Analysis of a Complex Kind

Week 4

Lecture 3: Möbius transformations, Part 1

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Möbius Transformations

Definition

A Möbius transformation (also called fractional linear transformation) is a function of the form

$$f(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$.

Notes:

- As $z \to \infty$, $f(z) \to \dfrac{a}{c}$ if $c \neq 0$ and $f(z) \to \infty$ if $c = 0$.

  We thus allow $z = \infty$ and define $f(\infty) = \dfrac{a}{c}$ if $c \neq 0$ and $f(\infty) = \infty$ if $c = 0$.
- Similarly, $f\left(-\dfrac{d}{c}\right) = \infty$, if $c \neq 0$.
- We thus regard f as a mapping from the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to the extended plane $\hat{\mathbb{C}}$.

Properties of $f(z) = \frac{az+b}{cz+d}$, $\quad a, b, c, d \in \mathbb{C}$, $\quad ad - bc \neq 0$.

- $f'(z) = \dfrac{(cz+d)a - (az+b)c}{(cz+d)^2} = \dfrac{ad-bc}{(cz+d)^2}$. The condition $ad - bc \neq 0$ thus simply guarantees that f is non-constant.
- If we multiply each parameter a, b, c, d by a constant $k \neq 0$, we obtain the same mapping, so a given mapping does not uniquely determine a, b, c, d.
- A Möbius transformation is one-to-one and onto from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$.
  To prove this, pick $w \in \hat{\mathbb{C}}$ and observe

$$\begin{aligned} f(z) = w \quad &\Longleftrightarrow \quad az + b = w(cz+d) \\ &\Longleftrightarrow \quad z(a - wc) = wd - b \\ &\Longleftrightarrow \quad z = \frac{wd-b}{-wc+a}. \end{aligned}$$

For each $w \in \hat{\mathbb{C}}$ there is one and only one $z \in \hat{\mathbb{C}}$ such that $f(z) = w$.

Möbius transformations are thus conformal mappings from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. In fact, Möbius transformations are the only conformal mappings from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$.

Examples:

- $c = 0, d = 1$: Then $f(z) = az + b$ ($a \neq 0$). These are also called affine transformations. They map ∞ to ∞ and therefore map $\mathbb{C}$ to $\mathbb{C}$. They are hence conformal mappings from $\mathbb{C}$ to $\mathbb{C}$ (and in fact, these are the only conformal mappings from $\mathbb{C}$ to $\mathbb{C}$).
    - $f(z) = az$ (i.e. $b = 0$) is a rotation & dilation.
    - $f(z) = z + b$ (i.e. $a = 1$) is a translation.

More Examples

- $a = 0, b = 1, c = 1, d = 0$: Then $f(z) = \dfrac{1}{z}$. This is an inversion.
  - If $z = re^{i\theta}$ then $\dfrac{1}{z} = \dfrac{1}{r}e^{-i\theta}$.
  - f interchanges outside and inside of the unit circle.
  - A circle, centered at 0, is clearly mapped to a circle, centered at 0, of reciprocal radius.
  - How about other circles?

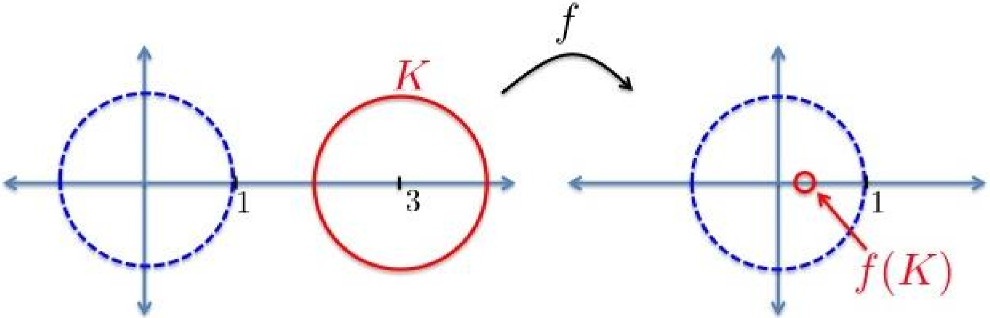
The Inversion $f(z) = 1/z$

What are images of circles under f?

