Generalized Coordinates

and Generalized Velocities and Constraints

mpatil@gatech.edu

Configuration Variables (N)

- The configuration of a set of particles or a set of rigid bodies (which themselves are sets of particles) or a set of flexible bodies in a reference frame A is the set of position vectors relative to a point fixed in A
  - Configuration Variables tells us where each and every particle is located

Configuration Constraints: Holonomic Constraints (M)

- Configuration constraints are constraints on the position

$$f(q_1, q_2, q_3, \cdots, t) = 0$$

  - q_1, q_2, q_3 are the configuration variables
  - They reduce the number of independent configuration variables (generalized coordinates) required to represent a system
  - These constraints can be linear or nonlinear
  - These constraints are simpler to take into account in the derivation of equations of motions and to impose during simulation

Generalized Coordinates (n)

- The generalized coordinates are the minimum set of (scalar) independent variables or coordinates required to fully determine the configuration
- The number of generalized coordinates required to represent a system depends on the number of particles/rigid bodies and the number of constraints
- Generalized coordinates are typically represented by: $q_1, q_2, q_3, \ldots$

    … see for example, most of the examples in the book and the HW problems

Configuration Constraints: From Particles to Rigid Bodies

Motion Constraints: Non-Holonomic Constraints (m)

- Motion constraints are non-integrable constraints on the velocity

$$f(q_1, q_2, q_3, \cdots, \dot{q}_1, \dot{q}_2, \dot{q}_3, \cdots, t) = 0$$

  - Motion constraints cannot be integrable, i.e., any holonomic constraint can be differentiated to get a ‘motion’ constraint but it can be integrated to give a configuration constraint (thus it is not a non-holonomic constraint)
  - If the constraint is linear (in velocity) then it is a **simple** non-holonomic constraint (the constraint can be nonlinear in generalized coordinates)

$$\sum f_i(q_1, q_2, q_3, \cdots, t)\dot{q}_i = 0$$

  - If the constraint in nonlinear in velocity (i.e., nonlinear in time derivatives of the generalized coordinates) then it is **complex** non-holonomic constraint
    - Complex non-holonomic systems are quite difficult to solve!

Non-holonomic Constraint: Parallel Parking

Motion Variables!

- Motion variables are generalized velocity variables defined by Kane

$$u_r = \sum_s Y_{rs}(q_1, q_2, q_3, \cdots, t)\dot{q}_i + Z_r(q_1, q_2, q_3, \cdots, t)$$

  - this (or rather it's inverse) are called the kinematic differential equations
  - linear relation between u_r and $\dot{q}_i$ (can be nonlinear in q_i)
  - the above set of equations has to be invertible
- This is where Kane's methods start to diverge from Newton-Euler's or Lagrange's
  - Motion variables lead to partial velocities
  - Partial velocities help us calculate generalized forces (active and inertial)
  - Generalized forces leads to (Kane's) equations of motion
- Selection of motion variables is up to the analyst in Kane's method
  - The equations of motion and thus the complexity of equations of motion depends on the choice of the motion variables
  - The motion variables are directions along which the equations of motion are derived (one can think of a projection of the equations of motion along that direction)

Selection and Use of Motion Variable

- A default set of motion variables could be:
$$u_i = \dot{q}_i$$
  - this would lead to equations similar to Lagrange's Equations
  - as will be shown later, it is better to uncouple the two – the choice of motion variables and choice of generalized coordinates – for optimal (simplest) equations
- Select the motion variables such that your velocities and angular velocities of interest can be written as simply as possible
- Any non-Holonomic constraints can be written in terms of the motion variables – and will thus reduce the number of independent motion variables (generalized velocities)
- The use of the motion variables leads to simplification of the equations for the velocities and angular velocities and thus simplified calculation and expressions of the acceleration and angular acceleration (in terms of derivatives of motion variables)
  - The equations of motion can be written in terms in terms of the generalized coordinates, motion variables (generalized velocities) and derivatives of the motion variables (generalized velocities)
  - The kinematic differential equations are written in terms of the generalized coordinates, motion variables and derivatives of the generalized coordinates
  - This is similar to a state-space model, i.e., first order ODEs in terms of generalized coordinated and velocities (rather than second order in terms of generalized coordinates only)
  - Required for Kane's method but can (should) be used in Newton-Euler equations as well

Number of Coordinates and Velocities

- Number of Configuration Variables: N
- Number of Holonomic Constraints: M
- Number of Generalized Coordinates: $n = N - M$
- Number of Motion Variables: n
- Number of (Simple) Non-Holonomic Constraints: m
- Number of Constraint Forces: m
- Number of Generalized Velocities: $p = n - m$
- Number of kinematic differential equations: n
- Number of equations of motion: p (Kane) or n (Newton-Euler)
- Number of equations equal to the number of unknowns: $n+p$ or $2*n+m$
- For Holonomic system: $p = n,\ m = 0$