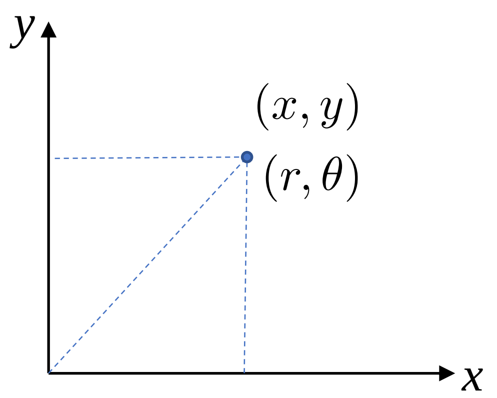
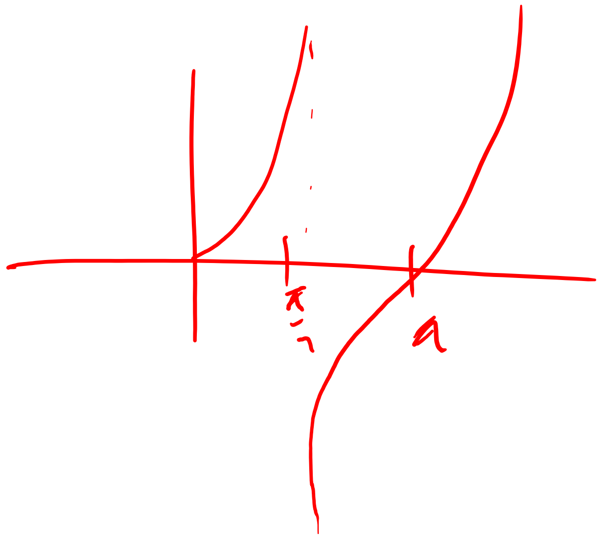
Polar/Cylindrical and Natural Coordinate Systems

Polar Coordinates (2D): Coordinate Transformation





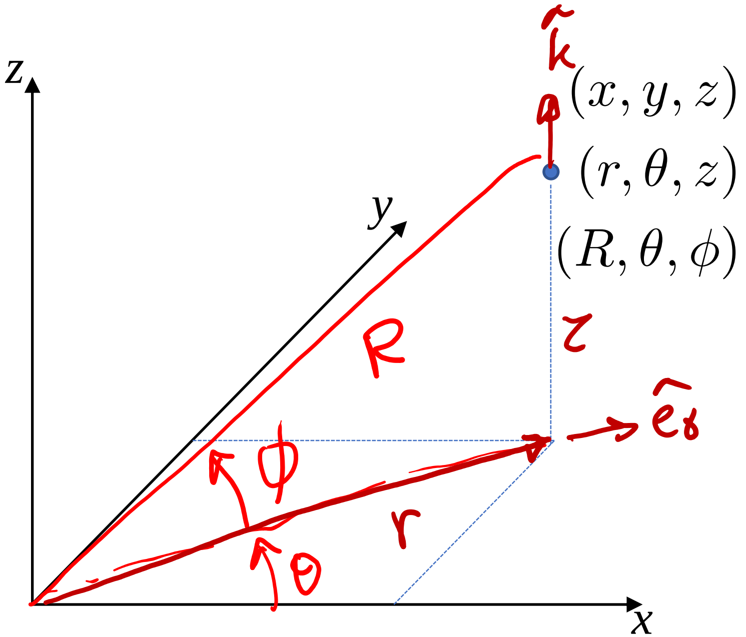
- The position of a point in 2D space can be represented by coordinates
  - Cartesian or Rectangular (x,y)
  - Polar (r,θ)
- The coordinates can be related to one another

$$x = r\cos\theta \qquad \theta = \tan^{-1}\frac{y}{x}$$

$$y = r\sin\theta \qquad r = +\sqrt{x^2 + y^2}$$

... the arctan should take into account the correct quadrant (`atan2`)!

