Euler's Dynamical Equations

Parallel Axis Theorem
Principle Moments of Inertia

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Euler’s Dynamical Equations (about center of mass)

- About Center of Mass
- Written in terms of the components in body frame

Rotational Kinematics

$\omega_1 \quad \omega_2 \quad \omega_3$
are components
of $\vec{\omega}$ in B-frame

- Consider a Newtonian reference frame N
- Consider the reference frame fixed in the body B: $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$
- Angular Velocity: $${}^N\vec{\omega}^B = \omega_1 \hat{b}_1 + \omega_2 \hat{b}_2 + \omega_3 \hat{b}_3$$
- Angular Acceleration: $${}^N\vec{\alpha}^B = \dot{\omega}_1 \hat{b}_1 + \dot{\omega}_2 \hat{b}_2 + \dot{\omega}_3 \hat{b}_3$$

$${}^N\vec{\alpha}^B = \frac{{}^N d\, {}^N\vec{\omega}^B}{dt}$$
$$= \frac{{}^B d\, {}^N\vec{\omega}^B}{dt}$$

- Moment of Inertia (about the center of mass C):

  … always relative to the body axes!

$$[I^C] = \begin{bmatrix} I^C_{11} & I^C_{12} & I^C_{13} \\ I^C_{12} & I^C_{22} & I^C_{23} \\ I^C_{13} & I^C_{23} & I^C_{33} \end{bmatrix}$$

components of $\overset{\Rightarrow}{I}$ in B-frame

Angular Momentum

- Angular Momentum (about the center of mass C):

$$\vec{H}^C = H_1^C \hat{b}_1 + H_2^C \hat{b}_2 + H_3^C \hat{b}_3$$

… in terms of moment of inertia and angular velocity

$$\begin{Bmatrix} H_1^C \\ H_2^C \\ H_3^C \end{Bmatrix} = \begin{bmatrix} I_{11}^C & I_{12}^C & I_{13}^C \\ I_{12}^C & I_{22}^C & I_{23}^C \\ I_{13}^C & I_{23}^C & I_{33}^C \end{bmatrix} \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix}$$

$$\vec{H}^c = \vec{\vec{I}}^c \cdot \vec{\omega}$$

- Applied Torque: $\vec{M}^C = M_1^C \hat{b}_1 + M_2^C \hat{b}_2 + M_3^C \hat{b}_3$

Euler's Dynamical Equations
(matrix form using components in the B-frame)

- Equations of Motion:

$$\vec{M}^C = \frac{{}^N d\vec{H}^C}{dt} = \frac{{}^B d\vec{H}^C}{dt} + {}^N\vec{\omega}^B \times \vec{H}^C$$

$$\begin{Bmatrix} M_1^C \\ M_2^C \\ M_3^C \end{Bmatrix} = [I^C] \begin{Bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{Bmatrix} + \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} \times \begin{Bmatrix} H_1^C \\ H_2^C \\ H_3^C \end{Bmatrix}$$

$$\boxed{\begin{Bmatrix} M_1^C \\ M_2^C \\ M_3^C \end{Bmatrix} = \begin{bmatrix} I_{11}^C & I_{12}^C & I_{13}^C \\ I_{12}^C & I_{22}^C & I_{23}^C \\ I_{13}^C & I_{23}^C & I_{33}^C \end{bmatrix} \begin{Bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{Bmatrix} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} I_{11}^C & I_{12}^C & I_{13}^C \\ I_{12}^C & I_{22}^C & I_{23}^C \\ I_{13}^C & I_{23}^C & I_{33}^C \end{bmatrix} \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix}}$$

Euler’s Dynamical Equations (about a fixed point)

- About a point fixed in the inertial frame
  - We will talk about a general point later
- Again will be written in terms of the components in body frame

Angular Momentum Expression for Rigid Bodies about non-CM point on the Body

- Consider a point P on the body :

$$\vec{H}^P = \int (\vec{r}^{PQ} \times \vec{v}^Q)\, dm$$

where Q is a point on dm on the body

- For a $\boxed{\text{rigid body}}$, we can write the velocity at any point in terms of velocity at one point and the angular velocity of the body

$$\vec{v}^Q = \vec{v}^P + \vec{\omega} \times \vec{r}^{PQ}$$

- Thus:

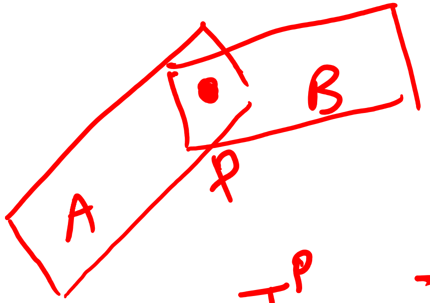
$$\vec{H}^P = \int [\vec{r}^{PQ} \times (\vec{v}^P + \vec{\omega} \times \vec{r}^{PQ})]\, dm$$

$$\vec{H}^P = \underline{m\, \vec{r}^{PC} \times \vec{v}^P} + \int (\vec{r}^{PQ} \times \vec{\omega} \times \vec{r}^{PQ})\, dm$$

Angular Momentum Expression about non-CM point on Body which is Fixed in Inertial Frame

- Such a point would be a "pivot" point (or socket) and would translate to a fixed axis (or hinge) in 2D
- For such a point O fixed in the inertial frame:

$$\vec{H}^O = \int \left[\vec{r}^{OQ} \times \left(\vec{\omega} \times \vec{r}^{OQ}\right)\right] dm$$



$I_B^P \quad I_A^P$

$$\boxed{\sum \vec{M}^C = \frac{d\vec{H}^C}{dt}} \qquad \boxed{\sum \vec{M}^O = \frac{d\vec{H}^O}{dt}}$$

$$\vec{H}^C = \int \vec{r}^{CQ} \times (\vec{\omega} \times \vec{r}^{CQ})\, dm$$

$$\vec{H}^O = \int \vec{r}^{OQ} \times (\vec{\omega} \times \vec{r}^{OQ})\, dm$$

Angular Momentum Expression about non-CM point on Body which is Fixed in Inertial Frame

- Consider points O and Q on the rigid body B : $\vec{r}^{OQ} = x^{OQ}\hat{b}_1 + y^{OQ}\hat{b}_2 + z^{OQ}\hat{b}_3$
- The angular velocity can be written in terms of its components in body axis

$$\vec{\omega} = \omega_x \hat{b}_1 + \omega_y \hat{b}_2 + \omega_z \hat{b}_3$$

- The Angular Momentum can be calculated as:

$$\begin{aligned}
\vec{H}^O &= \int (\vec{r}^{OQ} \times \vec{\omega} \times \vec{r}^{OQ})\, dm \\
&= \left[\int ({y^{OQ}}^2 + {z^{OQ}}^2) dm\ \omega_x - \int (x^{OQ} y^{OQ}) dm\ \omega_y - \int (x^{OQ} z^{OQ}) dm\ \omega_z\right] \hat{b}_1 \\
&+ \left[-\int (x^{OQ} y^{OQ}) dm\ \omega_x + \int ({x^{OQ}}^2 + {z^{OQ}}^2) dm\ \omega_y - \int (y^{OQ} z^{OQ}) dm\ \omega_z\right] \hat{b}_2 \\
&+ \left[-\int (x^{OQ} z^{OQ}) dm\ \omega_x - \int (y^{OQ} z^{OQ}) dm\ \omega_y + \int ({x^{OQ}}^2 + {y^{OQ}}^2) dm\ \omega_z\right] \hat{b}_3
\end{aligned}$$

Moments and Products of Inertia about O

$$I^O_{xx} = \int (y^{OQ^2} + z^{OQ^2})dm \qquad I^O_{xy} = -\int (x^{OQ} y^{OQ})dm$$

$$\{H^O\} = [I^O]\{\omega\}$$

$$I^O_{yy} = \int (x^{OQ^2} + z^{OQ^2})dm \qquad I^O_{xz} = -\int (x^{OQ} z^{OQ})dm$$

$$I^O_{zz} = \int (x^{OQ^2} + y^{OQ^2})dm \qquad I^O_{yz} = -\int (y^{OQ} z^{OQ})dm$$

$$I^O = \begin{bmatrix} I^O_{xx} & I^O_{xy} & I^O_{xz} \\ & & \\ & & \end{bmatrix}$$

