# Newton-Euler Method for Rigid Body Dynamics

… about a point that is not the center of mass

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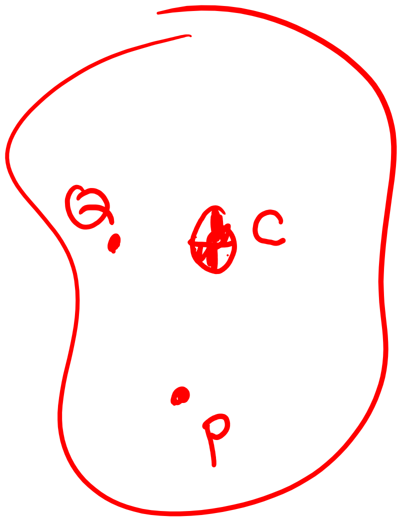
# Notations

- There are only two frames, inertial frame $I$ and body frame $B$
- All time derivatives are w.r.t. the inertial frame unless indicated
- All angular velocities and angular accelerations are of the body relative to the inertial frame
- $Q$ is a general point on the rigid body (for integration)
- $C$ is the center of mass of the rigid body
  - We already know the equations of motion using the velocities at $C$ and moments about $C$
- $P$ is a fixed point on the rigid body that is not the center of mass
  - We are trying to write the equations in terms of the velocities at $P$ and moments about $P$
- $O$ is a point fixed in the inertial frame

# Euler’s First Law




$$\sum \vec{F} = \frac{d\vec{L}}{dt}$$

- The above equation does not reference any point.
- Now consider a rigid body:

$$\vec{L} = \int \vec{v}^{\,Q} dm \quad = \sum m_i \vec{v}_i$$

- We can write the velocity at $Q$ relative to a reference point, $C$ or $P$ (using the two-point formula for velocity):

$$\vec{v}^{\,Q} = \vec{v}^{\,C} + \vec{\omega} \times \vec{r}^{\,CQ} \qquad\qquad \vec{v}^{\,Q} = \vec{v}^{\,P} + \vec{\omega} \times \vec{r}^{\,PQ}$$

- Linear Momentum:

$$\vec{L} = m\vec{v}^C$$

$\vec{r}^{\,PC}$

$$\vec{L} = m\vec{v}^P + m\vec{\omega} \times \vec{r}^{PC}$$
$$= m\vec{v}^P - m\vec{r}^{PC} \times \vec{\omega}$$

- Equations of Motion (assuming constant *m* and constant $\vec{r}^{PC}$ [in *B* frame]):

$$\sum \vec{F} = \frac{d\left(m\vec{v}^C\right)}{dt}$$
$$= m\dot{\vec{v}}^C$$
$$= m\vec{a}_c$$

$$\sum \vec{F} = \frac{d\left(m\vec{v}^P + m\vec{\omega} \times \vec{r}^{PC}\right)}{dt}$$
$$= m\dot{\vec{v}}^P + m\dot{\vec{\omega}} \times \vec{r}^{PC} + m\vec{\omega} \times \dot{\vec{r}}^{PC}$$
$$= m\dot{\vec{v}}^P - m\vec{r}^{PC} \times \dot{\vec{\omega}} + m\vec{\omega} \times \left(\omega \times \vec{r}^{PC}\right)$$

- We could have also derived the right equation from the left equation by using the two-point formula for acceleration

$$\boxed{\vec{a}^C = \vec{a}^P + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}^{PC}\right) + \vec{\alpha} \times \vec{r}^{PC}}$$

two point formula

# Euler's Second Law

think 2 pt formula for kinetics

$$\sum \vec{M}^O = \frac{d\vec{H}^O}{dt}$$

- The above equation is valid for any point fixed in the inertial frame.
- Now, the moment and angular moment about one point can be related to the vectors about another point …

$$\vec{M}^O = \vec{M}^C + \vec{r}^{OC} \times \vec{F} \qquad \vec{M}^O = \vec{M}^P + \vec{r}^{OP} \times \vec{F}$$

$$\vec{H}^O = \vec{H}^C + \vec{r}^{OC} \times \vec{L} \qquad \vec{H}^O = \vec{H}^P + \vec{r}^{OP} \times \vec{L}$$