Cylindrical Coordinates (3D): Coordinate Transformation



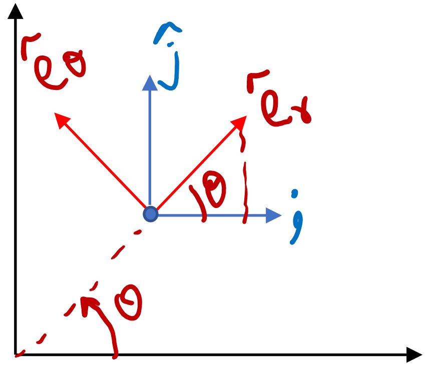
- The position of a point in 3D space can also be represented by coordinates
  - Cartesian or Rectangular (x,y,z)
  - Cylindrical (r,θ,z)
  - Spherical (R,θ,ϕ)
- The coordinates can be related as:

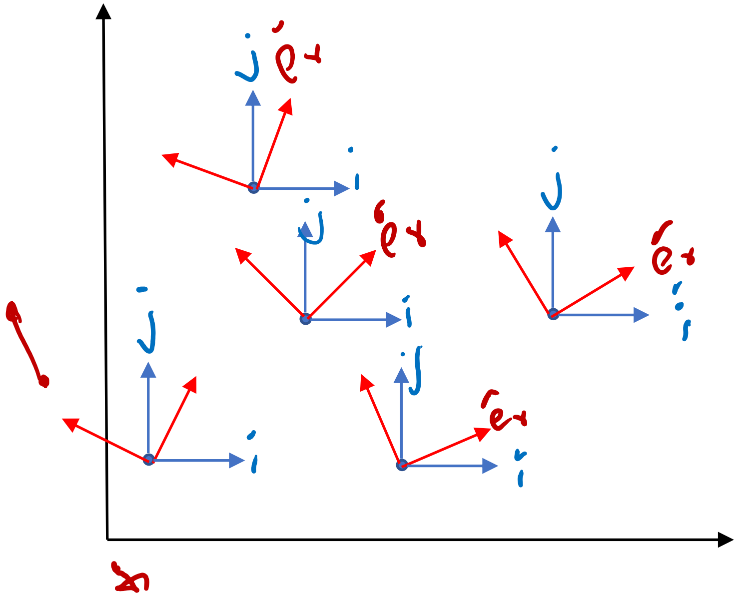
$$x = r\cos\theta \qquad y = r\sin\theta$$

$$z = z$$

$$\theta = \tan^{-1}\frac{y}{x} \qquad r = +\sqrt{x^2+y^2}$$

Polar Coordinates: Vector Transformation





- Cartesian coordinate system unit vectors: $\hat{i} \quad \hat{j}$
- Polar coordinate system unit vectors: $\hat{e}_r \quad \hat{e}_\theta$
- The unit vectors are related

$$\hat{e}_r = \cos\theta\ \hat{i} + \sin\theta\ \hat{j}$$

$$\hat{e}_\theta = -\sin\theta\ \hat{i} + \cos\theta\ \hat{j}$$

- Unit vectors in polar coordinate system are not the same at all points in the reference frame (they depends on θ)
  - The unit vectors depend on space (though not on time)
  - As a particle moves in space, the unit vectors change, and so a vector corresponding to a moving particle (say, position or velocity or acceleration) ends up being represented by unit vectors changing with time

