Particle Dynamics

Newton's Second Law

- A body acted upon by a force moves in such a manner that the time rate of change of linear momentum equals the force

$$\vec{F} = \frac{{}^{I}d}{dt}(m\vec{v}) = m\vec{a}$$

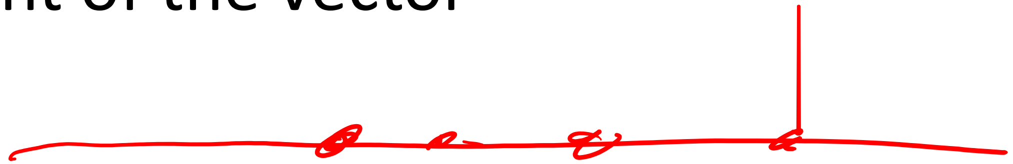
- Assumptions above
  - Frame is inertial (Newtonian, Galilean)
  - Constant mass (for the equation in terms of acceleration)
  - Valid for particles (not rigid bodies), though everything is relative, e.g., sometimes a planet can be considered as a particle
- In dynamics we relate the force to acceleration to velocity to position
  - Sometimes we use the prescribed motion to calculate the required force

Easy

Rectilinear Motion

- We are analyzing only one component of the vector

  … no need for vectors



- Position: *r(t)*
- Velocity (speed): *v(t)*
- Acceleration: *a(t)*
- Equations:

$$v(t) = \dot{r}(t)$$
$$a(t) = \dot{v}(t) = \ddot{r}(t)$$

kinematics
particles
rectilinear

Kinetics: $F = ma$

Differentiation

- Given $r(t)$, find $v(t)$ and $a(t)$
- Given $v(t)$, find $a(t)$
- Differentiation is easy (or at least easier than integration)!

Integration can be Difficult

$a = \frac{F(r)}{m}$

- Given *a(t)*, find *v(t)* and *r(t)*
- Given *v(t)*, find *r(t)*
- Multiple ways to think about it:

$\frac{dv}{dt} = a$

$v = \int a\,dt$

$v(t) - v(0) = \int_0^t a\,dt$ $\qquad$ $v(t) = \int a\,dt + C$

Definite Integral	Indefinite Integral and IC	Diff Eq and IC	2nd Diff Eq and IC
$v(t) - v(0) = \int_0^t a(t)dt$	$v(t) = \int a(t)dt + C$ $v(0) = v_0$	$\dot{v}(t) = a(t)$ $v(0) = v_0$	$\boxed{\begin{aligned} \ddot{r}(t) &= a(t) \\ r(0) &= r_0 \\ \dot{r}(0) &= v_0 \end{aligned}}$ norm
$r(t) - r(0) = \int_0^t v(t)dt$	$r(t) = \int v(t)dt + C_2$ $r(0) = r_0$	$\dot{r}(t) = v(t)$ $r(0) = r_0$ $a(r, v)$	

Differential Equations

- What if acceleration is a function of velocity and/or position *a(r,v,t)*?

  ….. then we get a "real" differential equation which cannot be solved

  using simple integration

- Example second order ODE
  - linear or nonlinear depending on the function *a(r,v,t)*

$$\ddot{r}(t) = a(r(t), \dot{r}(t), t) = \frac{F(r, \dot{r}, t)}{m}$$

$$r(0) = r_0$$

$$v(0) = \dot{r}(0) = v_0$$

  - differential equations can easily become complicated enough that analytical solutions are not possible (very common in real applications)

Numerical Solutions may be the Only Option

- Solving differential equations is what is using up more than 50% of supercomputer power in the world
- You can find numerical solutions quickly using Matlab (e.g., ode45) and Mathematica (e.g., NDSolve)

Consider Motion in 3D Space

- We are working with vectors now!
- Position Vector in an Inertial Frame: $\vec{r}(t) = x(t)\underline{\hat{i}} + y(t)\underline{\hat{j}} + z(t)\underline{\hat{k}}$
- Velocity Vector and Acceleration Vector are given by:

Inertial Cartesian

$$\vec{v}(t) = {}^{I}\dot{\vec{r}}(t)$$

$$\vec{a}(t) = {}^{I}\dot{\vec{v}}(t) = {}^{I}\ddot{\vec{r}}(t)$$

- The above time derivative notation is quite vague as it stands – we need to be more precise …

… we are assuming that we are differentiating in an inertial frame

Inertial Frame and Cartesian Coordinates

- Because we are differentiating relative to an Inertial Frame and we are representing the vectors using Cartesian Coordinates in an Inertial Frame
  - If $\hat{\imath}, \hat{\jmath}, \hat{k}$ are not changing with time …

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$
$$\vec{v}(t) = \dot{x}(t)\hat{i} + \dot{y}(t)\hat{j} + \dot{z}(t)\hat{k}$$
$$\vec{a}(t) = \ddot{x}(t)\hat{i} + \ddot{y}(t)\hat{j} + \ddot{z}(t)\hat{k}$$