- Using the above (and Euler's first law) we get:

$$\sum \vec{M}^C = \dot{\vec{H}}^C \qquad \sum \vec{M}^P = \dot{\vec{H}}^P + \underline{\dot{\vec{r}}^{OP} \times \vec{L}}$$

- Now for a rigid body …

$$\vec{H}^C = \int \vec{r}^{\,CQ} \times \vec{v}^{\,Q} dm \qquad \vec{H}^P = \int \vec{r}^{\,PQ} \times \vec{v}^{\,Q} dm$$

$$\vec{v}^{\,Q} = \vec{v}^{\,C} + \vec{\omega} \times \vec{r}^{\,CQ} \qquad \vec{v}^{\,Q} = \vec{v}^{\,P} + \vec{\omega} \times \vec{r}^{\,PQ}$$

- Integrate and use the definition of the center of mass …

$$\vec{H}^C = \vec{\vec{I}}^C \cdot \vec{\omega}$$

$$\vec{\vec{I}}^C \cdot \vec{\omega} = \int \left[\vec{r}^{\,CQ} \times \left(\vec{\omega} \times \vec{r}^{\,CQ}\right)\right] dm$$

$$= \int \left[\left(\vec{r}^{\,CQ} \cdot \vec{r}^{\,CQ}\right)\vec{\omega} - \vec{r}^{\,CQ}\left(\vec{r}^{\,CQ} \cdot \vec{\omega}\right)\right] dm$$

$$= \int \left[\left(\vec{r}^{\,CQ} \cdot \vec{r}^{\,CQ}\right)\vec{\vec{U}} - \vec{r}^{\,CQ}\vec{r}^{\,CQ}\right] dm \cdot \vec{\omega}$$

$$\vec{\vec{I}}^C = \int \left[\left(\vec{r}^{\,CQ} \cdot \vec{r}^{\,CQ}\right)\vec{\vec{U}} - \vec{r}^{\,CQ}\vec{r}^{\,CQ}\right] dm$$

$$\vec{H}^P = m\vec{r}^{\,PC} \times \vec{v}^{\,P} + \vec{\vec{I}}^P \cdot \vec{\omega}$$

$$\vec{\vec{I}}^P = \int \left[\left(\vec{r}^{\,PQ} \cdot \vec{r}^{\,PQ}\right)\vec{\vec{U}} - \vec{r}^{\,PQ}\vec{r}^{\,PQ}\right] dm$$

$$\vec{\vec{I}}^P \cdot \vec{\omega} = \vec{\vec{I}}^C \cdot \vec{\omega} - m\vec{r}^{\,PC} \times \left(\vec{r}^{\,PC} \times \vec{\omega}\right)$$

- Equations of Motion (assuming constant $m$ and constant $\vec{r}^{PC}$ $\vec{\vec{I}}^C$ $\vec{\vec{I}}^P$ [in $B$ frame]):

$$\boxed{\sum \vec{M}^C = \vec{\vec{I}}^C \cdot \dot{\vec{\omega}} + \vec{\omega} \times \left( \vec{\vec{I}}^C \cdot \vec{\omega} \right)}$$

— no $I^P$

$$\sum \vec{M}^P = m\dot{\vec{r}}^{PC} \times \vec{v}^P + m\vec{r}^{PC} \times \dot{\vec{v}}^P + \vec{\vec{I}}^P \cdot \dot{\vec{\omega}} + \vec{\omega} \times \left( \vec{\vec{I}}^P \cdot \vec{\omega} \right) + \dot{\vec{r}}^{OP} \times \vec{L}$$

$$= m\vec{r}^{PC} \times \dot{\vec{v}}^P + \vec{\vec{I}}^P \cdot \dot{\vec{\omega}} + \vec{\omega} \times \left( \vec{\vec{I}}^P \cdot \vec{\omega} \right)$$

— no $I^C$

- You could also derive the second equation from the first by substituting for the moment about $C$ in terms of the moment about $P$ and the force, and then substituting for the force using the equation of motion in translation

