Analysis of a Complex Kind

Week 2

Lecture 1: Complex Functions

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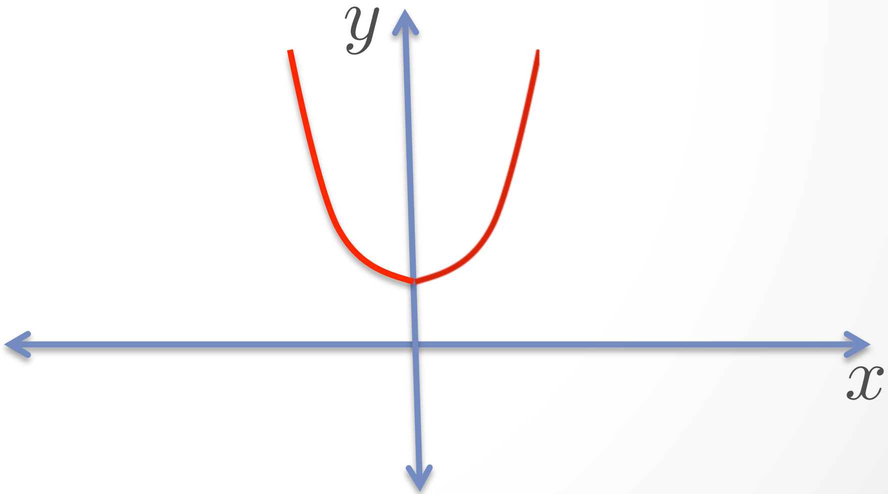
Welcome To Week Two!

This week:

- Julia sets for quadratic polynomials.
- The Mandelbrot set.
- We laid the ground last week!
- Just a little more preparation:
    - Complex functions (Lecture 1).
    - Sequences and limits (Lecture 2).
- We'll need to study quadratic polynomials of the form $f(z) = z^2 + c$.

Functions

- Recall: *A function* $f : A \to B$ *is a rule that assigns to each element of* A *exactly one element of* B.
- Example: $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2 + 1$
- The graph helps us understand the function.



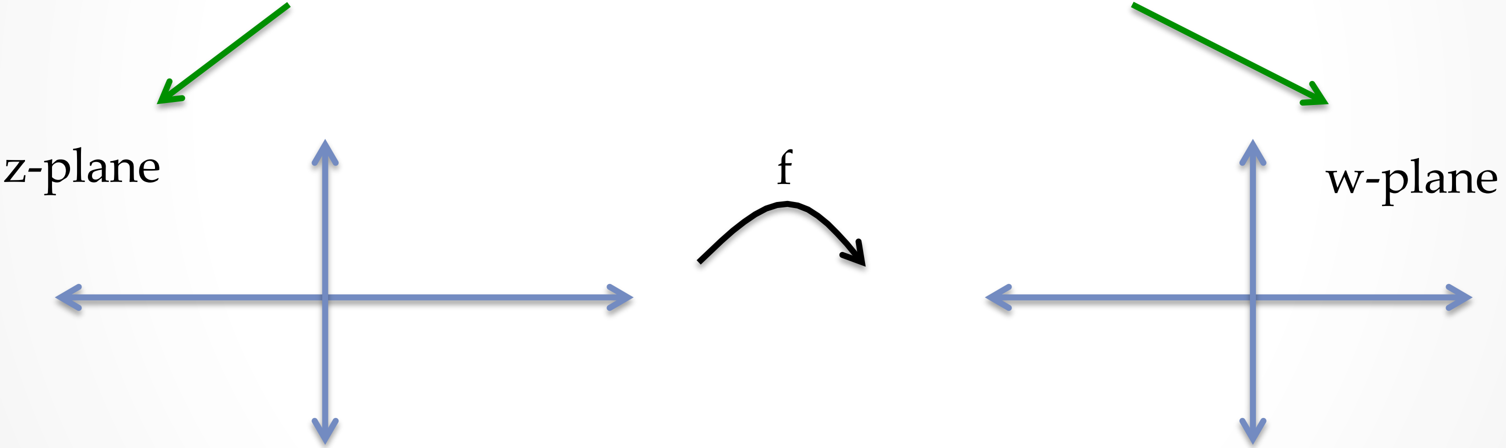
Complex Functions

- Now: $f : \mathbb{C} \to \mathbb{C},\ f(z) = z^2 + 1$
- How do we graph this? Need 4 dimensions?
- Writing $z = x + iy$ we see:

$$\begin{aligned} w = f(z) &= (x+iy)^2 + 1 \\ &= (x^2 - y^2 + 1) + i \cdot 2xy \\ &= u(x,y) + iv(x,y) \\ &\quad \text{where } u, v : \mathbb{R}^2 \to \mathbb{R}. \end{aligned}$$

Graphing Complex Functions

- Idea: Consider 2 complex planes:
  one for the domain, one for the range.



