Problem 1. Doubling strategy.

(1) $\tau < \infty$ *a.s.*

Proof. Let us consider for convenience a sequence of *i.i.d.* random variables $\{\xi_k\}_{k=1}^{\infty}$,

$$\xi_k = \begin{cases} +1, & p = \frac{1}{2}; \\ -1, & q = \frac{1}{2}. \end{cases}$$

That is, we receive money at step k if $\xi_k = +1$ and lose it if $\xi_k = -1$. The moment of the first win is defined as $\tau = \inf_k \{\xi_k = 1\}$. In particular, $\tau = \infty$ iff $\xi_k = -1$ for all k.

However, this situation occurs only with probability zero as the following calculation shows:

$$\mathbb{P}\{\tau = \infty\} = \mathbb{P}\{\xi_k = -1 \ \forall k = 1, 2, \dots\} = \mathbb{P}\Big\{\bigcap_{k=1}^{\infty}\{\xi_k = -1\}\Big\} =$$
$$\lim_n \mathbb{P}\Big\{\bigcap_{k=1}^{n}\{\xi_k = -1\}\Big\} = \lim_n \Big(\mathbb{P}\{\xi_1 = -1\}\Big)^n = \lim_n \Big(\frac{1}{2}\Big)^n = 0.$$

□

(2) $\mathbb{E}[\pi_\tau] = 1$.

Proof. First of all, our profit (or debt) after n-th turn can be expressed in terms of ξ_k and $V_k = 2^{k-1}$:

$$\pi_0 = 0,$$
$$\pi_n = \sum_{k=1}^{n} V_k \xi_k = \sum_{k=1}^{n} 2^{k-1}\xi_k.$$

Substituting a (random) stopping moment τ instead of n in the formula above, taking into account the definition of τ and the fact that $\tau < \infty$ *a.s.*, we obtain:

$$\pi_\tau = \sum_{k=1}^{\tau} 2^{k-1}\xi_k = 2^{\tau-1} - (2^{\tau-2} + 2^{\tau-3} + \ldots + 2^0) = 1 \ a.s.$$

Thus, $\pi_\tau = 1$ *a.s.* Evidently, $\mathbb{E}\pi_\tau = 1$. □

(3) $\mathbb{E}[\pi_k] = 0$.

Proof. To begin with, $\{\pi_k : k \geqslant 0\}$ is a martingale with respect to a natural filtration $\{\mathcal{F}_k : k \geqslant 0\}$, $\mathcal{F}_k = \sigma(\xi_1, \ldots, \xi_k)$, as a sum of independent random variables $2^{k-1}\xi_k$ with zero mean ($\mathbb{E}[2^{k-1}\xi_k] = 0$).

Particularly, $\mathbb{E}[\pi_k] = \mathbb{E}[\pi_{k-1}]$ for all $k \geqslant 1$. Hence, $\mathbb{E}[\pi_k] = \mathbb{E}[\pi_0] = 0$. □

(4) Comparison of the results.

Proof. To sum it up, $\mathbb{E}[\pi_\tau] = 1$ (moreover, $\tau < \infty$ *a.s.* and $\pi_\tau = 1$ *a.s.*), but $\mathbb{E}[\pi_k] = 0$ for all $k \geqslant 0$.

The following interpretation could be suggested. If we had infinite capital (and infinite time to bet), the lottery would almost surely be won (and, a.s., with profit of exactly 1).

On the opposite, after any finite number of turns, on average we will remain with a zero gain.

For further information, it is necessary to recall Doob's optional theorem stating the martingale property is preserved by uniformly bounded stopping times.

Theorem. *Suppose that π_n is a martingale with respect to a filtration $\mathcal{F}_n$ and $S \leqslant T$ are stopping times bounded above by some constant m. Then $\mathbb{E}[\pi_T|\mathcal{F}_S] = \pi_S$.*

As a consequence, if $\tau \leqslant m$ *a.s.* for some constant m one could take $S = 0$ and obtain $\mathbb{E}[\pi_\tau] = \mathbb{E}[\pi_0]$.

Nevertheless, in our specific case τ is not bounded from above, so the theorem itself cannot be applied. Moreover, our issue illustrates that at least *some* requirement of *regularity* of τ (not mandatory in boundedness from above), generally speaking, is essential.

