Analysis of a Complex Kind

Week 3

Lecture 5: First Properties of Analytic Functions

Petra Bonfert-Taylor

Analytic Functions with Zero Derivative

Theorem

If f is analytic on a domain D, and if $f'(z) = 0$ for all $z \in D$, then f is constant in D.

Recall the 1-dimensional analog:

Fact

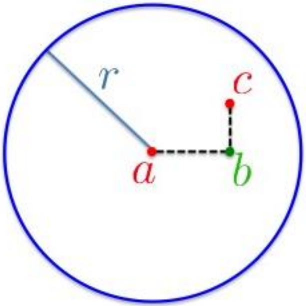
If $f : (a, b) \to \mathbb{R}$ is differentiable and satisfies $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on (a, b).

Idea of proof:

1. First show that f is constant in any disk $B_r(a)$, contained in D, using the 1-dimensional fact.
2. Second, use the fact that D is connected and the first fact to show that f is constant in D.

Here are the main steps in these proofs:

1. Let $B_r(a)$ be a disk contained in D, and let $c \in B_r(a)$. Pick the point $b \in B_r(a)$ as in the picture.

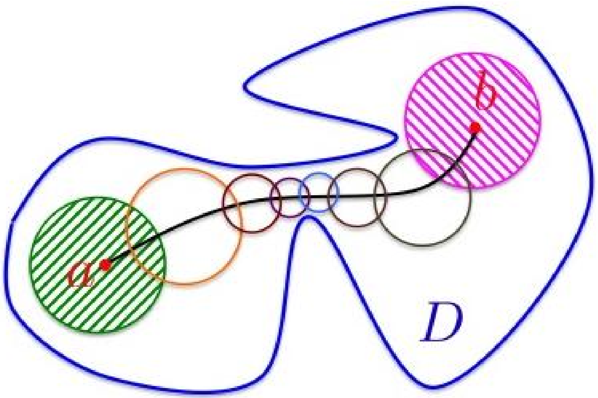


Since $f'(z) = 0$ in D we have $u_x = u_y = v_x = v_y = 0$ in D. In particular, look at u on the horizontal line segment from a to b. It depends only on one parameter (namely x) there, and $u_x = 0$. By the 1-dimensional fact, we find that u is constant on the line segment, in particular, $u(a) = u(b)$. Similarly, $u(b) = u(c)$, thus $u(a) = u(c)$.. Since c was an arbitrary point in $B_r(a)$, u is thus constant in $B_r(a)$. Similarly, v, is constant in $B_r(a)$, thus f is constant in $B_r(a)$.

Analytic Functions with Zero Derivative, cont.

Here is the second step of the proof:

2. Let a and b be two arbitrary points in D. Since D is connected, there exist a nice curve in D, joining a and b.



By the previous step, f is constant in the disk around the point a (see picture). Furthermore, f is also constant in the neighboring disk. Since these two disks overlap, the two constants must agree. Continue on in this matter until you reach b. Therefore, $f(a) = f(b)$. Thus f is constant in D.

Consequences

The previous theorem, together with the Cauchy-Riemann equations, has strong consequences.

- Suppose that $f = u + iv$ is analytic in a domain D. Suppose furthermore, that u is constant in D. Then f must be constant in D.

  Proof: u constant in D implies that $u_x = u_y = 0$ in D. Since f is analytic, the Cauchy-Riemann equations now imply that $v_x = v_y = 0$ as well. Thus $f' = u_x + iv_x = 0$ in D. By our theorem, f is constant in D.
- Similarly, if $f = u + iv$ is analytic in a domain D with v being constant, then f must be constant in D.
- Suppose next that $f = u + iv$ is analytic in a domain D with $|f|$ being constant in D. This too implies that f itself must be constant!

