Derivatives

….. of Scalars and Vectors

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Derivatives of Scalar Functions

- Definition: $$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

- Common notation: $$\frac{df(x)}{dx} = f'(x) \qquad \frac{df(t)}{dt} = \dot{f}(x)$$

  where, x is a spatial coordinate and t is time.

- Physical relevance: slope, velocity, rate, strain …

  most of the world we inhabit can be represented by (partial) differential equations

Sum/Product of Functions and Composite Functions

- **Sum Rule:** $f(x) = f_1(x) + f_2(x) \quad \Rightarrow \quad f'(x) = f_1'(x) + f_2'(x)$
- **Product Rule:** $f(x) = f_1(x) f_2(x) \quad \Rightarrow \quad f'(x) = f_1(x) f_2'(x) + f_1'(x) f_2(x)$
- **Chain Rule:** $f(x) = g(h(x)) \quad \Rightarrow \quad f'(x) = \frac{dg}{dh} h'(x)$

$$f(x) = \sin(x^2) + e^x \cos x$$

$$f'(x) = \frac{d}{dx}\left(\sin(x^2)\right) + \frac{d}{dx}\left(e^x \cos x\right)$$

$$= (\cos x^2)\, 2x - e^x \sin x + e^x \cos x$$

Functions of Multiple Variables

- Partial Derivatives …

  when differentiating w.r.t. a given variable, assume that all other variables are constants

$$\frac{\partial f(x,y,z)}{\partial x} = \frac{d\bar{f}(x)}{dx} \qquad \bar{f}(x) = f(x,y,z)|_{\substack{y=y_0\\ z=z_0}}$$

- Total Derivative (when all other variables are functions on one variable)

$$\boxed{f(q_1(t), q_2(t), \cdots, t)} \qquad \boxed{\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial f}{\partial q_2}\frac{dq_2}{dt} + \cdots}$$

Derivatives of Vector Functions

- Definition: $$\frac{d\vec{v}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\vec{v}(t + \Delta t) - \vec{v}(t)}{\Delta t}$$
- Expression for derivatives of vectors functions is similar to that for scalars functions
- Derivative of a vector is a vector
- Derivatives are always with respect to scalar variables!
  - though we may define derivative of a function or set of functions w.r.t. set of variables, e.g., gradient, Hessian

$$f(x_1, \ldots\ldots x_n) \longrightarrow \nabla f = \begin{Bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{Bmatrix} \qquad \nabla f = \begin{bmatrix} \quad \end{bmatrix}$$

Sum/Product of Vector Functions

- Sum: $\vec{v}(t) = \vec{v}_1(t) + \vec{v}_2(t) \quad \Rightarrow \quad \dot{\vec{v}}(t) = \dot{\vec{v}}_1(t) + \dot{\vec{v}}_2(t)$
- Product with Scalar: $\vec{v}(t) = f(t)\vec{u}(t) \quad \Rightarrow \quad \dot{\vec{v}}(t) = f(t)\dot{\vec{u}}(t) + \dot{f}(t)\vec{u}(t)$
- Dot Product: $f(t) = \vec{v}_1(t) \cdot \vec{v}_2(t) \quad \Rightarrow \quad \dot{f}(t) = \vec{v}_1(t) \cdot \dot{\vec{v}}_2(t) + \dot{\vec{v}}_1(t) \cdot \vec{v}_2(t)$

  vec · vec = scal
- Cross Product: $\vec{v}(t) = \vec{v}_1(t) \times \vec{v}_2(t) \quad \Rightarrow \quad \dot{\vec{v}}(t) = \vec{v}_1(t) \times \dot{\vec{v}}_2(t) + \dot{\vec{v}}_1(t) \times \vec{v}_2(t)$

Derivatives of Vector Function

$\vec{F} = m\vec{a} = m\ddot{\vec{x}}$

$\vec{F} = m\,{}^{I}\ddot{\vec{x}}$ indicates that the time deriv are in inertial frame