Finally, the theorem implies the absence of *any* winning strategy in the simplest fair lottery with a *fixed end time m*.

Indeed, whatever our strategy $\{V_k : k = 1, \dots, m\}$ is, $\{\pi_n = \sum_{k=1}^n V_k \xi_k : n = 1, \dots, m\}$ is a martingale again and due to Doob's theorem for $\tau \leqslant m$: $\mathbb{E}[\pi_\tau] = \mathbb{E}[\pi_0] = 0$. That is, in real life our winnings will average 0. □

Problem 2. L-estimator.

(1) $\mathcal{U}[a, b]$.

Proof. It is obvious that if $X \sim \mathcal{U}[a, b]$ then $X \in [a, b]$ *a.s.* Since $\{x_i\}_{i=1}^n$ is a sample from $\mathcal{U}[a, b]$ and $\{x_{(i)}\}_{i=1}^n$ is its variational series then we know for sure that $a \leqslant x_{(1)}$ and $b \geqslant x_{(n)}$. Hereby, it is quite natural to estimate parameters a, b of uniform distribution by looking at the smallest and the largest order statistics,

$$\hat{a} \stackrel{\text{def}}{=\!=} x_{(1)},$$
$$\hat{b} \stackrel{\text{def}}{=\!=} x_{(n)}.$$

Speaking in terms of $L(x) = \sum_{i=1}^n l_i x_{(i)}$ which we shall call as *L-estimator*,

$$l_1 = 1, l_2 = 0, \dots, l_{n-1} = 0, l_n = 0, \text{ for } \hat{a}$$
$$l_1 = 0, l_2 = 0, \dots, l_{n-1} = 0, l_n = 1, \text{ for } \hat{b}.$$

We are now concentrated on strict justification of our intuition with the use of maximum likelihood estimator, which is good enough due to its clarity.

Let us consider the likelihood function and try to maximize it:

$$\mathcal{L}(\mathbf{x}; a, b) = \prod_{i=1}^n f(x_i; a, b) = \prod_{i=1}^n \frac{1}{b-a} \cdot [a \leqslant x_i \leqslant b] = \Big(\frac{1}{b-a}\Big)^n \cdot [a \leqslant x_{(1)}, b \geqslant x_{(n)}] \longrightarrow \max_{a,b}.$$

The closer a and b to each other, the greater the value of the function $(a, b) \mapsto (\frac{1}{b-a})^n$ is, so the optimum of $\mathcal{L}(\mathbf{x}; a, b)$ is a point $\hat{a} = x_{(1)}$, $\hat{b} = x_{(n)}$ as promised.

It is well known from the general theory of MLE that our estimations $\hat{a}, \hat{b}$ for $\mathcal{U}(a, b)$ possesses a number of attractive properties. Among many other things, for example the are **invariant**. That is, if $g(a, b)$ is any transformation of (a, b), then the MLE for $\alpha = g(a, b)$ is $\hat{\alpha} = g(\hat{a}, \hat{b})$. It can be useful for instance when applying inverse transform sampling.

Therefore, we are lucky enough that maximum likelihood estimation got into our class of *L-estimators.* □

(2) $\mathcal{P}(\lambda)$.

Proof. Poisson distribution $\mathcal{P}(\lambda)$ is notable for the fact that its parameter coincides with the mean and with the dispersion i.e. $\mathbb{E}[X] = \mathbb{D}[X] = \lambda$ for $X \sim \mathcal{P}(\lambda)$. So it seems logical now to use method of moments.

According to it, we should find $\hat{\lambda}$ from the equation $\mathbb{E}[X] = \overline{\mathbf{x}}$, where $\overline{\mathbf{x}} = \frac{1}{n}\sum_{i=1}^{n} x_{(i)}$ is a sample average. In that way, $\hat{\lambda} = \overline{\mathbf{x}}$.

The sample mean is unequivocally best estimator for a Poisson distribution because of the following properties:

(a) **Unbiasedness.** The sample mean is an unbiased estimate of the mean, that is $\mathbb{E}[\overline{X}_n] - \lambda = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[\overline{X}_i] - \lambda = \lambda - \lambda = 0$.

