translation — Particle Dynamics

Rigid Body Dynamics — translation & rotation

Newton-Euler Method for Rigid Body Dynamics

mpatil@gatech.edu

Newton's Laws of Motion

- Newton's Laws were published in 1687 in *Principia*
- Newton's 2nd Law

  The force applied on a particle is proportional to the time derivative of the particle's (linear) momentum

$$ {}^{I}\frac{d\vec{L}}{dt} = \vec{F} = \frac{{}^{I}d}{dt}(m\vec{v}) = m\,{}^{I}\vec{a} $$

- Assumptions above
  - Frame is inertial (Newtonian, Galilean)
  - Constant mass (for the equation in terms of acceleration)
  - Valid for particles (not rigid bodies), though everything is relative, e.g., sometimes a planet can be considered as a particle
- Extension of Newton's 2nd law to bodies was done by Leonhard Euler
  - Bodies have size, shape, and mass distribution

$\{M\} = [I]\{\dot{\omega}\} + \{\omega\} \times [I]\{\omega\}$

Rotation
Euler's Dynamical Eqns

Leonhard Euler

- Euler's laws of motion - Wikipedia
- Euler angles – Wikipedia
- Euler–Lagrange equation - Wikipedia
- Euler equations (fluid dynamics) – Wikipedia
- Euler–Bernoulli beam theory - Wikipedia
- Euler's critical load (buckling) - Wikipedia
- Euler method (Numerical ODE) - Wikipedia
- Euler's formula – Wikipedia
- Optics, Number Theory, Graph Theory, Logic

Dynamics

Functional Lagrangian → Hamilton's principle → Lagrange's Eqn

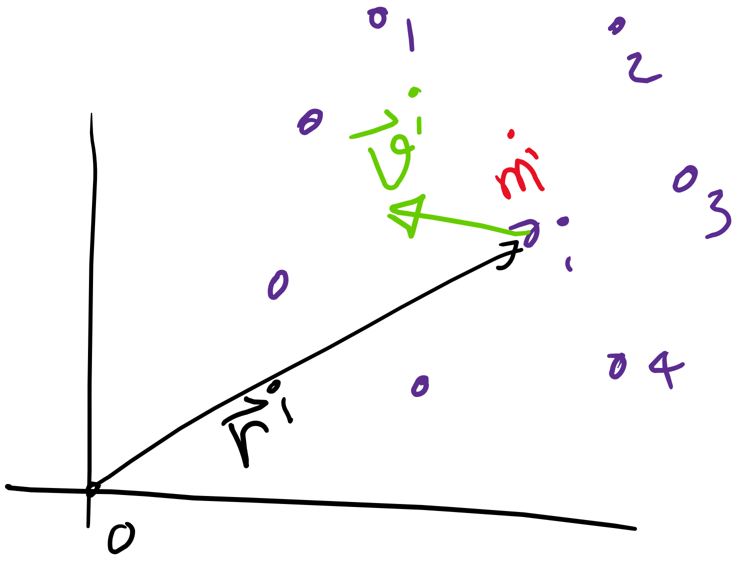
→ Calculus of Variation (Math)

Functional Minimization

Diff Eq ← Euler-Lagrange Eq. ←

https://en.wikipedia.org/wiki/Leonhard_Euler

Linear Momentum (Vector)



- For a particle i:

$$\vec{L}^i = m^i \vec{v}^i$$

- For a set of particles or a rigid body:

$$\vec{L} = \sum_i m^i \vec{v}^i \quad = \int_{\Theta} \rho \, v \, dx\, dy\, dz$$

- In terms of Center of Mass of a set of particles or a rigid body:

$$\vec{L} = \sum_i m^i \vec{v}^i = \sum_i m^i \frac{d\vec{r}^i}{dt} = \frac{d \sum_i m^i \vec{r}^i}{dt} = \frac{d(m\vec{r}^{OC})}{dt} = m\vec{v}^C$$

<u>All derivatives are in an inertial frame</u>!

Center of Mass defined in the next slide …

$$\sum m^i \vec{r}^i = m\vec{r}^{OC}$$

Center of Mass (Set of Particles)

- The linear momentum of the particles is related to the position vector of the particles

$$\sum_i m^i \vec{v}^i = \sum_i m^i \frac{d\vec{r}^i}{dt} = \sum_i \frac{d(m^i \vec{r}^i)}{dt} = \frac{d(\sum_i m^i \vec{r}^i)}{dt} = \frac{d(m\,\vec{r}^{OC})}{dt}$$