Consider Motion in 3D Space

- Position in an Inertial Frame: $\vec{p}(t) = r(t)\hat{e}_r + z(t)\hat{k}$

$$\vec{p}(t) = r(t)\hat{e}_r(\theta(t)) + z(t)\hat{k}$$

- Velocity Vector and Acceleration given by:

$$\vec{v}(t) = \dot{\vec{p}}(t) \quad = \dot{r}\,\hat{e}_r + r\,\dot{\hat{e}}_r + \dot{z}\,\hat{k} + z\,\cancel{\dot{\hat{k}}}^{\,0}$$

$$\vec{a}(t) = \dot{\vec{v}}(t) = \ddot{\vec{p}}(t)$$

- Because $\hat{e}_r$ $\hat{e}_\theta$ are function of *θ(t)*

$$\vec{v}(t) = \dot{r}(t)\hat{e}_r + r(t)\dot{\hat{e}}_r + \dot{z}(t)\hat{k}$$

$$\vec{a}(t) = \ddot{r}(t)\hat{e}_r + 2\dot{r}(t)\dot{\hat{e}}_r + r(t)\ddot{\hat{e}}_r + \ddot{z}(t)\hat{k}$$

Differentiation of Unit Vectors

- The unit vectors are related as:

$$\hat{e}_r = \cos\theta(t)\ \hat{i} + \sin\theta(t)\ \hat{j}$$

$$\hat{e}_\theta = -\sin\theta(t)\ \hat{i} + \cos\theta(t)\ \hat{j}$$

- Differentiating w.r.t. time, knowing that $\hat{i}$ $\hat{j}$ are not function of time:

$$\dot{\hat{e}}_r = -\sin\theta \cdot \dot{\theta}\,\hat{i} + \cos\theta \cdot \dot{\theta}\,\hat{j} = \dot{\theta}\,(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$$

$$= \dot{\theta}\,\hat{e}_\theta$$

$$\dot{\hat{e}}_\theta = -\dot{\theta}\,\hat{e}_r$$

- The derivative of any constant magnitude vector (e.g., unit vector) is always perpendicular to the vector.

Velocity and Acceleration in Cylindrical Coordinates

- Expressions for velocity and acceleration:

$$\vec{v}(t) = \dot{r}(t)\hat{e}_r + r(t)\dot{\hat{e}}_r + \dot{z}(t)\hat{k}$$

$$\vec{a}(t) = \ddot{r}(t)\hat{e}_r + 2\dot{r}(t)\dot{\hat{e}}_r + r(t)\ddot{\hat{e}}_r + \ddot{z}(t)\hat{k}$$

- Using:

$$\dot{\hat{e}}_r = \dot{\theta}(t)\hat{e}_\theta$$

$$\ddot{\hat{e}}_r = \ddot{\theta}(t)\hat{e}_\theta + \dot{\theta}(t)\underbrace{\dot{\hat{e}}_\theta}_{= -\dot{\theta}\hat{e}_r} = \ddot{\theta}(t)\hat{e}_\theta - \dot{\theta}(t)^2\hat{e}_r$$

- Final expressions:

$$\vec{a} = \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}$$

$$\vec{v}(t) = \dot{r}(t)\hat{e}_r + r(t)\dot{\theta}(t)\hat{e}_\theta + \dot{z}(t)\hat{k}$$

$$\vec{a}(t) = [\ddot{r}(t) \underbrace{\boxed{- r(t)\dot{\theta}(t)^2}}_{\text{Centripetal acc}}]\hat{e}_r + [r(t)\ddot{\theta}(t) \underbrace{\boxed{+ 2\dot{r}(t)\dot{\theta}(t)}}_{\text{Coriolis's accel}}]\hat{e}_\theta + \ddot{z}(t)\hat{k}$$

Centripetal and Coriolis Acceleration

- Part of the acceleration dependent on the first-order time derivatives
  - Present even if second-order time derivatives are zero
- Nonlinear (in terms of $r(t)$ and $\theta(t)$ and their time derivatives)

