Friction, Rolling, and Slipping in 2D

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Euler's Second Law for 2D Rigid Bodies (about CM)

- For 2D rigid bodies

$$\sum \vec{M}^C = \frac{d(I^C_{zz}\,\omega_z\,\hat{k})}{dt}$$

- The equation for the z-component of the moment gives us a scalar equation:

$$\sum M^C_z = \frac{d(I^C_{zz}\,\omega_z)}{dt} = I^C_{zz}\frac{d\omega_z}{dt}$$

$$\sum M^C_z = I^C_{zz}\dot{\omega}_z = I^C_{zz}\ddot{\theta} = I^C_{zz}\alpha$$

- For 2D rigid bodies, we do not have the nonlinear cross-product term because the moment of inertia I^C_{zz} does not change with rotation about z-axis and the products of inertia are assumed zero.

Equations of Motion of 2D Rigid Body

- The equations relative to the center of mass:

$$\sum F_x = m\ddot{x}_c \qquad \ddot{x}_p$$

$$\sum F_y = m\ddot{y}_c$$

$$\sum M_z^C = I_{zz}^C \ddot{\theta}$$

$P \qquad \vec{r} \times \vec{a}$

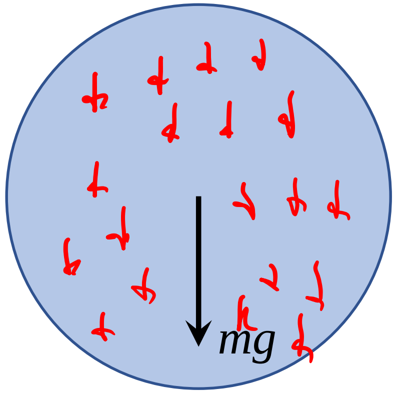
Note: that the position is of the center of mass, the moment and moment of inertia are about the center of mass, and the derivatives are all in the inertial frame.

	$x_c\ y_c\ \theta$	$x_p\ y_p\ \theta$
ΣM^c	?	?
ΣM^p	?	?

Forces on a Rolling or Slipping Cylinder

- Gravity
  - Acts at the center of mass
  - For rigid/flexible body assuming we are interested only in the overall motion of the body

$-\gamma\, dv$



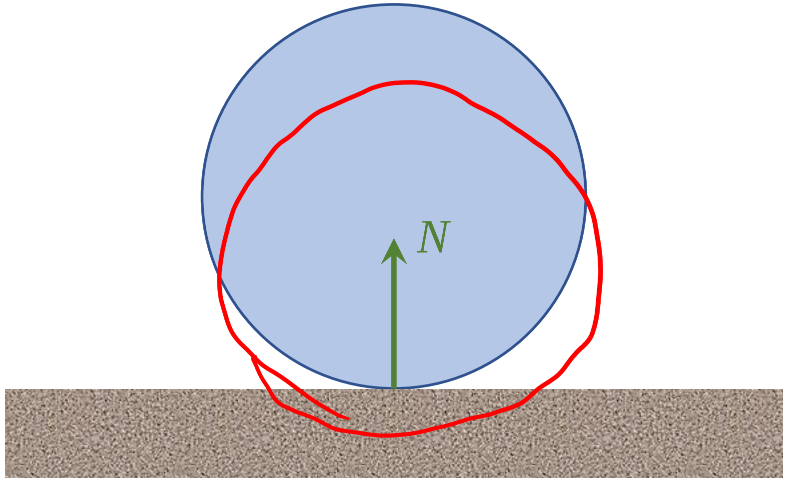
$$\int \rho g \, dv = mg$$

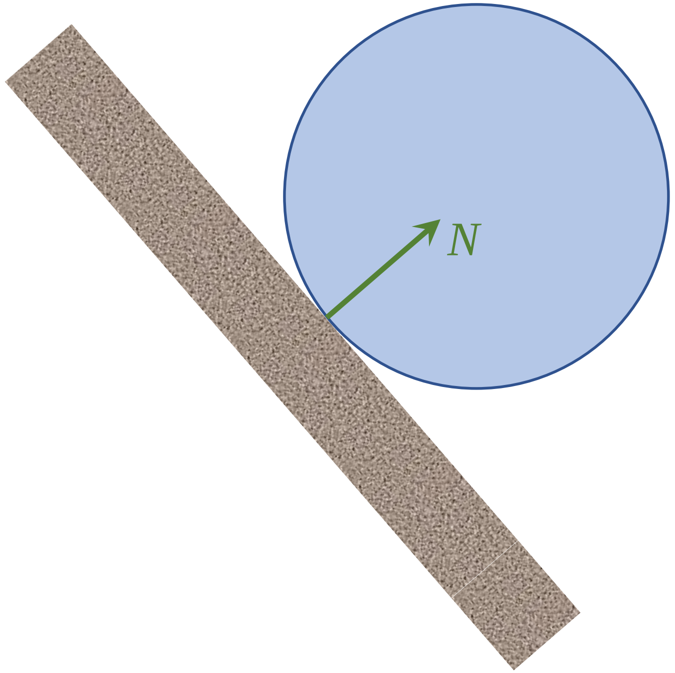
Forces on a Rolling or Slipping Cylinder

