are separated: For example, choose $U = B_1(0)$ and $V = B_1(2)$. Thus

$$X \cup Y = [0, 2] \setminus \{1\}$$

is not connected. It is hard to check whether a set is connected!!!

Connectedness for Open Sets in $\mathbb{C}$

For open sets, there is a much easier criterion to check whether or not a set is connected:

Theorem

Let G be an open set in $\mathbb{C}$. Then G is connected if and only if any two points in G can be joined in G by successive line segments.

Bounded Sets

Definition

A set A in $\mathbb{C}$ is *bounded* if there exists a number $R > 0$ such that $A \subset B_R(0)$. If no such R exists then A is called *unbounded*.

The Point at Infinity

- In $\mathbb{R}$, there are two directions that give rise to $\pm\infty$.

$$1, 2, 3, 4, 5, \ldots \to \infty; \qquad -1, -2, -3, -4, -5, \ldots \to -\infty.$$

- In $\mathbb{C}$, there is only one ∞ which can be attained in many directions.

$$\left.\begin{array}{l} 1,\ 2,\ 3,\ \ldots \\ -1,\ -2,\ -3,\ \ldots \\ i,\ 2i,\ 3i,\ \ldots \\ 1,\ 2i,\ -3,\ -4i,\ 5,\ 6i,\ -7,\ \ldots \\ \vdots \end{array}\right\} \longrightarrow \infty$$