(b) **Strong Consistency.** The sample mean is a strong consistent estimate of the mean because of Kolmogorov's Strong Law of Large Numbers, that is $\overline{X}_n \xrightarrow{a.s.} \lambda$.

(c) **Efficiency.** It is also well known that the sample mean is an efficient estimation for Poisson distribution. That is, it has the best possible variance among all unbiased estimators of λ.

(d) **Asymptotic normality.** Due to Lindeberg-Levy's Central Limit Theorem and the fact that $\mathbb{E}[X] = \mathbb{D}[X] = \lambda$ we have that $\sqrt{n}(\overline{X_n} - \lambda) \xrightarrow{d} \mathcal{N}(0, \lambda)$.

(e) **Invariance.** Actually, the sample mean $\hat{\lambda}$ is also the solution of MLE method for Poisson distribution (we do not give this boring calculation). In particular, if $g(\lambda)$ is any transformation of λ, then the MLE for $\alpha = g(\lambda)$ is $\hat{\alpha} = g(\hat{\lambda})$.

□

(3) $\mathcal{N}(\mu, \sigma)$.

Proof. To start with, it is a great choice to use just sample mean to estimate μ of the normal distribution because of various reasons. The situation here is extremely similar to the one of Poisson distribution described above. Therefore, you can safely take $\hat{\mu} = \overline{\mathbf{x}}$ as in the previous case and calm down.

However, we are asked to use unique estimator for every distribution. So we shall abandon the average and consider another measure of location. More precisely, the sample median (which is obviously an *L-estimator*) will be applied:

$$\hat{\mu} = \mathrm{med}(\mathbf{x}) = \begin{cases} x_{(n+1)/2}, & n \text{ is odd;} \\ \frac{1}{2}(x_{(n/2)} + x_{((n/2)+1)}), & n \text{ is even.} \end{cases}$$

Such an estimator has an intuitive explanation: normal distribution is symmetric with respect to its mean, that is $\mathbb{E}[X] = \mu = \mathrm{med}(X)$ for $X \sim \mathcal{N}(\mu, \sigma)$. Moreover, sample median has its own benefits in comparison with sample mean when the data is full of outliers and extreme values, namely:

(a) **Optimality property.** Sample median is a minimizer of MAE. To be more precise, $\mathrm{med}(\overline{\mathbf{x}}) = \operatorname{arginf}_a \sum_{i=1}^{n} |x_i - a|$.

(b) **Robustness.** The median is a robust measure of central tendency. Specifically, median has a breakdown point of 50%, meaning that half the points must be outliers before the median can be moved outside the range of the non-outliers, while the mean has a breakdown point of 0, as a single large observation can throw it off.

(c) **Efficient computation.** Blum-Floyd-Pratt-Rivest-Tarjan algorithm enable us to compute sample median with only $\Theta(n)$ operations.

Even more interesting is the situation with σ. Usually, it is approximated by sample standard deviation which is not suitable for us as it is not a linear combination of ordered statistics. Instead of using method of moments that lead us to such a sad result we shall consider so-called *L-moments* and their approximation by *L-statistics.*

Let X be a real-valued random variable ($X \sim \mathcal{N}(\mu, \sigma)$ in our case), and let (in new conventions) $X_{1:n} \leqslant X_{2:n} \leqslant \ldots \leqslant X_{n:n}$ be the order statistics of a random sample of size n drawn from the distribution of X. Define the *L-moments* of X to be quantities

$$\lambda_r \stackrel{\text{def}}{=\!=} r^{-1} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \mathbb{E}[X_{r-k:r}].$$

The complete theory of *L-moments* and *L-statistics* can be found in Hosking, 1989. Instead, we are concentrated only on the first and on the second *L-moments*,

$$\begin{aligned}\lambda_1 &= \mathbb{E}[X],\\ \lambda_2 &= \frac{1}{2}\mathbb{E}[X_{2:2} - X_{1:2}].\end{aligned}$$

It is easy to see that λ_2 is a measure of the scale or dispersion of the distribution. Clearly, if the two values of a sample of size 2 typically tend to be close together, then λ_2 will be smaller than if they are usually far apart. So λ_2 can be named the *L-scale.*

Similarly, the first *L-moment*, λ_1, is a measure of the location of the random variable (and coincides with the mean).