- Normal Reaction — Constraint Force — Constraint
  - Acts at the point of contact
  - Acts perpendicular to the plane of contact

force unknown

Kinematic Equation (extra)

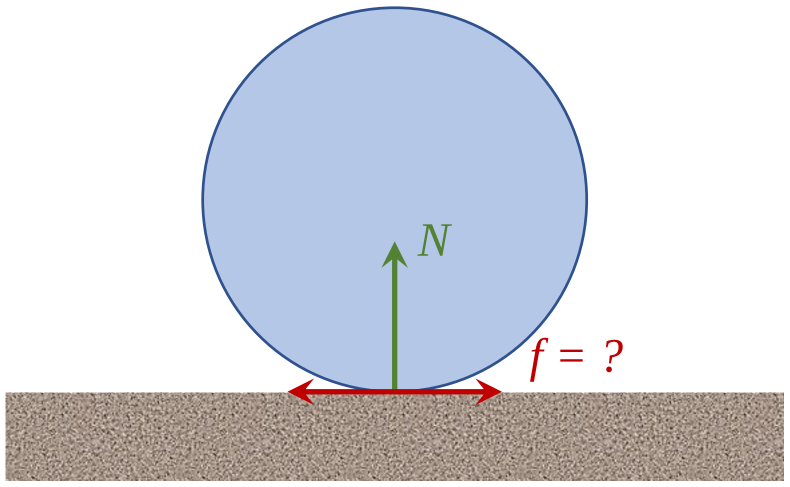


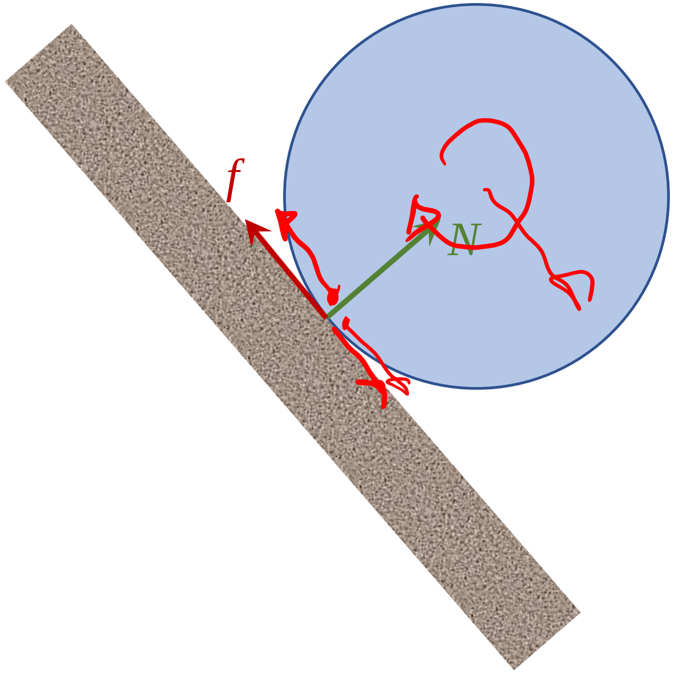


Forces on a Rolling or Slipping Cylinder

- Friction
  - Acts at the point of contact
  - Acts along the plane of contact – opposes slip

    … opposite in direction to the relative velocity if there were no friction
  - Is related to the normal reaction



