Hamiltonicity in randomly perturbed digraphs and hypergraphs

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December 15, 2014

Abstract

We give several results showing that different discrete structures become Hamiltonian after a modest random perturbation. First, we show that adding linearly many random edges to a dense hypergraph ensures the (asymptotically almost sure) existence of a matching or a loose Hamilton cycle. This involves a nonstandard application of Szemerédi's Regularity Lemma. We next show that digraphs with certain strong expansion properties are pancyclic, and use this to show that adding a linear number of random edges makes a dense digraph pancyclic. Finally, we show that perturbing a certain (minimum-degree-dependent) number of random edges in a tournament ensures the existence of multiple edge-disjoint Hamilton cycles. All our results are tight.

We say that a graph is *Hamiltonian* if it has a *Hamilton cycle*: a simple cycle containing every vertex in the graph. Hamiltonicity is a central notion in graph theory and has been extensively studied in a wide range of contexts. In particular, due to a seminal paper by Karp [9], it has become a canonical NP-complete problem to determine whether an arbitrary graph is Hamiltonian. There are nevertheless a variety of easily-checkable conditions that guarantee Hamiltonicity. The most famous of these is given by a classical theorem of Dirac [4], which states that any n-vertex graph with minimum degree at least $n/2$ is Hamiltonian.

Dirac's theorem requires a very strong density condition, but in a certain asymptotic sense "almost all" dense graphs are Hamiltonian. If we fix $\alpha > 0$ and select a graph uniformly at random among the (labelled) graphs with n vertices and $\alpha\binom{n}{2}$ edges, then the degrees will probably each be about αn. Such a random graph is Hamiltonian with probability $1 - o(1)$ (we say it is Hamiltonian *asymptotically almost surely*, or *a.a.s.*). This follows from a stronger result [18] that gives a *threshold* for Hamiltonicity: a random n-vertex, m-edge graph is Hamiltonian a.a.s. if $m \gg n \log n$, and fails to be Hamiltonian a.a.s. if $m \ll n \log n$. Here and from now on, all asymptotics are as $n \to \infty$, and we implicitly round large quantities to integers.

In [2], the authors studied the random graph model that starts with a dense graph and adds m random edges (this model has since been studied in a number of other contexts; see for example [1, 12]). They found that to ensure Hamiltonicity in this model we only need m to be linear, saving a logarithmic factor over the standard model where we start with nothing. To be precise, [2, Theorem 1] says that for every $\alpha > 0$ there is $c(\alpha)$ such that if we start with a graph with minimum degree at least αn and add $c(\alpha)n$ random edges, then the resulting graph will a.a.s. be Hamiltonian. Note that some dense graphs require a linear number of extra edges to become Hamiltonian (consider the complete bipartite graph with partition sizes $n/3$ and $2n/3$), so this result is asymptotically tight.

We can interpret this theorem as quantifying the "fragility" of the few dense graphs that are not Hamiltonian, by determining the amount of random perturbation that is necessary to make a dense graph Hamiltonian. A comparison can be drawn to the notion of *smoothed analysis* of algorithms introduced in [21], which involves studying the performance of algorithms on randomly perturbed inputs.

Our first contribution in this paper is to generalize the aforementioned theorem to hypergraphs (and to give a corresponding result for perfect matchings, which is nontrivial in the hypergraph setting). Unfortunately, there is no single most natural notion of a cycle or of minimum degree in hypergraphs. A k-uniform *loose* cycle is a k-uniform hypergraph with a cyclic ordering on its vertices such that every edge consists of k consecutive vertices and every pair of consecutive edges intersects in exactly one vertex. The degree of a set of vertices is the number of edges that include that set, and the *minimum* $(k-1)$*-degree* δ_{k-1} is the minimum degree among sets of $k-1$ vertices.

Theorem 1. *Let* $\mathbb{H}_k(n, m)$ *be the uniform distribution on* m*-edge* k*-uniform hypergraphs on the vertex set* $[n]$*.*

(a) There is $c_k^{\mathrm{M}}(\alpha)$ *such that if* H *is a* k*-uniform hypergraph on* $[kn]$ *with* $\delta_{k-1}(H) \geq \alpha n$*, and* $R \in \mathbb{H}_k\big(kn, c_k^{\mathrm{M}}(\alpha)n\big)$*, then* $H \cup R$ *a.a.s. has a perfect matching.*

(b) There is $c_k^{\mathrm{H}}(\alpha)$ *such that if* H *is a* k*-uniform hypergraph on* $[(k-1)n]$*with* $\delta_{k-1}(H) \geq \alpha n$*, and* $R \in \mathbb{H}\big((k-1)n, c_k^{\mathrm{H}}(\alpha)n\big)$*, then* $H \cup R$ *a.a.s. has a loose Hamilton cycle.*

There is a corresponding Dirac-type theorem for loose Hamilton cycles in dense hypergraphs (see [10]), and the threshold for both perfect matchings and loose Hamilton cycles is $n^{k-1} \log n$ random edges (see [5] and [8, Corollary 2.6]), so "almost all" dense hypergraphs have Hamilton cycles and perfect matchings. However, for hypergraphs with large uniformity our conclusion is much more extreme than for graphs: starting from a dense hypergraph only $O(n)$ extra edges are requred to ensure Hamiltonicity, which is far fewer than the $\Omega\big(n^{k-1} \log n\big)$ edges required to reach Hamiltonicity from scratch.

We will prove Theorem 1, and show that it is tight, in Section 1. The methods usually employed to study Hamilton cycles and perfect matchings in random graphs are largely ineffective in the hypergraph setting, so we need a very different proof. In particular, we cannot easily manipulate paths for Pósa-type arguments, and we do not have an analogue of Hall's marriage theorem allowing us to deduce the existence of a perfect matching from an expansion property. Our proof involves reducing the theorem to the existence of a perfect matching in a certain randomly perturbed bipartite graph. The "reason" for a perfect matching in this perturbed graph seems to be quite different depending on the structure of the initial bipartite graph, so we make critical use of the structural description given by Szemerédi's regularity lemma.