- Vector are typically represented in terms of measure numbers
  - Needs a reference frame
  - Needs a coordinate system in the reference frame

$$\vec{v}(t) = v_x(t)\hat{i} + v_y(t)\hat{j} + v_z(t)\hat{k}$$

- Unit vectors are vector

$$\dot{\vec{v}}(t) = \dot{v}_x(t)\hat{i} + v_x(t)\dot{\hat{i}} + \dot{v}_y(t)\hat{j} + v_y(t)\dot{\hat{j}} + \dot{v}_z(t)\hat{k} + v_z(t)\dot{\hat{k}}$$

If $\hat{i}, \hat{j}, \hat{k}$ are unit vectors in an inertial frame then $\dot{\hat{i}} = \dot{\hat{j}} = \dot{\hat{k}} = 0$

if observer is also fixed in the inertial frame

- Are the unit vectors changing?
  - That is why references frames and coordinate systems are important
  - Inertial frames with Cartesian coordinates are the easiest to differentiate
  - Non-inertial frames and/or non-Cartesian coordinate systems make certain problems easier to solve overall (even though the differentiation is more difficult)

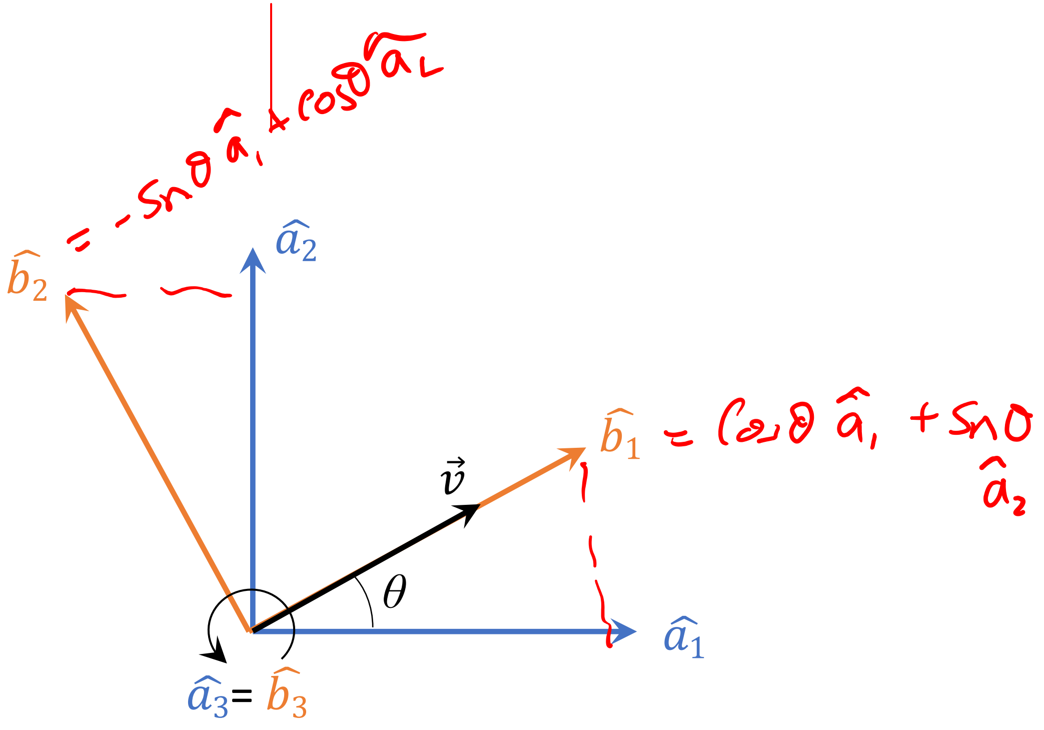
Inertial is important physics $\Rightarrow$ Newton

Reference Frames

- “A reference frame can be regarded as a massless rigid body and a rigid body can serve as a reference frame.”
- On a reference frame one can define one or more coordinate systems

Example

- Lets consider two reference frames A and B with corresponding coordinate systems – Cartesian, right-handed, mutually perpendicular, unit vectors fixed in the reference frame
    - $(\hat{a}_1, \hat{a}_2, \hat{a}_3)$ $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$



$$\vec{v} = v_0\,\hat{b}_1 + 0\,\hat{b}_2 + 0\,\hat{b}_3 = v_0 \cos\theta\,\hat{a}_1 + v_0 \sin\theta\,\hat{a}_2 + 0\,\hat{a}_3$$