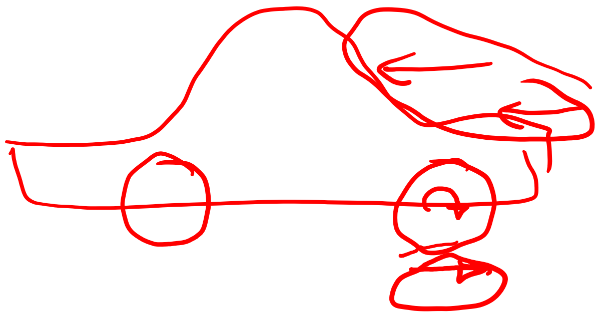


Friction on Rolling vs Slipping body

- Max friction force depends on normal reaction and the coefficient of friction between two objects

$$f_{\max} = \mu N$$

- During rolling: $|f| \leq \mu N$ $\rightarrow$ kinematic constraint $\vec{v}_p = 0$ Equator
- During slipping: $\boxed{|f| = \mu N}$ nice equation

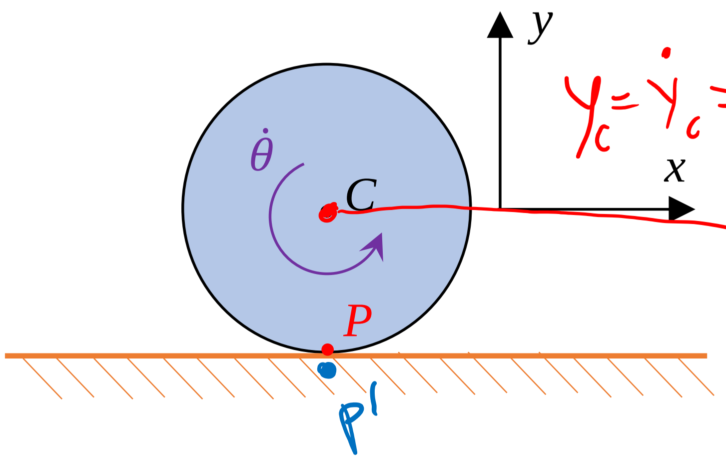


- We either know (i) the friction force (if slipping)

  or have (ii) a kinematic constraint (if rolling)

… in 3D we know the friction force magnitude and direction for slipping and the two components of velocity for rolling.

Kinematics of Rolling



$y_c = \dot{y}_c = \ddot{y}_c = 0$

- Assume the wheel has:
  - Angular Velocity: $\vec{\omega} = \dot{\theta}\hat{k}$
  - Angular Acceleration: $\vec{\alpha} = \ddot{\theta}\hat{k}$
- Assume the center of the wheel has
  - Velocity: $\vec{v}_C = \dot{x}_C\hat{\imath}$
  - Acceleration: $\vec{a}_C = \ddot{x}_C\hat{\imath}$

$\boxed{\vec{v}_p = \vec{v}_{p'}}$ velocity constraint

- Rolling constraint: $\vec{v}_P = 0$
- Constraint relates the velocity (of center of mass) and angular velocity (of wheel)

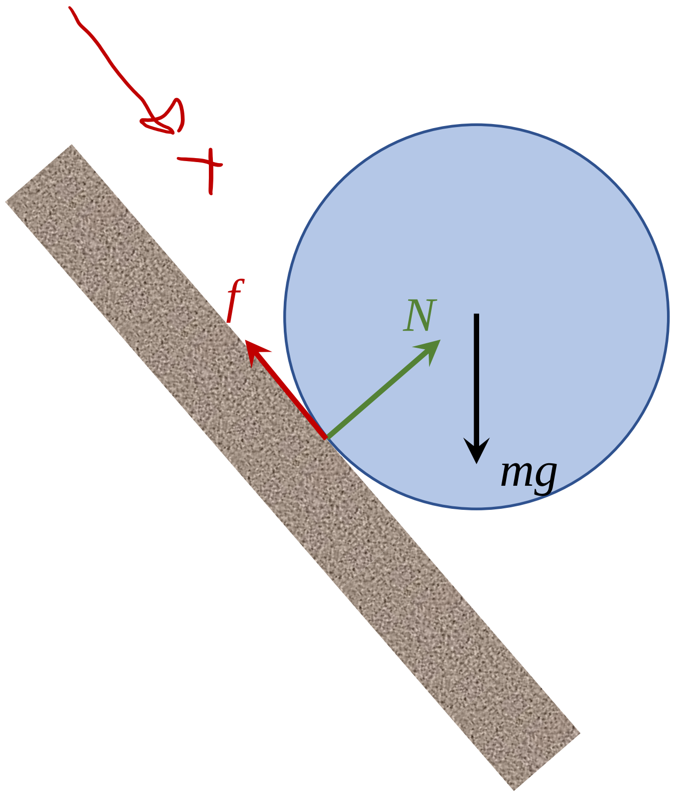
Rolling Const $\boxed{\dot{x}_C = -r\dot{\theta}} \quad \Rightarrow \quad \vec{v}_C = -r\dot{\theta}\hat{i} \qquad \vec{a}_C = -r\ddot{\theta}\hat{i}$