… the above assumed that the mass of the particles is not changing

$m = \Sigma m^i$

- We thus define the center of mass location and total mass via:

$$\boxed{m\,\vec{r}^{OC} = \sum_i m^i \vec{r}^i}$$

$m \vec{r}^{oc} = \Sigma_i m^i \vec{r}^i$

$i \leftarrow$ rigid body

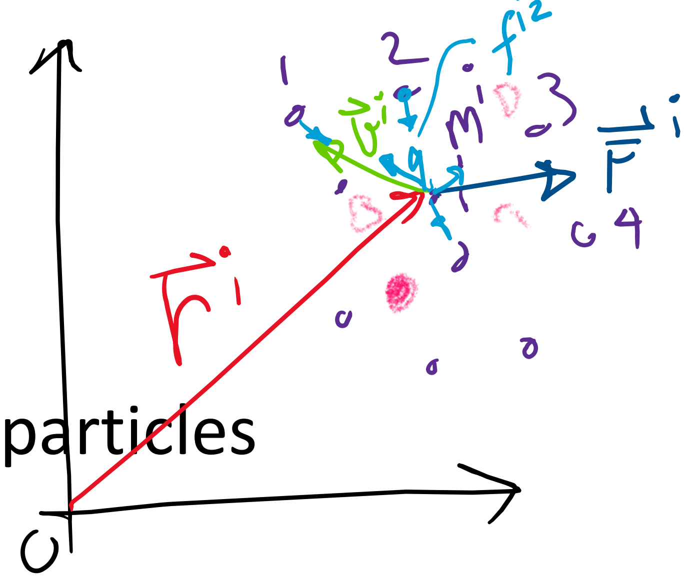
where,

$$\vec{r}^{OC} = \frac{\sum_i m^i \vec{r}^i}{m}$$

$$m = \sum m^i$$

$I^C_{zz} = \Sigma I^{C_i}_{zz_i} + m_i ({x^{CC_i}}^2 + {y^{CC_i}}^2)$

Euler’s First Law



- Newton’s 2nd law for i^{th} particle in a system of particles

$$\vec{F}^i + \sum_j \vec{f}^{ij} = \frac{d(m^i \vec{v}^i)}{dt}$$

- Summing the equations for all particles

$$\sum_i \vec{F}^i = \sum_i \frac{d(m^i \vec{v}^i)}{dt} = \frac{d \sum_i (m^i \vec{v}^i)}{dt} = \frac{d\vec{L}}{dt}$$

… we used the fact that the interaction forces are equal and opposite (Newton’s 3rd Law) and thus do not contribute to the dynamics of the system as a whole

- Euler’s First Law in terms of Linear Momentum:

$$\sum \vec{F}^i = \frac{d\vec{L}}{dt} = \frac{d(m\vec{v}_C)}{dt} = m\vec{a}_C$$

… valid for set of particles, rigid bodies, flexible bodies, mechanisms etc.

Extension to Rigid (or flexible) Bodies

- Replace the summations with integrations

$$\sum \vec{F}^i = \frac{d(\int \vec{v} dm)}{dt} \qquad \sum \vec{F}^i = \frac{d(\int \rho \vec{v} dV)}{dt}$$

- Equation for the Center of Mass

$$\boxed{\vec{r}^{OC} = \frac{\int \vec{r}^{OP} dm}{m}} \qquad \boxed{\vec{r}^{OC} = \frac{\int \rho \vec{r}^{OP} dV}{m}}$$

$$m = \int dm \qquad m = \int \rho dV$$

- Euler's First Law of Motion for translation of rigid bodies

$$\sum \vec{F}^i = \frac{d\vec{L}}{dt} = \frac{d(m\vec{v}_C)}{dt} = m\vec{a}_C$$

… same as for the set of particles

Angular Momentum

Think of moment as a general term
moment of a vector

(vector, depends on the point about which it is calculated)

moment that I use specifically in statics and dynamics
⇒ moment of force

- For a particle i, angular momentum about ant point P:

$$\vec{H}^{i/P} = \vec{r}^{Pi} \times \underbrace{m^i \vec{v}^i}_{\vec{L}^i}$$

Angular momentum is moment of (linear) momentum!

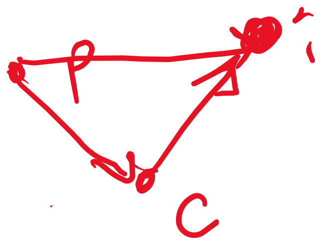
- For a set of particles or a rigid body:

$$\vec{H}^P = \sum_i (\vec{r}^{Pi} \times m^i \vec{v}^i) = \int_D \rho \, \vec{r}^{PQ} \times \vec{v}^Q \, dx\,dy\,dz$$

general point in the rigid body

- Angular momentum about any point P is related to angular momentum about the center of mass C and the moment of linear momentum about P:

$$\vec{H}^P = \vec{H}^C + \vec{r}^{PC} \times \vec{L}$$



$$\vec{H}^P = \vec{r}^{Pi} \times m^i \vec{v}^i = (\vec{r}^{PC} + \vec{r}^{Ci}) \times m^i \vec{v}^i$$

$$\vec{H}^{P} = \vec{H}^{C} + \vec{r}^{PC} \times \vec{L}$$

$$\vec{M}^{P} = \vec{M}^{C} + \vec{r}^{PC} \times \vec{F}$$

$$\vec{v}^{P} = \vec{V}^{C} + \vec{\omega} \times \vec{r}^{CP}$$

$$= \vec{V}^{C} + \vec{r}^{PC} \times \vec{\omega}$$

Euler's Second Law

- Newton's 2nd law for i^{th} particle in a system of particles:

$$\vec{F}^i + \sum_j \vec{f}^{ij} = \frac{{}^{I}d(m^i \vec{v}^i)}{dt}$$

$${}^{I}\vec{v}^{i}$$

- Moment about any point P :

$$\vec{r}^{Pi} \times \vec{F}^i + \vec{r}^{Pi} \times \sum_j \vec{f}^{ij} = \vec{r}^{Pi} \times \frac{{}^{I}d(m^i \vec{v}^i)}{dt}$$

