Analysis of a Complex Kind

Week 2

Lecture 2: Sequences and Limits of Complex Numbers

Petra Bonfert-Taylor

Consider the following sequences of complex numbers. What happens far out along the sequence?

- $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots, \frac{1}{n}, \ldots \to ?$
- $i, \frac{i}{2}, \frac{i}{3}, \frac{i}{4}, \frac{i}{5}, \frac{i}{6}, \ldots, \frac{i}{n}, \ldots \to ?$
- $i, \frac{-1}{2}, \frac{-i}{3}, \frac{1}{4}, \frac{i}{5}, \frac{-1}{6}, \ldots, \frac{i^n}{n}, \ldots \to ?$

Informally, a sequence $\{s_n\}$ converges to a limit s if the sequence eventually lies in any (every so small) disk centered at s.

How do you make this mathematically precise?

Limits

Definition

A sequence $\{s_n\}$ of complex numbers *converges to* $s \in \mathbb{C}$ if for every $\varepsilon > 0$ there exists an index $N \geq 1$ such that

$$|s_n - s| < \varepsilon \quad \text{for all } n \geq N.$$

In this case we write

$$\lim_{n\to\infty} s_n = s.$$

Examples:

- $\displaystyle\lim_{n\to\infty} \frac{1}{n} = 0.$

Examples

- $\lim_{n\to\infty} \frac{1}{n^p} = 0$ for any $0 < p < \infty$.
- $\lim_{n\to\infty} \frac{c}{n^p} = 0$ for any $c \in \mathbb{C}$, $0 < p < \infty$.
- $\lim_{n\to\infty} q^n = 0$ for $0 < q < 1$.
- $\lim_{n\to\infty} z^n = 0$ for $|z| < 1$.
- $\lim_{n\to\infty} \sqrt[n]{10} = 1$.
- $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Rules for Limits

1. Convergent sequences are bounded.

2. If $\{s_n\}$ converges to s and $\{t_n\}$ converges to t, then
   - $s_n + t_n \to s + t$.
   - $s_n \cdot t_n \to s \cdot t$ (in particular: $a \cdot s_n \to a \cdot s$ for any $a \in \mathbb{C}$.)
   - $\frac{s_n}{t_n} \to \frac{s}{t}$, provided $t \neq 0$.

Examples

- $\dfrac{n}{n+1} = \dfrac{1}{1+\frac{1}{n}} \to 1$ as $n \to \infty$.
- $\dfrac{3n^2+5}{in^2+2in-1} = \dfrac{3+\frac{5}{n^2}}{i+\frac{2i}{n}-\frac{1}{n^2}} \to \dfrac{3}{i} = -3i$ as $n \to \infty$.
- $\dfrac{n^2}{n+1} = \dfrac{n}{1+\frac{1}{n}}$ not bounded.
- $\dfrac{3n+5}{in^2+2in-1} = \dfrac{\frac{3}{n}+\frac{5}{n^2}}{i+\frac{2i}{n}-\frac{1}{n^2}} \to \dfrac{0}{i} = 0$ as $n \to \infty$.

How about . . . ?

Consider the sequence

$$\left\{\frac{i^n}{n}\right\} = i, \frac{-1}{2}, \frac{-i}{3}, \frac{1}{4}, \frac{i}{5}, \frac{-1}{6}, \dots$$

This sequence seems to converge to 0, but how do we show this? The previous rules don't seem to apply.

Facts:

- A sequence of complex numbers, $\{s_n\}$, converges to 0 if and only if the sequence $\{|s_n|\}$ of absolute values converges to 0.
- A sequence of complex numbers, $\{s_n\}$, with $s_n = x_n + iy_n$, converges to $s = x + iy$ if and only if $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

Some Facts about Sequence of Real Numbers

Here is a really neat fact, often called the “Squeeze Theorem”:

Theorem

Suppose that $\{r_n\}$, $\{s_n\}$ and $\{t_n\}$ are sequences of real numbers such that $r_n \leq s_n \leq t_n$ for all n. If both sequences $\{r_n\}$ and $\{t_n\}$ converge to the same limit, L, then the sequence $\{s_n\}$ has not choice but to converge to the limit L as well.

And here is the equivalent of a sequence running against a wall:

Theorem

A bounded, monotone sequence of real numbers converges.

Applying These New Facts...

Let's apply the facts that we learned on the last two slides to the sequence $\left\{ \frac{i^n}{n} \right\}$.

- $\left| \frac{i^n}{n} \right| = \frac{|i|^n}{n} = \frac{1}{n} \to 0$ as $n \to \infty$. Thus $\lim\limits_{n\to\infty} \frac{i^n}{n} = 0$. Or:
- $\frac{i^n}{n} = x_n + iy_n$, $x_n = \frac{1}{n} \begin{cases} 0, & n \text{ odd} \\ 1, & n = 4k \\ -1, & n = 4k+2 \end{cases}$ and $y_n = \frac{1}{n} \begin{cases} 0, & n \text{ even} \\ 1, & n = 4k+1 \\ -1, & n = 4k+3 \end{cases}.$

  Since $-\frac{1}{n} \le x_n \le \frac{1}{n}$ and $-\frac{1}{n} \le y_n \le \frac{1}{n}$ for all n, the Squeeze Theorem implies that $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = 0$, hence $\lim\limits_{n\to\infty} \frac{i^n}{n} = 0$.

Limits of Complex Functions

Definition

The complex-valued function $f(z)$ has limit L as $z \to z_0$ if the values of $f(z)$ are near L as $z \to z_0$.

(More formally: $\lim_{z \to z_0} f(z) = L$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(z) - L| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.)

Note: Of course $f(z)$ needs to be defined near z_0 for this definition to make sense (but not necessarily at z_0). Examples:

- $f(z) = \dfrac{z^2 - 1}{z - 1}$, $z \neq 1$. Then

$$\lim_{z \to 1} f(z) = \lim_{z \to 1} \frac{(z-1)(z+1)}{z-1} = \lim_{z \to 1} z + 1 = 2.$$

Examples

Let $f(z) = \operatorname{Arg} z$. Then:

- $\lim_{z \to i} \operatorname{Arg} z = \frac{\pi}{2}$.
- $\lim_{z \to 1} \operatorname{Arg} z = 0$.
- $\lim_{z \to -1} \operatorname{Arg} z = ?$

Some Facts About Limits of Functions

The previous facts about limits of sequences imply the following facts about limits of functions:

- If f has a limit at z_0 then f is bounded near z_0.
- If $f(z) \to L$ and $g(z) \to M$ as $z \to z_0$ then
  - $f(z) + g(z) \to L + M$ as $z \to z_0$,
  - $f(z) \cdot g(z) \to L \cdot M$ as $z \to z_0$,
  - $\dfrac{f(z)}{g(z)} \to \dfrac{L}{M}$ as $z \to z_0$, provided that $M \neq 0$.

Continuity

Definition

The function f is continuous at z_0 if $f(z) \to f(z_0)$ as $z \to z_0$.

Note: This definition implicitly says that:

- f is defined at z_0.
- f has a limit as $z \to z_0$.
- The limit equals $f(z_0)$.

Examples of continuous functions:

- constant functions
- $f(z) = z$
- polynomials
- $f(z) = |z|$
- $f(z) = \dfrac{p(z)}{q(z)}$, wherever $q(z) \neq 0$ (p and q are polynomials).