To compare λ_2 with the more familiar scale measure σ, the standard deviation, write

$$\begin{aligned}\lambda_2 &= \frac{1}{2}\mathbb{E}[X_{2:2} - X_{1:2}],\\ \sigma^2 &= \frac{1}{2}\mathbb{E}[X_{2:2} - X_{1:2}]^2.\end{aligned}$$

Both quantities measure the difference between two randomly drawn elements of a distribution, but σ^2 gives relatively more weight to the largest differences.

This is just an illustration of the main advantage of *L-moments* over conventional ones. Being linear functions of the data, *L-moments* suffer less from the effects of sampling variability: they are more robust to outliers and extreme values in the data. As a result, *L-moments* enable more secure inferences to be made from small samples about an underlying probability distribution and its parameters.

Going back to our sample $\{x_i\}_{i=1}^n$ and its variational series $x_{1:n} \leqslant x_{2:n} \leqslant \ldots \leqslant x_{n:n}$ we can naturally estimate λ_2 in the following way:

$$l_2 \stackrel{\text{def}}{=\!=} \frac{1}{2}\binom{n}{2}^{-1} \sum_{i>j} (x_{i:n} - x_{j:n}).$$

This is a special case of U-statistics that were introduced by Hoeffding (1948). Their properties of **unbiasedness**, **asymptotic normality**, and some modest **resistance to the influence of outliers** make them particularly attractive for statistical inference.

Further, rearranging the terms we can represent l_2 in the form of linear combination of sample ordered statistics,

$$l_2 = \frac{1}{2}\binom{n}{2}^{-1}\sum_{i=1}^{n}\Big[\binom{i-1}{1} - \binom{n-i}{1}\Big]x_{(i)}.$$

It is well known that $\sigma = \sqrt{\pi}\lambda_2$ for normal random variable, so we define our estimator of σ simply by plugging in (by equating the first 2 sample L-moments, $l_1 \overset{\text{def}}{=} \frac{1}{n}\sum_{i=1}^{n} x_{(i)}$ and l_2, to the corresponding population quantities, λ_1 and λ_2, analogously to the usual method of moments),

$$\hat{\sigma} \overset{\text{def}}{=} \sqrt{\pi}l_2.$$

Obviously, $\hat{\sigma}$ lies in the class of *L-estimators* required by the task condition.

It should be noted, that the sample L-scale, l_2, is a scalar multiple of Gini's mean difference statistic, that is $l_2 = \frac{1}{2}G$, where

$$G \overset{\text{def}}{=} \binom{n}{2}^{-1}\sum_{i>j}(x_{i:n} - x_{j:n}).$$

It is also known that the statistic $\hat{\sigma} = \sqrt{\pi}l_2 = \frac{1}{2}\sqrt{\pi}G$ is a 98% efficient estimator of the scale parameter of a normal distribution (Downton, 1966; David, 1968).

So our estimator $\hat{\sigma}$ is really good, at least with a sufficient sample size. □

Problem 3. Options.

(1) Pricing options on EUR/USD exchange rate.

Proof. We use notation as follows:

S = the spot FX rate denoted in domestic units per unit of foreign currency;
K = the strike price of option (domestic units per foreign unit);
T = the time remaining until maturity of option (in years);
C = the price of an FX call option (domestic units per foreign unit);
r_d = the domestic (riskless) interest rate;
r_f = the foreign (riskless) interest rate;
σ = the volatility of the spot currency price;
$\mathcal{N}(\cdot)$ = the cumulative normal distribution function.

Our goal is to price call options on EUR/USD exchange rate. We assume that our foreign (underlying) currency (FOR) is EUR and our domestic (numeraire, base) currency (DOM) is USD.

The first step to implement the model is to set its parameters. We need to identify S, r_d, r_f, σ using external data. It is known from the task conditions that $T = 1$ while K can be expressed in terms of S for both types of call options suggested.

Note that the unit of S is $\frac{[\text{DOM}]}{[\text{FOR}]}$ (i.e. in our model S is the number of DOM unites needed to buy one FOR unit).

There are a lot of applications providing currency exchange rates, but European Central Bank is one of the most credible sources.