Our second contribution in this paper is a theorem giving a general expansion condition for *pancyclicity*. We say an n-vertex (di-)graph is pancyclic if it contains cycles of all lengths ranging from 3 to n.

Theorem 2. *Let* D *be a directed graph on* n *vertices with all in- and out- degrees at least* $8k$*, and suppose for all disjoint* $A, B \subseteq V(D)$ *with* $|A| = |B| \geq k$*, there is an arc from* A *to* B*. Then* D *is pancyclic.*

We hope this theorem could be of general interest, but our particular motivation is that it implies a number of results about randomly perturbed graphs and digraphs. In particular it provides very simple proofs of the theorems in [2] concerning Hamiltonicity in randomly perturbed graphs and digraphs, and allows us to extend these theorems to pancyclicity. We will prove Theorem 2 and its corollaries in Section 2.

Our final theorem concerns randomly perturbed tournaments. The model that starts with a fixed (di-/hyper-)graph and adds random edges is not suitable for studying random perturbation in tournaments, because we want our perturbed tournament to remain a tournament. There are several other models of random perturbation we could consider that do allow us to make sense of randomly perturbed tournaments, or are more natural in certain contexts. However, the types of results in this paper are not sensitive to the model used. We will briefly describe a few different models here.

First, note that for most practical purposes, models that involve the selection of m random edges are equivalent to models that involve the selection of each edge with independent probability p, where $m = p\binom{n}{2}$. One perspective or the other can be more intuitive or result in cleaner proofs; we will use both interchangeably as is convenient without further discussion. In all the situations in this paper, equivalence can be proved with standard conditioning and coupling arguments.

As suggested in [20, Definition 1], one possible alternative model is to *change* random edges, instead of adding them. So, for our results so far, instead of taking the union of a fixed dense object with a random sparse object, we would take the symmetric difference. Our results still hold in this alternative model, basically because we can break up such a random perturbation into a phase that deletes edges (this does not destroy denseness), and a phase that adds edges. One undesirable quirk of this model is that it is not "monotonic": if we change too many edges then we "lose our randomness" and end up at the complement of our original object, which might not be Hamiltonian.

A second alternative model is to start with our fixed object and "make it more random" by interpolating slightly towards the corresponding uniform distribution. For example, in the graph case we could randomly designate a small number of pairs of vertices for "resampling" and then decide whether the corresponding edges should be present uniformly and independently at random. This is mostly equivalent to the symmetric difference model, and is the model in which we prefer to state our theorem about randomly perturbed tournaments.

It is known that almost all tournaments have a Hamiltonian cycle (see Section 3), so we do not need to restrict our attention to "dense" tournaments. Nevertheless, we can give stronger results for tournaments with large minimum in- and out- degrees. We are also able to show that randomly perturbed tournaments are not just Hamiltonian, but have multiple edge-disjoint Hamilton cycles.

Theorem 3. *Consider a tournament T with n vertices and all in- and out- degrees at least d. Independently designate $m = \omega(d + 1)/n$ edges as "random" then choose the orientations of the chosen edges independently and uniformly at random. The resulting perturbed tournament P a.a.s. has q arc-disjoint Hamilton cycles, for $q = O(1)$.*

We will prove Theorem 3 and show that it is tight in Section 3.

1 Perfect Matchings and Hamilton Cycles in Hypergraphs

We first make some observations about our minimum degree requirement. The *minimum q-degree* $\delta_q(H)$ of H is the minimum degree among all sets of q vertices. Note that this generalizes the two notions of denseness for graphs: in some contexts, we say graphs are dense if they have many edges (large δ_0), whereas in this paper we need a stronger notion of graph denseness based on minimum degree (δ_1). For a k-regular hypergraph H, a double-counting argument shows that if $q \le p$ then

$$\delta_q(H) \ge \delta_p(H)\binom{n-q}{p-q} \Big/ \binom{k-q}{p-q}.$$

So, imposing that a k-uniform hypergraph has large $(k-1)$-degree ensures that it has large q-degrees for all q. In particular, our requirement $\delta_{k-1}(H) = \Omega(n)$ actually implies $\delta_q(H) = \Omega\big(n^{k-q}\big)$ for all q.

Next, note that Theorem 1 is tight for essentially the same reason as [2, Theorem 1]: consider the "complete bipartite hypergraph" which has two partitions of sizes n and $2kn$, and has all possible k-edges that contain at least one vertex from each partition. Only $2n$ of these edges can contribute to a loose Hamilton cycle, so a linear number must be added to complete the necessary $(2k+1)n/(k-1)$ edges.

Now we proceed to the proof of Theorem 1, which will follow from a sequence of lemmas. The first step is to show that R almost gives the structure of interest on its own. Let a *sub-cycle* be a hypergraph which can be extended to a loose Hamilton cycle by adding edges.

Lemma 4. *For any $\varepsilon > 0$:*

(a) There is $b_k^{\mathrm{M}}(\varepsilon)$ such that $R \in \mathbb{H}_k\big(kn, b_k^{\mathrm{M}}(\varepsilon)\big)$ a.a.s. has a matching of size $(1-\varepsilon)n$.

(b) For any $\varepsilon > 0$, there is $b_k^{\mathrm{H}}(\varepsilon)$ such that $R \in \mathbb{H}_k\big((k-1)n, b_k^{\mathrm{H}}(\varepsilon)\big)$ a.a.s. has a sub-cycle of size $(1-\varepsilon)n$.

