Tensors and Dyadics

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3D Vector

- A quantity with magnitude and direction in 3D
- A quantity represented by three components along three (orthogonal) unit vectors in an axis system
- The components of the vector are transformed using a transformation (direction cosine) matrix.
- Using index notation (implied summation of repeated indices):

$$\vec{v} = v_i \hat{e}_i = v_i' \hat{e}_i'$$

- Transformation Equations (using index notation)

$$v_i' = C_{ij} v_j$$

  - Vector is a first order Tensor

Vector Multiplication

- Dot (scalar) Product:

$$s = \vec{u} \cdot \vec{v} \qquad\qquad \vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$$

$$\vec{v} = v_1 \hat{e}_1 + v_2 \hat{e}_2 + v_3 \hat{e}_3$$

- Cross (vector) Product:

$$\vec{w} = \vec{u} \times \vec{v} \qquad\qquad \vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$$

- Dyadic

$$\vec{\vec{D}} = \vec{u}\,\vec{v} \qquad\qquad \vec{u}\,\vec{v} \neq \vec{v}\,\vec{u}$$

- Dyadics do not have to be formed from just two vectors
  - Most dyadic are not just a product of two vector, they are the sum of products of two vectors.
  - Most generally written in terms of sum of product of all possible combinations of unit vectors

$$\vec{\vec{D}} = D_{11} \hat{e}_1 \hat{e}_1 + D_{12} \hat{e}_1 \hat{e}_2 \ldots$$

- $\hat{e}_i$ is a unit vector and $\hat{e}_i \hat{e}_j$ is a dyad (basis for dyadics)

Dyadic

$=u_1\hat{e}_1+u_2\hat{e}_2+u_3\hat{e}_3$

implied summation

- Consider two vectors

$$\vec{u} = u_i\hat{e}_i \qquad \vec{v} = v_i\hat{e}_i$$

- A dyadic can be written and expanded as

$$\vec{\vec{D}} = \vec{u}\,\vec{v} \quad \Longrightarrow \quad \vec{\vec{D}} = \underline{D_{ij}}\,\hat{e}_i\hat{e}_j$$

where,

$$D_{ij} = u_i v_j$$

- $\vec{\vec{D}} = D_{ij}\hat{e}_i\hat{e}_j$ is the most general representation of a dyadic
  - The components of the dyadic need not come from a product of components two vectors but may be obtained directly
- Vectors are represented by a column matrix of its components, while dyadic by a square matrix of its components
  - You should know the unit vectors or axis system being used to make sense of it

Various Orders of 3D Tensor

( )' is components in another frame

- Zeroth Order: Scalar

$$s \qquad\qquad s' = s$$

Invariants

s is invariant

$|\vec{v}|$ is invariant

det(D) is invariant

eig(D) is invariant

- First Order: Vector

$$\vec{v} = v_i \hat{e}_i = v_i' \hat{e}_i' \qquad\qquad v_i' = C_{ij} v_j$$

- Second Order: Second Order Tensor or Dyadic

$$\vec{\vec{D}} = D_{ij} \hat{e}_i \hat{e}_j = D_{ij}' \hat{e}_i' \hat{e}_j' \qquad\qquad D_{ij}' = C_{ik} C_{jl} D_{kl}$$

- Fourth Order: Fourth Order Tensor ☺

$$\vec{\vec{\vec{\vec{F}}}} = F_{ijkl} \hat{e}_i \hat{e}_j \hat{e}_k \hat{e}_l = F_{ijkl}' \hat{e}_i' \hat{e}_j' \hat{e}_k' \hat{e}_l' \qquad F_{ijkl}' = C_{im} C_{jn} C_{ko} C_{lp} F_{mnop}$$

Dyadic or 2nd Order 3D Tensor

- A quantity represented by 3^2 components along three unit vectors each of two sets of axis system
  - The two sets of unit vectors do not have to be the same but they most often are
  - The 9 numbers are typically written as a matrix with the first index giving the row number and the second index giving the column number
- The components of the 2^{nd} order tensor are transformed using a transformation matrix but we need two of them to take care of the two sets of axis systems
  - Again you can transform each axis system independently but most often you only have one set transforming into the new set
- The equations can be written in an index notation as well as matrix multiplication form as

$$D'_{ij} = C_{ik} C_{jl} D_{kl}$$

$$\boxed{[D'] = [C][D][C]^T}$$

Dyadic Algebra

- Sum of dyadics is a dyadic (commutative and associative)

$$\vec{\vec{D}} = \vec{\vec{E}} + \vec{\vec{F}} \qquad \rightarrow \qquad D_{ij} = E_{ij} + F_{ij}$$ (using same axis system)

$$\vec{\vec{D}} = \vec{\vec{E}} + \vec{\vec{F}} = \vec{\vec{F}} + \vec{\vec{E}}$$

$$\vec{\vec{D}} = \vec{\vec{E}} + (\vec{\vec{F}} + \vec{\vec{G}}) = (\vec{\vec{E}} + \vec{\vec{F}}) + \vec{\vec{G}}$$