According to ECB, EUR/USD=1.0934 (i.e. EUR 1 = USD 1.0934, we need 1.0934 USD to buy 1 EUR) up to 17 July 2024, so

$$S \stackrel{\text{def}}{=} 1.0934.$$

We are allowed to buy N_1 call options with 90% ATM strike and to sell N_2 call options with 110% ATM strike, so

$$K_1 = 0.9S = 0.9 \cdot 1.0934 = 0.98406,$$
$$K_2 = 1.1S = 1.1 \cdot 1.0934 = 1.20274.$$

One of the most modern techniques to estimate riskless interest rates is so-called short-term rate (or overnight financing rate) which is recommended to use by ECB, other central banks and various search groups. Exactly, we use SOFR published on 17 July 2024 by Federal Reserve Bank of New York for domestic rate and ESTR published on the same date by ECB for foreign rate, so

$$r_d \stackrel{\text{def}}{=} \text{SOFR} = 5.350\%,$$
$$r_f \stackrel{\text{def}}{=} \text{ESTR} = 3.662\%.$$

All the above variables are observable. The exception is volatility which is usually estimated from historical data. But the assumption that volatility of the past will hold in the future does not seem well-founded given the time to expiration is 1 year. According to research (Laurent et al. and Brownless et al.) GJR-GARCH models are the best in predicting the future pattern.

We shall take advantage of the Volatility Laboratory (V-Lab) of the NYO Stern (business school of New York University) that provides real time forecasting financial volatility.

In obedience to its US Dollar to Euro GJR-GARCH Volatility Analysis, 1 Year Pred is equal to 5.56%, so

$$\sigma \stackrel{\text{def}}{=} 5.56\%.$$

Now, when the parameters are defined, all we have to do is substitute them into the formulas proposed by Garman and Kohlhagen, which states that

$$C = Se^{-r_f T}\mathcal{N}(d_1) - Ke^{-r_d T}\mathcal{N}(d_2),$$
$$d_1 = \frac{\ln(S/K) + (r_d - r_f + \sigma^2/2)T}{\sigma\sqrt{T}},$$
$$d_2 = d_1 - \sigma\sqrt{T}.$$

The resulting option costs of each type are:

$$C_1 = 0.12156,$$
$$C_2 = 0.00218.$$

Since our domestic currency is USD, C_1 and C_2 are the values (premiums) in USD of the corresponding options with 1 EUR notional. Having memorized that result we are now moving to the most exciting part of our work. $\square$

(2) Optimal proportions of capital allocations.

Proof. Let us write delta, gamma and theta for each type of options from (1) calculated as corresponding partial derivatives of C, namely $\Delta = \frac{\partial C}{\partial S}$, $\Gamma = \frac{\partial^2 C}{\partial S^2}$ and $\Theta = -\frac{\partial C}{\partial t}$. The exact formulas are described in the attached code. Calculations (as well as all subsequent ones) are also carried out there. In short,

$$\Delta_1 = 0.95151,\ \Gamma_1 = 0.53066,\ \Theta_1 = -0.01204,$$
$$\Delta_2 = 0.08036,\ \Gamma_2 = 2.43180,\ \Theta_2 = -0.00586.$$

The above deltas are so-called spot deltas of the options. In the standard arbitrage theory Δ_{opt} of the EUR-USD call option we own denotes the number of EUR we are to *sell* (if it is positive) or to *buy* (if it is negative) for the delta hedge. Therefore, $S\Delta_{\text{opt}}$ is the number of USD to *"buy"* or to *"sell"* correspondingly.

To find delta of the whole portfolio we need to add up the total deltas of each of the options and the underlying asset, namely

$$\Delta_{\text{all}} = N_1 \cdot \Delta_1 - N_2 \cdot \Delta_2 + N_3.$$

Here the first term is positive since it corresponds to long options, the second term is negative since it corresponds to short options, the last term is equal to $N_3 \cdot 1$ as delta of the underlying asset is always equal to 1 and N_3 stands for the number of euros bought.

