$$\boxed{{}^{A}\frac{d\vec{p}}{dt} = {}^{B}\frac{d\vec{p}}{dt} + {}^{A}\vec{\omega}^{B} \times \vec{p}}$$

Angular Velocity

….. and Acceleration

mpatil@gatech.edu

Angular Velocity

- Angular velocity is a vector!

  … even though the general 3D orientation cannot be represented as a vector.

- Angular velocity is derived from small changes in the angle

  … small (infinitesimal) angles are additive and commutative and can be represented as a vector

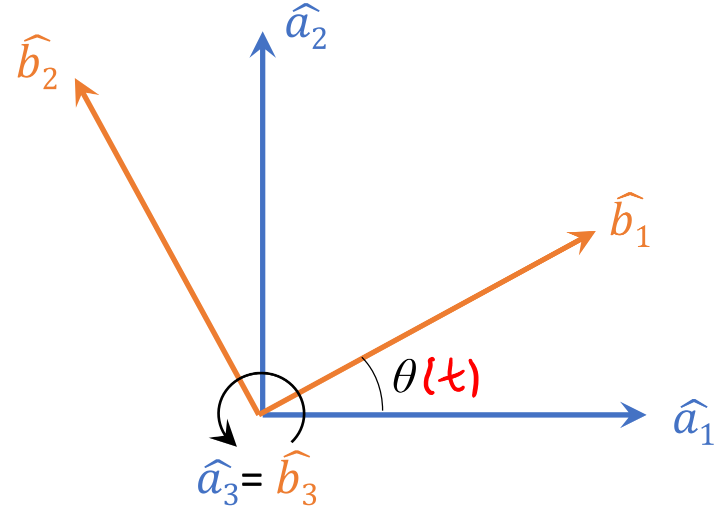
- For small-angle, linearized, dynamics, we may not need representation of general 3D orientation

  e.g., linearized attitude dynamic of aircraft including linearized stability analysis

Simple Rotation and Simple Angular Velocity

- Let's consider two reference frames A and B
- If there is a rotation of frame B relative to frame A, about a fixed axis in both frames (simple rotation), say the z-axis, of θ

$$\boxed{\cancel{\vec{\omega} = \dot{\theta}\hat{k}}} \qquad \vec{w} = \dot{\theta}\,\hat{b}_3 = \dot{\theta}\,\hat{a}_3$$



- This is simple angular velocity of B in A for a rotation about a fixed axis in A and B

$$\hat{b}_1 = \cos\theta\,\hat{a}_1 + \sin\theta\,\hat{a}_2$$
$$\hat{b}_2 = \cos\theta\,\hat{a}_2 - \sin\theta\,\hat{a}_1$$
$$\hat{b}_3 = \hat{a}_3$$

$$\begin{Bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{Bmatrix} = \begin{bmatrix} C_\theta & S_\theta & 0 \\ -S_\theta & C_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{Bmatrix}$$

DCM, Rotation, Transform

Mathematical Definition of Angular Velocity

$$
\begin{aligned}
{}^{A}\vec{\omega}^{B} &\triangleq \hat{b}_1 \frac{{}^{A}d\hat{b}_2}{dt} \cdot \hat{b}_3 + \hat{b}_2 \frac{{}^{A}d\hat{b}_3}{dt} \cdot \hat{b}_1 + \hat{b}_3 \frac{{}^{A}d\hat{b}_1}{dt} \cdot \hat{b}_2 \\
&= \left(\frac{{}^{A}d\hat{b}_2}{dt} \cdot \hat{b}_3 \right) \hat{b}_1 + \left(\frac{{}^{A}d\hat{b}_3}{dt} \cdot \hat{b}_1 \right) \hat{b}_2 + \left(\frac{{}^{A}d\hat{b}_1}{dt} \cdot \hat{b}_2 \right) \hat{b}_3
\end{aligned}
$$

- The angular velocity is written in terms of the rate of change of unit vectors of B in A
- All the angular velocity expression can be derived from the above equation
- The expression can also be used to derive other very useful relations

Addition Theorem for Angular Velocities

- Lets consider two reference frames A and B and auxiliary or intermediate frames $A_1, A_2, \ldots, A_{n-1}, A_n$

$$\boxed{{}^{A}\vec{\omega}^{B} = {}^{A}\vec{\omega}^{A_1} + {}^{A_1}\vec{\omega}^{A_2} + \ldots {}^{A_{n-1}}\vec{\omega}^{A_n} + {}^{A_n}\vec{\omega}^{B}}$$

- Thus:

$${}^{B}\vec{\omega}^{A} = -{}^{A}\vec{\omega}^{B}$$

- We will use this in calculation of the aircraft angular velocity in terms of intermediate frames C and D
- For later, angular acceleration does not satisfy any addition theorem!