# Final Equations of Motion

degrees of freedom at P (not C)

$$\sum \vec{F} = m\dot{\vec{v}}^{P} - m\vec{r}^{PC} \times \dot{\vec{\omega}} + m\vec{\omega} \times (\omega \times \vec{r}^{PC})$$

$$\sum \vec{M}^{P} = m\vec{r}^{PC} \times \dot{\vec{v}}^{P} + \vec{\vec{I}}^{P} \cdot \dot{\vec{\omega}} + \vec{\omega} \times \left(\vec{\vec{I}}^{P} \cdot \vec{\omega}\right)$$

moment about P (not C)

- Note that these are vector equations and all derivatives are in the inertial frame
- The above equations will be solved in terms of components in a frame
- All the vectors except velocity are typically written in the body frame
- The velocity vector may be written in the inertial or body frame and you may need …

$$\dot{\vec{v}}^{P} = {}^{I}\dot{\vec{v}}^{P} = {}^{B}\dot{\vec{v}}^{P} + \vec{\omega} \times \vec{v}^{P}$$

- You can mix and match, i.e., use velocity at $P$ but write moment equation about $C$
- For 2D problems, it is possible to easily write all the vectors except $\vec{r}^{PC}$ in the inertial frame.

# Matrix Form (from wikipedia)

- From https://en.wikipedia.org/wiki/Newton-Euler_equations

With respect to a coordinate frame located at point **P** that is fixed in the body and *not* coincident with the center of mass, the equations assume the more complex form:

$r_{cp}$ $[\ ]^\times$ $[\sim]$

$$\begin{pmatrix}\mathbf{F}\\ \boldsymbol{\tau}_{\rm p}\end{pmatrix} = \underbrace{\begin{pmatrix} m\mathbf{I}_3 & -m[\mathbf{c}]^\times \\ m[\mathbf{c}]^\times & \mathbf{I}_{\rm cm} - m[\mathbf{c}]^\times[\mathbf{c}]^\times\end{pmatrix}}_{\text{inertial}} \begin{pmatrix}\mathbf{a}_{\rm p}\\ \boldsymbol{\alpha}\end{pmatrix} + \underbrace{\begin{pmatrix} m[\boldsymbol{\omega}]^\times[\boldsymbol{\omega}]^\times\mathbf{c} \\ [\boldsymbol{\omega}]^\times(\mathbf{I}_{\rm cm} - m[\mathbf{c}]^\times[\mathbf{c}]^\times)\,\boldsymbol{\omega}\end{pmatrix}}_{\text{nonlinear}},$$

where **c** is the location of the center of mass expressed in the body-fixed frame, and

$$[\mathbf{c}]^\times \equiv \begin{pmatrix} 0 & -c_z & c_y \\ c_z & 0 & -c_x \\ -c_y & c_x & 0 \end{pmatrix} \qquad [\boldsymbol{\omega}]^\times \equiv \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}$$

denote skew-symmetric cross product matrices.

# Linear Equations of Motion

- For small rotations, we can derive linear equations of motion.
- We no longer need to differentiate between the inertial and body frame.
- The equations have a "nice" symmetric form.

$$\boxed{\begin{Bmatrix} \sum F \\ \sum M^P \end{Bmatrix} = \begin{bmatrix} m\Delta & m\tilde{\xi}^T \\ m\tilde{\xi} & I^P \end{bmatrix} \begin{Bmatrix} \dot{v}^P \\ \dot{\omega} \end{Bmatrix}}$$

... where, $\xi$ are the component of $\vec{r}^{PC}$

... $\tilde{\xi}$ is the dual or cross-product matrix of $\xi$

... $\Delta$ is a 3 x 3 identity matrix

... also

$$I^P = I^C - m\tilde{\xi}\tilde{\xi} = I^C + m\tilde{\xi}\tilde{\xi}^T$$

... finally, the kinetic energy can we written as

$$T = \frac{1}{2} \begin{Bmatrix} v^P \\ \omega \end{Bmatrix}^T \begin{bmatrix} m\Delta & m\tilde{\xi}^T \\ m\tilde{\xi} & I^P \end{bmatrix} \begin{Bmatrix} v^P \\ \omega \end{Bmatrix}$$