Proof of Lemma 4(a). To realize the distribution $\mathbb{H}_k(kn, \beta_k'(\varepsilon)n)$, imagine that independently uniformly random edges are repeatedly dropping into R, until we have seen $b_k^{\mathrm{M}}(\varepsilon)$ distinct edges. We iteratively build a matching M greedily. If a new edge e drops into R which does not share any vertices with the edges in M so far, then add e to M. If we have already added i edges to M, then the probability we will add the next random edge is

$$p_i = \binom{kn-ki}{k} \Big/ \binom{kn}{k} = \Omega\left(\left(\frac{n-i}{n}\right)^k\right).$$

The number of random edges required to build a matching of size $(1-\varepsilon)n$ therefore has the distribution of

$$X = \sum_{i=1}^{(1-\varepsilon)n} X_i,$$

where each X_i is independently geometrically distributed with success probability p_i. By Chebyshev's inequality or a Chernoff bound, X is a.a.s. close to its expectation, which is at most

$$O\left(\sum_{i=1}^{(1-\varepsilon)n} n^k/(n-i)^k\right) = O\left(n^k \int_{\varepsilon n}^{\infty} \frac{1}{x^k}dx\right) = O(n).$$

So, if $b_k^{\mathrm{M}}(\varepsilon)$ is large enough then a.a.s. $|M| \ge (1-\varepsilon)n$. □

Proof of Lemma 4(b). A sub-cycle is a loose Hamilton cycle or a disjoint union of "loose paths", but a union of disjoint paths may not be a sub-cycle if it has so many paths that there are not enough vertices left to link them together. A union of j paths (which have length greater than zero) with a total of i edges is a sub-cycle precisely when $i + j \le n$. Call such sub-cycles *proper* sub-cycles.

As in the proof of Lemma 4, imagine that independently uniformly random edges are repeatedly dropping into R. We build a proper sub-cycle S greedily. Suppose we are at a point where S has i edges and j paths, with $i \le (1-\varepsilon)n$. There are $(k-1)i + j \le (k-2)i + n$ vertices in the paths in S.

There are 3 kinds of edges we can add to S: an edge that starts a new loose path ("type 1"), an edge that extends an existing path ("type 2"), or an edge that joins two paths together ("type 3"). We can only add a type 1 edge when $i + j \le n - 2$, and can only add a type 2 edge when $i + j \le n - 1$.

If $i + j \le n - 2$ then there are at least

$$\binom{(k-1)n - ((k-2)i + n)}{k} = \binom{(k-2)(n-i)}{k} = \Omega\Big((n-i)^k\Big)$$

type-1 edges that could possibly be added. Otherwise, we must have $j > n - 2 - i \ge \varepsilon n - 2$, so (identifying one vertex in an extremal edge of each path as "eligible to be joined"), there are at least

$$\binom{j}{2}\binom{(k-2)(n-i)}{k-2} = \Omega\Big(n^2(n-i)^{k-2}\Big) = \Omega\Big((n-i)^k\Big)$$

type-3 edges that could be added. Combining both cases, we can see that if S has $i \le (1-\varepsilon)n$ edges, then the next edge will be added with probability at least

$$\Omega\Big((n-i)^k\Big) \Big/ \binom{(k-1)n}{k} = \Omega\left(\left(\frac{n-i}{n}\right)^k\right).$$

We conclude with the same reasoning as the proof of Lemma 4(a). $\square$

The second step to prove Theorem 3 is to show that a dense hypergraph plus a large partial structure a.a.s. gives the structure we are looking for. For both theorems, we will be able to reduce this step to the following lemma.

Lemma 5. *There is $\xi(\alpha) > 0$ such that the following holds. Let G be a bipartite graph with partitions A, B of equal size n, and suppose $\delta(G) \ge \alpha n$. Let M be a uniformly random matching with $(1 - \xi(\alpha))n$ edges between A and B. Then $G \cup M$ has a perfect matching.*

The immediate naïve approach to prove this lemma would be to apply Hall's marriage theorem, by showing that each set of vertices expands with high probability and then applying a union bound. However, the probability of failure to expand is not small enough for this to work. We can gain some insight into the problem by considering two "extremal" cases for G. First, consider the case where the edges of G are not evenly-distributed, and are "concentrated" in certain spots. For example, identify sets $A' \subset A$ and $B' \subset B$ with $|A'|, |B'| = \alpha n$, and let G contain only those edges incident to a vertex in A' or B'. The addition of the near-perfect matching M gives a near-perfect matching between $A \backslash A'$ and $B \backslash B'$, and we can match the unmatched vertices from A (respectively B) with any element of B' (respectively A'). That is, if our graph is not well-distributed, then the more concentrated parts help us augment M into a perfect matching in $G \cup M$. On the other extreme, if G is a random-like, well-distributed dense graph then we cannot augment M in the same way. But this is not necssary, because a random dense graph G contains a perfect matching on its own! In order to apply these ideas to prove the lemma for an arbitrary graph G, we use the structural description of G provided by Szemerédi's regularity lemma.

We will use a bipartite version of the regularity lemma (which can be deduced from say [22, Theorem 2.3] in a similar way to [11, Theorem 1.10]). Let $\alpha' = \alpha/2$ and let $\varepsilon > 0$ be a small constant depending on α that will be determined later (assume for now that $\varepsilon < \alpha/8$). There is a large constant K depending only on α such that there exist partitions $A = V_0^1 \cup \cdots \cup V_k^1$ and $B = V_0^2 \cup \cdots \cup V_k^2$ with $k \le K$, in such a way that the following conditions are satisfied. The "exceptional" clusters V_0^1 and V_0^2 both have fewer than εn vertices, and the non-exceptional clusters in A and B have equal size: $\left|V_1^1\right| = \cdots = \left|V_k^\ell\right| = an$ and $\left|V_1^1\right| = \cdots = \left|V_k^\ell\right| = bn$. There is a subgraph $G' \subseteq G$ with minimum degree at least $(\alpha' + \varepsilon)n$ such that each pair of distinct clusters V_i^1, V_j^2 $(i, j \ge 1)$ is ε-regular in G' with density zero or at least 2ε.

Define the cluster graph C as the bipartite graph whose vertices are the non-exceptional clusters V_i^ℓ, and whose edges are the pairs of clusters between which there is nonzero density in G'. The fact that G is dense implies that C is dense as well: note that in G' each V_i^ℓ has at least $(\alpha' + \varepsilon)n\left|V_i^\ell\right|$ edges to other clusters. There are at most $\varepsilon n\left|V_i^\ell\right|$ edges to the exceptional cluster V_0 and at most $\left|V_i^\ell\right|^2$ edges to each other cluster. So, $d_C\left(V_i^\ell\right) \ge \left((\alpha' + \varepsilon)n\left|V_i^\ell\right| - \varepsilon n\left|V_i^\ell\right|\right)/\left|V_i^\ell\right|^2 \ge \alpha' k$ and C has minimum degree at least $\alpha' k$.