- Analyze how geometric configurations in the z-plane are mapped under f to the w-plane.

Example

$$f(z) = z^2, \text{ so } w = (x+iy)^2 = (x^2 - y^2) + 2ixy$$

Not so useful?

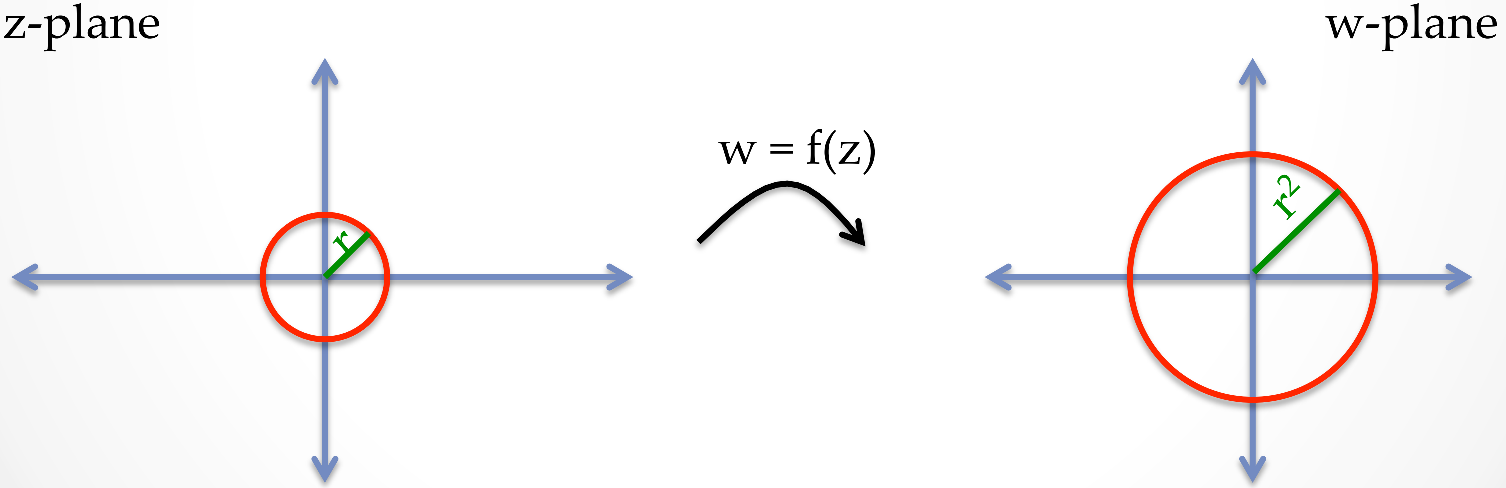
More useful in this case: polar coordinates!