Angular Velocity in terms of Body Axis Measure Numbers

- It is common to write angular velocity as

$$\vec{\omega} = p\,\hat{b}_1 + q\,\hat{b}_2 + r\,\hat{b}_3$$

body frame

- For small angles:

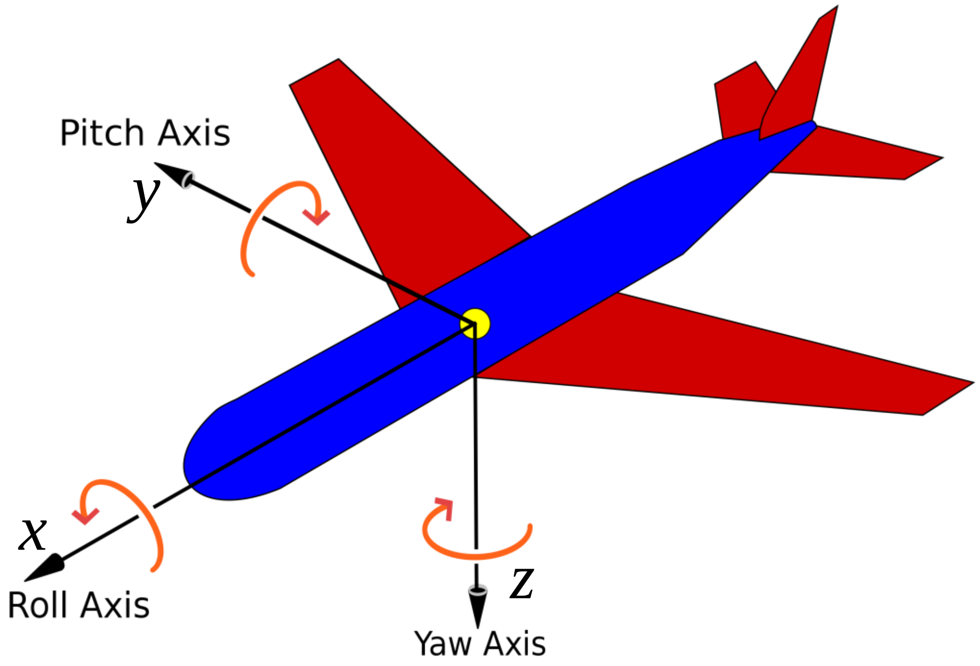
$$p = \dot{\overline{\delta\phi}}$$

$$\delta\phi = \int p\,dt$$

$$q = \dot{\overline{\delta\theta}}$$

$$r = \dot{\overline{\delta\psi}}$$

- Above angular velocities cannot be integrated to get corresponding angles for finite rotations



Simple rotation + Ang Vel Addition Theorem → Life is good

Angular Velocity in terms of Body Axis Measure Numbers – in terms of finite angles

- Consider angles (ϕ,θ,ψ) representing (Body 3-2-1) orientation from Earth frame A to aircraft body frame B:
  - Rotate about Earth z-axis by ψ to get intermediate reference frame C
  - Rotate about ref frame C y-axis by θ to get intermediate ref frame D
  - Rotate about ref frame D x-axis by ϕ to get aircraft body frame B
  - There are singularities but not if we restrict the angles

$$A \xrightarrow[\psi]{} C \xrightarrow[\theta]{} D \xrightarrow[\phi]{} B$$

$$\vec{\omega} = \dot{\phi}\,\hat{d}_1 + \dot{\theta}\,\hat{c}_2 + \dot{\psi}\,\hat{a}_3$$
$$= \dot{\phi}\,\hat{b}_1 + \dot{\theta}\,\hat{d}_2 + \dot{\psi}\,\hat{c}_3$$