Proof of Lemma 5. We use Hall's marriage theorem: we need to show that a.a.s. $|N_{G \cup M}(W)| \ge |W|$ for all $W \subseteq A$. If $|W| \le \alpha n$, then $|N_{G \cup M}(W)| \ge |N_G(W)| \ge \alpha n \ge |W|$ by the degree condition on G. Similarly, if $|W| \ge (1 - \alpha)n$ then every $b \in B$ has an edge to W in G, so $|N_{G \cup M}(W)| = |B| \ge |W|$. The difficult case is where $\alpha n \le |W| \le (1 - \alpha)n$.

Apply our version of the regularity lemma. We first want to prove that the edges of M a.a.s. "spread out evenly" between the $O(1)$ clusters. Fix some V_i^1 and V_j^2, with $i, j \ge 1$. The vertices of each V_i^1 are matched by M to at most an vertices; conditioning on their number, these are uniformly randomly chosen from B. The number of such vertices in V_j^2 is then hypergeometrically distributed, so with probability $1 - e^{\Omega(n)}$ (see for example [7, Theorem 2.10]) there are at least $(ab + \xi(\alpha))n$ edges of M between V_i^1 and V_j^2. By a union bound, a.a.s. there are at most $(ab + \xi(\alpha))n$ edges of M between every pair V_i^1, V_j^2. That is, M spreads out evenly between the clusters. We assume this holds for the remainder of the proof.

Consider any $W \subset A$ with $\alpha n \le |W| \le (1 - \alpha)n$. For each i let $\pi_i = \left|V_i^1 \cap W\right|/(an)$, and let D be the set of clusters V_i^1 with $\pi_i \ge \varepsilon$. If ε is small, D must be nonempty. Now, if $V_j^2 \in N_C(D)$ then by ε-regularity there are edges in G' from W to at least $(1 - \varepsilon)bn$ vertices of V_j^2. It follows that if $|N_C(D)| = k$ (as would occur if G was well-distributed) then $|N_{G \cup M}(W)| \ge |N_{G'}(W)| \ge (1 - \varepsilon)\left(n - \left|V_0^2\right|\right) \ge |W|$ for small ε, and we are done.

Otherwise, choose $E \subseteq N(D)$ with $|E| = \alpha' k$ and let S contain the vertices in the clusters of E. By the same ε-regularity argument, $|N_{G'}(W) \cap S| \ge \alpha' k(1 - \varepsilon)bn \ge (1 - \varepsilon)^2\alpha' n$. Since $|N_C(D)| < k$, there is V_j^2 outside $N_C(D)$, and this V_j^2 must have $\alpha' k$ neighbours outside D in C, so $|D| \le (1 - \alpha')k$. Since $|M| = (1 - \xi(\alpha))n$, each $V_i^1 \cap W$ has at least $(\pi_i a - \xi(\alpha))n$ neighbours in M, and at most

$\alpha' k(ab+\xi(\alpha))n$ of these are in S. So,

$$\begin{aligned}|N_M(W)\backslash S| &\geq \sum_{V_i^1 \in D} (\pi_i a - \xi(\alpha))n - |D|\big(\alpha' k(ab+\xi(\alpha))\big)n \\ &\geq |W| - \varepsilon n - \big(1-\alpha'\big)\alpha'(1-\varepsilon)^2 - k^2\xi(\alpha)n\end{aligned}$$

and

$$|N_{G\cup M}(W)| \geq |W| + \Big(\alpha'(1-\varepsilon)^2 - \varepsilon - \big(1-\alpha'\big)\alpha'(1-\varepsilon)^2 - k^2\xi(\alpha)\Big)n.$$

If ε is chosen to be small enough relative to α and $\xi(\alpha)$ chosen to be small enough relative to K, then this gives $|N_{G\cup M}(W)| \geq |W|$. □

Now we describe the reduction of Theorem 3 to Lemma 5. Consider a k-uniform hypergraph H. Suppose A is a set of n vertices and B is a $(k-1)$-uniform hypergraph on the remaining vertices. Then we define a bipartite graph $G_{A,B}(H)$ as follows. The vertices of $G_{A,B}(H)$ are the vertices in A, as well as the edges in B (we abuse notation and identify the hypergraph B with its edge set). We put an edge between $a \in A$ and $\{b_1, \ldots, b_{k-1}\} \in B$ if $\{a, b_1, \ldots, b_{k-1}\}$ is an edge in H.

The significance of $G_{A,B}(H)$ is that if B is a perfect matching or loose Hamilton cycle on $V(H)\backslash A$, and if $G_{A,B}(H)$ has a large matching M, then the edges of M correspond to a large matching (respectively, a large sub-cycle) in H. Conversely, if H has a large matching or Hamilton cycle (as provided by Lemma 4), then there is a set A and a perfect matching (respectively, loose Hamilton cycle) B such that $G_{A,B}(H)$ has a large matching.

By the symmetry of the distribution of R, we can assume A and B are uniformly random. We give one final lemma, establishing that the density of H corresponds a.a.s. to density of $G_{A,B}(H)$.

Lemma 6.

(a) There is $\beta^{\mathrm{M}}(\alpha) > 0$ such that the following holds. Let H satisfy the conditions of Theorem 1(a), let A be a uniformly random set of n vertices, and let B be a uniformly random perfect matching on $V(H)\backslash A$. Then a.a.s. $G_{A,B}(H)$ has minimum degree $\beta^{\mathrm{M}}(\alpha)n$.

(b) There is $\beta^{\mathrm{H}}(\alpha) > 0$ such that the following holds. Let H satisfy the conditions of Theorem 1(b), let A be a uniformly random set of n vertices, and let B be a uniformly random loose Hamilton cycle on $V(H)\backslash A$. Then a.a.s. $G_{A,B}(H)$ has minimum degree $\beta^{\mathrm{H}}(\alpha)n$.

