Solutions:
A Book of Abstract Algebra (2e)
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June 20, 2018

Many of these solutions are incorrect. Please let me know if you find an error!

Chapter 2 Operations

A. Examples of Operations

1.
$$a * b = \sqrt{|ab|}, \text{ on the set } \mathbb{Q}$$

This is not an operation because many roots are irrational

2.
$$a * b = \ln b, \text{ on the set } \{x \in \mathbb{R} : x > 0\}$$

This is an operation

3.
$$a * b \text{ is a root of the equation } x^2 - a^2b^2 = 0, \text{ on the set } \mathbb{R}$$

This is not an operation because it is not uniquely defined for any $a \neq 0$ $b \neq 0$ because $x = a * b = \pm ab$

4.
$$\text{Subtraction, on the set } \mathbb{Z}$$

This is an operation.

5.

$$\text{Subtraction, on the set } \{n \in \mathbb{Z} \geq 0\}$$

This is not an operation when $a - b \leq 0$

6.

$$a * b = |a - b|, \text{ on the set } \{n \in \mathbb{Z} \geq 0\}$$

This is an operation

B. Properties of Operations

1.

$$x * y = x + 2y + 4$$

i $y * x = y + 2x + 4 \neq x + 2y + 4$
Not commutative

ii $x * (y * z) = x(y + 2z + 4) = xy + 2xz + 4x \neq$
$(x * y) * z = (x + 2y + 4)z = xz + 2yz + 4z$
Not associative

iii $x * e = x$ for $e : x * e = x + 2e + 4 = x$; therefore $e = 2 \neq$
$e * y = y$ for $e : e * y = e + 2y + 4 = y$; therefore $e = -y - 4$
Identity does not exist

iiii Inverse can't exist without an identity

2.

$$x * y = x + 2y - xy$$

i $y * x = y + 2x - yx \neq x + 2y - xy = x * y$
Not commutative

ii $x*(y*z) = x*(y+2z-yz) = x+2(y+2z-yz)-x(y+2z-yz) =$
$x + 2y + 4z - 2yz - xy - 2xz + xyz \neq$
$(x*y)*z = (x+2y-xy)*z = (x+2y-xy)+2z-(x+2y-xy)z =$
$x + 2y - xy + 2z - xz - 2yz + xyz$
Not associative

iii $x * e = x + 2e - xe = x$, therefore $e = 0$
$e * y = e + 2y - ey = y$, therefore $e = \frac{-y}{1-y}$
No identity

iiii No inverse

3.

$$x * y = |x + y|$$

i $y * x = |y + x| = |x + y| = x + y$
Commutative

ii $x * (y * z) = x * |y + z| = |x + |y + z|| =$
$(x * y) * z = |x + y| * z = ||x + y| + z|$
Not associative (for example let $x = 2$, $y = -3$, $z = 5$)

iii $x * e = |x + e| = x$
No identity

iiii No inverse

4.

$$x * y = |x - y|$$

i $y * x = |y - x| = |x - y| = x * y$
Commutative

ii $x * (y * z) = x * |y - z| = |x - |y - z||$
$(x * y) * z = |x - y| * z = ||x - y| - z|$
Not associative (for example let $x = 2$, $y = -3$, $z = 5$)

iii $x * e = |x - e| = x$
No identity

iiii No inverse

5.

$$x * y = xy + 1$$

i $y * x = yx + 1 = xy + 1 = x * y$
Commutative

ii $x * (y * z) = x * (yz + 1) = x(yz + 1) + 1 = xyz + x + 1 \neq$
$(x * y) * z = (xy + 1) * z = (xy + 1)z + 1 = xyz + z + 1$
Not associative

iii $x * e = xe + 1 = x$, therefore $e = \frac{x-1}{x}$
No identity

iiii No inverse

6.

$$x * y = max\{x, y\}$$

i $y * x = max\{y, x\} = max\{x, y\} = x * y$
Commutative

ii $x*(y*z) = x*max\{y, z\} = max\{x, max\{y, z\}\} = max\{x, y, z\} =$
$(x * y) * z = max\{x, y\} * z = max\{max\{x, y\}, z\} = max\{x, y, z\}$
Associative

iii $x * e = max\{x, e\} = x$, therefore $e = -\infty$
No identity

iiii No inverse

7.

$$x * y = \frac{xy}{x + y + 1}\text{(on the set of positive real numbers)}$$

i $y * x = \frac{yx}{y+x+1} = \frac{xy}{x+y+1} = x * y$
Commutative

ii

$$x*(y*z) = x*\frac{yz}{y + z + 1} = \frac{x\frac{yz}{y+z+1}}{x + \frac{yz}{y+z+1} + 1} = \frac{yzx}{1 + x + y + z + xy + xz + yz} =$$

$$(x*y)*z = \frac{xy}{x + y + 1}*z = \frac{\frac{xy}{x+y+1}z}{\frac{xy}{x+y+1} + z + 1} = \frac{xyz}{1 + x + y + z + xy + xz + yz}$$

