We are now in the approximation of a very large reduced velocity where the dynamics of the solid is so slow that we may neglect the velocity of the interface in the fluid dynamics. The fluid dynamics is fast, and the solid dynamics is slow, comparatively. Under this approximation, can we say anything in general on the effect of the fluid loading on the dynamics of the solids?

If we use the simplest framework for the dynamics of the solid, that of a single mode approximation, what will be the form of the fluid loading and what would be the consequences on the dynamics of the solid? In the single mode approximation the position of the interface depends on the single parameter q. So considering that interface is fixed at the position defined by q, we have in the fluid domain a steady-state problem with a boundary condition dependent on q. The pressure and the velocity in the fluid will depend on that value q, and so will the fluid loading projected on the model $F_F S$. We may state that the fluid loading at a given time will depend on the position of the interface at that same time.

The projected fluid loading is actually proportional to the Cauchy number Cy as noted before. So I write the force as the Cauchy number times the function of the other parameters such as the Reynolds number and my interface displacement, q.

For small motion, D much smaller than 1, we may now expand the fluid loading F as a function of the small parameter, D. The first term is the permanent loading corresponding to the reference position of $q = 0$. The next term is D times the fluctuation of the fluid loading force, $f_F S$. This is the flow induced force resulting from the motion of the interface. It is proportional to the modal displacement on q. This is a flow induced stiffness force as a spring force.

So from the point of view of the dynamics of a solid, the coupling to the flow is identical to the coupling with the spring. But what is the stiffness k_F of this spring? We have

$$k_F = C_Y(\frac{dF}{dq})^0$$

. First, we may note that it may be positive as well as negative. Its sign just depends on the way the flow reacts to the displacement of the boundary, as will be shown in examples. Moreover, the magnitude of the stiffness depends on the flow velocity being proportional to the Cauchy number. Remember that the Cauchy varies as u squared. So we have a flow induced stiffness with a sign and a magnitude that depends totally on the fluid dynamics.

What is the consequence of such a flow-induced stiffness on the dynamics of our solid? The dynamics of the mode is governed by an oscillator equation that we obtained in dimensionless form. If we are interested in the dynamics of the solid, we should now change the reference time and use the dimensionless time t bar based on t fluid. Thus we have a simple oscillator equation for the quantity q of t. In the presence of flow, the oscillator equation is modified by the right hand side force f_{FS}, which we just derived. As this force is proportional to the model displacement q, we can incorporate it in the oscillator equation. So in the presence of flow we have a new oscillator with a total stiffness that varies with the flow velocity. Depending on the sign of the fluid induced stiffness, the total stiffness may decrease as the flow velocity is increased. At some stage, all stiffness is lost. At 0 stiffness, instability occurs because the frequency becomes imaginary. This instability is often referred to as the static instability in the sense that it only involves displacements term, not inertia or velocity. It is also called buckling or divergence. Now what happens in practice? If we consider the response to a perturbation, here's what will happen in this approximation. At 0 flow velocity the solid oscillates. For velocity below the instability threshold, the perturbation also results in an oscillation, but the frequency is smaller. Above the threshold, any perturbation will be exponentially amplified in time.

Let us summarize. Here's what the dynamics of the oscillator and the flow is going to be in the case where the flow induced stiffness is negative. At very small Cauchy number, the flow is unable to affect system and the dimensionless frequency of free oscillation is going to be 1, as expected. As the flow velocity is increased, the frequency decreases down to the point of static instability.

We can illustrate this on the very simple case of an airfoil, which may deform in torsion around an axis. Let θ be the angle of torsion with a zero reference position. Without flow, the wing section angle would follow an oscillator equation where J is the moment of inertia and C is the torsion stiffness. Using a time based on Tsolid, the period of the oscillator, the equation becomes, the elementary oscillator equation. Let us consider now a flow such that our assumption of quasi-static aeroelasticity applies. This means that the reduced velocity is high enough. Here, the reduced velocity UR reads as the ratio between the time of oscillation, T_{solid}, and a fluid convection time, L/U. The displacement number would be here the magnitude of the displacement over the size of the wing section, which means $D = \frac{\theta_0}{L}$, which is theta naught, where theta naught is the angle of the motion of interest. The condition for the use of an approximation is that UR is much larger than D. If we consider oscillations of the order of degrees, say, theta naught equals pi over 100, on a typical small aircraft wing, we have UR on the order of 100, which is much larger than pi over 100. Under this approximation we may use as model of the fluid force acting on the airfoil, exactly the same model as if the airfoil was fixed with a time independent angle of torsion theta. This is the well known configuration of a fixed airfoil with an angle of attack. For small enough angles, before stall, the flow around the airfoil results mainly in a lift force acting about at the quarter chord. The force is proportional to U squared and to C_L, the lift coefficient, which depends on the angle of attack, theta. Let x be the distance between the point where lift applies and the axis of rotation of the airfoil. Now, dimensionless equations. We may define a Cauchy number, C_Y, as the ratio between the fluid loading and the flexibility in torsion. The moment exerted by this force acts on the rotation of the wing at right hand side term in the oscillator equation. Let us now expand the r.h.s for small angles and obtain a stiffness term related to the derivative of the lift coefficient with respect to the angle of attack. There is a lot of data on the dependence of the lift coefficient with the angle of attack, depending on the shape of the airfoil. For instance, for a symmetric NACA 0012 section, here is a schematic view of the curve. All we need here is the slope $\frac{dC_L}{d\theta}$. It is typically equal to 2π in the thin airfoil theory. We may write now the equation that governs the torsion angle, taking into account the flow induced stiffness. We see that when x is positive, for instance for a rotation axis at the mid chord, the stiffness destabilizes the torsion dynamics. The total stiffness vanishes at a critical Cauchy number, which we can simply compute. Now that we have the critical Cauchy number, we can go back to dimensional quantities. The critical velocity is found to depend simply on the rotation stiffness, C, on the chord of the wing, L, and on the position of the axis of rotation of the wing, x. Here is an illustration of the response of the wing to an initial pertubation. For velocity below the critical value, the airfoil oscillates. Above the critical velocity, the amplitude increases exponentially.

Actually, these instabilities were a problem for early planes made of rather soft materials. For more recent planes, materials are stiffer and more importantly the point of the axis of rotation may be moved upward, which totally suppresses instability. This instability was observed here, on a plane without engines, a very special case. You may also find some of these instabilities in torsion in very flexible foot bridges. To summarize, we have shown that the very simple approximation on the reduced velocity could be used to predict some kinds of interactions between a fast flow and a solid. And some of these interactions could lead to instabilities of the buckling type. This is very important in practice and remember that predictions of the critical velocities only require standard data on fluid loading on structures.