Proof of Lemma 6(a). It is helpful to conceptualize the randomness in a certain way. Let

$$a_1, \ldots, a_n,\ b_1^1, \ldots, b_k^1,\ b_1^2, \ldots, b_k^n$$

be a uniformly random ordering of $V(H)$. Let $A = \{a_1, \ldots, a_n\}$, let $b^i = \big\{b_1^i, \ldots, b_k^i\big\}$ and let $B = \big\{b^1, \ldots, b^n\big\}$.

First, condition on some $b = b^i$. Note that $\Pr(b \cup \{a_1\} \in E(H)) \geq \delta_{k-1}(H)/n$ and more generally

$$\Pr(b \cup \{a_j\} \in E(H) \mid a_1, \ldots, a_j) \geq (\delta_{k-1}(H) - j)/n \geq \alpha/2$$

for $j \leq \alpha n/2$. By a Chernoff bound, $d_{G_H}(b) \geq \alpha^2 n/8$ with probability $1 - e^{-\Omega(n)}$. With a union bound, this a.a.s. holds for all b.

Now, $\delta_1(H) \geq \alpha' n^{k-1}$ for some α' depending on α. Condition on some $a = a_i$. Each b^i_j shares at most $\binom{n}{k-2}$ edges with a_i, so if $j \leq 2\sqrt{\beta^{\mathrm{M}}(\alpha)}n$ for sufficently small $\beta^{\mathrm{M}}(\alpha)$ and large n, then

$$\Pr\big(\{a\} \cup b^{j+1} \in E(H) \mid b^1, \ldots, b^j\big) \geq \left(\delta_1(H) - kj\binom{n}{k-2}\right) \Big/ \binom{n}{k-1} \geq \sqrt{\beta^{\mathrm{M}}(\alpha)}.$$

By a Chernoff bound and a union bound, a.a.s. each $d_{G_H}(a) \geq \beta^{\mathrm{M}}(\alpha)n$. □

Proof of Lemma 6(b). We give essentially the same proof as for Lemma 6(a). Choose a uniformly random ordering of $V(H)$:

$$a_1, \ldots, a_n, b_0, \ldots, b_{(k-2)n-1}.$$

Let $A = \{a_1, \ldots, a_n\}$, let $b^i = \left\{b_{(k-2)i}, b_{(k-2)i+1}, \ldots, b_{(k-2)(i+1)}\right\}$ (where the subscripts are interpreted modulo $(k-2)n$), and let $B = \left\{b^1, \ldots, b^n\right\}$.

With exactly the same proof as for Lemma 6(a), there is small $\beta^{\mathrm{H}}(\alpha)$ such that a.a.s. $d_{G_H}(b) \geq \beta^{\mathrm{H}}(\alpha)n$ for all b. Next, condition on some $a = a_i$. For small $\zeta(\alpha)$ and $j \leq 2\sqrt{\beta^{\mathrm{H}}(\alpha)}n$,

$$\Pr\big(\{a\} \cup b^j \in E(H) \mid b^0, \ldots, b^{j-1}\big) \geq \left(\delta_2(H) - (k-2)j\binom{n}{k-3}\right) \Big/ \binom{n}{k-2} \geq \sqrt{\beta^{\mathrm{H}}(\alpha)},$$

so a.a.s. each $d_{G_H}(a) \geq \beta^{\mathrm{H}}(\alpha)n$. □

We have established that $G_{A,B}(H)$ is a bipartite graph with linear minimum degree. If we condition on A and B, then $G_{A,B}(R)$ a.a.s. provides a uniformly random large matching between A and B, so Lemma 5 ensures the existence of a perfect matching in $G_{A,B}(H \cup R)$, corresponding to a perfect matching or loose Hamilton cycle in $H \cup R$.

2 Pancyclicity in dense digraphs

In this section we prove Theorem 2, and as a corollary we also prove the following theorem.

Theorem 7. *Let $\mathbb{D}(n,m)$ be the uniform distribution on (not necessarily oriented) n-vertex, m-arc digraphs. For each $\alpha > 0$, there is $c(\alpha)$ such that if D is a digraph with all in- and out- degrees at least αn, and $R \in \mathbb{D}(n, c(\alpha)n)$, then $D \cup R$ is a.a.s. pancyclic.*

One motivation for this theorem is an observation by Bondy (see [3]), that almost all known nontrivial conditions that ensure Hamiltonicity also ensure pancyclicity. He made an informal "metaconjecture" that this was always the case; our Theorem 7 verifies his metaconjecture in the setting of randomly perturbed dense graphs and digraphs.

Theorem 7 obviously implies [2, Theorems 1 and 3]. We do not fight very hard to optimize constants, but we note that if we make some simplifications in the proof of Theorem 7, the resulting values of $c(\alpha)$ seem to be better than those found in [2], for most values of α.

We now turn to the proof of Theorem 2, which will follow from the corresponding result for Hamiltonicity.

Lemma 8. *Let D be a directed graph with all in- and out- degrees at least $4k$, and suppose for all disjoint $A, B \subseteq V(D)$ with $|A| = |B| \geq k$, there is an arc from A to B. Then D is Hamiltonian.*

Remark 9. With some effort, the ideas in the proof of Lemma 8 can be used directly to prove Theorem 2 with a weaker degree condition. We do not know whether the condition can be weakened all the way to $4k$, as Bondy's metaconjecture would suggest. The constants in Theorem 2 and Lemma 8 can both be halved for the undirected case, just by simplifying the main argument in the proof of Lemma 8.

The idea of the proof is to start with a longest path P and manipulate it into a cycle C on the same vertex set. We will show that D is strongly connected, so if C were not Hamiltonian, there would be an arc from $V(C)$ to its complement, which could be combined with C to give a longer path than P, contradicting maximality. This type of argument goes back to the proof of Dirac's theorem [4, Theorem 3]. It also bears some resemblance to the "rotation-extension" idea introduced in [18], and a variation for directed graphs in [6, Section 4.3].

Proof of Lemma 8. First we acknowledge some immediate consequences of the condition on D. Note that if A and B are disjoint sets with size at least k, then in fact there are $|A| - k$ vertices of A with an arc into B. To see this, note that for any fewer number of such vertices in A, we can delete those vertices and at least k will remain, one of which has an arc to B. Also, D is strongly connected. To see this, note that for any v, w, both of $N^+(v)$ and $N^-(w)$ have size at least $4k > k$. If they intersect then there is a length-2 path from v to w; otherwise there must be an arc from $N^+(v)$ to $N^-(w)$ giving a length-3 path.

