Analysis of a Complex Kind

Week 3

Lecture 3: The Complex Exponential Function

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Definition of the Complex Exponential Function

Recall from the last lecture that the function

$$f(z) = e^x \cos y + ie^x \sin y$$

(where again, $z = x + iy$) is an entire (= analytic in $\mathbb{C}$) function.
Let's look at some of its properties:

- If $y = 0$, then $f(z) = f(x + i \cdot 0) = f(x) = e^x$, so f agrees with the "regular" exponential function on $\mathbb{R}$.
- $f(z) = e^x(\cos y + i \sin y) = e^x e^{iy}$.

Definition

The complex exponential function, e^z, sometimes also denoted $\exp(z)$, is defined by

$$e^z = e^x \cdot e^{iy}, \quad \text{where } z = x + iy.$$

Properties

$$e^z = e^x \cdot e^{iy}, \quad \text{where } z = x + iy.$$

- $|e^z| = |e^x||e^{iy}| = e^x$.
- $\arg e^z = \arg(e^x e^{iy}) = y$.
- $e^{z+2\pi i} = e^x e^{i(y+2\pi)} = e^x e^{iy} = e^z$.
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$$\begin{aligned} e^{z+w} = e^{(x+iy)+(u+iv)} &= e^{(x+u)+i(y+v)} \\ &= e^{x+u} e^{i(y+v)} \\ &= e^x e^u e^{iy} e^{iv} \\ &= (e^x e^{iy})(e^u e^{iv}) \\ &= e^z e^w. \end{aligned} \qquad (z = x + iy,\ w = u + iv)$$

Further Properties of $e^z = e^x e^{iy}$

- $\dfrac{1}{e^z} = e^{-z}$ since $e^z e^{-z} = e^{z-z} = e^0 = 1$.
- e^z is an entire function (we already showed this).
- What is its derivative? Recall

$$u(x,y) = e^x \cos y \quad v(x,y) = e^x \sin y$$

and

$$\begin{aligned} u_x(x,y) &= e^x \cos y & v_x(x,y) &= e^x \sin y \\ u_y(x,y) &= -e^x \sin y & v_y(x,y) &= e^x \cos y \end{aligned}$$

Thus $f'(z) = u_x(x,y) + iv_x(x,y) = e^x \cos y + ie^x \sin y = e^z$!

So the derivative of e^z is e^z, in symbols, $\dfrac{d}{dz} e^z = e^z$.

Even More Properties of $e^z = e^x e^{iy}$

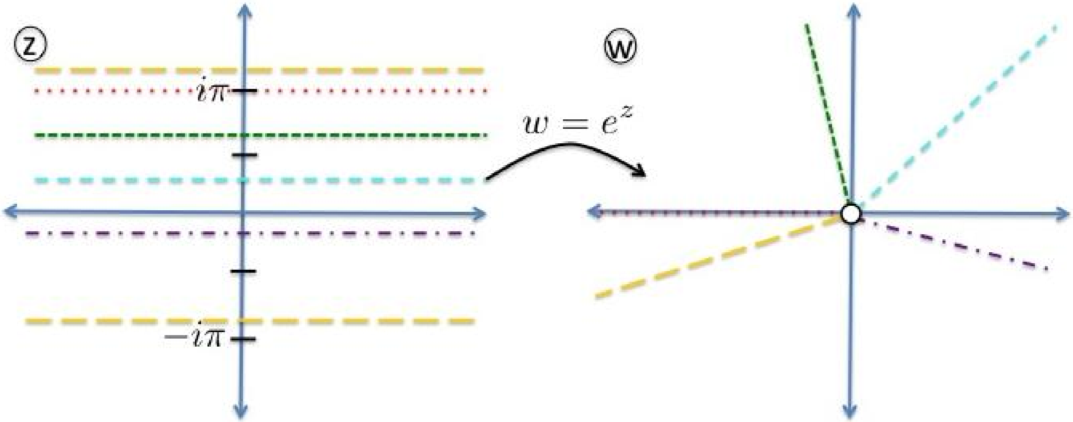
- $\dfrac{d}{dz} e^{az} = a \cdot e^{az}$ $(a \in \mathbb{C})$ by the chain rule.
- $e^{\bar{z}} = e^{x-iy} = e^x e^{-iy} = e^x \overline{e^{iy}} = \overline{e^x e^{iy}} = \overline{e^z}$.
- $e^z = 1$ if and only if $e^x e^{iy} = 1$. The complex number in polar form, $e^x e^{iy}$, equals 1, when its length equals 1 and its argument equals 0, i.e. when $e^x = 1$ and $y = 2k\pi$, i.e. $x = 0$ and $y = 2k\pi$, $k \in \mathbb{Z}$. Thus