- Centripetal Acceleration: $-r(t)\dot{\theta}(t)^2\hat{e}_r$

- Coriolis Acceleration: $2\dot{r}(t)\dot{\theta}(t)\hat{e}_\theta$

Analysis in 2D/3D

- Again, differentiation is very much easier than integration
  - Given displacement, we can easily calculate the velocity and acceleration
  - Analytical solutions of the derivatives is typically possible
- Integration in Cylindrical coordinate system is not straightforward …
  - Even if the acceleration is only a function of time, the equations are <u>nonlinear</u> and <u>coupled</u> - analytical solutions of the integral typically not possible
    - Given acceleration, calculation of displacement typically needs numerical integration
  - If the acceleration is a function of the states ($r(t)$ and $\theta(t)$ and their time derivatives), the equations are nonlinear and coupled ODEs and we need numerical ODE solvers

Satellite Simulation
(Polar coordinate system)

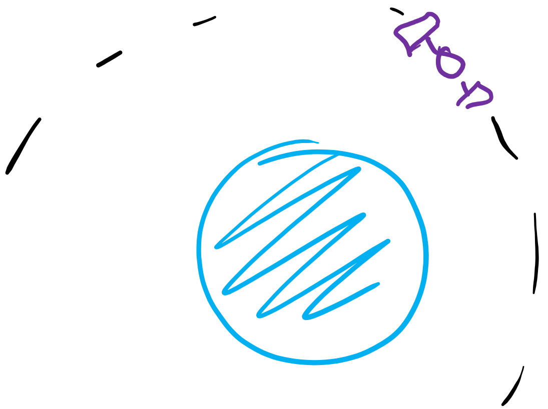
- Equations of Motion (for acceleration)

$$\vec{F} = -\frac{GM_E m \hat{e}_r}{r^2}$$

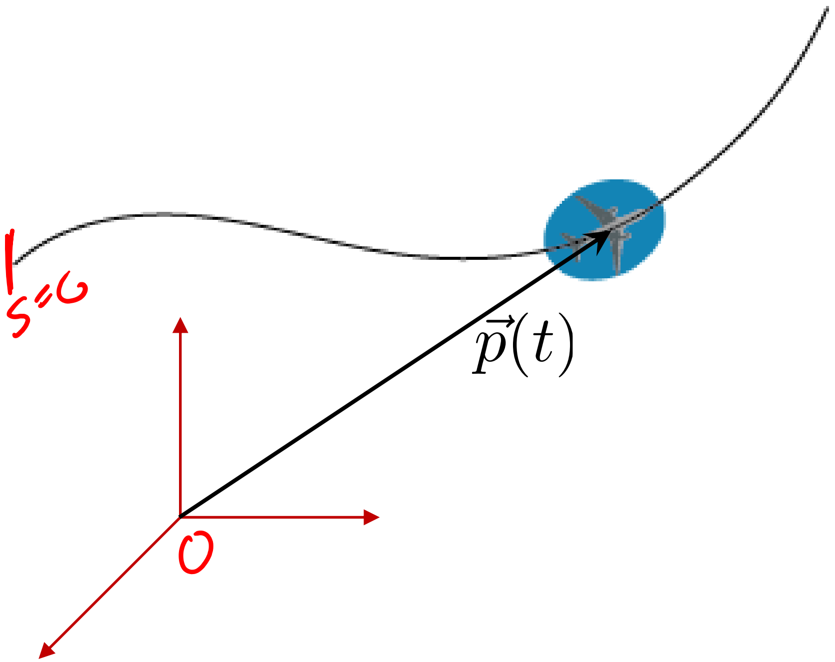
$$m\vec{a} = \vec{F}$$

$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{GM_E m}{r^2}$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0$$