- For a point O fixed in the inertial frame:

$$\boxed{\{M^O\} = [I^O]\{\dot{\omega}\} + [\tilde{\omega}][I^O]\{\omega\}}$$

… similar to the equation about the center of mass, but we need to use the MoI about P

Parallel Axis Theorem

- Relates moment/product of inertia about any point fixed on the body to the moment/product of inertia about the center of mass

$$I_{xx}^{O} = \int \left(y^{OQ^2} + z^{OQ^2} \right) dm$$

$$= \int \left[\left(y^{OC} + y^{CQ} \right)^2 + \left(z^{OC} + z^{CQ} \right)^2 \right] dm$$

$$= \int \left(y^{OC^2} + z^{OC^2} \right) dm + 2 y^{OC} \cancelto{0}{\int y^{CQ} dm} + 2 z^{OC} \cancel{\int z^{CQ} dm} + \int \left(y^{CQ^2} + z^{CQ^2} \right) dm$$

$$= \left(y^{OC^2} + z^{OC^2} \right) m + I_{xx}^{C}$$

Parallel Axis Theorem

- Are Moments of Inertia about two points related? For example, is I_{zz}^{O} related to I_{zz}^{C}? Yes …

$$I_{zz}^{O} = I_{zz}^{C} + md^2$$

$I_{zz}^{C} \neq I_{zz}^{O} + md^2$

… where, d is the distance between the two z-axis passing though O and C

$$d^2 = x^{CO^2} + y^{CO^2}$$

… the MoI is always positive and minimum about the CoM

- Are Products of Inertia about two points related? For example, is I_{xy}^{O} related to I_{xy}^{C}? Yes again.

$$I_{xy}^{O} = I_{xy}^{C} - m\, x^{CO} y^{CO}$$

… the PoI can be positive or negative

Use of Parallel Axis Theorem

- Parallel axis theorem is commonly used to calculate the moment of inertia about the CM for a single rigid body composed of multiple parts.
- If the mass and moment of inertia of each part abouts it's own CM is given as: m_i $\quad I_{zz_i}^{C_i}$
- The moment of inertia of each part about the body CM will be:

$$I_{zz_i}^{C} = I_{zz_i}^{C_i} + m_i d_i^2 \qquad d_i^{c} = x^{cc_i\,2} + y^{cc_i\,2}$$

- The moment of inertia of the whole body about it's CM will be:

$I_{zz}^{C} + I_{zz}^{C}$

$$I_{zz}^{C} = \sum I_{zz_i}^{C} = \sum (I_{zz_i}^{C_i} + m_i d_i^2)$$

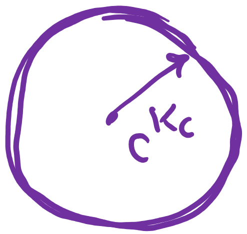
$$I_{zz}^{P} = I_{zz}^{C} + md^2$$

You can only sum MoI which are ① about same axis ② about same point

Radius of Gyration

- A measure of how the mass is distributed about an axis and is defined for MoI as:

$$I_{zz}^{C} = mk_C^2$$
$$I_{zz}^{O} = mk_O^2$$



… commonly used in 2D problems

- The two radii of gyration are related (using the parallel axis theorem):

$$k_O^2 = k_C^2 + d^2$$

Principal Axis and Principal Moments of Inertia

- For any rigid body, there exist a set of three orthogonal body fixed axes about which all the Products of Inertia are zero
  - No need for symmetry
  - The Euler's dynamical equations in this axis system are significantly simpler (less terms)

$\vec{n} \cdot \vec{v} = v_n$

Rotation of Moment of Inertia Tensor

- Moment of inertia about any axis can be calculated as:

$$I_{nn} = \underbrace{\{n\}^T[I]\{n\}}_{\text{Projection}}$$

$$\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}^T \begin{bmatrix} I_{xx} & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}^T \begin{Bmatrix} I_{xx} \\ I_{xy} \\ I_{xz} \end{Bmatrix} = I_{xx}$$