Let $P = u \ldots w$ be a maximal-length directed path in D. We will use the notation v^+ (respectively v^-) for the successor (respectively predecessor) of a vertex v on P, and also write V^+, V^- for the set of successors or predecessors of a set of vertices V.

By maximality, $N^+(w) \subset P$ and $N^-(u) \subset P$. Let U_1 be the first $3k$ elements of $N^-(u)$ on P, and let U_2 be the last k (note $U_1 \cap U_2 = \varnothing$). Similarly let W_1 be the first k and W_2 the last $3k$ elements of $N^+(w)$. We will now show that there is a cycle on the vertex set $V(P)$.

First, consider the case where each vertex of W_1 precedes each vertex of U_2. If wu is in D then we can immediately close P into a cycle. Otherwise, $\left|W_1^-\right| = |W_1| = \left|U_2^+\right| = |U_2| = k$, so there is an arc w_1u_2 from W_1^- to U_2^+. This is enough to piece together a cycle on $V(P)$: start at u_2 and move along P to w, from where there is a shortcut back to w_1^+. Now move along P from w_1^+ to u_2^-, from where we can jump back to u, then move along P to w_1, then jump to u_2. See Figure Figure 1 for an illustration.

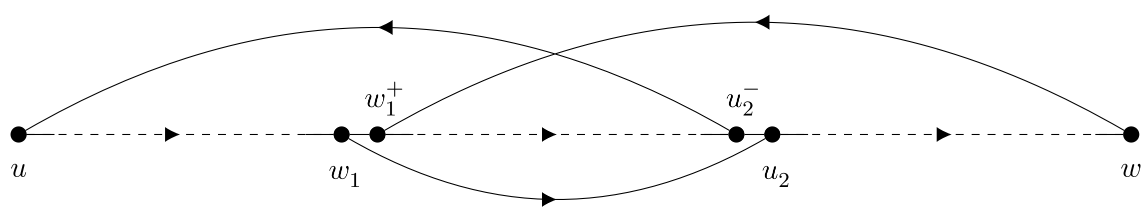

Figure 1. The case where the vertices of W_1 precede the vertices of U_2. The horizontal line through the center is P; the broken lines indicate subpaths.

Otherwise, each vertex of U_1 precedes each vertex of W_2. Let U_{12} contain the k elements of U_1 furthest down the path. Let U_{11} be the set of vertices among the first $2k$ vertices of P which have an arc to U_{12}^+. By the discussion at the beginning of the proof, $|U_{11}| \geq k$. Similarly, let W_{21} contain the k elements of W_2 first appearing on the path, and let W_{22} be the set of at least k vertices among the last $2k$ on P which have an arc from W_{21}^-. By the condition on D, there is an arc $w_{22}u_{11}$ from W_{22}^- to U_{11}^+. By definition, there is $u_{12} \in U_{12}^+$ and $w_{21} \in W_{21}^-$ such that the arcs $u_{11}^-u_{12}$ and $w_{21}w_{22}^+$ are in D. We can piece everything together to get a cycle on the vertices of P: start at u_{11}, move along P until u_{12}^-, then jump back to u. Move along P until u_{11}^-, then take the shortcut to u_{12}. Continue along P to w_{21}, jump to w_{22}^+, continue to w, jump back to w_{21}^+, and continue to w_{22}. From here there is a shortcut back to u_{11}. See Figure Figure 2.

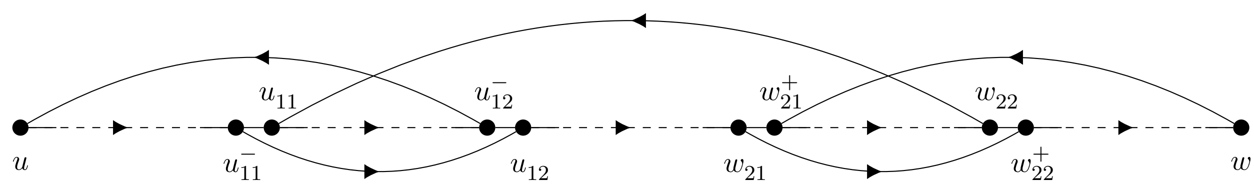

Figure 2. The case where the vertices of U_1 precede the vertices of W_2.

As outlined, the fact that D is strongly connected, combined with the fact that the vertex set of P induces a cycle C, implies that C is a Hamilton cycle. $\square$

Proof of Lemma 8. Fix a vertex v. Let U^+ and U^- be arbitrary k-subsets of $N^+(v)$ and $N^-(v)$ respectively. There is an arc from U^+ to U^- which immediately gives a 3-cycle.

Next, let W^+ be the set of (fewer than k) vertices with no arc into U^+, and similarly let W^- be the set of vertices with no arc from U^-. Now consider the induced digraph D' obtained from D by deleting v and the vertices in U^+, U^-, W^+, W^-. Since we have removed fewer than $4k$ vertices, D' satisfies the conditions of Lemma 8 so has a Hamilton cycle. In particular, for every ℓ satisfying $4 \leq \ell \leq n-4k$, there is a path $P_\ell = u_\ell \dots w_\ell$ in D' of length $\ell - 4$. By construction, there is an arc from U^+ to u_ℓ and from w_ℓ to U^-, which we can combine with arcs to and from v to get a cycle of length ℓ.

Finally, for every $\ell > n-4k$, arbitrarily delete vertices from D to obtain an induced digraph D'' with ℓ vertices which satisfies the conditions of Lemma 8. Since D'' has a Hamilton cycle, D has a cycle of length ℓ. $\square$

Proof of Theorem 7. In view of the discussion in the introduction, we assume each possible arc is present in R with independent probability $p = 2c(\alpha)/n$.

