Taylor ~~Swift~~ Series (Maclaurin Series)

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Made with LaTeX

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Outline

What is it?

- Approximation of a function with an infinite series
- Approximates near 0

Why?

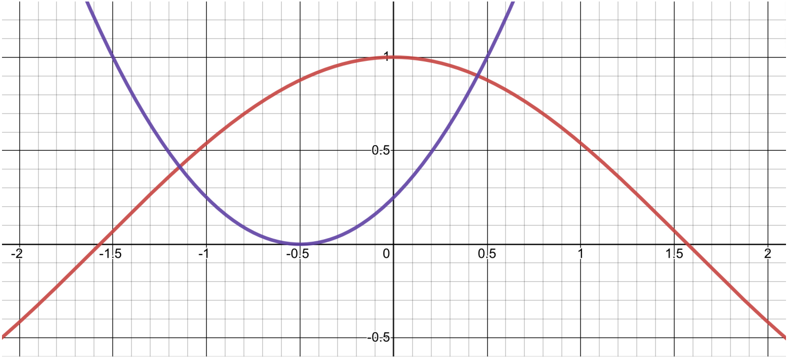
- To compute $\sin x$, $\cos x$, and e^x *fast*
- Calculators (your TI) use this technique
- To simplify equations/functions
- In simple pendulum, we *approximated* $\sin x$ with x

Derivation

- Calculators can multiply, add, subtract, divide, and take powers of whole numbers *quickly*
- Let us use *polynomials*
- Polynomials are just multiplications, additions, and exponentiations

Derivation

Figure: The Function $\cos x$



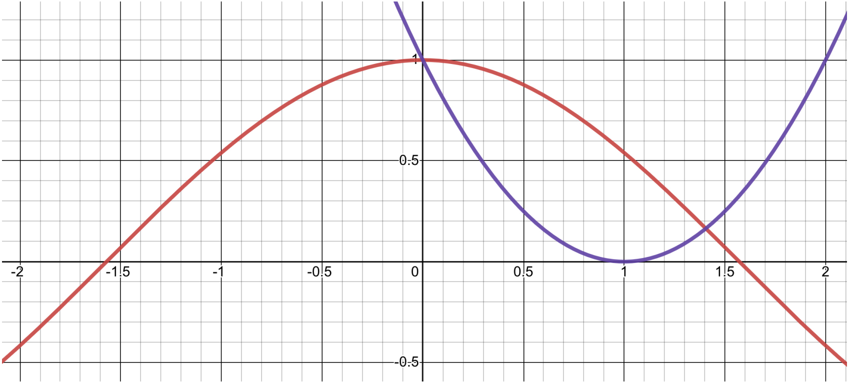
- Approximate to two degrees
- Find real numbers for c_0, c_1, and c_2 that approximate $\cos x$ the *best*

$$\cos x \approx c_0 + c_1 x + c_2 x^2$$

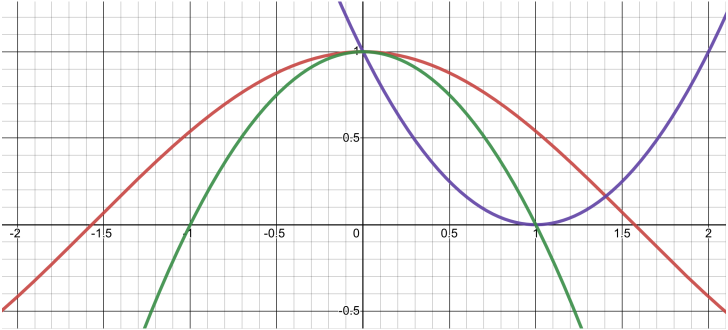
Derivation

- We want to approximate *near* $x = 0$
- $\cos x = c_0 + c_1 x + c_2 x^2$ at $x = 0$

$$\cos 0 = c_0 + c_1 \cdot 0 + c_2 \cdot 0^2$$
$$1 = a$$



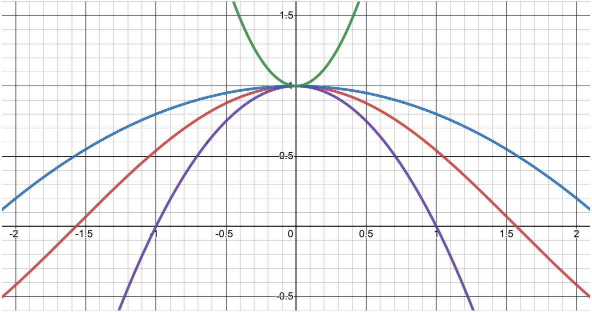
Derivation



- The green function is better, but why?
- The rate of change is the same as $\cos x$ at $x = 0$
- Our approximation must have the same derivative at $x = 0$
- $\cos'(x) = -\sin x$ and $(c_0 + c_1 x + c_2 x^2)' = c_1 + 2c_2 x$

$$-\sin 0 = 0 = c_1 + 2c_2 \cdot 0$$
$$b = 0$$

Derivation



- $\cos x$ *curves* downwards at $x = 0$
- So, the second derivative is negative
- Which means the rate of change is decreasing
- Same second derivative will ensure that they curve at the same rate

