Orientation Angles

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Three Angles

- Orientation of body B in reference frame A can be represented in terms of three angles …
- Consider unit vectors: $(\hat{a}_1, \hat{a}_2, \hat{a}_3)$ and $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$
- Starting with $(\hat{a}_1, \hat{a}_2, \hat{a}_3)$ and $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$ aligned, one can rotate about three angles about three particular axes to get from $(\hat{a}_1, \hat{a}_2, \hat{a}_3)$ to $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$
- These are three orientation angles
- See Appendix I and II in the book for all the possible angle combinations

Example: Body 1-2-3

- Consider the body $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$ is initially aligned with $(\hat{a}_1, \hat{a}_2, \hat{a}_3)$
- Consider a rotation of θ_1 about $\hat{b}_1' = \hat{a}_1$
- This takes us to a new reference frame, say $(\hat{b}_1', \hat{b}_2', \hat{b}_3')$

$$\boxed{\begin{Bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{Bmatrix} = [C_1(\theta_1)] \begin{Bmatrix} \hat{b}_1' \\ \hat{b}_2' \\ \hat{b}_3' \end{Bmatrix}}$$

$$\begin{tikzcd} A \arrow[r, "\theta_1"'] \arrow[rrr, bend left] & B' \arrow[r, "\theta_2"'] & B'' \arrow[r, "\theta_3"] & B \end{tikzcd}$$

$\hat{a}_1 = \hat{b}_1'$

$C_1(\theta_1)$

rotation about 1

$\hat{b}_2' = \hat{b}_2''$

$C_2(\theta_2)$

$\hat{b}_3'' = \hat{b}_3$

$C_3(\theta_3)$

- Next, consider a rotation of θ_2 about $\hat{b}_2' = \hat{b}_2''$
- This takes us to a new reference frame, say $(\hat{b}_1'', \hat{b}_2'', \hat{b}_3'')$

$$\begin{Bmatrix} \hat{b}_1' \\ \hat{b}_2' \\ \hat{b}_3' \end{Bmatrix} = [C_2(\theta_2)] \begin{Bmatrix} \hat{b}_1'' \\ \hat{b}_2'' \\ \hat{b}_3'' \end{Bmatrix}$$

- Finally, consider a rotation of θ_3 about $\hat{b}_3'' = \hat{b}_3$
- This takes us to the body reference frame system $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$

$$\begin{Bmatrix} \hat{b}_1'' \\ \hat{b}_2'' \\ \hat{b}_3'' \end{Bmatrix} = [C_3(\theta_3)] \begin{Bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{Bmatrix}$$

- The direction cosine matrix can be written as:

$\cos(\theta_2)$ $\sin(\theta_3)$

$$\left[{}^{A}C^{B}\right] = \left[C_1(\theta_1)\right]\left[C_2(\theta_2)\right]\left[C_3(\theta_3)\right] = \begin{bmatrix} c_2 c_3 & -c_2 s_3 & s_2 \\ s_1 s_2 c_3 + s_3 c_1 & -s_1 s_2 s_3 + c_3 c_1 & -s_1 c_2 \\ -c_1 s_2 c_3 + s_3 s_1 & c_1 s_2 s_3 + c_3 s_1 & c_1 c_2 \end{bmatrix}$$

$$\begin{Bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{Bmatrix} = \left[{}^{A}C^{B}\right] \begin{Bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{Bmatrix}$$

- One can go from the angles to direction cosine matrix and one can go from the direction cosine matrix back to angles* (see section 10.3)

   * except at singularities

Angular Velocity

- Use the addition theorem and the simple angular velocities to calculate the angular velocity vectors

$= {}^{A}\vec{\omega}^{B'} + {}^{B'}\vec{\omega}^{B''} + {}^{B''}\vec{\omega}^{B}$

$$
\begin{aligned}
{}^{A}\vec{\omega}^{B} &= \dot{\theta}_1 \hat{b}_1' + \dot{\theta}_2 \hat{b}_2'' + \dot{\theta}_3 \hat{b}_3 \rightarrow \text{in mixed basis} \\
&= \omega_1 \hat{b}_1 + \omega_2 \hat{b}_2 + \omega_3 \hat{b}_3 \rightarrow \text{in body frame}
\end{aligned}
$$

- The measure numbers of the angular velocity vector can be represented in any reference frame, specifically we can find the measure numbers in the B reference frame

angular velocity components

$$
\boxed{\begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} c_2 c_3 & s_3 & 0 \\ -c_2 s_3 & c_3 & 0 \\ s_2 & 0 & 1 \end{bmatrix}}_{M} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{Bmatrix}}
$$

$\{\dot{\theta}\} = [M]^{-1}\{\omega\}$

rate of change of Euler angles

... these are the kinematical differential equations

... the above matrix is not always invertible – singularity!

Gimbal lock.

Other Orientation Angle Representations

- Other axis sequence, e.g., Body 3-2-1 used in aircraft flight mechanics
  - The direction cosine matrix and kinematical differential equations for Body 3-2-1 were presented in Lecture 3

$$\vec{\omega} = p\,\hat{b}_1 + q\,\hat{b}_2 + r\,\hat{b}_3 \qquad \begin{Bmatrix} p \\ q \\ r \end{Bmatrix} = \begin{bmatrix} 1 & 0 & -s_\theta \\ 0 & c_\phi & c_\theta s_\phi \\ 0 & -s_\phi & c_\theta c_\phi \end{bmatrix} \begin{Bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{Bmatrix}$$

- Angles measured about the space fixed axes, e.g, Space 1-2-3
- 2-axis (not 2-angle) instead of 3-axis angles, e.g., Space 1-2-1

  → a student is going to figure it out
- See Appendix I and II for the direction cosine matrices and kinematical differential equations for 24 different orientation angle sequences