$$z = re^{i\theta}, \text{ then } w = r^2 e^{2i\theta}, \text{ so}$$

$$|w| = |z|^2 \text{ and } \arg w = 2 \arg z.$$

$$w = f(z) = z^2, \text{ so } w = r^2 e^{2i\theta}$$

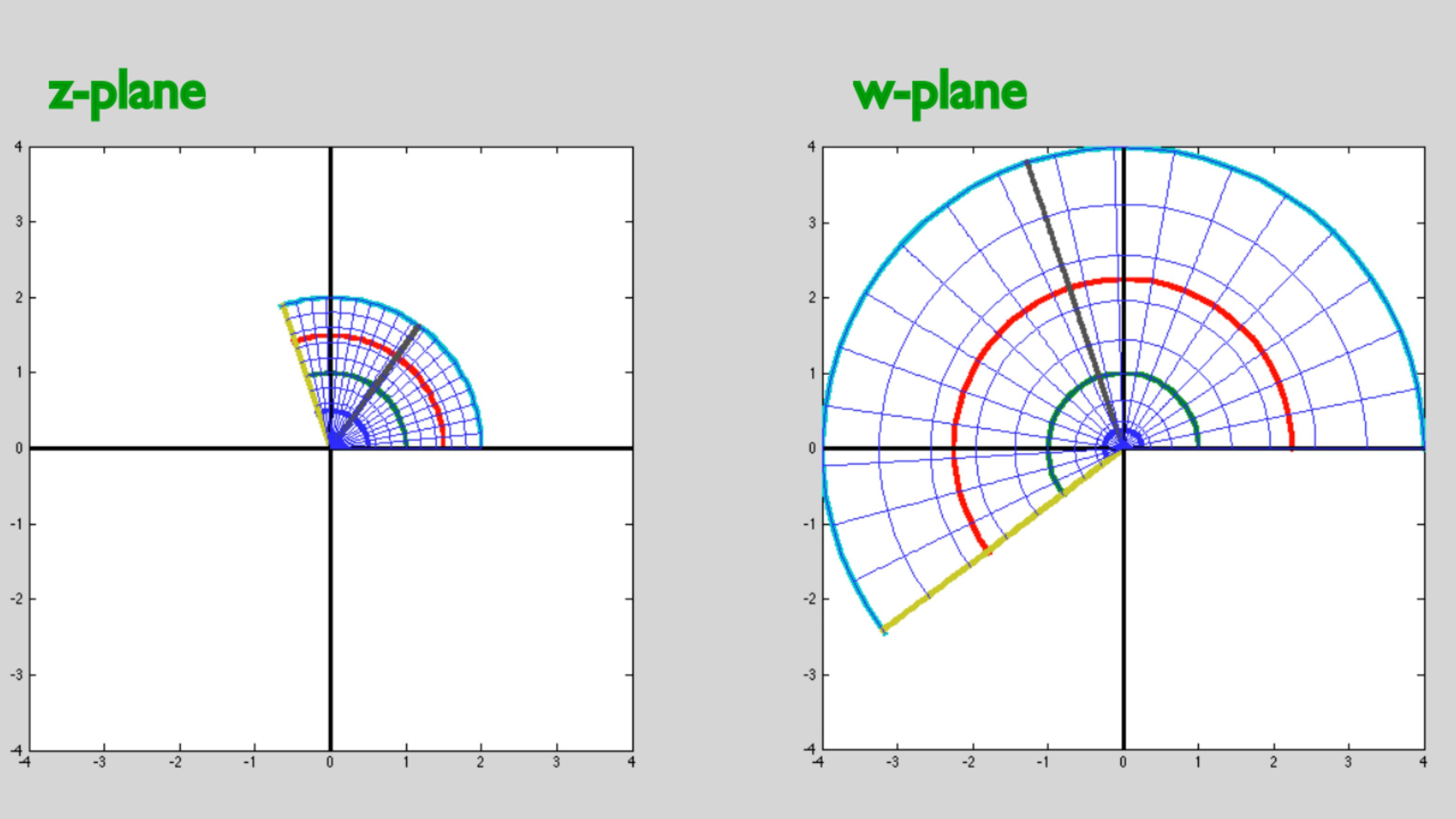
- As z moves around a circle of radius r once, w moves around the circle of radius r^2 at double speed, twice.