If $A, B \subseteq V(D)$ are disjoint sets with $|A| = |B| = \alpha n/8$, the probability that there are no arcs from A to B in $D \cup R$ is at most

$$(1-p)^{(\alpha n/8)^2} \leq e^{-p\alpha^2 n^2/64} = e^{-c(\alpha)\alpha^2 n/32}.$$

The number of choices of such pairs of disjoint sets A, B is

$$\binom{n}{\alpha n/4}\binom{\alpha n/4}{\alpha n/8} = O\left(\frac{1}{\sqrt{n}} e^{n(H(\alpha/4)+\alpha H(1/2)/4)}\right),$$

where $H(\alpha) = -\alpha \log \alpha - (1-\alpha)\log(1-\alpha)$ (this can be proved with Stirling's approximation). By a union bound, the probability that $D \cup R$ does not satisfy the condition of Lemma 8 is at most

$$O\left(\frac{1}{\sqrt{n}} e^{n\left(H(\alpha/4)+\alpha H(1/2)/4 - c(\alpha)\alpha^2 n/32 + o(1)\right)}\right).$$

This converges to zero if $c(\alpha)$ is chosen to be sufficiently large. $\square$

Remark 10. If we are only interested in Hamiltonicity, we can use Lemma 8 directly in the proof of Theorem 7, which gives a better constant $c(\alpha)$ than in [2, Theorem 3], for all α. If we make the adjustment for undirected graphs mentioned in Remark 9, then we also beat the corresponding constant in [2, Theorem 1] except for small values of α.

3 Hamilton cycles in tournaments

There are several seemingly different conditions that are equivalent to Hamiltonicity for tournaments (see [15, Chapters 2-3]). A tournament is Hamiltonian if and only if it is irreducible (cannot be divided into two partitions with all arcs between the two partitions in the same direction), if and only if it is strongly connected (has a directed path from every vertex to every other), if and only if it is pancyclic (contains cycles of all lengths). All tournaments have a Hamilton path, and it was first proved in [16] that a uniformly random tournament is a.a.s. irreducible, hence Hamiltonian.

It was more recently proved in [13] that if a tournament is t-strongly connected (it remains strongly connected after the deletion of $t-1$ vertices), then it has $\Omega\left(\sqrt{t}/\log t\right)$ arc-disjoint Hamilton cycles (this was improved to $\Omega\left(\sqrt{t}\right)$ in [17]). It is not difficult to show that a random tournament is a.a.s. t-connected for fixed t, which implies that a random tournament contains q arc-disjoint Hamilton cycles for $q = O(1)$. We will use this to prove Theorem 3.

Before we proceed to the proof, we first explain why Theorem 3 is sharp. The "obvious" worst case for T is a *transitive* tournament (corresponding to a linear order on the vertices). In this case, a superlinear number of edges must be flipped in order to a.a.s. flip one of the arcs pointing away from the least element of the linear order. Actually, the model where random edges are flipped in a transitive tournament has already been studied in [14], by analogy to the evolution of the random graph.

More generally, consider a "transitive cluster-tournament" T on $n = rk$ vertices defined as follows. Let R be a regular tournament on $2d+1$ vertices (this means the indegree and outdegree are equal for each vertex). To construct T, start with r disjoint copies $R_1, \ldots, R_r$ of R, then put an arc from v to w for every $v \in R_i$, $w \in R_j$ with $i < j$. In order for the perturbed tournament P to be Hamiltonian, there must be an arc entering R_1, so one of the $O(n(d+1))$ arcs exiting R_1 must be changed. This will not happen a.a.s. unless $m = \omega(d+1)/n$.

We now prove Theorem 3. In accordance with the discussion in the introduction, we will work with the model where each edge is flipped with independent probability p, where $2p\binom{n}{2} = m$. (Designating an edge for resampling with probability $2p$ is the same as flipping it with probability p).

Fix t; we will prove that P is t-strongly connected. The idea of the proof is to choose a set S of t vertices with a large indegree and outdegree, then show that with high probability almost every vertex has many paths to and from each vertex in S. The probability that a vertex v has paths from S is smallest if v has small indegree, so we need to show that not many vertices can have small indegree.

Lemma 11. *In any tournament, there are less than k vertices with indegree (respectively outdegree) less than $(k-1)/2$.*

Proof. The sum of indegrees (respectively outdegrees) of a tournament on k vertices is $\binom{k}{2}$, because each arc contributes 1 to this sum. Therefore in every set of k vertices of a tournament, there is a vertex of outdegree (indegree) at least $(k-1)/2$ in the induced tournament. □

A consequence of Lemma 11 is that the set of all vertices with indegree (respectively outdegree) less than $n/6$ has size smaller than $n/3$. So, there are at least $n/3$ vertices with indegree and outdegree at least $n/6$. We can therefore choose a set S of t such vertices.

Now, we prove that the random perturbation does not change the in- and -out degrees very much.

Lemma 12. *If a vertex v has outdegree (respectively indegree) k in T, then it has outdegree (respectively indegree) at least $k/3$ in P, with probability $e^{-\Omega(\sqrt{n})}$ uniformly over k.*

Proof. We only prove the statement where v has outdegree k; the indegree case is identical. If $k = 0$ the lemma is trivial, so assume $k \geq 1$. First consider the cases where $k > \sqrt{n}$ or $p < 1/\sqrt{n}$. There are k arcs pointing away from v in T, and a Cherhoff bound says that the probability more than $2k/3$ arcs are changed by the perturbation is

$$e^{-\Omega\left(k^2/\mathbb{E}X\right)} = e^{-\Omega\left(\sqrt{n}\right)}.$$

That is, with the required probability, $k/3$ of the original out-neighbours survive the perturbation.

