Hamiltonian Mechanics

Phase Space (q, p) gen cood & gen momenta

Symplectic

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Lagrangian Mechanics

Lagrange's Equations for Conservative Systems

- Lagrangian:

$$\mathcal{L}(q_i, \dot{q}_i) = T(q_i, \dot{q}_i) - V(q_i)$$

- Action:

$$S = \int_{t_1}^{t_2} \mathcal{L}(q_i, \dot{q}_i) dt$$

- Hamilton's Principle: for a system with prescribed initial and final values of the generalized coordinates, the path taken has an extremum action

$q(t)$

$$\delta S = 0$$

- Gives us Lagrange's Equations:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0$$

Write integral form → integration - weak form → direct int for soln

Diff form - gives you EoM diff eqns

Phase Space

\# of dof
of gen coord

- Lagrange's equations are n 2nd order ODEs in terms of the configuration space q_i
- One could write the system as $2n$ 1st order ODEs in terms of q_i and $\dot{q}_i$
- $(q_i, \dot{q}_i)$ is considered the phase space
- A better way to represent the phase space is in terms of (p_i, q_i) where p_i are the generalized momenta defined as:

$$\boxed{p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i}}$$

p_i's in terms of $\dot{q}_i$'s

1 2nd order $m\ddot{x} + kx = 0$

2 1st order $\left[\begin{array}{l} \dot{x} = v \\ m\dot{v} + kx = 0 \end{array}\right.$

- p_i is the conjugate to q_i

Hamiltonian

- We can define the Hamiltonian in terms of the phase space (column) vectors $\bar{p}$ and $\bar{q}$ as:

$$\mathcal{H}(\bar{p},\bar{q},t) = \bar{p}\cdot\dot{\bar{q}} - \mathcal{L}(\bar{q},\dot{\bar{q}},t)$$

where, the $\dot{\bar{q}}$ are written in terms of the $\bar{p}$ using the expression for the generalized momenta:

$$\mathcal{H}(\bar{p},\bar{q},t) = \bar{p}\cdot\dot{\bar{q}}(\bar{p},\bar{q},t) - \mathcal{L}(\bar{q},\dot{\bar{q}}(\bar{p},\bar{q},t),t) = \bar{p}(t)\cdot\dot{\bar{q}}(\bar{p}(t),\bar{q}(t),t)$$

- For time invariant systems, i.e., the constraints or potential are not a functions of time, the Hamiltonian of the system is equal to the total energy

$$\mathcal{H} = T + V$$

otherwise, it is an energy-type functional that is conserved.

$\dot{x} = Ax$

$\dot{x} = A(t)\,x$

$A(t+T) = A(t)$

Hamilton's Equations

Lagrange's Eqns have some symmetry fr symm M & K matrix

- The Hamilton's Equations (kinematic and motion) can be derived using the Hamiltonian as:

Hamilton's Eqns have skew-symmetry

$$\boxed{\begin{aligned} \dot{q}_i &= \frac{\partial \mathcal{H}}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial \mathcal{H}}{\partial q_i} \end{aligned}}$$

if H is not a function of p_j then $\dot{q}_j = 0$ or $q_j = \text{const}$

q_j is ignorable coord.

… these are in terms of the phase space variables and lead to first order ODEs (state space form)

- In addition we have:

$$\frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}$$

Look into a change of p variables and from there get q variable which may not seem natural but could have advantages.

Conservation of Hamiltonian

$$\frac{d\,H(q,p,t)}{dt} = \frac{\partial H}{\partial q}\cdot\dot{q} + \frac{\partial H}{\partial p}\cdot\dot{p} + \frac{\partial H}{\partial t}$$

$$= \cancel{\frac{\partial H}{\partial q}\cdot\frac{\partial H}{\partial p}} + \cancel{\frac{\partial H}{\partial p}\cdot\left(-\frac{\partial H}{\partial q}\right)} + \frac{\partial H}{\partial t}$$

- The rate of change of the Hamiltonian is:

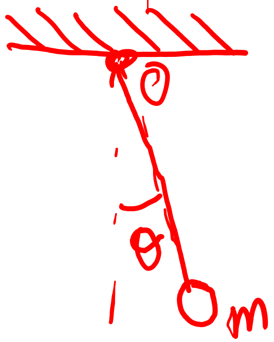
$$\frac{d\mathcal{H}}{dt} = \frac{\partial\mathcal{H}}{\partial\bar{q}}\dot{\bar{q}} + \frac{\partial\mathcal{H}}{\partial\bar{p}}\dot{\bar{p}} + \frac{\partial\mathcal{H}}{\partial t}$$

$$= \frac{\partial\mathcal{H}}{\partial t} = -\frac{\partial\mathcal{L}}{\partial t}$$

… if Hamiltonian (or Lagrangian) is not an explicit function of time, the Hamiltonian (Energy) is conserved.

Flow in the Phase Space

- One can plot the trajectories of any simulation – the flow – in the $2n$ dimensional phase space
  - Only possible to visualize for $n = 1$ $(2n = 2)$
  - Example pendulum …



$$T = \frac{1}{2} m (L\dot{\theta})^2 = \frac{1}{2} m L^2 \dot{\theta}^2$$

$$p = \frac{\partial T}{\partial \dot{\theta}} = m L^2 \dot{\theta}$$

$$\dot{\theta} = \frac{1}{mL^2} p$$

$$T = \frac{1}{2} \frac{1}{mL^2} p^2$$

$$V = -mgL\cos\theta$$

$$H = p \cdot \left(\frac{1}{mL^2} p\right) - \left(\frac{1}{2}\frac{1}{mL^2} p^2 + mgL\cos\theta\right)$$

$$= \frac{1}{2}\frac{1}{mL^2} p^2 - mgL\cos\theta = E$$

Time invariant: $H = T + V$

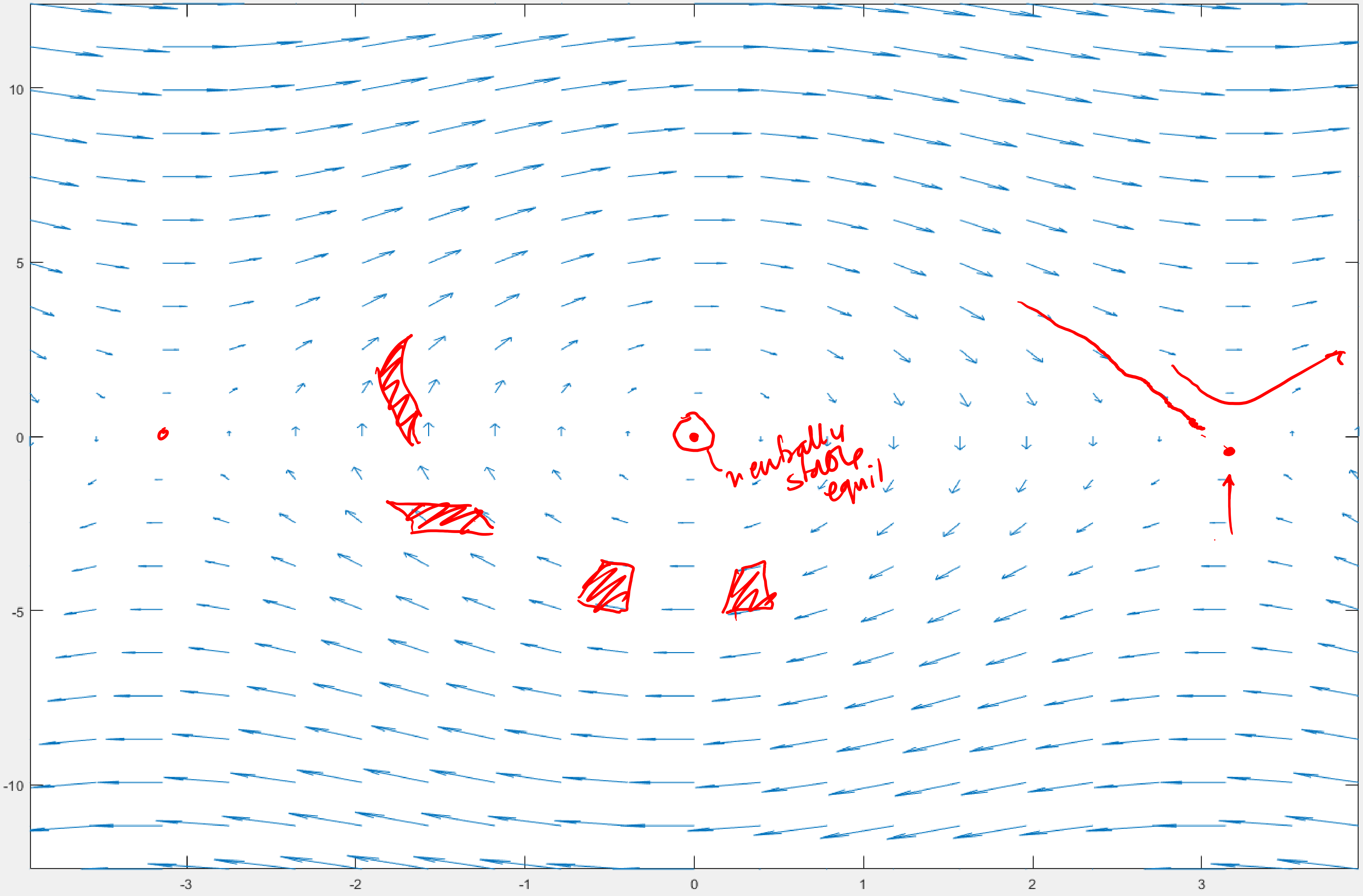
$$\dot{q} = \frac{\partial H}{\partial p}$$

