Analysis of a Complex Kind

Week 1

Lecture 2: Algebra and Geometry in the Complex Plane

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The Complex Plane

- Complex numbers: expressions of the form $z = x + iy$, where
  - x is called the *real part* of z; $x = \operatorname{Re} z$, and
  - y is called the *imaginary part* of z; $y = \operatorname{Im} z$.
- Set of complex numbers: $\mathbb{C}$ (the *complex plane*).
- Real numbers: subset of the complex numbers (those whose imaginary part is zero).
- The complex plane can be identified with $\mathbb{R}^2$.

Adding Complex Numbers

Definition

$$\underbrace{(x+iy)}_{z}+\underbrace{(u+iv)}_{w}=\underbrace{(x+u)}_{\mathrm{Re}(z+w)}+i\underbrace{(y+v)}_{\mathrm{Im}(z+w)}$$

Thus

$$\mathrm{Re}(z+w)=\mathrm{Re}\,z+\mathrm{Re}\,w \quad \text{and} \quad \mathrm{Im}(z+w)=\mathrm{Im}\,z+\mathrm{Im}\,w.$$

Graphically, this corresponds to vector addition:

The Modulus of a Complex Number

Definition

The *modulus* of the complex number $z = x + iy$ is the length of the vector z:

$$|z| = \sqrt{x^2 + y^2}.$$

Multiplication of Complex Numbers

- Motivation: $(x+iy)\cdot(u+iv) = xu + ixv + iyu + i^2yv$
- So we define:

Definition

$$(x+iy)\cdot(u+iv) = (xu - yv) + i(xv + yu) \in \mathbb{C}.$$

- Example:

$$(3+4i)(-1+7i) = (-3-28) + i(21-4) = -31 + 17i.$$

- The usual properties hold:
  - $(z_1z_2)z_3 = z_1(z_2z_3)$ (*associative*)
  - $z_1z_2 = z_2z_1$ (*commutative*)
  - $z_1(z_2+z_3) = z_1z_2 + z_1z_3$ (*distributive*)

So What is i?

$$i = 0 + 1i,$$

so

$$i^2 = (0 + 1i)(0 + 1i) = (0 \cdot 0 - 1 \cdot 1) + i(0 \cdot 1 + 1 \cdot 0) = -1.$$

- $i^3 = i^2 \cdot i = -1 \cdot i = -i$
- $i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$
- $i^5 = i^4 \cdot i = i$
- $i^6 = -1 \ldots$

How Do You Divide Complex Numbers?

Suppose that $z = x + iy$ and $w = u + iv$. What is $\dfrac{z}{w}$ (for $w \neq 0$)?

$$\begin{aligned}
\frac{z}{w} &= \frac{x+iy}{u+iv} \\
&= \frac{(x+iy)(u-iv)}{(u+iv)(u-iv)} \\
&= \frac{(xu+yv)+i(-xv+yu)}{u^2+v^2+i(-uv+vu)} \\
&= \frac{xu+yv}{u^2+v^2} + i\frac{yu-xv}{u^2+v^2}.
\end{aligned}$$

In particular:

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}, \text{ as long as } z \neq 0.$$

The Complex Conjugate

Note the importance of the quantity $x - iy$ in the previous calculation!

Definition

If $z = x + iy$ then $\overline{z} = x - iy$ is the *complex conjugate* of z.

Properties:

- $\overline{\overline{z}} = z$
- $\overline{z + w} = \overline{z} + \overline{w}$
- $|z| = |\overline{z}|$
- $z\overline{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$
- $\dfrac{1}{z} = \dfrac{\overline{z}}{z\overline{z}} = \dfrac{\overline{z}}{|z|^2}$

More Properties of the Complex Conjugate

- When is $z = \overline{z}$?
- $z + \overline{z} = (x + iy) + (x - iy) = 2x$, so

$$\operatorname{Re} z = \frac{z + \overline{z}}{2}, \quad \text{similarly} \quad \operatorname{Im} z = \frac{z - \overline{z}}{2i}.$$

- $|z \cdot w| = |z| \cdot |w|$
- $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$, $(w \neq 0)$
- $|z| = 0$ if and only if $z = 0$.

Some Inequalities

- $-|z| \leq \operatorname{Re} z \leq |z|$
- $-|z| \leq \operatorname{Im} z \leq |z|$
- $|z + w| \leq |z| + |w|$ (*triangle inequality*)
- $|z - w| \geq |z| - |w|$ (*reverse triangle inequality*)

The Fundamental Theorem of Algebra

Theorem

If $a_0, a_1, \ldots, a_n$ are complex numbers with $a_n \neq 0$, then the polynomial

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

has n roots $z_1, z_2, \ldots z_n$ in $\mathbb{C}$. It can be factored as

$$p(z) = a_n(z - z_1)(z - z_2) \cdots (z - z_n).$$

We will be able to prove this theorem later in this course!
Consider the polynomial $p(x) = x^2 + 1$ in $\mathbb{R}$. It has no real roots!
But in $\mathbb{C}$ it can be factored: $z^2 + 1 = (z + i)(z - i)$!