Linearization and Stability

General Nonlinear Equations of Motion

- Equations of motion have the form of:

$$f_i(\bar{q}(t), \dot{\bar{q}}(t), \ddot{\bar{q}}(t), t, \bar{p}) = 0$$

or

$$g_i(\bar{q}(t), \dot{\bar{q}}(t), \bar{u}(t), t, \bar{p}) = 0$$

$$f_i(\bar{q}(t), \bar{u}(t), \dot{\bar{u}}(t), t, \bar{p}) = 0$$

where, $\bar{q}(t)$ is the column of generalized coordinates, $\bar{u}(t)$ is the column of generalized speeds, and $\bar{p}$ is the set of parameters of the problem.

Equilibrium or Steady-State Solution

- One can calculate a nonlinear equilibrium solution of the equations of motion by solving the algebraic (non-time-dependent) equation:

multiple solns or Equilibria

Solve nonlinear algebraic eqs

$$f_i(\bar{q}(t), \cancelto{0}{\dot{\bar{q}}(t)}, \cancelto{0}{\ddot{\bar{q}}(t)}, \cancel{t}, \bar{p}) = f_i(\bar{q}_0, 0, 0, \times, \bar{p}) = 0 \implies \bar{q}_0$$

- For some problems where equations of motion do not depend of the generalized coordinates, we can calculate a steady-state solution by solving an algebraic equation for the constant rate or velocity:

$$f_i(\cancel{\bar{q}(t)}, \dot{\bar{q}}(t), \cancelto{0}{\ddot{\bar{q}}(t)}, \cancel{t}, \bar{p}) = f_i(\times, \dot{\bar{q}}_0, 0, \times, \bar{p}) = 0$$

Linear vs Linearized Equations

Equations linearized about $q=0, \dot{q}=0$ are the linear equation.

- Linear: All equations are derived assuming that the magnitudes of the generalized coordinates and generalized speeds are small.
  - Velocities, angular velocities, accelerations and angular accelerations are linear function of generalized coordinates/speeds and their derivatives. $V = [\;]\{q\}$
  - Equations – KDE, EoE, constraint equations – are all linear in terms of generalized coordinates/speeds.
  - Energy expressions are all quadratic in terms of generalized coordinates/speeds.
- Linearized: Nonlinear equations are derived for all the above. Linearized dynamic equations are derived by assuming that:

$\sin(q_0+q') = \sin q_0 \cos q' + \cos q_0 \sin q'$

$\sin q_0 = \sin q_0$, $\sin q' = q'$

$$\bar{q}(t) = \bar{q}_0 + \bar{q}'(t)$$

where, the $()_0$ indicates a (possibly large) equilibrium or steady-state solution that has been calculated by solving the nonlinear equations of motion and $()'$ indicated the small perturbation about this nonlinear steady-state.

$\sin(q_0+q') \approx \sin q_0 + \cos q_0 \cdot q'$

Linear

Linearized Equations of Motion

- Linear/linearized equations of motion have the form of:

$$[M]\{\ddot{\bar{q}}'(t)\} + [C]\{\dot{\bar{q}}'(t)\} + [K]\{\bar{q}'(t)\} = \{\bar{f}(t)\}$$ Struc Dyn

or

$$\begin{Bmatrix} \dot{\bar{q}}'(t) \\ \dot{\bar{u}}'(t) \end{Bmatrix} = [A] \begin{Bmatrix} \bar{q}'(t) \\ \bar{u}'(t) \end{Bmatrix} + \{\bar{f}(t)\}$$ Controls

where, $[M]$, $[C]$, $[K]$, and $[A]$ are typically constant matrices which are functions of your problem parameters.