More accurately, if $\operatorname{sign}(N_3) = +1$ we are to buy $|N_3|$ EUR (i.e. to sell $|N_3| \cdot S$ USD) and if $\operatorname{sign}(N_3) = -1$ we are to sell $|N_3|$ EUR (i.e. to buy $|N_3| \cdot S$ USD, that is to keep $|N_3| \cdot S$ USD in our portfolio and not to buy any EUR).

Similarly, the portfolio gamma can be calculated

$$\Gamma_{\text{all}} = N_1 \cdot \Gamma_1 - N_2 \cdot \Gamma_2 + 0.$$

Here the last term is zero since the gamma of the underlying asset is always equal to 0.

We want to find such N_1, N_2, N_3 that $\Delta_{\text{all}} = 0$ and $\Gamma_{\text{all}} = 0$ under the condition that $N_1 + N_2 + N_3 = N$ is fixed. Denoting the proportions $n_i \stackrel{\text{def}}{=} \frac{N_i}{N}$ for $i = 1, 2, 3$ and dividing all the three equations by N we obtain a linear system

$$\begin{cases} \Delta_1 n_1 - \Delta_2 n_2 + n_3 = 0, \\ \Gamma_1 n_1 - \Gamma_2 n_2 = 0, \\ n_1 + n_2 + n_3 = 1. \end{cases}$$

Solving it with respect to n_1, n_2, n_3 we come to the answer

$$\begin{aligned} n_1 &= 3.51819628, \\ n_2 &= 0.76772331, \\ n_3 &= -3.28591958. \end{aligned}$$

□

(3) Concrete capital investing.

Proof. Let us denote our total capital in dollars by V. We manage $V = N_{\text{USD}} = 100,000$ dollars and fully invest this capital in the optimal strategy from (2).

We are spending some part of the capital V on buying (longing) N_1 call options for price C_1 (i.e. we pay the premium of C_1 USD for each of N_1 calls): $-N_1 \cdot C_1$. We also get some dollars on selling (shorting) N_2 call options for price C_2 (i.e. we gain the premium of C_2 USD for each of N_2 calls): $+N_2 \cdot C_2$. Finally, due to negative value of N_3, after completing these operations we must keep $|N_3| \cdot S$ USD in our portfolio. Thereby we obtain an equation

$$V - N_1 \cdot C_1 + N_2 \cdot C_2 = |N_3| \cdot S. \tag{1}$$

Further, we know that

$$\begin{aligned} N_1 \quad &= \frac{n_1}{n_2} N_2 \quad = 4.5826 \cdot N_2, \\ |N_3| &= \frac{|n_3|}{n_2} N_2 = 4.2801 \cdot N_2. \end{aligned}$$

Substituting this into the equation (1) and expressing N_2 from it we obtain

$$N_2 = \frac{V}{\frac{n_1}{n_2} C_1 - C_2 + \frac{|n_3|}{n_2} S}.$$

For calculated earlier concrete values of n_1, n_2, n_3, C_1, C_2 and given values of S, V the result is $N_2 = 19103.23$. Thus, using above expressions for N_1 and N_2 we see that

$$\begin{aligned} \hat{N}_1 &= 87543.14, \\ \hat{N}_2 &= 19103.23, \\ \hat{N}_3 &= 81763.41. \end{aligned}$$

To sum it up, according to our delta-gamma neutral strategy we should buy $\hat{N}_1$ call options for the price of C_1 USD, sell $\hat{N}_2$ call options for the price of C_2 USD and to keep $|\hat{N}_3| \cdot S$ USD in our portfolio.

In other words, our portfolio would consists of exactly

$$\begin{aligned} \hat{N}_1 \quad\quad &= 87543.14 \ \text{ long EUR-USD call options,} \\ \hat{N}_2 \quad\quad &= 19103.23 \ \text{ short EUR-USD call options,} \\ \hat{N}_3 \cdot S &= 89400.11 \ \text{ US dollars.} \end{aligned}$$

Next, to calculate theta for an entire portfolio we need to sum up the theta of each option multiplied by its number just like we counted the portfolio delta and portfolio gamma, that is

$$\Theta_{\text{all}} = N_1 \cdot \Theta_1 - N_2 \cdot \Theta_2.$$

Note that here the first term is taken with a plus sign since it is a long position (negative Θ_1 works against us) and the second term is taken with a minus sign since it is a short position (negative Θ_2 works for us).

