Orientation Parameters

… and Euler Rotation

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Euler Rotation

axis-angle rep

- Orientation of body B in a reference frame A can be represented as a single rotation about an axis of rotation – Euler's theorem on rotation
- Consider that the above rotation is represented by θ and the axis by unit vector by $\hat{\lambda}$ → unit vector
- Note the rotation is about $\hat{\lambda}$ and thus the measure numbers of $\hat{\lambda}$ in both the A and B reference frames are the same:

$$\lambda_i = \hat{\lambda} \cdot \hat{a}_i = \hat{\lambda} \cdot \hat{b}_i$$

- Thus orientation is represented by 4 parameters: $\theta, \lambda_1, \lambda_2, \lambda_3$, with 1 constraint: $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$, i.e., 3 independent parameters
- The axis of rotation and the angle of rotation may change with time
  - The orientation can be represented at an instant of time as a simple rotation
  - The axis of rotation can change and thus the angular velocity cannot in general be represented by a simple angular velocity about the axis

DCM from Euler Rotation

- Consider $\{\lambda\}$ to be the column of measure numbers of $\hat{\lambda}$, and $[\tilde{\lambda}]$ the skew-symmetric matrix based on the column
- The DCM can be written as:

$$\boxed{[{}^{A}C^{B}] = [I]\cos\theta + [\tilde{\lambda}]\sin\theta + \{\lambda\}\{\lambda\}^{T}(1-\cos\theta)}$$

Rodrigues

$$[\tilde{v}] = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$$

$\theta\hat{\lambda}$

$C = e^{-\theta[\tilde{\lambda}]}$

$\dot{\theta}$

$\dot{\lambda}$ $\leftrightarrow$ ω_1 ω_1 ω_2

Euler Parameter (Quaternion)

- Euler Vector:

$$\vec{\epsilon} = \hat{\lambda} \sin \frac{\theta}{2}$$

- First three Euler parameters:

$$\epsilon_i = \vec{\epsilon} \cdot \hat{a}_i = \vec{\epsilon} \cdot \hat{b}_i$$

- Fourth Euler parameter:

$$\epsilon_4 = \cos \frac{\theta}{2}$$

- Constraint:

$$\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2 = 1$$

$$\beta_1 \quad \beta_2 \quad \beta_3 \quad \beta_0$$

DCM from Euler Parameters

- Consider $\{\epsilon\}$ to be the column of measure numbers of $\hat{\epsilon}$, and $[\tilde{\epsilon}]$ the skew-symmetric matrix based on the column
- The DCM can be written as:

$$[{}^{A}C^{B}] = (1 - 2\{\epsilon\}^T\{\epsilon\})[I] + 2\{\epsilon\}\{\epsilon\}^T + 2\epsilon_4[\tilde{\epsilon}]$$

- The DCM is "nice" and quadratic in terms of Euler parameters
- The Euler parameters can be calculated from the DCM

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{Bmatrix}$$

Angular Velocity in terms of Euler Parameters

- The measure numbers of the angular velocity vector can be represented in any reference frame, specifically we can find the measure numbers in the B reference frame

$$\begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ 0 \end{Bmatrix} = 2 \begin{bmatrix} \epsilon_4 & \epsilon_3 & -\epsilon_2 & -\epsilon_1 \\ -\epsilon_3 & \epsilon_4 & \epsilon_1 & -\epsilon_2 \\ \epsilon_2 & -\epsilon_1 & \epsilon_4 & -\epsilon_3 \\ \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 \end{bmatrix} \begin{Bmatrix} \dot{\epsilon}_1 \\ \dot{\epsilon}_2 \\ \dot{\epsilon}_3 \\ \dot{\epsilon}_4 \end{Bmatrix} \begin{matrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_0 \end{matrix}$$

$= 2[E]\{\}$

$\frac{1}{2}[E]^T$

  - Only quadratic nonlinearity!
- The above equation can be inverted (the matrix is orthogonal and the inverse is the transpose) to represent the time derivatives of Euler parameters in terms of the angular velocity measure numbers
  - Nicely invertible, no singularities, no ill-conditioning!!!