- If derived from a general nonlinear system, the matrices are functions of the steady-state solution about which the problem is linearized. $\bar{q}_0$

Nonlinear System

Nonlinear ODE

Optimization

minimize $f(x_i)$

gradient $\frac{\partial f}{\partial x_i}$

Hessian $\frac{\partial^2 f}{\partial x_i \partial x_k}$

Calculate Equilibrium/SS

Solve nonlinear algebra

(Newton-Raphson)

$f_i(x_j) = 0$

Jacobian $= \frac{\partial f_i}{\partial x_j}$

Simulate for specific parameters, forcing, initial condition

Analysis

sometimes bright mathematicians will give you insight

simulation for insight

Linearize

(linear dynamics system)

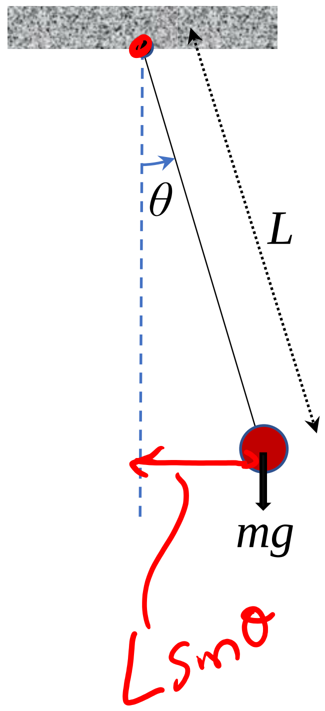
trivial

eigenvalues / stability

stable (region of attraction)

unstable (Limit cycle oscillation)

Simple Pendulum (Point Mass)



$$\cancel{m}L^{\cancel{2}}\ddot{\theta} = -\cancel{m}g\cancel{L}\sin\theta$$

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0$$

Equilibrium $\ddot{\theta} = 0$

$$\frac{g}{L}\sin\theta = 0 \Rightarrow \theta_0 = 0 \quad \text{or} \quad \theta_0 = \pi$$

Linearize

$$\sin\theta = \sin(\theta_0 + \theta')$$

$$= \sin\theta_0 + \cos\theta_0\,\theta'$$

$\theta_0 = 0$: $= \theta'$

$\theta_0 = \pi$: $= -\theta'$

Linearization and Frequency Domain Analysis

$\underline{\theta_0 = 0}$

$$\ddot{\theta}' + \frac{g}{L}\theta' = 0$$

$$\downarrow$$

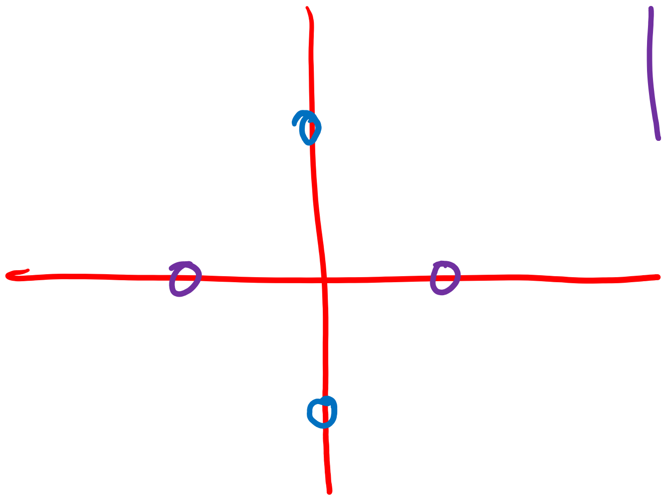
$$\theta' = c\,e^{\lambda t}$$

$$\boxed{\lambda = \pm i\sqrt{\frac{g}{L}}}$$

$\underline{\theta_0 = \pi}$

$$\ddot{\theta}' - \frac{g}{L}\theta' = 0$$

$$\boxed{\lambda = \pm\sqrt{\frac{g}{L}}}$$



Euler Equations (about Principal Axes)

- If $I_{12}^C = I_{13}^C = I_{23}^C = 0$

$$\begin{aligned}
M_1^C &= I_{11}^C\dot{\omega}_1 + (I_{33}^C - I_{22}^C)\omega_2\omega_3 \\
M_2^C &= I_{22}^C\dot{\omega}_2 + (I_{11}^C - I_{33}^C)\omega_1\omega_3 \\
M_3^C &= I_{33}^C\dot{\omega}_3 + (I_{22}^C - I_{11}^C)\omega_1\omega_2
\end{aligned}$$

Linearization and Stability Analysis