$$\cos''(x) = -\cos x$$

$$\left(c_0 + c_1 x + c_2 x^2\right)'' = 2c_2$$

Derivation

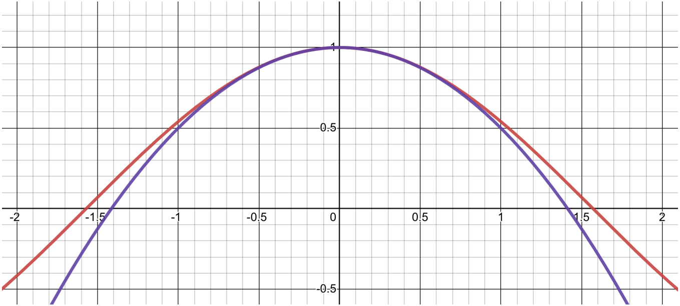
- $\cos''(x) = -\cos x$, and $\left(c_0 + c_1 x + c_2 x^2\right)'' = 2c_2$

$$-\cos 0 = 2c_2$$

$$-1 = 2c_2$$

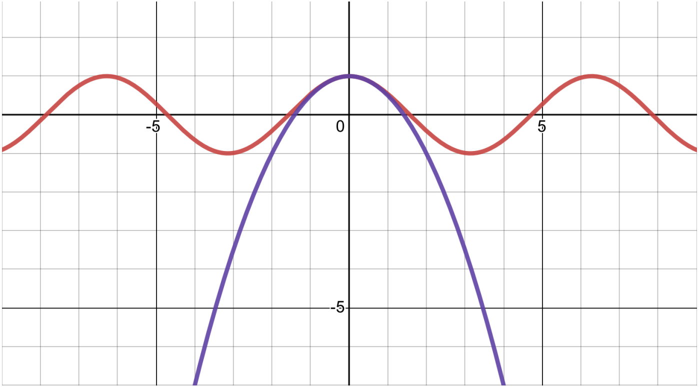
$$c_2 = -\frac{1}{2}$$

$$\cos x \approx 1 - \frac{1}{2}x^2$$



Derivation

- Okay, but how *good* is our approximation?
- For $x = 0.1$, $\cos x = 0.99500417$, and our approximation, $1 - \frac{1}{2}x^2 = 0.995$
- For $x = 0.25$, $\cos x = 0.9689124$, and our approximation, $1 - \frac{1}{2}x^2 = 0.96875$



The More the Merrier

- But why stop at x^2? Why not go further?
- More terms will give us more *control* over the approximation
- Let us add another term c_3x^3 to our approximation

$$\cos x \approx 1 - \frac{1}{2}x^2 + c_3x^3$$

- Taking the third derivative of a polynomial, all the terms that has a power less than 3 will vanish
- And, $\cos'''(x) = \sin x$
- Taking the derivative,

$$\cos'''(x) = \sin x = \left(-x + 3c_3x^2\right)'' = \left(-1 + 2 \cdot 3c_3x\right)' = 1 \cdot 2 \cdot 3 \cdot c_3$$
$$\sin 0 = 1 \cdot 2 \cdot 3 \cdot c_3$$
$$c_3 = 0$$

The More the Merrier

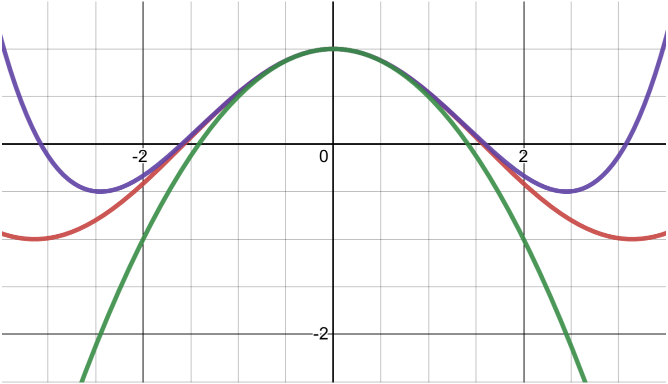
$$\cos x \approx 1 - \frac{1}{2}x^2$$

- Our approximation is the best for all cubic polynomials too
- But, we can do better if we extended to another term
$$\cos x \approx 1 - \frac{1}{2}x^2 + c_4 x^4$$
- Taking the fourth derivative,

$$\begin{aligned} \cos^{(4)}(x) &= \cos x \\ \left(1 - \frac{1}{2}x^2 + c_4 x^4\right)'''' &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot c_4 \\ \cos 0 &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot c_4 \\ c_4 &= \frac{1}{24} \end{aligned}$$

The More the Merrier

$$\cos x \approx 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$



- This is a really good approximation of $\cos x$
- For most physics problems, this would be fine
- But, we are dealing with maths

