

Taylor Swift Series (Maclaurin Series)

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What is it?

- Approximation of a function with an infinite series
- Approximates near $x = 0$

Why?

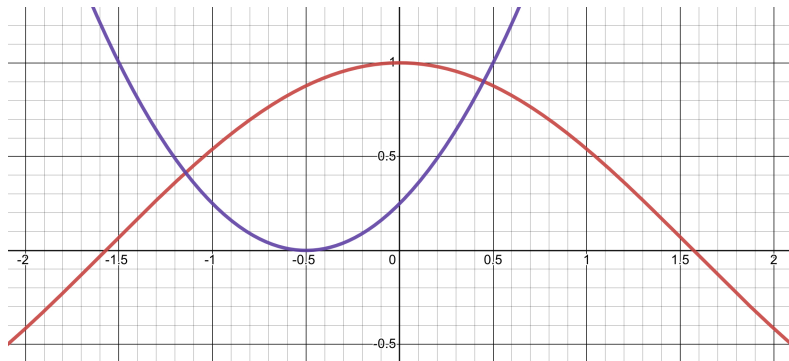
- To compute $\sin x$, $\cos x$, and e^x *fast*
- Calculators (your TI) use this technique
- To simplify equations/functions
- In simple pendulum, we *approximated* $\sin x$ with x

Derivation

- Calculators can multiply, add, subtract, divide, and take powers of whole numbers *quickly*
- Using *polynomials* will be efficient
- Since polynomials are just multiplications, additions, and exponentiations of numbers

Derivation

Figure: The Function $\cos x$



- Approximate to two degrees
- Find real numbers for c_0, c_1 , and c_2 that approximate $\cos x$ the *best*

