Ang Vel ($\vec{\omega}$): Vector (R^3)

Ang Orientation (?): Not a vector (SO3) for $\boxed{\text{FINITE ROTATION}}$

↳ for infinitesimal angles: it can be represented as a vector

↳ small (engineering)

Kinematics of Orientation

Direction Cosines

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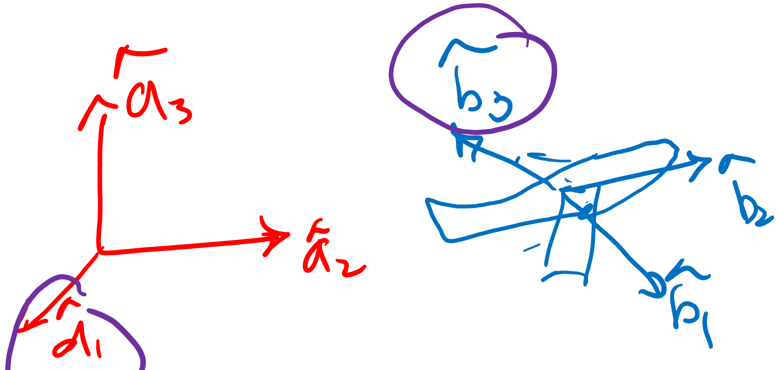
Direction Cosines

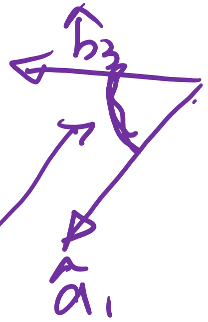


right handed

- Consider two dextral sets of orthogonal unit vectors, the direction cosines are:

$$C_{ij} = \hat{a}_i \cdot \hat{b}_j$$

$$= |\hat{a}_i|\,|\hat{b}_j| \cos(\angle \hat{a}_i \hat{b}_j)$$



- The direction cosine matrix is:

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

C is not sym, but is orthogonal

… the first index is always the row number and second is the column number.

- The above direction cosine matrix can be more precisely written as: ${}^{A}C^{B}$

$${}^{A}C^{B}_{ij} = \hat{a}_i \cdot \hat{b}_j \qquad {}^{B}C^{A}_{ij} = \hat{b}_i \cdot \hat{a}_j$$

$$= \hat{a}_j \cdot \hat{b}_i$$

$${}^{A}C^{B}_{ij} = {}^{B}C^{A}_{ji}$$

What does DCM give us?

$$C_{ij} = \hat{a}_i \cdot \hat{b}_j$$

$$C = \begin{array}{c|ccc} & b_1 & b_2 & b_3 \\ \hline a_1 & C_{11} & C_{12} & C_{13} \\ a_2 & C_{21} & C_{22} & C_{23} \\ & C_{31} & C_{32} & C_{33} \end{array}$$

projection

$$\hat{a}_i = (\hat{a}_i \cdot \hat{b}_1)\hat{b}_1 + (\hat{a}_i \cdot \hat{b}_2)\hat{b}_2 + (\hat{a}_i \cdot \hat{b}_3)\hat{b}_3$$

$$= \sum_j C_{ij} \hat{b}_j$$

$$\{{}^B\hat{a}_1\} = \begin{Bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{Bmatrix} \qquad \{{}^B\hat{a}_2\} = \begin{Bmatrix} C_{21} \\ C_{22} \\ C_{23} \end{Bmatrix}$$

$$\{{}^B\hat{a}_1\} \cdot \{{}^B\hat{a}_2\} = 0$$

$$\{{}^B\hat{a}_1\} \cdot \{{}^B\hat{a}_1\} = 1$$

$$\hat{b}_i = (\hat{b}_i \cdot \hat{a}_1)\hat{a}_1 + (\hat{b}_i \cdot \hat{a}_2)\hat{a}_2 + (\hat{b}_i \cdot \hat{a}_3)\hat{a}_3$$