  Let me show you why:

Let $f = u + iv$ be analytic in a domain D, and suppose that $|f|$ is constant in D. Then $|f|^2$ is also constant, i.e. there exists $c \in \mathbb{C}$ such that

$$|f(z)|^2 = u^2(z) + v^2(z) = c \quad \text{for all } z \in D.$$

- If $c = 0$ then u and v must be equal to zero everywhere, and so f is equal to zero in D.
- If $c \neq 0$ then in fact $c > 0$. Taking the partial derivative with respect to x (and similarly with respect to y) of the above equation yields:

  $$2uu_x + 2vv_x = 0 \quad \text{and} \quad 2uu_y + 2vv_y = 0.$$

  Substituting $v_x = -u_y$ in the first and $v_y = u_x$ in the second equation gives

  $$uu_x - vu_y = 0 \quad \text{and} \quad uu_y + vu_x = 0.$$

  Multiplying the first equation by u and the second by v we find

  $$u^2u_x - uvu_y = 0 \quad \text{and} \quad uvu_y + v^2u_x = 0.$$

Analytic Functions with Constant Norm, cont.

$$u^2 u_x - uvu_y = 0 \quad \text{and} \quad uvu_y + v^2 u_x = 0.$$

We now add these two equations! The result:

$$u^2 u_x + v^2 u_x = 0.$$

Since $u^2 + v^2 = c$, this last equation becomes

$$cu_x = 0.$$

But $c > 0$, so it must be the case that $u_x = 0$ in D. We can similarly find that $u_y = 0$ in D, and using the Cauchy-Riemann equations we also obtain that $v_x = v_y = 0$ in D. Hence $f'(z) = 0$ in D, and our theorem yields that f is constant in D.

Note: The assumption of D being connected is important!

A Strange Example

Let's look at a strange example: Let

$$f : \mathbb{C} \to \mathbb{C}, \quad f(z) = \begin{cases} e^{-\frac{1}{z^4}}, & z \neq 0 \\ 0, & z = 0. \end{cases}$$

- One can find u, v, u_x, u_y, v_x, v_y and they actually satisfy the Cauchy-Riemann equations in $\mathbb{C}$.
- Clearly, f is analytic in $\mathbb{C} \setminus \{0\}$.
- At the origin, one can show that $u_x(0) = u_y(0) = v_x(0) = v_y(0) = 0$.
- However, f is not differentiable at 0. How is this possible?
- The function f isn't even continuous at the origin!

A Strange Example, cont.

$$f : \mathbb{C} \to \mathbb{C}, \quad f(z) = \begin{cases} e^{-\frac{1}{z^4}}, & z \neq 0 \\ 0, & z = 0. \end{cases}$$

Why is f not continuous at the origin?

- Consider z approaching the origin along the real axis, i.e. $z = x + i \cdot 0 \to 0$. Then
$$f(z) = f(x) = e^{\frac{-1}{x^4}} \to 0.$$
- Next, consider z approaching the origin along the imaginary axis, i.e. $z = 0 + iy \to 0$. Then
$$f(z) = f(iy) = e^{\frac{-1}{i^4 y^4}} \to 0.$$
So far so good?
- However... Consider $z = re^{i\frac{\pi}{4}} \to 0$. Then $z^4 = r^4 e^{i\frac{\pi}{4}\cdot 4} = -r^4$, so
$$f(z) = e^{-\frac{1}{-r^4}} = e^{\frac{1}{r^4}} \to \infty \neq f(0).$$

A Strange Example, cont.

What happened? The functions u and v satisfy the Cauchy-Riemann equations, yet, f is not differentiable at the origin?

Recall our theorem:

Theorem

Let $f = u + iv$ be defined on a domain $D \subset \mathbb{C}$. Then f is analytic in D if and only if $u(x, y)$ and $v(x, y)$ have continuous first partial derivatives on D that satisfy the Cauchy-Riemann equations.

In our strange example, the partial derivatives are not continuous at 0, so the assumptions of the theorem are not satisfied.

Next...

Next week:

Conformal mappings and Möbius transformations.
Bonus: The Riemann mapping theorem.