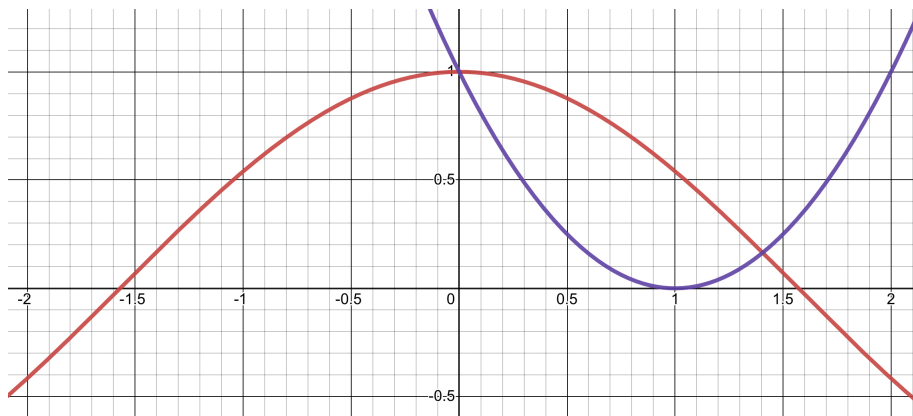
$$\cos x \approx c_0 + c_1x + c_2x^2$$

Derivation

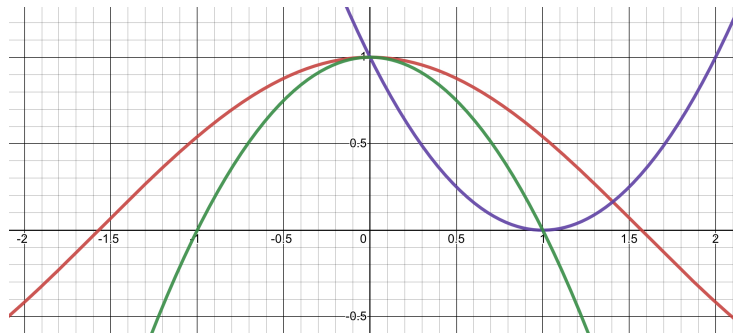
- Approximation *near* $x = 0$

$$\cos 0 = c_0 + c_1 \cdot 0 + c_2 \cdot 0^2$$

$$c_0 = 1$$



Derivation

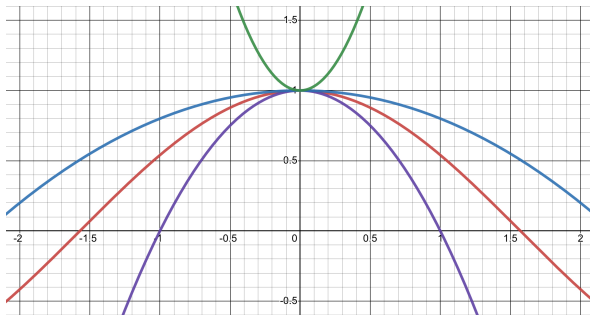


- The green function is better, but why?
- The rate of change is the same as $\cos x$ at $x = 0$
- Approximation must have the same rate of change at $x = 0$
- $\cos'(x) = -\sin x$, and $(c_0 + c_1x + c_2x^2)' = c_1 + 2c_2x$

$$-\sin 0 = 0 = c_1 + 2c_2 \cdot 0$$

$$c_1 = 0$$

Derivation



- $\cos x$ curves downwards at $x = 0$
- So, the second derivative is negative
- So, the rate of change is decreasing
- Same second derivative will ensure that they curve at the same rate

$$\cos''(x) = -\cos x$$

$$(c_0 + c_1x + c_2x^2)'' = 2c_2$$

Derivation

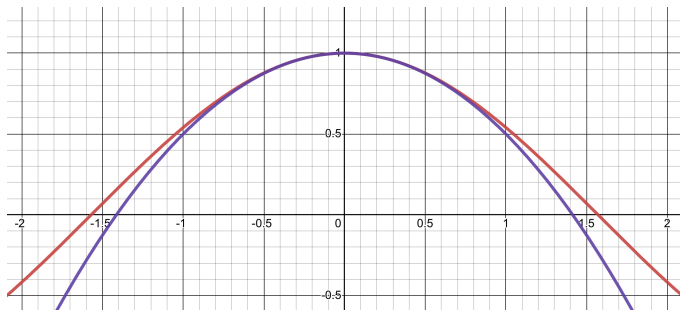
- $\cos''(x) = -\cos x$, and $(c_0 + c_1x + c_2x^2)'' = 2c_2$

$$-\cos 0 = 2c_2$$

$$-1 = 2c_2$$

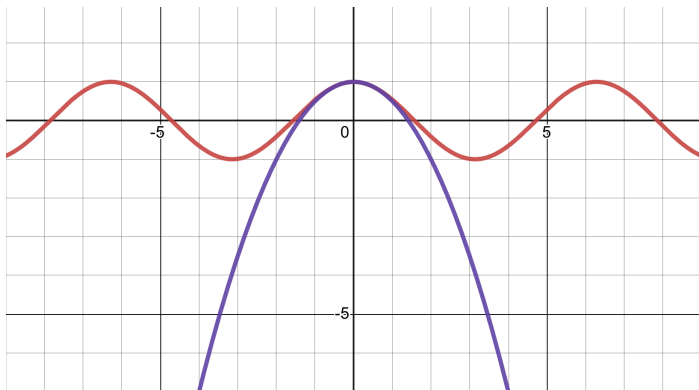
$$c_2 = -\frac{1}{2}$$

$$\cos x \approx 1 - \frac{1}{2}x^2$$



Derivation

- Okay, but how *good* is the approximation?
- For $x = 0.1$, $\cos x = 0.99500417$, and the approximation,
 $1 - \frac{1}{2}x^2 = 0.995$
- For $x = 0.25$, $\cos x = 0.9689124$, and the approximation,
 $1 - \frac{1}{2}x^2 = 0.96875$