Natural Coordinates: Path and Coordinate along the Path



- For a particle (or later the center of mass of a rigid body) in motion, the path of the particle is given by $\vec{p}(t)$
- We can write the position vector in terms of the coordinate along the path – arclength coordinate – instead of time

$$\vec{p}(t) = \vec{p}(s(t))$$

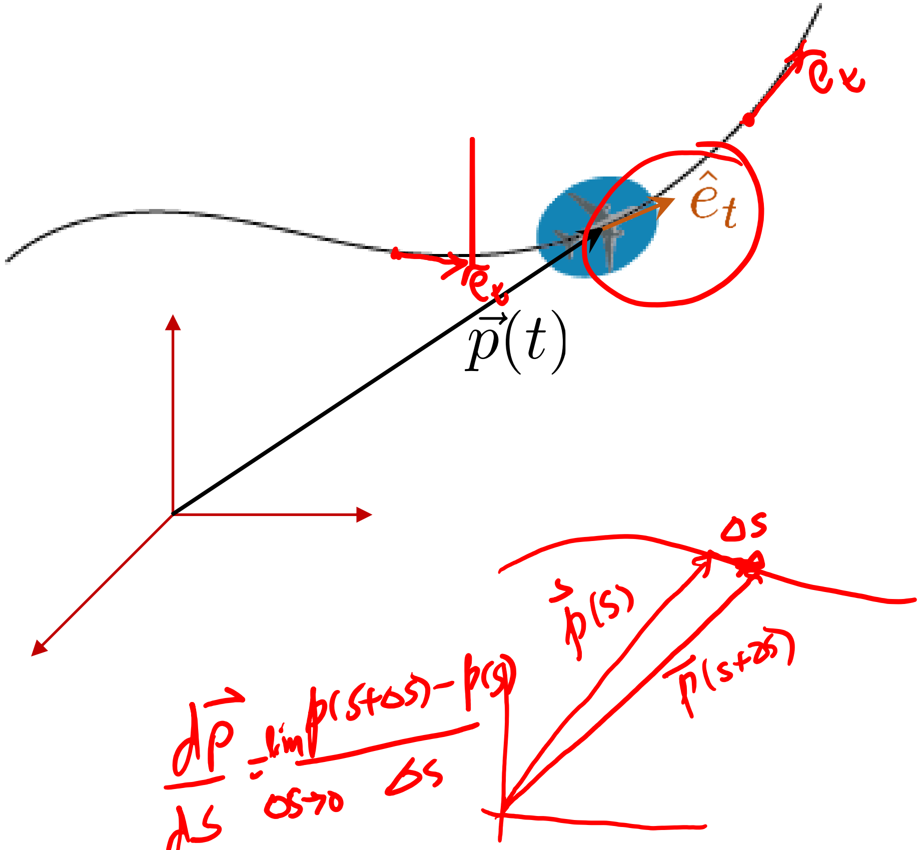
- *s(t)* is the arclength coordinate which is a function of time – it is the distance travelled along the (curved) path

Tangential Unit Vector and Velocity

- The tangential unit vector is along the path
- The velocity can be written as:

$$\vec{v}_P = \frac{d\vec{p}}{dt} = \frac{d\vec{p}}{ds}\frac{ds}{dt} = \dot{s}\hat{e}_t$$