$$\boxed{\dot{\theta} = \frac{1}{mL^2} p}$$

$$\dot{p} = -\frac{\partial H}{\partial q}$$

$$\boxed{\dot{p} = -mgL\sin\theta}$$

$$\begin{Bmatrix} \dot{\theta} \\ \dot{p} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{mL^2} p \\ -mgL\sin\theta \end{Bmatrix} \overset{\text{linearize}}{\approx} \begin{bmatrix} 0 & \frac{1}{mL^2} \\ -mgL & 0 \end{bmatrix} \begin{Bmatrix} \theta \\ p \end{Bmatrix}$$



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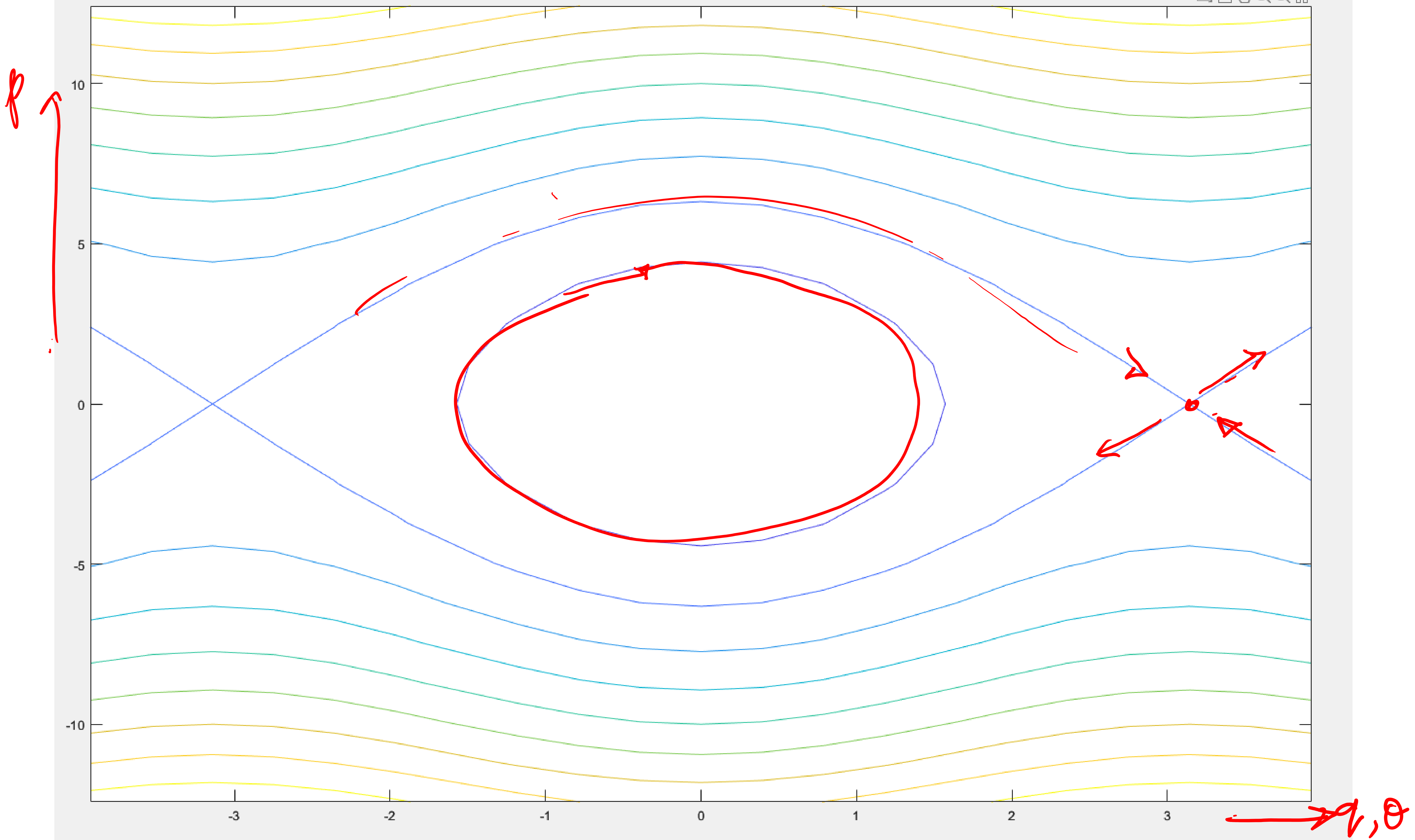
neutrally stable equil



10
5
0
-5
-10
-3
-2
-1
0
1
2
3
p
q, θ

Integral Form

- Hamilton's principle of least action for conservative systems (using Lagrangian):

$$\delta \int_{t_1}^{t_2} \mathcal{L}(q_i, \dot{q}_i) dt = 0$$

- Using the phase space coordinates we can derive based on Hamiltonian:

$$\int_{t_1}^{t_2} \left[\left(\dot{\bar{q}} - \frac{\partial \mathcal{H}}{\partial \bar{p}} \right) \delta \bar{p} - \left(\dot{\bar{p}} + \frac{\partial \mathcal{H}}{\partial \bar{q}} \right) \delta \bar{q} \right] dt = 0$$

  - You can solve or time march the Hamilton's equations approximately using the above integral form rather than the differential form given earlier. This form is also called the weighted residual or weak form.

Liouville's Theorem and Symplectic Structure

- If we consider $\bar{p}$ and $\bar{q}$ to be coordinates and $\dot{\bar{p}}$ and $\dot{\bar{q}}$ to be the velocities. The divergence of the $(\dot{\bar{p}}, \dot{\bar{q}})$ is zero:

$$\bar{\nabla}(\dot{\bar{p}}, \dot{\bar{q}}) = \frac{\partial \dot{\bar{p}}}{\partial \bar{p}} + \frac{\partial \dot{\bar{q}}}{\partial \bar{q}} = 0$$

$$\rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \frac{\partial \rho}{\partial t}$$

- Liouville's theorem states that the phase space flow is divergence-free or that Hamiltonian systems are symplectic
  - Incompressible flow in fluids is divergence-free: the area/volume of a blob of fluid is maintained over time though the shape may change
  - The set of initial conditions in the phase space are like the blob, which flow in a 2D (for single DoF system) or 2*n*-dimensional space while conserving the volume in that space
- One could use the above to derive time marching schemes that conserve the area/volume in the phase space (thus conserving the underlying mathematical structure of the physical system) – leading to symplectic integrators

Example (2 DoF)