$v_0(t)$ $\theta(t)$

Representing the vector in different reference frames does not change the **vector**

Differentiating w.r.t. generalized coordinate

- What is $\frac{d\vec{v}}{d\theta}$?
- Depends on the reference frame …
  - … even though $\vec{v}$ does not depend on the reference frame and θ does not depend on the reference frame, the change or derivative depends on the reference frame (observer)

$$\frac{{}^{A}d\vec{v}}{d\theta} \neq \frac{{}^{B}d\vec{v}}{d\theta} \neq \frac{{}^{C}d\vec{v}}{d\theta}$$

  - This change or derivative (in a particular reference frame) is a **vector** and may be represented in any reference frame

$$\frac{{}^{A}d\vec{v}}{d\theta} = \alpha_1\,\hat{a}_1 + \alpha_2\,\hat{a}_2 + \alpha_3\,\hat{a}_3 = \beta_1\,\hat{b}_1 + \beta_2\,\hat{b}_2 + \beta_3\,\hat{b}_3$$

Example: Find derivative w.r.t. θ

$$\vec{v} = v_0\,\hat{b}_1 + 0\,\hat{b}_2 + 0\,\hat{b}_3$$

$$\vec{v} = v_0\cos\theta\,\hat{a}_1 + v_0\sin\theta\,\hat{a}_2 + 0\,\hat{a}_3$$

Differentiating w.r.t. time

- Is $\vec{v}$ a function of time
  - … depends on if v_0 is a function of time
  - … and on the observer

$$^{B}\dot{\vec{v}} = \dot{v}_0 \hat{b}_1 + v_0 \, {}^{B}\dot{\hat{b}}_1$$
$$= \dot{v}_0 \hat{b}_1$$

$$^{A}\dot{\vec{v}} = \dot{v}_0 \cos\theta \, \hat{a}_1 - v_0 \sin\theta \cdot \dot{\theta} \, \hat{a}_1 + v_0 \cos\theta \, {}^{A}\dot{\hat{a}}_1$$
$$+ \dot{v}_0 \sin\theta \, \hat{a}_2 + v_0 \cos\theta \, \dot{\theta} \, \hat{a}_2 + v_0 \sin\theta \, {}^{A}\dot{\hat{a}}_2$$
$$= \dot{v}_0 (\cos\theta \, \hat{a}_1 + \sin\theta \, \hat{a}_2)$$
$$+ v_0 \dot{\theta} (-\sin\theta \, \hat{a}_1 + \cos\theta \, \hat{a}_2)$$
$$= \dot{v}_0 \hat{b}_1 + v_0 \dot{\theta} \hat{b}_2$$

$$^{B}\dot{\vec{v}} \neq {}^{A}\dot{\vec{v}}$$

Order of Differentiation: Higher Order Derivatives

$$\frac{d}{d\theta_2}\frac{df}{d\theta_1} = \frac{d}{d\theta_1}\frac{df}{d\theta_2}$$

- Just do one at a time
- Can we interchange the order of differentiation?
  - … yes, if the two differentiations are in the same frame of reference

$$\frac{{}^A d}{d\theta_2}\frac{{}^A d\vec{v}}{d\theta_1} = \frac{{}^A d}{d\theta_1}\frac{{}^A d\vec{v}}{d\theta_2} \qquad \frac{{}^A d}{d\theta}\frac{{}^A d\vec{v}}{dt} = \frac{{}^A d}{dt}\frac{{}^A d\vec{v}}{d\theta}$$

  - … no, if the two differentiations are in different frames of reference

$$\frac{{}^A d}{dt}\frac{{}^B d\vec{v}}{dt} \neq \frac{{}^B d}{dt}\frac{{}^A d\vec{v}}{dt} \qquad \frac{{}^A d}{d\theta}\frac{{}^B d\vec{v}}{dt} \neq \frac{{}^B d}{dt}\frac{{}^A d\vec{v}}{d\theta} \qquad \frac{{}^A d}{d\theta_2}\frac{{}^B d\vec{v}}{d\theta_1} \neq \frac{{}^B d}{d\theta_1}\frac{{}^A d\vec{v}}{d\theta_2}$$