Associative

iii $x * e = \frac{xe}{x+e+1} = x$, therefore $x = -1$
No identity

iiii No inverse

C. Operations on a Two-Element Set

1

(x,y)	x*y
(a,a)	a
(a,b)	a
(b,b)	a
(b,a)	a

2

(x,y)	x*y
(a,a)	a
(a,b)	a
(b,b)	a
(b,a)	b

3

(x,y)	x*y
(a,a)	a
(a,b)	a
(b,b)	b
(b,a)	b

4

(x,y)	x*y
(a,a)	a
(a,b)	a
(b,b)	b
(b,a)	a

5

(x,y)	x*y
(a,a)	a
(a,b)	b
(b,b)	b
(b,a)	a

6

(x,y)	x*y
(a,a)	a
(a,b)	b
(b,b)	b
(b,a)	b

7

(x,y)	x*y
(a,a)	a
(a,b)	b
(b,b)	a
(b,a)	b

8

(x,y)	x*y
(a,a)	a
(a,b)	b
(b,b)	a
(b,a)	a

9

(x,y)	x*y
(a,a)	b
(a,b)	b
(b,b)	a
(b,a)	a

10

(x,y)	x*y
(a,a)	b
(a,b)	b
(b,b)	a
(b,a)	b

11

(x,y)	x*y
(a,a)	b
(a,b)	b
(b,b)	b
(b,a)	b

12

(x,y)	x*y
(a,a)	b
(a,b)	b
(b,b)	b
(b,a)	a

13

(x,y)	x*y
(a,a)	b
(a,b)	a
(b,b)	b
(b,a)	a

14

(x,y)	x*y
(a,a)	b
(a,b)	a
(b,b)	b
(b,a)	b

15

(x,y)	x*y
(a,a)	b
(a,b)	a
(b,b)	a
(b,a)	b

16

(x,y)	x*y
(a,a)	b
(a,b)	a
(b,b)	a
(b,a)	a

1.

2. Commutative: 1, 4, 6, 7, 10, 11, 13, 16
   For all of these $(a, b) = (b, a)$

3. Associative: 1, 3, 4, 5, 6, 7, 11, 13
   I used the following Clojure code to solve:

```
(def tables
  (for [aa ['a 'b]
        ab ['a 'b]
        bb ['a 'b]
        ba ['a 'b]]
    {['a 'a] aa
     ['a 'b] ab
     ['b 'b] bb
     ['b 'a] ba}))


(defn assoc?
  [table]
  (every? identity
          (for [x ['a 'b]
                y ['a 'b]
                z ['a 'b]]
            (= (table [(table [x y]) z])
               (table [x (table [y z])])))))

(->> tables
     (map (fn [table] [table (assoc? table)]))
     (filter second))
```

2. Have identity: 4, 6, 7, 13
   I used the following Clojure code to solve:

   $((a,a) = a \cap (a,b) = b \cap (b,a) = b) \cup ((b,b) = b \cap (a,b) = a \cap (b,a) = a)$

```
(for [aa ['a 'b]
      ab ['a 'b]
      bb ['a 'b]
      ba ['a 'b]
      :when (or (and (= aa 'a)
                     (= ab 'b)
                     (= ba 'b))
```

```
                  (and (= bb 'b)
                       (= ab 'a)
                       (= ba 'a)))]
  {['a 'a] aa
   ['a 'b] ab
   ['b 'b] bb
   ['b 'a] ba})
```

2. Have inverse: 7 and 13
   4 has identity b: there exists no x where $a * x = b$
   6 has identity a: there exists no x where $b * x = a$
   7 has identity b: $a * b = b = e$ and $b * a = b = e$
   13 has identity a: $a * b = a = e$ and $b * a = a = e$

D. Automata: The Algebra of Input/Output Sequences

1. Let $\boldsymbol{a} = a_1..a_n, \boldsymbol{b} = b_1..b_m, \boldsymbol{c} = c_1..c_k$
   $\boldsymbol{a} * (\boldsymbol{b} * \boldsymbol{c}) = \boldsymbol{a} * (b_1..b_m c_1..c_k) = a_1..a_n b_1..b_m c_1..c_k$
   $(\boldsymbol{a} * \boldsymbol{b}) * \boldsymbol{c} = (a_1..a_n b_1..b_m) * \boldsymbol{c} = a_1..a_n b_1..b_m c_1..c_k$
2. Concatenation is not commutative because placing elements on to the beginning versus the end of a sequence leads to different sequences.
3. Let $\boldsymbol{a} = a_1..a_n$
   $\boldsymbol{a}\lambda = \lambda\boldsymbol{a} = \boldsymbol{a}$