Substituting the concrete values $N_1, N_2, \Theta_1, \Theta_2$ from above we see that

$$\Theta_{\text{all}} = -941.98.$$

Negative Theta indicates that our portfolio is losing value over time, even if there are no changes in the underlying asset or market conditions.

Theta is calculated in years, but if we divide it by 252 (the precise number of trading days in the year), we get the daily decline solely due to time decay. In our particular case, in days theta is

$$\Theta_{\text{days}} = \frac{-941.98}{252} = -3.738.$$

Thus, we lose 3.738 USD per trading day. In other words, time is finally working against us. This is not surprising as we bought much more options then we sold and time is usually an enemy rather than an ally for the options buyer.

Simultaneously, our delta-gamma-neutral strategy minimized the risks associated with market movements. So, in some sense, negative theta is the price we pay for such an absence of these type of risks.

Furthermore, creating neutrality **in both** delta, gamma and theta, that is in **three dimensions**, requires at least three different option contracts since we need at least as many option contracts to make as parameters we are trying to hedge.

This is why it would require an additional new option contract to fix the problems with theta and create a delta-, gamma-, and theta-neutral portfolio if we want to avoid losses caused by the passage of time.

Unless we correct our strategy in this way we can expect *negative revenue*, that is **we are expected to lose** approximately 942 dollars in a year.

To interpret the results mathematically we can also look at a Taylor series expansion of the change in the portfolio value in a short period of time. The volatility of the underlying asset is assumed to be constant, so the value Π of the portfolio is a function of the asset price S (the only one stochastic variable) and time t. The Taylor series expansion gives

$$\Delta\Pi = \frac{\partial\Pi}{\partial S}\Delta S + \frac{\partial\Pi}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2\Pi}{\partial S^2}\Delta S^2 + \frac{1}{2}\frac{\partial^2\Pi}{\partial t^2}\Delta t^2 + \frac{\partial^2\Pi}{\partial S\partial t}\Delta S\Delta t + \cdots$$

where $\Delta\Pi$ and ΔS are the change in Π and S in a small time interval Δt.

For our delta and gamma neutral portfolio the first and the third term are made zero, the second term is nonstochastic and other terms are of order higher than Δt, so that (when terms of order higher than Δt are ignored)

$$\Delta\Pi = \Theta\Delta t.$$

As a result, our delta-hedged portfolio will retain its total value regardless of which direction the price of the underlying asset moves. Albeit, by default for only small movements of the underlying, a short amount of time and not-withstanding in other market conditions such as volatility and the rate of return for a risk-free investment. Further, neutralized gamma ensures that the hedge will be effective over a wider range of underlying price movements.

In other words, under the assumption that volatility σ and risk-free rates r_d, r_f are constant, the total value of our portfolio will only decrease over time with a proportionality coefficient equal to Θ.

Unfortunately, in real life *volatility* σ (and *riskless rates* r_d, r_f) will inevitably fluctuate meaning that Vega $= \frac{\partial\Pi}{\partial S}$ and Pho $= \frac{\partial\Pi}{\partial r}$ will enter the stage to eat our money. The following section is devoted to a slightly more detailed description of the problems of the model. $\square$

(4) Review and analysis.

The Garman–Kohlhagen model is an extension of the Black–Scholes model for stock options. Thus, fundamental hypotheses are the usual ones and can be divided into two types: some are about the market behavior and some are about the underlying assets.

The assumptions about the market:

- **No arbitrage.** It is impossible to make a riskless profit by simultaneously buying and selling identical assets in different markets. This means that there is an equilibrium between supply and demand in the market, and asset prices are determined only by fundamental factors.
- **No restrictions on short sales.** You can buy and sell any amount of currency without covering (short), which allows to fully hedge your position on the option.
- **Frictionless markets.** The above transactions do not incur any fees, costs or other expenses, associated with buying or selling currency or options.