$$= C_{1i}\hat{a}_1 + C_{2i}\hat{a}_2 + C_{3i}\hat{a}_3 = \sum_j C_{ji} a_j$$

$$\{{}^A\hat{b}_1\} = \begin{Bmatrix} C_{11} \\ C_{21} \\ C_{31} \end{Bmatrix}$$

Properties of DCM

- Transpose:

$$^{B}C^{A} = (^{A}C^{B})^{T}$$

- Orthogonal:

$$CC^{T} = C^{T}C = I_3$$

$\boxed{\text{... six constraints}}$

- Determinant (for Dextral orthogonal system):

$$|C| = +1$$

Unit Vector can be Related

- The unit vectors are related as:

$$\lfloor \hat{a}_1 \quad \hat{a}_2 \quad \hat{a}_3 \rfloor = \lfloor \hat{b}_1 \quad \hat{b}_2 \quad \hat{b}_3 \rfloor \, [{}^A C^B]^T$$

$$\boxed{\begin{Bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{Bmatrix} = [{}^A C^B] \begin{Bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{Bmatrix}}$$

difficult to use in computer code

Transformations: Vector Components

- Components of a vector in two axis systems:

3D

$$^{A}v_i = \vec{v} \cdot \hat{a}_i$$

$$^{B}v_i = \vec{v} \cdot \hat{b}_i$$

$$\vec{v} = {}^{A}v_1 \hat{a}_1 + {}^{A}v_2 \hat{a}_2 + {}^{A}v_3 \hat{a}_3$$

$$= {}^{B}v_1 \hat{b}_1 + {}^{B}v_2 \hat{b}_2 + {}^{B}v_3 \hat{b}_3$$

- Transformation:

$$\boxed{\{^{A}v\} = [^{A}C^{B}]\{^{B}v\}}$$

3x3 3x1

$$\{^{B}v\} = [^{A}C^{B}]^{T}\{^{A}v\}$$

$$\lfloor ^{B}v \rfloor = \lfloor ^{A}v \rfloor [^{A}C^{B}]$$

1x3 3x3

$$\lfloor ^{A}v \rfloor = \lfloor ^{B}v \rfloor [^{A}C^{B}]^{T}$$

Simple Rotations

- Simple rotations of θ about axis $\hat{a}_1 = \hat{b}_1$

$$C_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

- Similarly:

$$C_2(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$C_3(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Angular Velocity

$\{{}^B M\} = [{}^B I]\{{}^B\dot{\omega}\} + [{}^B\tilde{\omega}][{}^B I]\{{}^B\omega\}$

dual

Applied Moment ⟷ related to rate of change of ang vel

- If angular velocity is written in the b-frame:

Now kinematic Equations will relate ang vel ⟷ rate of change of Orientation

$$
{}^A\vec{\omega}^B = \omega_1\hat{b}_1 + \omega_2\hat{b}_2 + \omega_3\hat{b}_3
$$

- The skew-symmetric dual matrix of the above angular velocity components is:

$$
\tilde{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}
$$

easy

$\dot{\omega} = I^{-1}M + I^{-1}\tilde{\omega}I\omega$

- The angular velocity can be derived in terms of DCM:

$$
\boxed{\tilde{\omega} = C^T\dot{C}}
$$

$$
\boxed{\dot{C} = C\tilde{\omega}} \rightarrow \text{matrix equation} \rightarrow 9 \text{ equations}
$$

... Poisson's kinematical equations

Solution in terms of DCM

- We can use the Euler's kinetic equations (3) with the Poisson's kinematic equations (9) to solve for the 12 states – 3 angular velocities and 9 DCs.
  - Note 6 of the 9 kinematic equations are differentiated constraints on DCs.
  - Have to be a bit careful numerically since 6 of 9 equations from Poisson's kinematic equations are time derivatives of orthogonality constraints
  - Only quadratic nonlinearities! No singularities!!! → awesome but so can Euler Paramet
- You can add the three ~~linear~~ translational kinetic and three kinematic equations for the six ~~linear~~ states

translation