The More the Merrier

- But why stop at x^2 ? Why not go further?
- More terms will give more *control* over the approximation
- Add another term c_3x^3 to the approximation

$$\cos x \approx 1 - \frac{1}{2}x^2 + c_3x^3$$

- Taking the third derivative of a polynomial, all the terms that have a power less than 3 will vanish
- And, $\cos'''(x) = \sin x$
- Taking the derivative,

$$\cos'''(x) = \sin x = (-x + 3c_3x^2)'' = (-1 + 2 \cdot 3c_3x)' = 1 \cdot 2 \cdot 3 \cdot c_3$$

$$\sin 0 = 1 \cdot 2 \cdot 3 \cdot c_3$$

$$c_3 = 0$$

The More the Merrier

$$\cos x \approx 1 - \frac{1}{2}x^2$$

- This approximation is the best for all cubic polynomials, as well as all the quadratic polynomials
- But, we can do better if we extend to another term

$$\cos x \approx 1 - \frac{1}{2}x^2 + c_4x^4$$

$$\cos^{(4)}(x) = \cos x$$

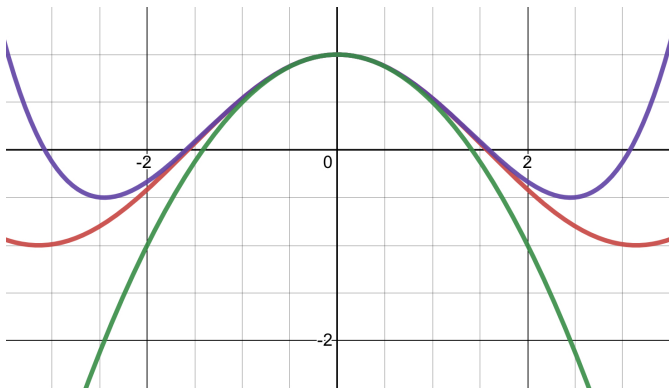
$$\left(1 - \frac{1}{2}x^2 + c_4x^4\right)^{(4)} = 1 \cdot 2 \cdot 3 \cdot 4 \cdot c_4$$

$$\cos 0 = 1 \cdot 2 \cdot 3 \cdot 4 \cdot c_4$$

$$c_4 = \frac{1}{24}$$

The More the Merrier

$$\cos x \approx 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$



- This is a really good approximation of $\cos x$
- For most physics problems, this would be fine
- But, we are dealing with maths

Notice a Few Things

- Firstly, factorials come up quite naturally from taking n successive derivatives of $c_n x^n$

$$\frac{d(c_n x^n)}{dx} = n \cdot c_n \cdot x^{n-1}$$

$$\frac{d^2(c_n x^n)}{dx^2} = n \cdot (n-1) \cdot c_n \cdot x^{n-2}$$

$$\frac{d^3(c_n x^n)}{dx^3} = n \cdot (n-1) \cdot (n-2) \cdot c_n \cdot x^{n-3}$$

\vdots

$$\frac{d^n(c_n x^n)}{dx^n} = n! \cdot c_n$$

- So, we have to divide by the appropriate factorial to cancel out this effect

$$c_n = \frac{\text{desired derivative value}}{n!}$$

Notice a Few Things

- Secondly, adding new terms does *not* mess up older terms
- Other higher-order terms that have x will not affect the lower order terms

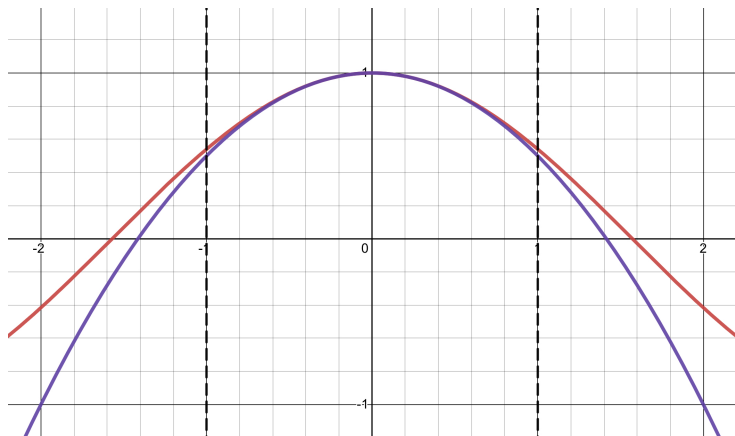
$$P(x) = 1 - \frac{1}{2}x^2 + c_4x^4$$
$$P''(0) = 2\left(-\frac{1}{2}\right) + 3 \cdot 4(0)^2$$