Notice a Few Things

- Firstly, factorials come up very naturally
- Taking n successive derivatives of $c_n x^n$,

$$\begin{aligned}
\frac{\mathrm{d}\left(c_n x^n\right)}{\mathrm{d}x} &= n \cdot c_n \cdot x^{n-1} \\
\frac{\mathrm{d}^2\left(c_n x^n\right)}{\mathrm{d}x^2} &= n \cdot (n-1) \cdot c_n \cdot x^{n-2} \\
\frac{\mathrm{d}^3\left(c_n x^n\right)}{\mathrm{d}x^3} &= n \cdot (n-1) \cdot (n-2) \cdot c_n \cdot x^{n-3} \\
&\vdots \\
\frac{\mathrm{d}^n\left(c_n x^n\right)}{\mathrm{d}x^n} &= n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1 \cdot c_n \\
\frac{\mathrm{d}^n\left(c_n x^n\right)}{\mathrm{d}x^n} &= n! \cdot c_n
\end{aligned}$$

Notice a Few Things

- So, we have to divide by the appropriate factorial to cancel out this effect

$$c_n = \frac{\text{desired derivative value}}{n!}$$

Notice a Few Things

- Secondly, adding new terms does *not* mess up older terms
- Other higher-order terms that have x will not affect the lower order terms

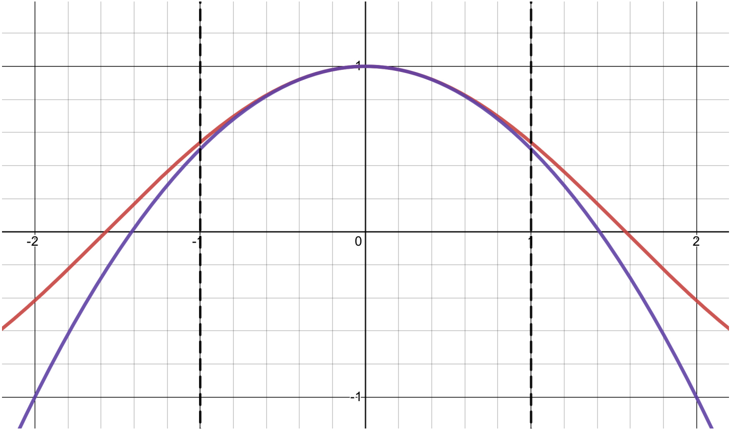
$$P(x) = 1 - \frac{1}{2}x^2 + c_4 x^4$$
$$P''(0) = 2\left(-\frac{1}{2}\right) + 3 \cdot 4(0)^2$$

- Each derivative of a polynomial at $x = 0$ is controlled by one and only one of the coefficients

$$p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4$$

Notice a Few Things

- Derivative information at $x = 0 \longrightarrow$ output information near $x = 0$



Notice a Few Things

$$\cos 0 = 1$$
$$\cos' 0 = 0$$
$$\cos'' 0 = -1$$
$$\cos''' 0 = 0$$
$$\cos^{(4)} 0 = 1$$
$$\vdots$$

$$P(x) = 1 + 0\frac{x^1}{1!} + -1\frac{x^2}{2!} + 0\frac{x^3}{3!} + 1\frac{x^4}{4!} + \cdots$$

- Those factorials are there to cancel out the cascading effect of derivatives

Maclaurin Series

- We can take the same approach for *any* function
- We can approximate $f(x)$ near $x = 0$ with any degree of accuracy we want

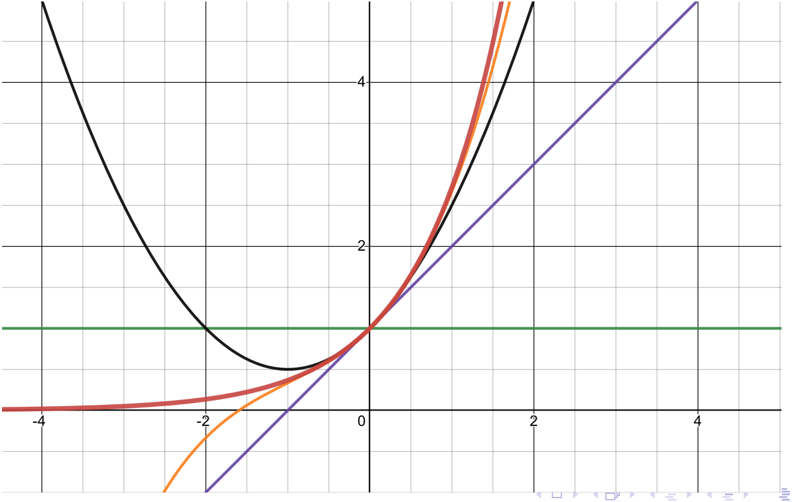
$$P(x) = f(0) + f'(0)\frac{x}{1} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!}$$

- This is called the *Maclaurin Series* of $f(x)$
- Let us approximate the function e^x (which is in the *Formula Booklet*)

Maclaurin Series

- Any derivative of e^x is e^x, so $e^0 = 1$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$



One More Thing

- We can, in fact, use this knowledge to prove what we kind of assumed before

$$\cos\theta + i\sin\theta = \mathrm{e}^{i\theta}$$