

Problem 1. (20 points) Consider the simplest fair lottery: you make a bet V and with probability $\frac{1}{2}$ win (receive $+V$) or lose (receive $-V$). Consider following strategy $\{V_t\}_{t=1}^{\infty}$:

$$V_t = \begin{cases} 2^{t-1}, & \text{if } t = 1 \text{ or you lost all tries before } t. \\ 0, & \text{otherwise.} \end{cases}$$

In other words, you start with $V_1 = 1$ and double your bet until your first win, say at turn τ , and don't play anymore after. Denote your profit (or debt) after t -th turn as π_t with initial condition $\pi_0 = 0$.

- 1) Prove that $\tau < \infty$ *a.s.*
- 2) Find $\mathbb{E}[\pi_\tau]$.
- 3) Find $\mathbb{E}[\pi_t]$.
- 4) Compare results in 2) and 3). Explain this in terms of Doob's optional sampling theorem.

Problem 2. (20 points) Consider a sample $\{x_i\}_{i=1}^n$ from some unknown distribution. Let's order it: $x_{(1)} \leq \dots \leq x_{(n)}$, so we have $\{x_{(i)}\}_{i=1}^n$. Also consider a class of estimators:

$$L(x) = \sum_{i=1}^n l_i x_{(i)},$$

i.e. a linear combination of order statistics. This estimator is fully defined by set of parameters $\{l_i\}_{i=1}^n$. *Note: it is possible that some $l_i = 0$ or $l_i = l_i(x)$.*

For each of following distributions propose (try to use unique estimator for every distributions) estimators from the above class of each parameter:

- (a) $\mathcal{U}[a, b]$,
- (b) $\mathcal{N}(\mu, \sigma)$,
- (c) $\text{Pois}(\lambda)$.

Problem 3. (60 points) Consider following option strategy. You allow to buy N_1 call options with 90% ATM strike, sell N_2 calls with 110% ATM strike and buy or sell some amount of underlying asset, N_3 . All options have the same maturity, T .

- 1) Use the Garman–Kohlhagen model to price all above options on USD/EUR exchange rate with $T = 1Y$. Explain how you set model parameters and why.
- 2) For fixed $N = \sum_{i=1}^3 N_i^*$, find optimal proportions $\left(\frac{N_1^*}{N}, \frac{N_2^*}{N}, \frac{N_3^*}{N}\right)$ of capital allocation which reset to zero delta and gamma of the strategy.
- 3) Suppose you manage USD100,000 and fully invest this capital in the optimal strategy from 2). Calculate theta of the portfolio and interpret the result. What strategy revenue can you expect?
- 4) What assumptions must be hold for the model 1) works? What can go wrong in real life? How to fix it?