Analysis of a Complex Kind

Week 1

Lecture 4: Roots of Complex Numbers

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The nth Root

Definition

Let w be a complex number. An nth root of w is a complex number z such that $z^n = w$.

- We'll see: If $w \neq 0$ there are exactly n distinct nth roots:
- Use the polar form for w and z: $w = \rho e^{i\varphi}$ and $z = re^{i\theta}$.
- The equation $z^n = w$ then becomes

$$r^n e^{in\theta} = \rho e^{i\varphi}, \quad \text{so} \quad r^n = \rho \text{ and } e^{in\theta} = e^{i\varphi}.$$

- Thus $r = \sqrt[n]{\rho}$ and $n\theta = \varphi + 2k\pi$, $k \in \mathbb{Z}$, so $\theta = \dfrac{\varphi}{n} + \dfrac{2k\pi}{n}$, $k = 0, 1, \ldots, n-1$.
- We write $w^{\frac{1}{n}} = \sqrt[n]{\rho} e^{i(\frac{\varphi}{n} + \frac{2k\pi}{n})}$, $k = 0, 1, \ldots, n-1$.

Examples of nth Roots

$$w = \rho e^{i\varphi}, \quad w^{\frac{1}{n}} = \sqrt[n]{\rho} e^{i(\frac{\varphi}{n} + \frac{2k\pi}{n})}, \quad k = 0, 1, \ldots, n-1.$$

- Square roots of $4i$:

$$4i = 4e^{i\frac{\pi}{2}}, \quad \text{so} \quad \rho = 4, \varphi = \frac{\pi}{2} \text{ and } n = 2.$$

$$\begin{aligned}
(4i)^{\frac{1}{2}} &= \sqrt{4} \cdot e^{i(\frac{\pi}{4} + \frac{2k\pi}{2})}, k = 0, 1 \\
&= \begin{cases} 2 \cdot e^{i\frac{\pi}{4}} & \text{if } k = 0 \\ 2 \cdot e^{i(\frac{\pi}{4} + \pi)} & \text{if } k = 1 \end{cases} \\
&= \pm(\sqrt{2} + i\sqrt{2}).
\end{aligned}$$

Examples of *n*th Roots

$$w = \rho e^{i\varphi}, \quad w^{\frac{1}{n}} = \sqrt[n]{\rho} e^{i\left(\frac{\varphi}{n} + \frac{2k\pi}{n}\right)}, \quad k = 0, 1, \ldots, n-1.$$

- Cubed roots of -8:

$$-8 = 8e^{i\pi}, \quad \text{so} \quad \rho = 8, \varphi = \pi \text{ and } n = 3.$$

$$\begin{aligned}
(-8)^{\frac{1}{3}} &= \sqrt[3]{8} \cdot e^{i\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right)}, k = 0, 1, 2 \\
&= \begin{cases} 2 \cdot e^{i\frac{\pi}{3}} & \text{if } k = 0 \\ 2 \cdot e^{i\pi} = -2 & \text{if } k = 1 \\ 2 \cdot e^{i\frac{5\pi}{3}} & \text{if } k = 2. \end{cases}
\end{aligned}$$

Roots of Unity

Definition

The *n*th roots of 1 are called the *nth roots of unity*.

Since $1 = 1e^{i\cdot 0}$, we find that

$$\begin{aligned} 1^{\frac{1}{n}} &= \sqrt[n]{1}\cdot e^{i\left(\frac{0}{n}+\frac{2k\pi}{n}\right)}, k = 0, 1, \ldots, n-1 \\ &= e^{i\frac{2\pi k}{n}}, k = 0, 1, \ldots, n-1. \end{aligned}$$