… where $\{n\}$ is a unit vector

- Product of Inertia about any two axis can be calculated as:

$$I_{nm} = I_{mn} = \{m\}^T[I]\{n\} = \{n\}^T[I]\{m\}$$

… where $\{n\}$ and $\{m\}$ are orthogonal unit vectors

- Moment of Inertia Matrix in two different axis systems B_1 and B_2 (both fixed in the body reference frame):

$$[^{B_2}I] = [C]^T[^{B_1}I][C]$$

$$\{^{B_2}v\} = [C]^T\{^{B_1}v\}$$

Principal Moment of Inertia and Principal Axes

- Is it possible to have an axis system with zero products of inertia?
- Yes, if:

$$\boxed{[I]\{n\} = I_p\{n\}}$$

$\{m\}^T[I]\{n\} \stackrel{?}{=} 0$ ✓

$I_p\{m\}^T\{n\} = 0$

... when multiplied by any orthogonal unit vector the above will give zero products of inertia.

- The above is an eigenvalue problem which has three eigenvalues - I_p^1, I_p^2, I_p^3, (principal moments of inertia) about three (orthogonal) axes – $\{n^1\}, \{n^2\}, \{n^3\}$.
- The three principal axes could be used as the body axis so that the moment of inertia matrix is diagonal

$$[I^C] = \begin{bmatrix} I_p^1 & 0 & 0 \\ 0 & I_p^2 & 0 \\ 0 & 0 & I_p^3 \end{bmatrix}$$

Eigenvalues Analysis $[A]\{x\} = \lambda\{x\}$ (Quick Cheat-Sheet)

- n x n matrix will have n eigenvalues and corresponding n eigenvectors
  - Lets assume distinct eigenvectors for now (you should read up on algebraic / geometric multiplicity)
- Eigenvectors do not have a magnitude, only "direction"
  - We typically choose the magnitude to be unity, especially to use as unit vectors $\{b_1\}^T\{b_1\} = 1$
- If matrix is real symmetric, the eigenvalues are real and then the eigenvectors are orthogonal! $\{b_1\}^T\{b_2\} = 0$
  - We can use the eigenvectors as basis vectors for a new axis system (yay!)
  - Be careful and check that $b_3 = b_1$ x b_2, if not choose negative b_3
  - The eigenvectors are orthogonal w.r.t. the matrix $\{b_i\}^T\ [A]\ \{b_j\} = 0$ (for $i \neq j$)
    - i.e., off-diagonal terms of MoI matrix are zero if the eigenvectors are used as basis vectors
- If matrix is symmetric and positive-definite (which MoI matrix is), the eigenvalues are all positive
- If the matrix is not symmetric then we have two sets of eigenvector – right and left
  - The right eigenvectors are orthogonal to the left eigenvector
- A generalized eigenvalue problem is of the form $[A]\{x\} = \lambda[B]\{x\}$
  - For symmetric generalized eigenvalue problems, the eigenvectors are orthogonal w.r.t. $[A]$ and $[B]$: $\{n_i\}^T\ [A]\ \{n_j\} = 0$ and $\{n_i\}^T\ [B]\ \{n_j\} = 0$ for $i \neq j$, but not directly orthogonal orthogonal $\{n_i\}^T\ \{n_j\} \neq 0$

Euler’s Dynamical Equations (about Principal Axes in Body frame)

- If $I_{12}^C = I_{13}^C = I_{23}^C = 0$

$$M_1^C = I_{11}^C \dot{\omega}_1 + (I_{33}^C - I_{22}^C)\omega_2\omega_3$$
$$M_2^C = I_{22}^C \dot{\omega}_2 + (I_{11}^C - I_{33}^C)\omega_1\omega_3$$
$$M_3^C = I_{33}^C \dot{\omega}_3 + (I_{22}^C - I_{11}^C)\omega_1\omega_2$$