Analysis of a Complex Kind

Week 1

Lecture 3: Polar Representation of Complex Numbers

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Polar Coordinates

- Consider $z = x + iy \in \mathbb{C}$, $z \neq 0$.
- z can also be described by the distance r from the origin ($r = |z|$) and the angle θ between the positive x-axis and the line segment from 0 to z.
- (r, θ) are the polar coordinates of z.
- Relation between Cartesian and polar coordinates:
  - $x = r\cos\theta$
  - $y = r\sin\theta$
- ... so: $$\begin{aligned} z &= x + iy \\ &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta). \end{aligned}$$
  This is called the *Polar representation of z.*

The Argument of a Complex Number

- $z = x + iy = r(\cos\theta + i\sin\theta)$.
- $r = |z|$ is easy to find, but how to find θ? Note: θ is not unique!

Definition

The *principal argument of z*, called $\operatorname{Arg} z$, is the value of θ for which $-\pi < \theta \leq \pi$.

- $\arg z = \{\operatorname{Arg} z + 2\pi k : k = 0, \pm 1, \pm 2, \ldots\}, z \neq 0$.
- Examples:
  - $\operatorname{Arg} i = \frac{\pi}{2}$,
  - $\operatorname{Arg} 1 = 0$,
  - $\operatorname{Arg}(-1) = \pi$,
  - $\operatorname{Arg}(1 - i) = -\frac{\pi}{4}$,
  - $\operatorname{Arg}(-i) = -\frac{\pi}{2}, \ldots$

Exponential Notation

- Convenient notation: $e^{i\theta} = \cos\theta + i\sin\theta$.
- So $z = r(\cos\theta + i\sin\theta)$ becomes $z = re^{i\theta}$, the *polar form of z*.
- Note: $e^{i(\theta+2\pi)} = e^{i\theta} = e^{i(\theta+4\pi)} = \cdots = e^{i(\theta+2k\pi)}, k \in \mathbb{Z}$.
- Examples
  - $e^{i\frac{\pi}{2}} = i$,
  - $e^{i\pi} = -1$,
  - $e^{i\frac{\pi}{4}} = \frac{1+i}{\sqrt{2}}$,
  - $e^{2\pi i} = 1, \ldots$

Properties of the Exponential Notation

- $|e^{i\theta}| = 1$.
- $\overline{e^{i\theta}} = e^{-i\theta}$.
- $\frac{1}{e^{i\theta}} = e^{-i\theta}$.
- $e^{i(\theta+\varphi)} = e^{i\theta} \cdot e^{i\varphi}$.

Conclusions for the Argument Function

- $\arg(\overline{z}) = -\arg z$.
- $\arg(\frac{1}{z}) = -\arg z$.
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$.
- Examples:
  - $\arg(i \cdot i) = \arg(-1) = \pi$, $\arg(i) + \arg(i) = \frac{\pi}{2} + \frac{\pi}{2} = \pi$.
  - $\arg((-1)(-1)) = \arg(1) = 0$, $\arg(-1) + \arg(-1) = \pi + \pi = 2\pi$.

Multiplication in Polar Form

- Consider $z_1 = r_1 e^{i\varphi_1}$ and $z_2 = r_2 e^{i\varphi_2}$. What is the polar form of $z_1 z_2$?
- $z_1 z_2 = r_1 e^{i\varphi_1} r_2 e^{i\varphi_2} = (r_1 r_2) e^{i(\varphi_1 + \varphi_2)}$.

De Moivre's Formula

- $e^{i\theta} \cdot e^{i\theta} = e^{i(\theta+\theta)} = e^{i \cdot 2\theta}$.
- $(e^{i\theta})^3 = e^{i \cdot 3\theta}$.

- $(e^{i\theta})^n = e^{in\theta}$ (also true for negative n).
- Recall that $e^{i\theta}$ is simply short for $\cos\theta + i \sin\theta$. Thus this last formula means:
- $(\cos\theta + i \sin\theta)^n = \cos(n\theta) + i \sin(n\theta)$.

Consequences of De Moivre's Formula

- $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$.
- This can be used to derive equations for sine and cosine. Ex: $n = 3$:
  - $\cos(3\theta) = \cos^3\theta - 3\cos\theta\sin^2\theta$.
  - $\sin(3\theta) = 3\cos^2\theta\sin\theta - \sin^3\theta$.