- Let $K = \{z : |z-3| = 1\}$ be the circle of radius 1, centered at 3. What is its image $f(K)$?

$$
\begin{aligned}
w \in f(K) \iff \frac{1}{w} \in K \quad &\iff \quad \left|\frac{1}{w} - 3\right| = 1 \\
&\iff \quad |1 - 3w|^2 = |w|^2 \\
&\iff \quad 1 - 3w - 3\overline{w} + 9|w|^2 = |w|^2 \\
&\iff \quad 8|w|^2 - 3w - 3\overline{w} = -1 \\
&\iff \quad \left(w - \frac{3}{8}\right)\left(\overline{w} - \frac{3}{8}\right) = \frac{9}{64} - \frac{1}{8} \\
&\iff \quad \left|w - \frac{3}{8}\right| = \frac{1}{8}.
\end{aligned}
$$

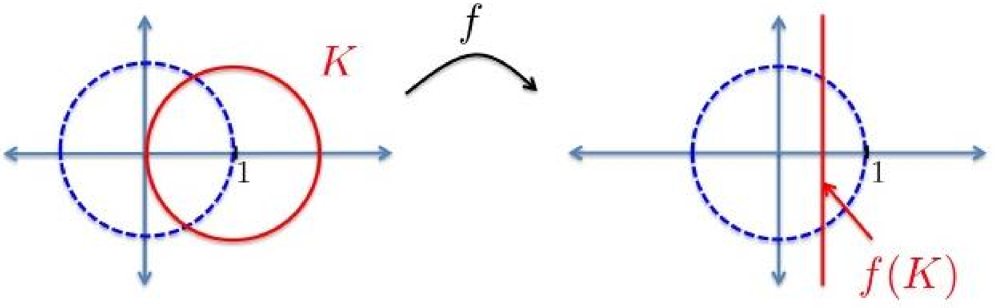
Thus the image of the circle $K = \{z : |z-3| = 1\}$ under $f(z) = \frac{1}{z}$ is another circle, namely the circle of radius $\frac{1}{8}$, centered at $\frac{3}{8}$.



Let now $K = \{z : |z-1| = 1\}$ be the circle of radius 1, centered at 1. What is $f(K)$?

$$\begin{aligned} w \in f(K) \iff \frac{1}{w} \in K \quad &\iff \quad \left|\frac{1}{w} - 1\right| = 1 \\ &\iff \quad |1-w|^2 = |w|^2 \\ &\iff \quad 1 - w - \overline{w} + |w|^2 = |w|^2 \\ &\iff \quad w + \overline{w} = 1 \\ &\iff \quad \operatorname{Re} w = \frac{1}{2}. \end{aligned}$$

Thus the image of the circle $K = \{z : |z - 1| = 1\}$ is the vertical line $f(K) = \{w : \operatorname{Re} w = \frac{1}{2}\}$.



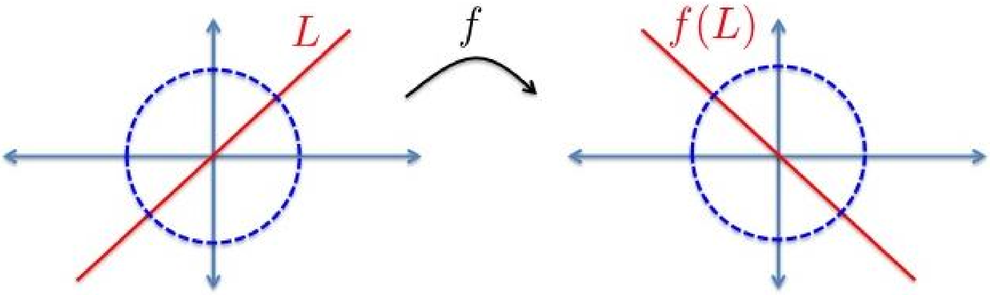
- Since $f(f(z)) = f(\frac{1}{z}) = z$, we also find:
  - f maps the line $\{z : \mathrm{Re}\, z = \frac{1}{2}\}$ to the circle $\{z : |z-1| = 1\}$.
  - f maps the circle $\{z : |z - \frac{3}{8}| = \frac{1}{8}$ to the circle $\{z : |z-3| = 1\}$.

- Let now L be the line $\{z : z = t + it, -\infty < t < \infty\}$.
  If $z = t + it$, then

$$f(z) = \frac{1}{z} = \frac{\overline{z}}{|z|^2} = \frac{t - it}{2t^2} = \frac{1}{2t} - i\frac{1}{2t} = s - is.$$

Images of Lines Through the Origin

Thus the image of the line $\{t + it \,:\, t \in \mathbb{R}\}$ is the line $\{s - is \,:\, s \in \mathbb{R}\}$.



Images of lines and circles seem to be lines or circles!

Facts About Möbius Transformations

Fact

Every Möbis transformation maps circles and lines to circles or lines.

Note: A line could be viewed as a "circle through infinity".

Fact

Given three distinct points $z_1, z_2, z_3 \in \hat{\mathbb{C}}$, there exists a unique Möbius transformation f such that $f(z_1) = 0$, $f(z_2) = 1$, and $f(z_3) = \infty$.

You can actually write this Möbius transformation down:

$$f(z) = \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}.$$

Next up: Constructing examples.