Taylor ~~Swift~~ Series (Maclaurin Series)

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Made with LaTeX

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Outline

What is it?

- Approximation of a function with an infinite series
- Approximates near 0

Why?

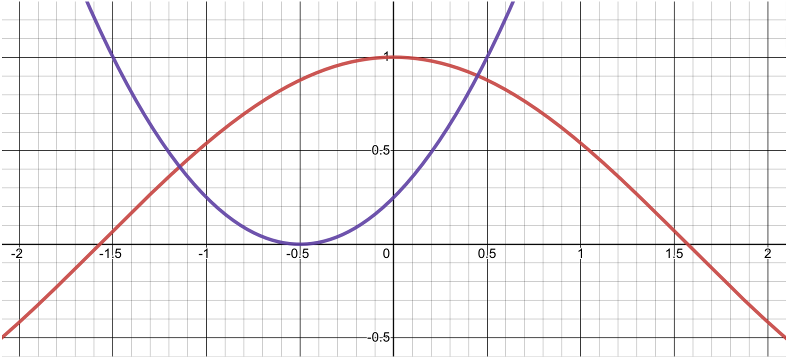
- To compute $\sin x$, $\cos x$, and e^x *fast*
- Calculators (your TI) use this technique
- To simplify equations/functions
- In simple pendulum, we *approximated* $\sin x$ with x

Derivation

- Calculators can multiply, add, subtract, divide, and take powers of whole numbers *quickly*
- Let us use *polynomials*
- Polynomials are just multiplications, additions, and exponentiations

Derivation

Figure: The Function $\cos x$



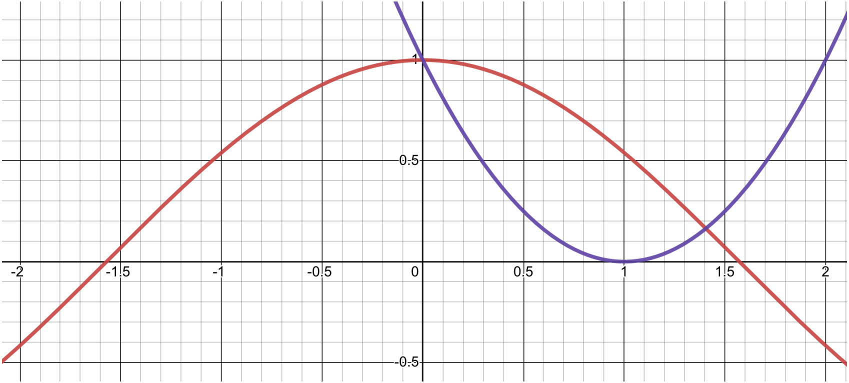
- Approximate to two degrees
- Find real numbers for $a, b,$ and c that approximate $\cos x$ the *best*