- Integrating the constraint

$$\boxed{x_C = -r\theta + \text{constant}}$$

Problem: Cylinder on an Incline

- Starting from rest
- Four possible Cases:
  - Frictionless surfaces - slip
  - Sufficient friction - roll
  - Insufficient friction - slip
  - Unknown - roll/slip



Cylinder on an Incline – Frictionless surfaces

E.N. Eq of Motion

$$\begin{cases} \sum F_x = m\ddot{x}_c \Rightarrow mg\sin\phi = m\ddot{x}_c & (1) \\ \sum F_y = m\ddot{y}_c \Rightarrow N - mg\cos\phi = m\ddot{y}_c & (2) \\ \sum M_z^c = I_{zz}\ddot{\theta} \Rightarrow 0 = I_{zz}\ddot{\theta} & (3) \end{cases}$$

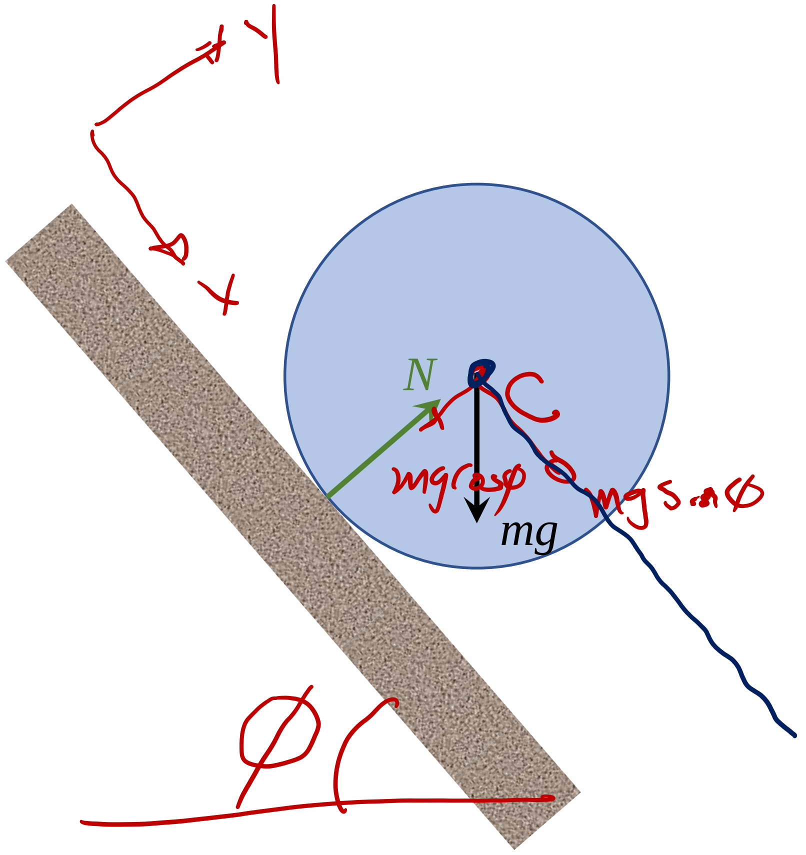
$$y_c = R \Rightarrow \ddot{y}_c = 0 \quad (4)$$