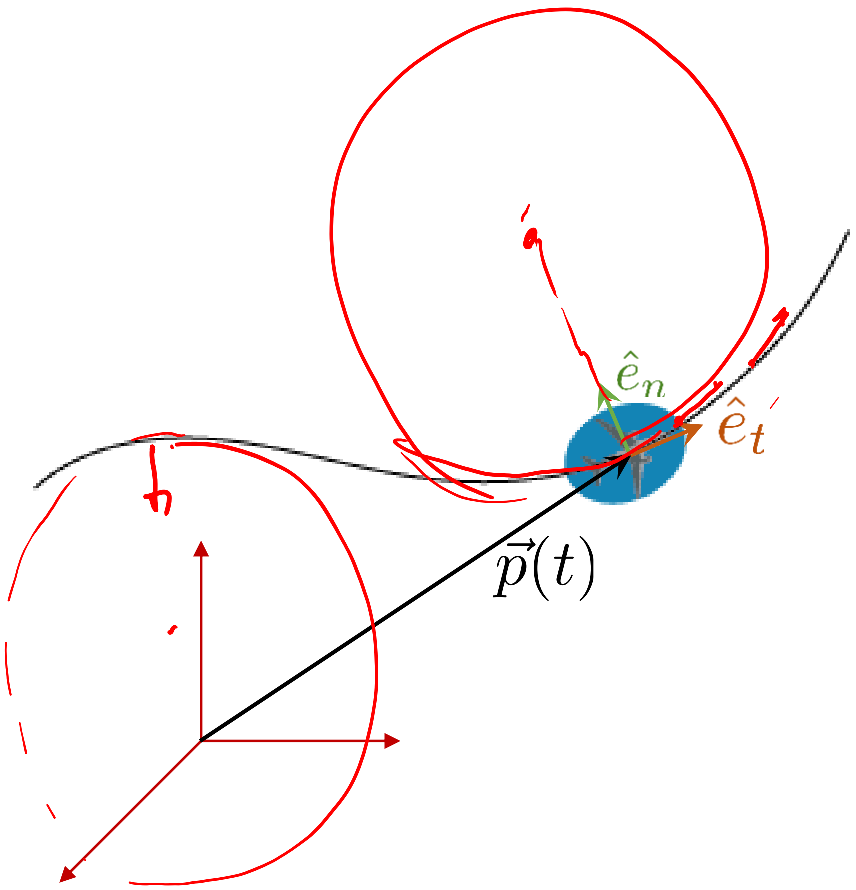
$$\hat{e}_t = \frac{d\vec{p}}{ds} = \vec{p}'$$



$$\frac{d\vec{p}}{ds} = \lim_{\Delta s \to 0} \frac{p(s+\Delta s) - p(s)}{\Delta s}$$

- The velocity in always in the tangential direction
- The unit vector is not in the body frame
  - For airplanes: the unit vector is not along the nose of the airplane (though typically close to it)
  - For airplanes: the angle of attack and sideslip are typically small and measure the body orientation relative to the tangential/velocity unit vector, e.g., wind axis.
- No easy way to write the position vector for a general case as a function of $s(t)$ and $\hat{e}_t$

Principal Normal



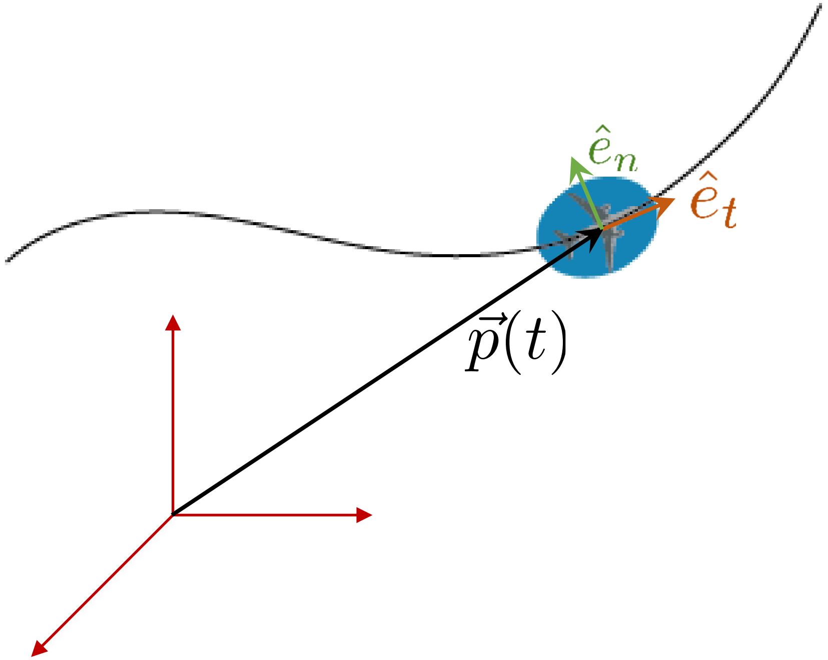
- The principal normal is not defined if the airplane is moving in a straight line
  - Natural coordinates are not used for rectilinear motion
- For curved path, the principal normal unit vector is directed towards the change in the direction of the unit tangent:

rate of change in space w.r.t. s

$$\frac{d\hat{e}_t/ds}{|d\hat{e}_t/ds|} = \hat{e}_n = \frac{\hat{e}_t'}{|\hat{e}_t'|} = \frac{\vec{p}''}{|\vec{p}''|} \qquad \vec{p}' = \hat{e}_t$$