- Product of dyadic with scalar is a dyadic (distributive)

$$\vec{\vec{D}} = s\vec{\vec{E}} = \vec{\vec{E}}s \qquad \rightarrow \qquad D_{ij} = sE_{ij} = E_{ij}s$$ (using same axis system)

$$\vec{\vec{D}} = s(\vec{\vec{E}} + \vec{\vec{F}}) = s\vec{\vec{E}} + s\vec{\vec{F}} \qquad \vec{\vec{D}} = (r+s)\vec{\vec{E}} = r\vec{\vec{E}} + s\vec{\vec{E}}$$

$$\vec{\vec{D}} = s\vec{u}\vec{v} = (s\vec{u})\vec{v} = \vec{u}(s\vec{v}) = s(\vec{u}\vec{v})$$

- Dot product of a dyadic with a vector is a vector (distributive and associative but not commutative)

$$\vec{v} = \vec{\vec{D}} \cdot \vec{u} \qquad \rightarrow \qquad v_i = D_{ij} u_j$$

$$\vec{v} = \vec{u} \cdot \vec{\vec{D}} \qquad \rightarrow \qquad v_i = u_j D_{ji} = D_{ji} u_j = D^T_{ij} u_j$$

$$\vec{\vec{D}} \cdot \vec{u} \neq \vec{u} \cdot \vec{\vec{D}}$$ (unless D is a symmetric tensor)

$$\vec{\vec{D}} \cdot (\vec{u} + \vec{v}) = \vec{\vec{D}} \cdot \vec{u} + \vec{\vec{D}} \cdot \vec{v} \qquad (\vec{\vec{D}} + \vec{\vec{E}}) \cdot \vec{u} = \vec{\vec{D}} \cdot \vec{u} + \vec{\vec{E}} \cdot \vec{u}$$

$$\vec{\vec{D}} = \vec{u}\,\vec{v} \qquad \Longrightarrow \qquad \vec{\vec{D}} \cdot \vec{w} = (\vec{u}\,\vec{v}) \cdot \vec{w} = \vec{u}(\vec{v} \cdot \vec{w}) = (\vec{v} \cdot \vec{w})\vec{u} = \vec{v} \cdot (\vec{w}\vec{u})$$

- Dot product of dyadic with a dyadic is a dyadic (associative and distributive but not commutative)

$$\vec{\vec{F}} = \vec{\vec{D}} \cdot \vec{\vec{E}} \qquad \rightarrow \qquad F_{ij} = D_{ik}E_{kj}$$ (its matrix multiplication)

$$\vec{\vec{D}} \cdot \vec{\vec{E}} \neq \vec{\vec{E}} \cdot \vec{\vec{D}}$$

$$\vec{\vec{D}} \cdot (\vec{\vec{E}} + \vec{\vec{F}}) = \vec{\vec{D}} \cdot \vec{\vec{E}} + \vec{\vec{D}} \cdot \vec{\vec{F}} \qquad (\vec{\vec{D}} + \vec{\vec{E}}) \cdot \vec{\vec{F}} = \vec{\vec{D}} \cdot \vec{\vec{F}} + \vec{\vec{E}} \cdot \vec{\vec{F}}$$

$$\vec{\vec{D}} = \vec{u}\,\vec{v} \quad \& \quad \vec{\vec{E}} = \vec{w}\,\vec{x} \implies$$

$$\vec{\vec{D}} \cdot \vec{\vec{E}} = (\vec{u}\,\vec{v}) \cdot (\vec{w}\,\vec{x}) = \vec{u}(\vec{v} \cdot \vec{w})\vec{x} = (\vec{v} \cdot \vec{w})(\vec{u}\,\vec{x}) = \vec{v} \cdot (\vec{w}\,\vec{u}\,\vec{x}) = (\vec{u}\,\vec{x}\,\vec{v}) \cdot \vec{w}$$