Rodrigues Parameters

- Various forms including classical and modified. Also called Cayley-Gibbs-Rodrigues parameters.
- Components of a vector!
- Only three parameters – $\boxed{\text{singularities shall exist!!!}}$

$$\boxed{\vec{\sigma} = 2\hat{\lambda}\tan\frac{\theta}{2}}$$

$$\vec{\sigma}' = \hat{\lambda}\tan\frac{\theta}{2} = \boxed{\frac{\vec{\epsilon}}{\epsilon_4}}$$

DCM and Angular Velocity in terms of Rodrigues Parameters

- The DCM can be written as:

$$[{}^{A}C^{B}] = \frac{(1 - \frac{1}{4}\{\sigma\}^T\{\sigma\})[I] + \frac{1}{2}\{\sigma\}\{\sigma\}^T + [\tilde{\sigma}]}{1 + \frac{1}{4}\{\sigma\}^T\{\sigma\}}$$

- The measure numbers of the angular velocity vector can be represented in the B reference frame

$$\{\omega\} = \frac{[I] - \frac{1}{2}[\tilde{\sigma}]}{1 + \frac{1}{4}\{\sigma\}^T\{\sigma\}}\{\dot{\sigma}\} \qquad \{\dot{\sigma}\} = \left([I] + \frac{1}{2}[\tilde{\sigma}] + \frac{1}{4}\{\sigma\}\{\sigma\}^T\right)\{\omega\}$$

- The above equation can be used and inverted except at singular points – rotation of π rad – when the Rodrigues parameters approach infinity!

Wiener-Milenkovic Parameters

- Can be considered generalization of Rodrigues parameters
- Again components of a vector!
- Only three parameters – singularities but a bit farther off, and can easily be avoided by offsetting the angle θ

$$\vec{\mu} = 4\hat{\lambda}\tan\frac{\theta}{4}$$

$\vec{\mu}$

Maybe it is possible to not need offsetting, just choosing between θ & $2\pi - \theta$

$\hat{\lambda}$ $-\hat{\lambda}$

$\underline{\underline{C}}^{AB}$ . . . $\mu(t_i)$

$\downarrow$

$\mu(t_{200}) = \mu_0$

$\downarrow$

$\underline{\underline{C}}_0$

$\underline{\underline{C}}^{AB} = \hat{\underline{\underline{C}}}\,\underline{\underline{C}}_0$

DCM and Angular Velocity in terms of Wiener-Milenkovic Parameters

- The DCM can be written as:

$$[{}^{A}C^{B}] = [F][F]$$

where,

$$[F] = \frac{(1 - \frac{1}{16}\{\mu\}^T\{\mu\})[I] + \frac{1}{8}\{\mu\}\{\mu\}^T + \frac{1}{2}[\tilde{\mu}]}{1 + \frac{1}{16}\{\mu\}^T\{\mu\}}$$

- The measure numbers of the angular velocity vector can be represented in the B reference frame

$$\{\omega\} = \frac{[F]^T}{1 + \frac{1}{16}\{\mu\}^T\{\mu\}}\{\dot{\mu}\} \qquad \{\dot{\mu}\} = \left(1 + \frac{1}{16}\{\mu\}^T\{\mu\}\right)[F]\{\omega\}$$

- The above equation can be used and inverted ($[F]$ is an orthogonal matrix) except at singular points – rotation of 2π rad.

Small Angle Assumption

- DCM:

θ^2

$$[{}^A C^B] = [I] + [\tilde{\lambda}]\theta = [I] + 2[\tilde{\epsilon}] = \begin{bmatrix} 1 & -\theta_3 & \theta_2 \\ \theta_3 & 1 & -\theta_1 \\ -\theta_2 & \theta_1 & 1 \end{bmatrix} \approx [I]$$

- Angular Velocity:

$$\begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{Bmatrix}$$

- For small (not finite) simple rotations, the order of rotation does not matter
- Used for linear analysis including stability analysis and control design