$$e^z = 1 \iff z = 2\pi i k, \quad k \in \mathbb{Z}.$$

- $e^z = e^w \iff e^{z-w} = 1 \iff z - w = 2\pi i k \iff z = w + 2\pi i k.$

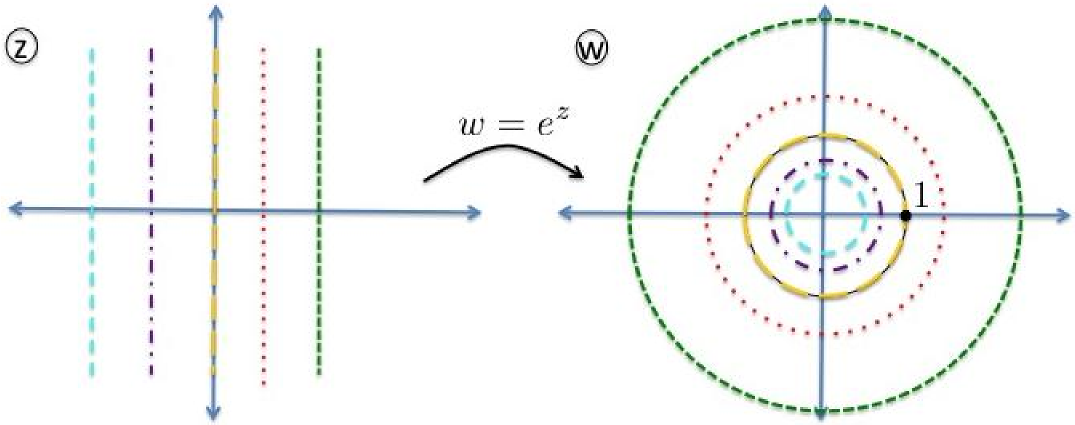
Understanding the Mapping $w = e^z$

The function $w = e^z$ is a mapping from $\underbrace{\mathbb{C}}_{z-\text{plane}}$ to $\underbrace{\mathbb{C}}_{w-\text{plane}}$. What are the images of horizontal lines? $L = \{x + iy_0 \mid x \in \mathbb{R}\}$ for fixed $y_0 \in \mathbb{R}$. Then $e^{x+iy_0} = e^x e^{iy_0}$.



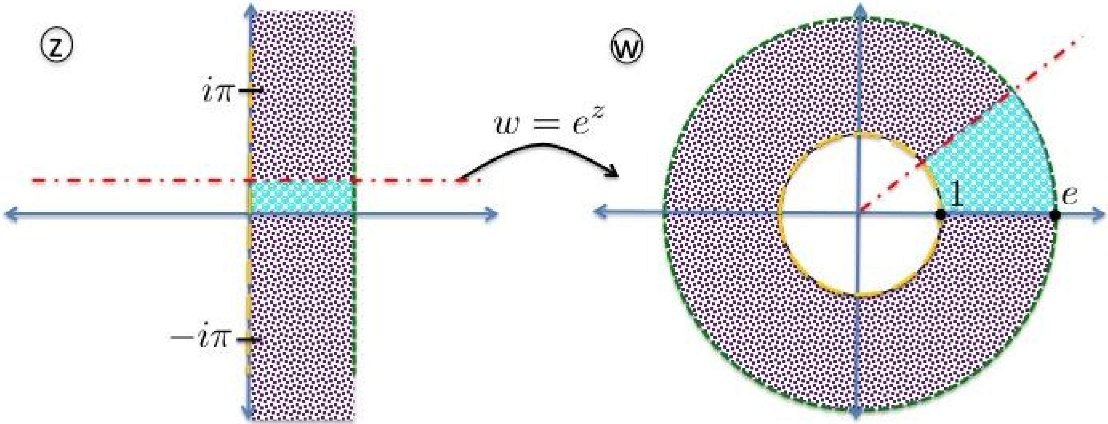
Understanding the Mapping $w = e^z$, cont.

What are the images of vertical lines? $L = \{x_0 + iy \mid y \in \mathbb{R}\}$ for fixed $x_0 \in \mathbb{R}$. Then $e^{x_0+iy} = e^{x_0} e^{iy}$.



Understanding the Mapping $w = e^z$, cont.

What is the image of a vertical strip? $S = \{z : 0 < \operatorname{Re} z < 1\}$.



Inverting e^z

- When is $e^z = 0$?

$$e^z = 0 \iff e^x \cdot e^{iy} = 0 \qquad \text{Note: } e^{iy} \text{ has absolute value 1!}$$
$$\iff e^x = 0$$
$$\iff \text{Never...!}$$

- For a given $z \in \mathbb{C} \setminus \{0\}$, is there a $w \in \mathbb{C}$ such that $e^w = z$? Writing $z = |z|e^{i\theta}$ and $w = u + iv$ this is equivalent to:

$$\begin{aligned} e^w = z &\iff e^u e^{iv} = |z|e^{i\theta} \\ &\iff e^u = |z| \quad \text{and} \quad e^{iv} = e^{i\theta} \\ &\iff u = \ln|z| \quad \text{and} \quad v = \theta + 2k\pi \\ &\iff w = \ln|z| + i\arg z. \end{aligned}$$

This is the complex logarithm!