4 unknowns: $\ddot{x}_c, \ddot{y}_c, \ddot{\theta}, N$

$$\ddot{x}_c = g\sin\phi$$

$$\ddot{y}_c = 0$$

$$\ddot{\theta} = 0 \qquad N = mg\cos\phi$$



Cylinder on an Incline: Rolling

$$\sum F_x = m\ddot{x}_c \Rightarrow mg\sin\phi - f = m\ddot{x}_c \quad (1)$$

$$N - mg\cos\phi = m\ddot{y}_c \quad (2)$$

$$-\boxed{f}R = I_{zz}\ddot{\theta} \quad (3)$$

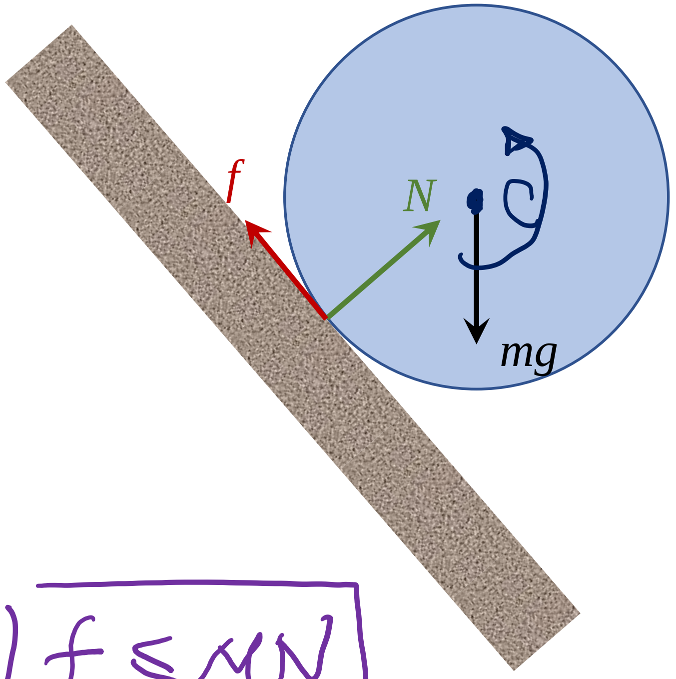
$$y_c = R \Rightarrow \ddot{y}_c = 0 \quad (4)$$

Rolling $\dot{x}_c = -R\dot{\theta} \Rightarrow \ddot{x}_c = -R\ddot{\theta} \quad (5)$

Unknowns: $\ddot{x}_c, \ddot{y}_c, \ddot{\theta}, N, f$

I assumed rolling, can I check my assumption $\boxed{f \leq \mu N}$

checked



Check: Is there sufficient friction?

Cylinder on an Incline: Slipping

$\sum F_x = m\ddot{x}_c \Rightarrow mg\sin\phi - f = m\ddot{x}_c$ (1)

$N - mg\cos\phi = m\ddot{y}_c$ (2)

$-\boxed{f}R = I_{zz}\ddot{\theta}$ (3)

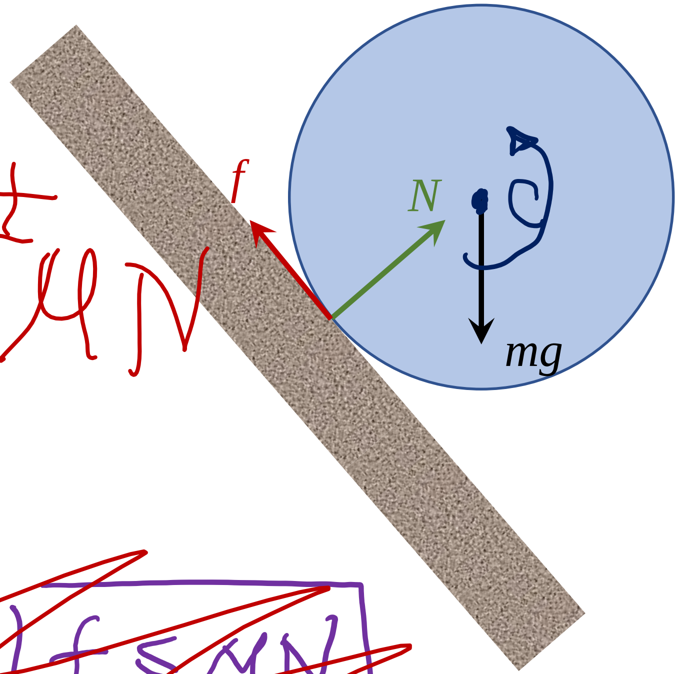
$y_c = R \Rightarrow \ddot{y}_c = 0$ (4)

~~Rolling~~ $\cancel{\dot{x}_c = -R\dot{\theta} \Rightarrow \ddot{x}_c = -R\ddot{\theta}}$ (5) $f = \mu N$

Unknowns: $\ddot{x}_c, \ddot{y}_c, \ddot{\theta}, N, f$

~~I assumed rolling, can I check my assumption~~ $\cancel{\boxed{f \le \mu N}}$

checked



Check: Is it slipping in the right direction?

Cylinder on an Incline: Unknown rolling/slipping