- Summing the equations for all particles:

$$\sum_i \vec{r}^{Pi} \times \vec{F}^i = \sum_i \vec{r}^{Pi} \times \frac{{}^{I}d(m^i \vec{v}^i)}{dt}$$

$$\sum_i \vec{M}^{i/P} = \sum_i \vec{r}^{Pi} \times \frac{{}^{I}d(m^i \vec{v}^i)}{dt}$$

… is the term on the right related to angular momentum?

Yes, for particular choice for point P!

Euler’s Second Law in terms of Angular Momentum about a Point Fixed in an Inertial Frame

- If the moment is taken about a point O, which is fixed in an inertial frame, the angular momentum is:

O

$$\vec{H}^O = \sum_i (\vec{r}^{Oi} \times m^i \vec{v}^i)$$

- The rate of change of angular momentum is:

$$\frac{d\vec{H}^O}{dt} = \sum_i \left(\frac{d\vec{r}^{Oi}}{dt} \times m^i \vec{v}^i + \vec{r}^{Oi} \times \frac{d(m^i \vec{v}^i)}{dt} \right)$$

$$\frac{d\vec{H}^O}{dt} = \sum_i \left(\cancelto{0}{\vec{v}^i \times m^i \vec{v}^i} + \vec{r}^{Oi} \times \frac{d(m^i \vec{v}^i)}{dt} \right) \qquad \Rightarrow \qquad \frac{d\vec{H}^O}{dt} = \sum_i \vec{r}^{Oi} \times \frac{d(m^i \vec{v}^i)}{dt}$$

- Thus, if the point O is fixed in an inertia frame, Euler’s Second Law becomes:

$$\boxed{\sum_i \vec{M}^{i/O} = \frac{d\vec{H}^O}{dt}}$$

$$\sum \vec{F}^i = \frac{d\vec{L}}{dt}$$

Euler's Second Law in terms of Angular Momentum about Center of Mass

- Euler's Second Law for angular momentum about O

$$\sum \vec{M}^{i/O} = \frac{d\vec{H}^O}{dt}$$

- Angular momentum about C is related to angular momentum about O :

$$\vec{H}^O = \vec{H}^C + \vec{r}^{OC} \times \vec{L}$$

- The applied moment about C is related to applied moment about O :

$$\vec{M}^{i/O} = \vec{M}^{i/C} + \vec{r}^{OC} \times \vec{F}^i$$

- Thus, if the point C is the center of mass, Euler's Second Law becomes:

$$\boxed{\sum_i \vec{M}^{i/C} = \frac{d\vec{H}^C}{dt}}$$

$$\sum M^{i/o} = \sum M^{i/c} + \boxed{\vec{r}^{\,oc} \times \vec{F}}$$

$$\frac{d\vec{H}^o}{dt} = \frac{d}{dt}\left(\vec{H}^c + \vec{r}^{\,oc} \times \vec{L}\right) = \frac{d\vec{H}^c}{dt} + \frac{d\vec{r}^{\,oc}}{dt} \times \vec{L} + \boxed{\vec{r}^{\,oc} \times \frac{d\vec{L}}{dt}}$$

$\vec{v}^c$, 0, $m\vec{v}^c$, $\vec{F}$

$$\sum M^{i/c} = \frac{d}{dt}\vec{H}^c$$

$$\vec{H}^o = \vec{\vec{I}}^{\,o}\,\vec{\omega} \overset{2D}{\Longrightarrow} H^o_z = I^o_{zz}\,\omega_z$$

$$I^o_{zz} = I^c_{zz} + m(x_{co}^2 + y_{co}^2)$$

$$H^c_z = I^c_{zz}\,\omega_z$$

$$\omega_z = \dot{\theta}$$

Angular Momentum forms of Euler’s Second Law

- Euler’s Second Law for set of particles or for a rigid body can be simplified and written in terms of the angular momentum about ….
  - point fixed in inertial frame
    - For a rigid body, it could be a pivot point fixed in inertial frame and the rigid body (but not be it’s center of mass), e.g., rotation about a fixed axis not passing through the center of mass in 2D
  - center of mass of the set of particle or of the rigid body (even if moving)

$$\sum_i \vec{M}^{i/O} = \frac{d\vec{H}^O}{dt}$$

$$\sum_i \vec{M}^{i/C} = \frac{d\vec{H}^C}{dt}$$

… valid for set of particles, rigid bodies, flexible bodies, mechanisms etc.

- Euler’s Second Law cannot be as easily written in terms of any general point:

$$\sum_i \vec{M}^{i/P} = \sum_i \vec{r}^{\,Pi} \times \frac{d(m^i \vec{v}^{\,i})}{dt} \neq \frac{d\vec{H}^P}{dt}$$