- Each derivative of a polynomial at $x = 0$ is controlled by one and only one of the coefficients

$$P(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4$$

Notice a Few Things

- Derivative information at $x = 0 \rightarrow$ output information near $x = 0$



Notice a Few Things

$$\cos 0 = 1$$

$$\cos' 0 = 0$$

$$\cos'' 0 = -1$$

$$\cos''' 0 = 0$$

$$\cos^{(4)} 0 = 1$$

$$\vdots$$

$$P(x) = 1 + 0\frac{x^1}{1!} + -1\frac{x^2}{2!} + 0\frac{x^3}{3!} + 1\frac{x^4}{4!} + \dots$$

- Those factorials are there to cancel out the cascading effect of derivatives

Maclaurin Series

- The same approach can be used for *any* function
- We can approximate $f(x)$ near $x = 0$ with any degree of accuracy we want

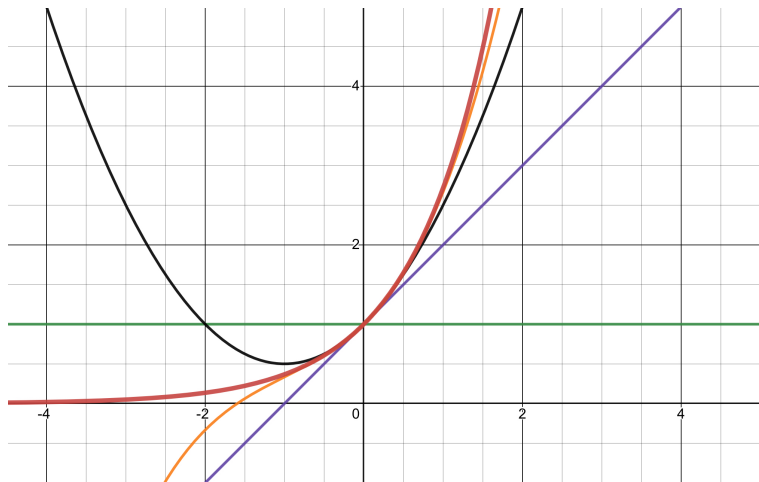
$$P(x) = f(0) + f'(0)\frac{x}{1} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!}$$

- This summation at infinity is the *Maclaurin series* of $f(x)$
- Let us approximate the function e^x (which is in the *Formula Booklet*)

Maclaurin Series

- Any derivative of e^x is e^x , so $e^0 = 1$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$



Euler's Formula

- We can, in fact, use this to prove

$$\cos \theta + i \sin \theta = e^{i\theta}$$

$$\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots$$

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots$$

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \dots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \dots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\right) + i \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\right)$$

$$= \cos \theta + i \sin \theta$$

$$= \operatorname{cis} \theta$$

Example

- Find the Maclaurin series of the function $f(x) = e^x \sin x$ up to the term x^3
- Two methods: multiply series expansions of e^x and $\sin x$, or rigor

$$f(x) = e^x \sin x$$

$$f'(x) = e^x \sin x + e^x \cos x$$

$$f''(x) = e^x \cos x - e^x \sin x + e^x \sin x + e^x \cos x = 2e^x \cos x$$

$$f'''(x) = 2e^x \cos x - 2e^x \sin x = 2e^x (\cos x - \sin x)$$

- $f(0) = 0, f'(0) = 1, f''(0) = 2, f'''(0) = 2$

$$\begin{aligned} f(x) &= 0 + 1x + \frac{2x^2}{2!} + \frac{2x^3}{3!} \\ &= x + x^2 + \frac{x^3}{3} \end{aligned}$$

Connection to Taylor Series

- Maclaurin series approximates a function near $x = 0$
- Can be approximated near any point $x = a$ using Taylor series

$$P(x) = f(a) + f'(a)(x - a) + f''(a)\frac{(x - a)^2}{2!} + f'''(a)\frac{(x - a)^3}{3!} + \dots$$

Figure: The Function $\ln x$ and Approximation