  - The derivative of a vector with constant magnitude is always perpendicular to the vector
- The above vector is directed towards the center of the curved path (based on the local curvature of the path at that point).

Curvature and Binormal



- On a path you have:
  - a point
  - a slope at that point (first derivative) $\hat{e}_t$
  - a curvature at that point (second derivative)
- The magnitude of the local curvature is inverse of the local radius of curvature of the path at that point: ρ

$$\underset{\text{curvature}}{\kappa} = \frac{1}{\rho} = |\vec{p}''| = |\hat{e}_t'|$$

radius of curvature

$$\hat{e}_n = \rho \hat{e}_t'$$

$$\hat{e}_t' = \frac{1}{\rho}\hat{e}_n$$

- Third unit vector is called the binormal vector is perpendicular to the other two:

$$\hat{e}_b = \hat{e}_t \times \hat{e}_n$$

Acceleration

- The acceleration can be calculated by differentiating the velocity

$$\vec{a}_P = \dot{\vec{v}}_P = \frac{d(\dot{s}\hat{e}_t)}{dt} = \ddot{s}\,\hat{e}_t + \dot{s}\,\dot{\hat{e}}_t$$

$$\frac{d\hat{e}_t}{dt} = \hat{e}_t' \cdot \dot{s} = \frac{1}{\rho}\dot{s}\,\hat{e}_n$$

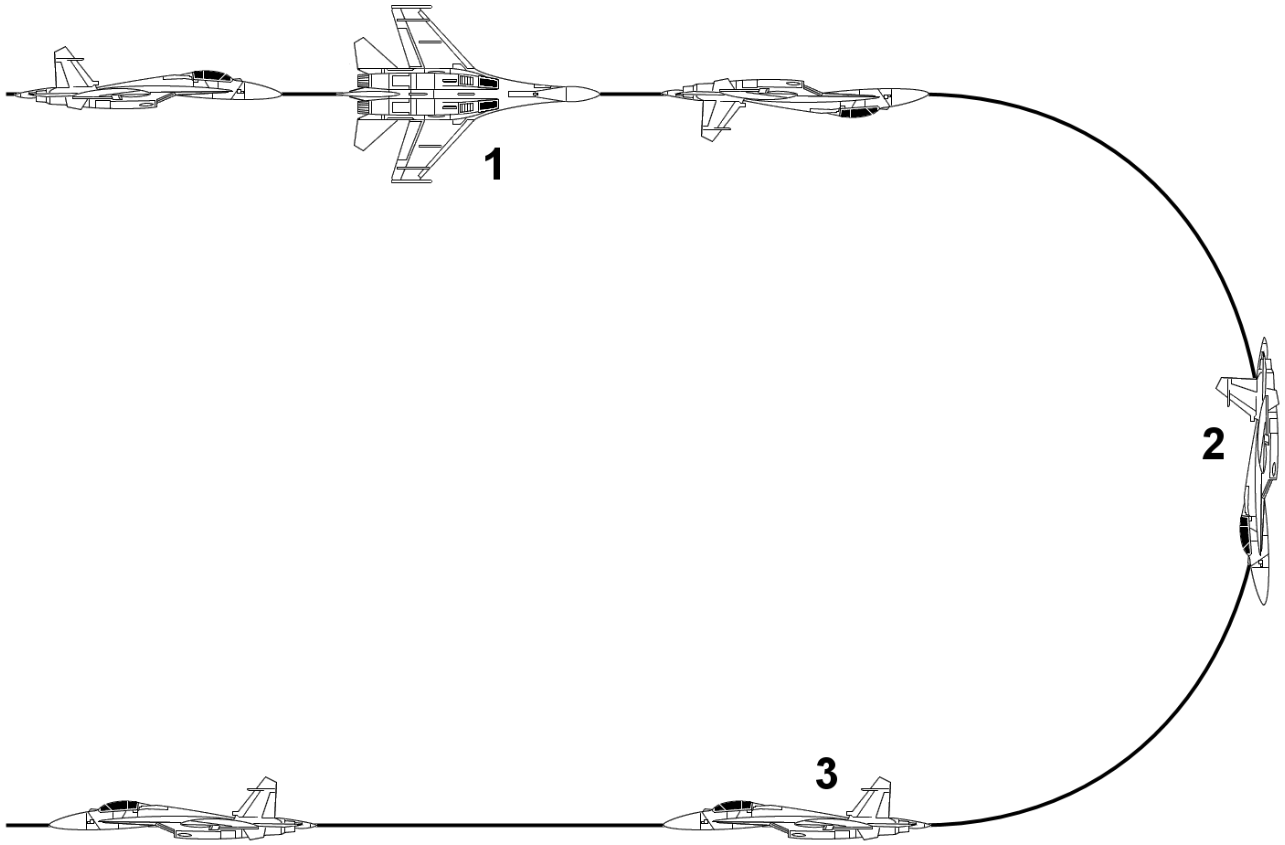
- The acceleration is only in the *t* and *n* directions and the expression is quite simple

