$$\sum \vec{M}^C = \frac{{}^I d\vec{H}^C}{dt}$$

center of mass

$$\sum \vec{M}^O = \frac{{}^I d\vec{H}^O}{dt}$$

fixed point

$$\sum \vec{M}^P \neq \frac{{}^I d\vec{H}^P}{dt}$$

Moment of Inertia and Euler's Dynamical Equations

Valid for rigid bodies, flexible bodies, system of particles & bodies

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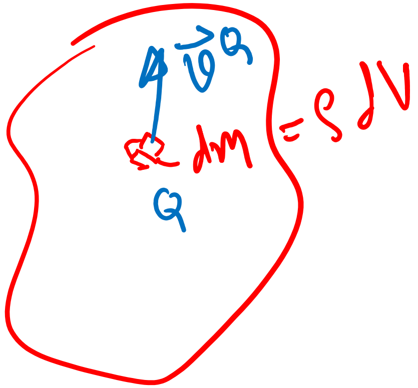
Angular Momentum Expression about CM



- For a set of particles:

$$\vec{H}^C = \sum (\vec{r}^{Ci} \times m^i \vec{v}^i)$$

- For a body:

$$\vec{H}^C = \int (\vec{r}^{CQ} \times \vec{v}^Q)\, dm$$

where Q is a point on dm on the body

- For a rigid body, we can write the velocity at any point in terms of velocity at one point and the angular velocity of the body

$$\vec{v}^Q = \vec{v}^C + \vec{\omega} \times \vec{r}^{CQ}$$

2 point formula

- Thus:

$$\vec{H}^C = \int [\vec{r}^{CQ} \times (\vec{v}^C + \vec{\omega} \times \vec{r}^{CQ})]\, dm$$

$$\vec{H}^C = \int [\vec{r}^{CQ} \times (\vec{\omega} \times \vec{r}^{CQ})]\, dm$$

geom

mass dist

$$\int \vec{r}^{\,CQ} \times \vec{v}^{\,C}\, dm + \int \vec{r}^{\,CQ} \times (\vec{\omega} \times \vec{r}^{\,CQ})\, dm$$

geom

const

$$\parallel$$

$$\left(\int \vec{r}^{\,CQ} dm\right) \times \vec{v}^{\,C}$$

$$\parallel$$

$$m\, \cancel{\vec{r}^{\,CC}}$$

$$\parallel$$

$$0$$

Angular Momentum Expression about CM in terms of components in the body frame

- Consider a point Q on the rigid body B with CM at C: $\vec{r}^{\,CQ} = x\hat{b}_1 + y\hat{b}_2 + z\hat{b}_3$
- The angular velocity can be written in terms of its components in body axis

$$\vec{\omega} = \omega_x \hat{b}_1 + \omega_y \hat{b}_2 + \omega_z \hat{b}_3$$

- The cross product gives:

$$\begin{aligned}\vec{H}^C &= \int \left[\vec{r}^{\,CQ} \times (\vec{\omega} \times \vec{r}^{\,CQ})\right] dm \\ &= \left[\boxed{\int (y^2+z^2)dm}\,\omega_x \boxed{-\int (xy)dm}\,\omega_y \boxed{-\int (xz)dm}\,\omega_z\right]\hat{b}_1 \\ &+ \left[\boxed{-\int (xy)dm}\,\omega_x \boxed{+\int (x^2+z^2)dm}\,\omega_y \boxed{-\int (yz)dm}\,\omega_z\right]\hat{b}_2 \\ &+ \left[\boxed{-\int (xz)dm}\,\omega_x \boxed{-\int (yz)dm}\,\omega_y \boxed{+\int (x^2+y^2)dm}\,\omega_z\right]\hat{b}_3\end{aligned}$$

- We have the measure numbers of the angular moment in the body axis – use of any other axis is prohibitively complicated