Unit Dyadic or Unit Tensor

- Defined as:

$$\vec{\vec{U}} = \hat{e}_i \hat{e}_i = \hat{e}_1 \hat{e}_1 + \hat{e}_2 \hat{e}_2 + \hat{e}_3 \hat{e}_3$$

- Dot product with the any vector is the same vector

$$\vec{v} = \vec{v} \cdot \vec{\vec{U}} = \vec{\vec{U}} \cdot \vec{v}$$

- Dot product with the any tensor is the same tensor

$$\vec{\vec{D}} = \vec{\vec{D}} \cdot \vec{\vec{U}} = \vec{\vec{U}} \cdot \vec{\vec{D}}$$

- The matric of components of the unit dyadic is the identity matrix in any axis system (assuming the basis is orthogonal unit vectors)

Vector Triple Product

- Known identity:

$$\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$$

- Write in terms of dyadic-vector dot product

$$\begin{aligned}
(\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w} &= (\vec{u} \cdot \vec{w})\vec{\vec{U}} \cdot \vec{v} - \vec{w}(\vec{u} \cdot \vec{v}) \\
&= (\vec{u} \cdot \vec{w})\vec{\vec{U}} \cdot \vec{v} - (\vec{w}\vec{u}) \cdot \vec{v} \\
&= \left[(\vec{u} \cdot \vec{w})\vec{\vec{U}} - \vec{w}\vec{u}\right] \cdot \vec{v}
\end{aligned}$$

Angular Momentum and Moment of Inertia in Tensor Form (Vector-Dyadic)

- Using the vector triple product identity

$$
\begin{aligned}
\vec{H}^C &= \int \left[\vec{r}^{CQ} \times \left(\vec{\omega} \times \vec{r}^{CQ}\right)\right] dm \\
&= \int \left[\left(\vec{r}^{CQ} \cdot \vec{r}^{CQ}\right) \vec{\omega} - \vec{r}^{CQ} \left(\vec{r}^{CQ} \cdot \vec{\omega}\right)\right] dm \\
&= \underbrace{\int \left[\left(\vec{r}^{CQ} \cdot \vec{r}^{CQ}\right) \vec{\vec{U}} - \vec{r}^{CQ}\vec{r}^{CQ}\right] dm}\cdot \vec{\omega} \\
&= \vec{\vec{I}}^C \cdot \vec{\omega}
\end{aligned}
$$

- Moment of Inertia is thus a Dyadic given by:

$$
\vec{\vec{I}}^C = \int \left[\left(\vec{r}^{CQ} \cdot \vec{r}^{CQ}\right) \vec{\vec{U}} - \vec{r}^{CQ}\vec{r}^{CQ}\right] dm
$$

- The components of the dyadic in different frames is given by

$$
I'_{ij} = C_{ik} C_{jl} I_{kl} \implies [I'] = [C][I][C]^T
$$

Rotation Dyadic or Rotation Tensor

- We can rotate a vector by taking a dot product with a rotation tensor.
- For two reference frames with unit vectors in the initial frame $\hat{e}_i$ and unit vectors in the deformed frame $\hat{e}'_i$:

$$\vec{\vec{C}} = \hat{e}'_i \hat{e}_i = \hat{e}'_1 \hat{e}_1 + \hat{e}'_2 \hat{e}_2 + \hat{e}'_3 \hat{e}_3$$

- We can write the components of the dyadic in a single frame as

$$\vec{\vec{C}} = \vec{\vec{U}} \cdot \vec{\vec{C}} = (\hat{e}_i \hat{e}_i) \cdot (\hat{e}'_j \hat{e}_j) = \hat{e}_i (\hat{e}_i \cdot \hat{e}'_j) \hat{e}_j = C_{ij} \hat{e}_i \hat{e}_j$$

where,

$$C_{ij} = \hat{e}_i \cdot \hat{e}'_j$$

- The components of the rotation tensor in either the initial or rotated frame give you the direction cosine matrix (or its transpose)

Higher Order Tensors

- Not as easy to represent as planar matrices – you represent them as arrays in n dimensions
- Not as easy to write dot products as just positioning to the left or right only for dot products along the first or last index
- We thus use index notation to do most of the work
- Example: Material Stiffness Tensor is a 4^{th} order tensor and thus the components of the stiffness can be written as an 3 x 3 x 3 x 3 array in a given axis system.
  - You can easily transform the tensor to be represented in another axis system using the tensor transformation equation – for example, it is common in structural analysis of composites to prescribe the stiffness in the material frame but then transform it to the structural analysis frame during analysis.