change in speed

$$\boxed{\vec{a}_P = \ddot{s}\hat{e}_t + \frac{\dot{s}^2}{\rho}\hat{e}_n}$$

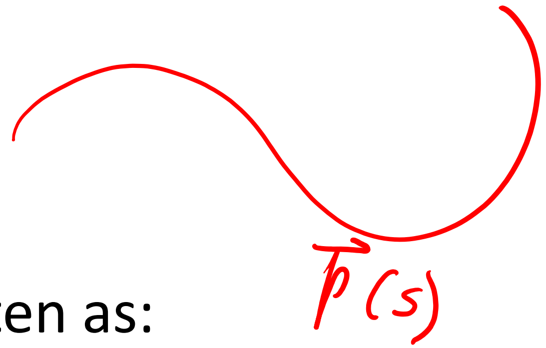
$\dot{s}$ is speed and $\ddot{s}$ is change in speed or change in magnitude of velocity

Example: Split-S Maneuver



https://en.wikipedia.org/wiki/Split_S

Curvature and Radius of Curvature



- For general 3D parameterized curves, the curvature can be written as:

$$\boxed{\kappa = \frac{1}{\rho} = \frac{|\vec{p}\,' \times \vec{p}\,''|}{|\vec{p}\,'|^3}}$$

  - where, the derivative is w.r.t. a parameter, which could be arclength, time, another coordinate (θ) or one of the cartesian coordinates (x, y, or z).
  - This will work for 2D curves as well

- For curves parameterized by arclength: $\vec{p}(s)$

$$\begin{matrix} |\vec{p}\,'| = 1 \\ |\vec{p}\,''| \perp |\vec{p}\,'| \end{matrix} \quad \Rightarrow \quad \boxed{\kappa = |\vec{p}\,''|}$$

- For curves parameterized by a cartesian coordinate: $y(x)$ and $z(x)$
  - Use the above equation with:

$$x' = 1 \qquad y' = \frac{dy}{dx} \qquad z' = \frac{dz}{dx}$$

$$x'' = 0 \qquad y'' = \frac{d^2y}{dx^2} \qquad z'' = \frac{d^2z}{dx^2}$$

https://en.wikipedia.org/wiki/Curvature