$$\cos x \approx a + bx + cx^2$$

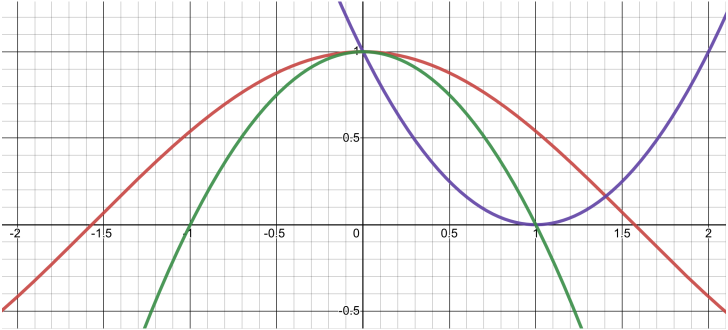
Derivation

- We want to approximate *near* $x = 0$
- $\cos x = a + bx + cx^2$ at $x = 0$

$$\cos 0 = a + b \cdot 0 + c \cdot 0^2$$
$$1 = a$$



Derivation

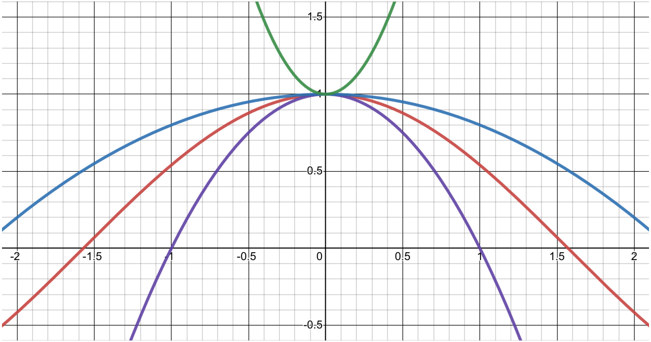


- The green function is better, but why?
- The rate of change is the same as $\cos x$ at $x = 0$
- Our approximation must have the same derivative at $x = 0$
- $\cos'(x) = -\sin x$ and $(a + bx + cx^2)' = b + 2cx$

$$-\sin 0 = 0 = b + 2c \cdot 0$$

$$b = 0$$

Derivation



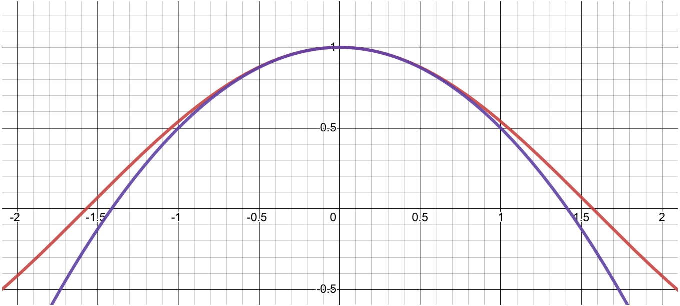
- $\cos x$ *curves* downwards at $x = 0$
- So, the second derivative is negative
- Which means the rate of change is decreasing
- Same second derivative will ensure that they curve at the same rate

$$\cos''(x) = -\cos x$$
$$(a + bx + cx^2)'' = 2c$$

Derivation

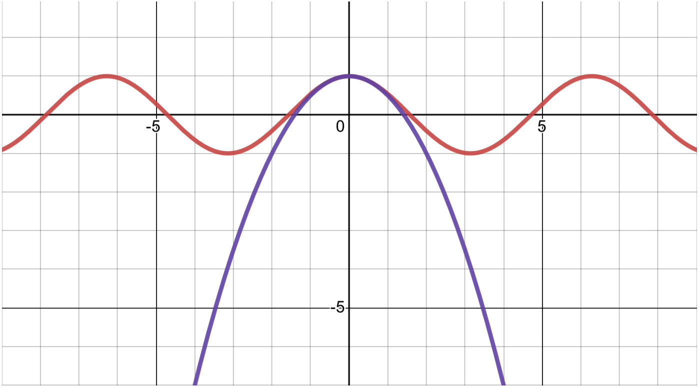
- $\cos''(x) = -\cos x$, and $(a + bx + cx^2)'' = 2c$

$$-\cos 0 = 2c$$
$$-1 = 2c$$
$$c = -\frac{1}{2}$$
$$\cos x \approx 1 - \frac{1}{2}x^2$$



Derivation

- Okay, but how *good* is our approximation?
- For $x = 0.1$, $\cos x = 0.99500417$, and our approximation, $1 - \frac{1}{2}x^2 = 0.995$
- For $x = 0.25$, $\cos x = 0.9689124$, and our approximation, $1 - \frac{1}{2}x^2 = 0.96875$



The More the Merrier

- But why stop at x^2? Why not go furhter?
- In fact, we can
- More tearms will give us more *control* over the approximation
- Let us add another term dx^3 to our approximation

$$\cos x \approx 1 - \frac{1}{2}x^2 + dx^3$$

- Taking the third derivative of a polynomial, all the terms that has a power less than 3 will vanish
- And, $\cos'''(x) = \cos x$
- For our new term, each derivative will change its coefficient

$$\cos x = (-x + 3dx^2)'' = (-1 + 2 \cdot 3dx)' = 1 \cdot 2 \cdot 3 \cdot d$$