$v_x\hat{i} + v_y\hat{j} + v_z\hat{k} = v'_x\hat{i}' + v'_y\hat{j}' + v'_z\hat{k}'$

Moments and Products of Inertia

$\begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix}$

Scalar - 0th order tensor

Vector - 1st order tensor

MoI, Stress, Strain - 2nd order tensor

Moments of Inertia

$$I^C_{xx} = \int (y^2 + z^2)dm$$

$$I^C_{yy} = \int (x^2 + z^2)dm$$

$$I^C_{zz} = \int (x^2 + y^2)dm$$

$I^C_{xx} > 0$

Products of Inertia

$$I^C_{xy} = -\int (xy)dm$$

$$I^C_{xz} = -\int (xz)dm$$

$$I^C_{yz} = -\int (yz)dm$$

$I^C_{yx} = -\int (yx)dm$

- The angular momentum can be written in terms of the **Moment on Inertia matrix** as:

$$I^C = \begin{bmatrix} I^C_{xx} & I^C_{xy} & I^C_{xz} \\ I^C_{xy} & I^C_{yy} & I^C_{yz} \\ I^C_{xz} & I^C_{yz} & I^C_{zz} \end{bmatrix} \qquad \vec{H}^C = \begin{Bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{Bmatrix}^T \begin{bmatrix} I^C_{xx} & I^C_{xy} & I^C_{xz} \\ I^C_{xy} & I^C_{yy} & I^C_{yz} \\ I^C_{xz} & I^C_{yz} & I^C_{zz} \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}$$

I^C_{yx}

$\vec{I}^C = I^C_{xx}\hat{i}\hat{i} + I^C_{xy}\hat{i}\hat{j} + I^C_{xz}\boxed{\hat{i}\hat{k}} \rightarrow$ dyadics $= I^C_{xy}\hat{i}'\hat{j}' + I^C_{xy}\hat{i}'\hat{j}'$

Product of Inertia and Symmetry

- Any rigid body (or part of a rigid body) will have a zero product of inertia about its CM if either of the two axis representing the product of inertia is an axis of symmetry.
  - If there is one axis of symmetry, two products of inertia are zero
  - If there are two or more axes of symmetry, all the three products of inertia will be zero

Aircraft: y is axis of symm $\Rightarrow I_{xy} = I_{yz} = 0$

xz is plane of symm

Euler's Dynamical Equations for Rotational Motion of Rigid Bodies

$\bar{a} \times \bar{b}$ $\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \times \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$

$$\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

- Euler's Second Law

$$\boxed{\sum \vec{M}^C = \frac{{}^I d\vec{H}^C}{dt}} = \frac{{}^B d\vec{H}^C}{dt} + \underline{\underline{{}^I\vec{\omega}^B}} \times \vec{H}^C$$

- Using the expression for angular momentum in terms of the measure number in body axis

assuming MoI constant in Body Frame

$$\sum \vec{M}^C = \begin{Bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{Bmatrix}^T \begin{bmatrix} I^C_{xx} & I^C_{xy} & I^C_{xz} \\ I^C_{xy} & I^C_{yy} & I^C_{yz} \\ I^C_{xz} & I^C_{yz} & I^C_{zz} \end{bmatrix} \begin{Bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{Bmatrix} + \begin{Bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{Bmatrix}^T \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \boxed{\begin{bmatrix} I^C_{xx} & I^C_{xy} & I^C_{xz} \\ I^C_{xy} & I^C_{yy} & I^C_{yz} \\ I^C_{xz} & I^C_{yz} & I^C_{zz} \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}}$$

... note that the second term is nonlinear!

- Also note the cross-product matrix or dual matrix denoted by $\tilde{\omega} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$

$$\begin{Bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{Bmatrix} \begin{bmatrix} \dot{I} & \dot{I} & \cdot \end{bmatrix} \begin{Bmatrix} \omega \\ \omega \\ \omega \end{Bmatrix}$$