$$w = f(z) = z^2$$

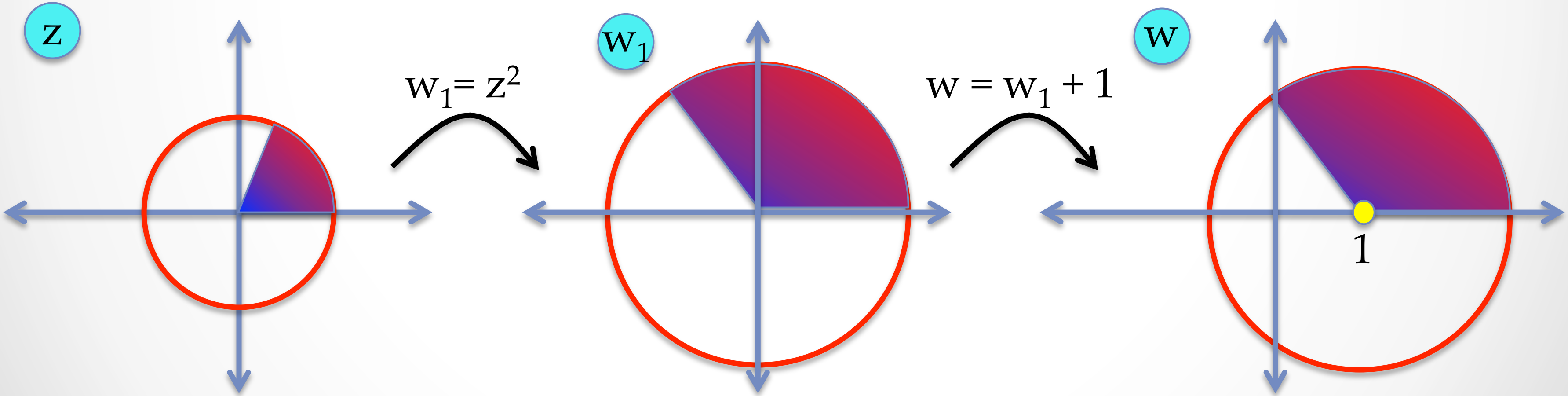
movie here about rotating twice

$$w = f(z) = z^2$$



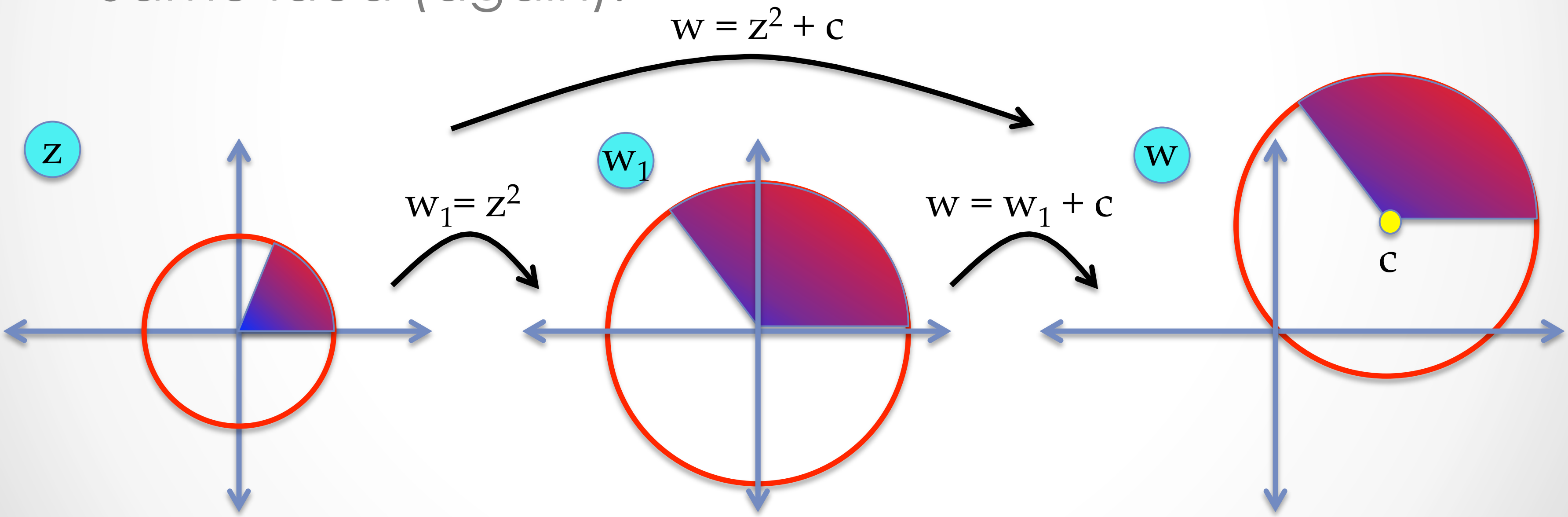
More Complicated Functions

- How do we understand more complicated functions, such as $f(z) = z^2 + 1$?
- Same idea!



How about…

- And how about $f(z) = z^2 + c,\ c \in \mathbb{C}$?
- Same idea (again)!



Iteration of Functions

Let $f(z) = z+1$. Then

- $f^2(z) = f(f(z)) = f(z+1) = (z+1) + 1 = z+2$.
- $f^3(z) = f(f^2(z)) = f(z+2) = (z+2) + 1 = z+3$.
- …
- $f^n(z) = z+n$.

- f^n (read: "Eff n") is called the nth iterate of f. *(Not to be confused with the nth power of f.)*

Another Example

Let $f(z) = 3z$. Then:

- $f^2(z) = f(f(z)) = f(3z) = 3*3z = 3^2z.$
- $f^3(z) = f(f^2(z)) = 3*3^2z = 3^3z.$
- …
- $f^n(z) = 3^nz$

Two More...

Let $f(z) = z^d$. Then:

- $f^2(z) = (z^d)^d = z^{(d^2)}$
- ...
- $f^n(z) = z^{(d^n)}$

Now let $f(z) = z^2 + 2$. Then:

- $f^2(z) = (z^2 + 2)^2 + 2 = z^4 + 4z^2 + 6$
- $f^3(z) = (z^4 + 4z^2 + 6)^2 + 2 = z^8 + \ldots$
- f^n is a polynomial of degree 2^n.

Julia Sets

- To study the Julia set of the polynomial $f(z) = z^2 + c$ we'll study the behavior of the iterates $f, f^2, f^3, f^4, \ldots, f^n, \ldots$ of this function.
- The Julia set of f is the set of points z in the complex plane at which this sequence of iterates behaves "chaotically".
- We thus need one more preparation: We need to study sequences of complex numbers. That's next!