Analysis in 3D

$\det(A - sI)$

$\vec{F} \rightarrow \vec{a} = \frac{\vec{F}}{m}$

$$\begin{Bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{Bmatrix} = \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{Bmatrix} + f$$

- Again, differentiation is “easy”
- Assuming inertial frame and Cartesian coordinate system
  - If the acceleration is only a function of time, integration in 3D is equivalent to three rectilinear motion integration problems
  - If acceleration is a function of position or velocity, you will get three ordinary differential equations which are coupled and need to be solved together (not exactly the same as solving to three independent rectilinear problems)
    - Linear coupled ODEs can be solved using eigenvalues and decoupling (linear systems theory)
    - Nonlinear coupled ODEs typically need numerical solutions
      - For stability (to small disturbance) you can linearize the problem and do stability analysis

[Set of coupled ODEs]

$\ddot{x} = f_x(x, y, z, \dot{x}, \dot{y}, \dot{z})$

$\ddot{y} = f_y($ $\quad \ddot{z} = f_z($

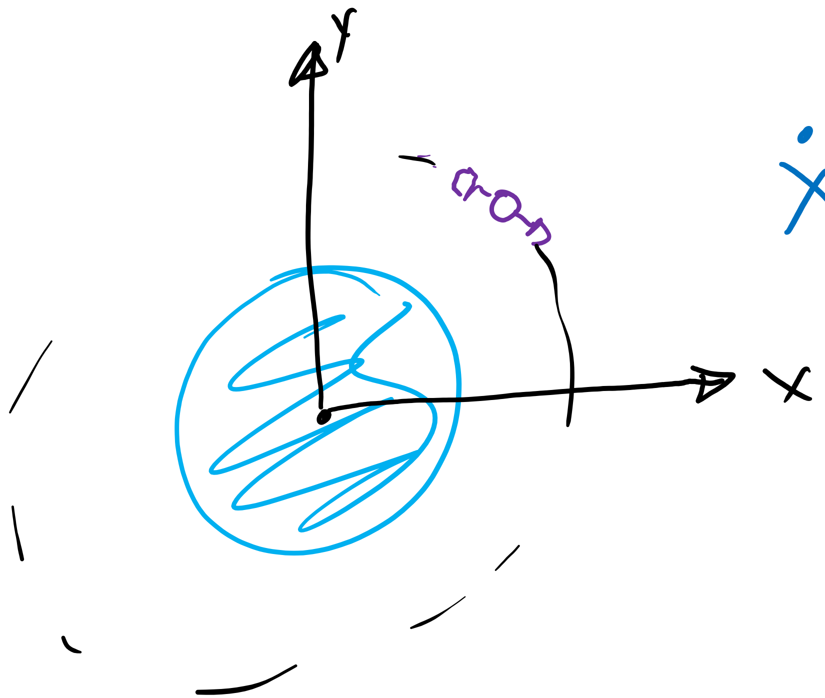
Satellite Simulation (Cartesian Coord System)

- Gravity Model:

$$\vec{g} = -\frac{G M_E \vec{r}}{|\vec{r}|^3} = -g_0 \frac{r_E^2 \vec{r}}{|\vec{r}|^3}$$

$$= -G M_E (x\hat{i} + y\hat{j}) / (x^2+y^2)^{3/2}$$

- Equations of Motion (for acceleration) – 2 coupled ODEs



$$m\vec{a} = -m\vec{g}$$

$$\dot{x} = u \qquad \ddot{x} = \frac{-G M_E x}{(x^2+y^2)^{3/2}}$$

$$\dot{y} = v \qquad \ddot{y} = \frac{-G M_E y}{(x^2+y^2)^{3/2}}$$

2 coupled nonlinear ODEs

2 2nd order

↳ 4 1st order

First Order Form and Initial Conditions

RK algorithms need first order form & ICs (two-point boundary value problems not solved)

$$dq/dt = \begin{Bmatrix} u \\ v \\ \dfrac{-GM_E x}{(x^2+y^2)^{3/2}} \\ \dfrac{-GM_E y}{(x^2+y^2)^{3/2}} \end{Bmatrix}$$

$$\underbrace{\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{u} &= -\frac{GM_E x}{(x^2+y^2)^{3/2}} \\ \dot{v} &= \frac{-GM_E y}{(x^2+y^2)^{3/2}} \end{aligned}}$$

$$q = \underbrace{\begin{Bmatrix} x \\ y \\ u \\ v \end{Bmatrix}}_{\text{state}} \qquad \underbrace{\begin{Bmatrix} x(0) \\ y(0) \\ u(0) \\ v(0) \end{Bmatrix}}_{\text{needed}}$$

$q(1) = x$

$q(2) = y$

MATLAB (ODE) Function and ODE Solvers