Otherwise (for large n), there is a set of $n/2$ arcs pointing towards v in T. A Chernoff bound says that the probability less than $k/3$ of these arcs are changed is $e^{-\Omega\left(\sqrt{n}\right)}$. That is, with the required probability, $k/3$ new out-neighbours are added by the perturbation. □

Lemma 13. *Suppose w has outdegree (respectively indegree) at least $n/6$, and $v \neq w$ is a vertex with indegree (respectively outdegree) at least k. Then with probability $1 - e^{-\Omega\left(\sqrt{n}\right)} - e^{-\Omega(np(k+1))}$ (uniformly over k), there are $t' = 3t$ internally vertex-disjoint paths of length at most 3 from w to v (respectively from v to w).*

Proof. We will only prove the statement where v has indegree at least k; the outdegree case is identical. By independence, we can condition on the outcome of the perturbation on individual arcs. Condition on the outcome for all arcs adjacent to w, and let $N_P^+(w)$ be the set of vertices to which there is an arc from w in P. By Lemma 12, we can assume $\left|N_P^+(w)\right| \geq n/18$.

We first prove the lemma for the case where $k \leq 12t'$. There are at least $n' = n/18 - 1$ arcs between $N_P^+(w)$ and v, each of which will be pointing towards v in P with independent probability at least p. By a Chernoff bound, the probability less than t' arcs will point from $N_P^+(w)$ to v in P is

$$e^{-\Omega(\mathbb{E}Z - t')} = e^{-\Omega(np)} = e^{-\Omega(np(k+1))}.$$

So, with the required probability there are t' suitable length-2 paths from w to v.

We can now assume $k > 12t'$. Condition on the result of the perturbation for the arcs adjacent to v (in addition to the arcs we have conditioned on so far). Let $N_P^-(v)$ be the set of vertices from which there is an edge into v in P; by Lemma 12, we can assume $\left|N_P^-(v)\right| \geq k/3$.

Now, if $\left|N_P^+(w) \cap N_P^-(v)\right| \geq t'$ then there are t' disjoint length-2 paths from w to v and we are done. So we can assume $U^+ = N_P^+(w) \backslash \left(N_P^-(v) \cup \{v\}\right)$ has at least $n' = n/18 - t' - 1$ vertices, and $U^- = N_P^-(v) \backslash \left(N_P^+(w) \cup \{w\}\right)$ has at least $k' = k/6 - t' - 1$ vertices (note $k' \geq t'$ by assumption).

Now, we would like to show that with the required probability there is a set of t' independent arcs from U^+ into U^- in P, which will give t' suitable length-3 paths. Partition U^+(respectively U^-) into subsets $U_1^+, \dots, U_{t'}^+$ (respectively $U_1^-, \dots, U_{t'}^-$), such that each $\left|U_1^+\right| > n'/(2t')$ and each $\left|U_1^-\right| > k'/(2t')$. For each i, the probability that there is no arc from U_i^+ into U_i^- after the perturbation is at most

$$(1 - p/2)^{n'k'/(4t')} \leq e^{-\Theta(np(k+1))},$$

so with the required probability there a set of t' suitable independent arcs, each between a pair U_i^+, U_i^-. ☐

Lemma 14. *Fix some $w \in S$. In P, there are a.a.s. t disjoint paths from w to each other vertex (respectively, from each other vertex to w).*

Proof. We only prove there are paths from w to each other vertex; the reverse case is identical. If there are $3t$ disjoint paths of length at most 3 from w to v then we say v is *safe*. It follows from Lemma 13 that a vertex with indegree k is safe with probability $1 - e^{-\Omega\left(\sqrt{n}\right)} - e^{-\Omega(np(k+1))}$.

By Lemma 11, there are at most $(2d+1)$ vertices with outdegree d, and the vertex with the $2k$th smallest outdegree has outdegree at least $k-1$. Let Q be the set of non-safe vertices, and note

$$\begin{aligned}
\mathbb{E}|Q| &= (2d+1)e^{-\Omega(np(d+1))} + \sum_{k=d+1}^{n} 2e^{-\Omega(np(k+1))} + ne^{-\Omega\left(\sqrt{n}\right)} \\
&= e^{-\Omega(np(d+1))}\left(O(d+1) + \frac{1}{1 - e^{-\Omega(np)}}\right) + o(1) \\
&= e^{-\Omega(np(d+1))}O(d+1).
\end{aligned}$$

where we have used the geometric series formula, the inequality $1 - e^{-x} \geq (x \wedge 1)/2$ (for positive x) and the fact that $d + 1 = \Omega(1)$.

By Markov's inequality, a.a.s. $|Q| \leq \sqrt{e^{-\Omega(pn(d+1))}}O(d+1) = o(d+1)$. If $d \leq 4t$ then $|Q| = 0$ for large n. Otherwise, for large n, $|Q| \leq 3d$, so every vertex $v \in Q$ has $3t$ safe neighbours $v_1, \dots, v_{3t}$. Now, fix a maximal set M of disjoint paths from w to v. If $|M| < t$ then the paths in M collectively use less than $3t$ vertices other than t, so there is some v_i not in any path of M, and there is some path from w to v_i that does not contain any vertex of M. But then $w \dots v_i v$ is a path from w to v disjoint with every path in M, contradicting maximality. ☐

It follows from Lemma 14 that a.a.s. every vertex outside S has t disjoint paths to and from every vertex in S. If we delete $t-1$ vertices, then there is at least one vertex w of S remaining, and w has least one path to and from every other vertex. That is, P is strongly connected. (In fact, we have also proved that P has diameter at most 8).

4 Concluding Remarks

We have determined the amount of random perturbation required to make a tournament, dense digraph or uniform hypergraph Hamiltonian. In the process, we have proved a general lemma about pancyclicity in highly connected digraphs, and demonstrated an interesting application of the Szemerédi regularity lemma. There are a few important questions this paper leaves open.

First, we have only studied loose Hamiltonicity. The other most popular notion of a hypergraph cycle is a *tight* cycle, in which every consecutive pair of edges in the cycle intersects in $k-1$ vertices. More generally, an ℓ-cycle has consecutive edges intersecting in $k-\ell$ vertices. Also, we have only studied hypergraphs with high $(k-1)$-degree, which is the strongest density assumption we could make. There are a large variety of Dirac-type theorems for different types of minimum degree and different types of cycles (see [19] for a survey), which would suggest that similar random perturbation results are possible in these settings. In addition, it is possible that our use of Szemerédi's regularity lemma could be avoided, thereby drastically improving the constants $c_k^{\mathrm{H}}(\alpha)$ and $c_k^{\mathrm{M}}(\alpha)$.

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