The assumptions about the currency exchange rate:

- **Stochasticity.** Option price is a function of only one stochastic variable, namely S, the spot price of the deliverable currency (domestic units per foreign unit).
- **Random walk.** Geometric Brownian motion governs the currency spot price: i.e., the differential representation of spot price movements is $dS_t = \mu S_t dt + \sigma S_t dW_t$, where W_t is the standard Wiener process.
- **Volatility.** Exchange rate volatility remains *constant* throughout the life of the option.
- **Riskless rates.** Risk free simple interest rates, both in the domestic and foreign markets, are *constant.*

Unfortunately, the reality is not that simple, so the assumptions are not always empirically valid, which surely will lead to inaccuracies in pricing.

The limitations of the model have different depths. Some of them stem from specific properties of parameters (and our capability to solve differential equations analytically). They only require cosmetic changes in working with already developed approach and can be solved mathematically. We shall call them *second-order issues.*

Others are much more complicated. Their root cause is in our approach itself. Exactly, in reality security prices do not follow a strict stationary log-normal process. So there is no apparent opportunity to deal with issues caused by this hypothesis staying within our model. As a result, we are forced to use completely different methods to eliminate these shortcomings. We shall call them *first-order issues.*

Let us list the most substantial issues and briefly refer to the ways to solve them.

The *second-order issues* are:

(a) **The obscurity of interest rates.** The risk free interest actually is not known in the strict sense of the word. Fortunately, new methods of its assessment with higher accuracy and reliability are being introduced. For example, ESTR comes to replace EONIA and EURIBOR in Europe and SOFR moves us away from Libor in the USA.

(b) **Ignoring interest rate differentials.** Interest rates practically are not constant over time. Fortunately, modern versions of the model account for dynamic interest rates. The same is true about transactions costs and taxes, and dividend payouts.

(c) **Ignoring volatility and volatility differentials.** Volatility can be highly variable and dependent on market conditions as well as drift in real life. Again, if drift and volatility are time-varying, a suitably modified Black–Scholes formula can be deduced. However, the model is not extensible in case of stochastic volatility. For the latter, you can refer to SABR and local volatility models.

The *first-order issues* are:

(a) The assumption of continuous time and continuous trading, yielding *gap risk*, which can be hedged with *gamma hedging* (and, in our strategy, it *is* gamma hedged).
(b) The assumption of a stationary process, yielding *volatility risk*, which can be hedged with *volatility hedging*.
(c) The assumption of instant, cost-less trading, yielding *liquidity risk*, which is *difficult to hedge*.
(d) The underestimation of extreme moves such as stock market crashes (GBM has continuous trajectories while jumps in stock prices are often observed in real life, caused by unpredictable events or news), yielding *tail risk*, which can be hedged with *out-of-the-money options*.

In conclusion, the model is one of the best for delta hedging, but it is suffering from other sources of risk including Vega and Theta.

References

[1] Hosking, J. R. M. (1990) L-Moments: Analysis and Estimation of Distributions Using Linear Combinations of Order Statistics. Journal of the Royal Statistical Society. Series B (Methodological), 52(1), 105–124.
[2] Hoeffding, W. (1948) A class of statistics with asymptotically Normal distribution. Ann. Math. Statist., 19, 293-325.
[3] Downton, F. (1966) Linear estimates with polynomial coefficients. Biometrika, 53, 129-141.
[4] David, H. A. (1968) Gini's mean difference rediscovered. Biometrika, 55, 573-575.
[5] Mark B. Garman, Steven W. Kohlhagen (1983) Foreign currency option values. Journal of International Money and Finance, Volume 2, Issue 3, 231-237, 0261-5606.
[6] MacBeth, James D. and Larry J. Merville (1979) An Empirical Examination of the Black-Scholes Call Option Pricing Model. Journal of Finance 34, 1173-1186.
[7] Basma Almisshal, Mustafa Emir (2021) Modelling exchange rate volatility using GARCH models. Gazi Journal of Economics and Business, 2548-0162.
[8] Brownlees, C. T., Engle, R. F., & Kelly, B. T. (2011). A practical guide to volatility forecasting through calm and storm. Available at SSRN 1502915.
[9] Laurent, S., Rombouts, J. V., & Violante, F. (2012). On the forecasting accuracy of multivariate GARCH models. Journal of Applied Econometrics, 27(6), 934-955.
[10] V-lab: GJR-GARCH Volatility documentation. https://vlab.stern.nyu.edu/docs/volatility/GJR-GARCH