$$\vec{\omega} = p\,\hat{b}_1 + q\,\hat{b}_2 + r\,\hat{b}_3$$

$$p = -\dot{\psi}\sin\theta + \dot{\phi}$$
$$q = \dot{\psi}\cos\theta\sin\phi + \dot{\theta}\cos\phi$$
$$r = \dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi$$

$${}^{A}\vec{\omega}^{B} = \underbrace{{}^{D}\vec{\omega}^{B}}_{\dot{\phi}\,\hat{d}_1} + \underbrace{{}^{C}\vec{\omega}^{D}}_{\dot{\theta}\,\hat{c}_2} + \underbrace{{}^{A}\vec{\omega}^{C}}_{\dot{\psi}\,\hat{a}_3}$$

Components of Vectors

- Use transformation matrix to represent a given vector in different reference frames

Differentiation in Two Reference Frames

- For any vector $\vec{v}$

$$\frac{{}^{A}d\vec{v}}{dt} = \frac{{}^{B}d\vec{v}}{dt} + {}^{A}\vec{\omega}^{B} \times \vec{v}$$

- This is an important relation used frequently
- Angular velocity is key to the differentiation of vectors in rotating frames

  … in addition to giving information about the angular motion of the body
- If vector is fixed in B

$$\frac{{}^{A}d\vec{v}}{dt} = {}^{A}\vec{\omega}^{B} \times \vec{v}$$

Angular Acceleration

- Lets consider two reference frames A and B, the angular acceleration of B in A is given by:

$$\boxed{{}^{A}}\vec{\alpha}\boxed{{}^{B}} = \frac{\boxed{{}^{A}}d\boxed{{}^{A}}\vec{\omega}\boxed{{}^{B}}}{dt}$$

- We can also differentiate the angular velocity in the B frame to get the angular acceleration

$$\boxed{{}^{A}\vec{\alpha}^{B} = \frac{{}^{A}d\,{}^{A}\vec{\omega}^{B}}{dt} = \frac{{}^{B}d\,{}^{A}\vec{\omega}^{B}}{dt}} + {}^{A}\vec{\omega}^{B} \times {}^{A}\vec{\omega}^{B}$$

$${}^{A}\vec{\omega}^{B} = p\,\hat{b}_1 + q\,\hat{b}_2 + r\,\hat{b}_3$$

$${}^{A}\vec{\alpha}^{B} = \dot{p}\,\hat{b}_1 + \dot{q}\,\hat{b}_2 + \dot{r}\,\hat{b}_3$$

- For simple rotation and angular velocity, we have

$$\vec{\omega} = \dot{\theta}\hat{k} \qquad \vec{\alpha} = \ddot{\theta}\hat{k}$$

Angular Acceleration

- You can differentiate the angular velocity in A or B but not C

$$\frac{{}^{A}d\,{}^{A}\vec{\omega}^{B}}{dt} = \frac{{}^{C}d\,{}^{A}\vec{\omega}^{B}}{dt} + \underbrace{{}^{A}\vec{\omega}^{C} \times {}^{A}\vec{\omega}^{B}}_{\text{not zero}}$$

- Angular Acceleration is not Additive! C is an intermediate axis

$${}^{A}\vec{\omega}^{B} = {}^{A}\vec{\omega}^{C} + {}^{C}\vec{\omega}^{B}$$

$${}^{A}\vec{\alpha}^{B} = \frac{{}^{A}d\,{}^{A}\vec{\omega}^{B}}{dt} = \frac{{}^{A}d\,{}^{A}\vec{\omega}^{C}}{dt} + \frac{{}^{A}d\,{}^{C}\vec{\omega}^{B}}{dt}$$

$$= {}^{A}\vec{\alpha}^{C} + \frac{{}^{C}d\,{}^{C}\vec{\omega}^{B}}{dt} + {}^{A}\vec{\omega}^{C} \times {}^{C}\vec{\omega}^{B} = {}^{A}\vec{\alpha}^{C} + {}^{C}\vec{\alpha}^{B} + {}^{A}\vec{\omega}^{C} \times {}^{C}\vec{\omega}^{B}$$