Honours Readings Summaries

Matthew Kwan

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1 Combinatorial estimates by the switching method

1.1 Summary

Consider a finite set Ω of objects. A switching is a (nondeterministic) operation that transforms one object into another (or more generally, a switching is a relation $R \subseteq \Omega \times \Omega$). We can partition Ω into subsets $\{C(v)\}_{v \in V}$ and put a directed graph structure (possibly with loops) on the index set V: if an element in $C(v)$ can switch to an element in $C(w)$ then there is an arc between v and w.

Hopefully, for each v we can find a good lower bound $a(v)$ on the number of ways an $\omega \in C(v)$ can be switched, and a good upper bound $b(v)$ on the number of switchings an $\omega \in C(v)$ can be produced by. If we imagine that all switchings are performed at once, a and b give bounds on the inflow and outflow at each vertex in terms of the size of each $C(v)$. (So, $a(v)$ and $b(v)$ can more generally be bounds on the *average* number of switchings per element in $C(v)$).

This gives some information about the relative sizes of the classes $C(v)$. Given a set of vertices X, let $N(X)$ be the total amount of elements in all $C(x)$, where $x \in X$. The objective of the paper is to bound $N(Y)/N(X)$ for given vertex sets X and Y.

In order to obtain bounds on $N(Y)/N(X)$, we write the constraints as a system of linear inequalities. Let $s'(v, w)$ be the (unknown) amount of switchings going from $C(v)$ to $C(w)$. For instance, for any vertex v, we have $\sum s'(vw) \geq a(v)N(v)$. A nonzero assignment of s and N satisfying the full set of inequalities is called a *feasible solution*, and a feasible solution which maximizes $N(Y)/N(X)$ is called an *optimal solution*.

The key result of the paper is to show that optimal solutions always take one of six standard forms. This is proved by a reduction of the inequalities to a linear program, and the fact that an optimal solution of a linear program always occurs at a vertex of the corresponding convex polytope. The analysis is amenable to a general bound α on the arcs of the digraph instead of the functions a, b (in that case we have $\alpha(v, w) = b(w)/a(v)$). Although α doesn't have a clear combinatorial interpretation, the paper suggests that different α may arise from richer information bounding the behaviour of the flow of switchings.

For a number of commonly-satisfied assumptions, the paper proves some alternative bounds for $N(Y)/N(X)$ that are slightly looser but easier to apply. In particular, a common use case is that we have some statistic on the objects in Ω and we believe that objects with a high value of that statistic make up a negligible fraction of all the objects in Ω. We would then partition the objects according to our statistic, and design a switching that tends to decrease the statistic, but by no more than some fixed amount. We would choose X to be the set of all vertices (partitions) and Y would be the set of all vertices (partitions) with a statistic value higher than some M.

1.2 Remarks

- I'm not sure why $i_{k-1} = \max\{M - (k-1)K, N+1\}$ is required for cases (a) and (b) of Lemma 3, instead of just $M - (k-1)K$ as in case (c).

- In Corollary 1, it isn't spelled out what assumptions X should satisfy, but it would seem A1 still has to hold.

- In Corollary 1, the somewhat inscrutable expression $k = \left\lceil \frac{M + \min\{0, K - \rho - 1\}}{K} \right\rceil$ can be more easily seen to be $k = \left\lceil \frac{M - \lfloor \rho \rfloor}{K} \right\rceil$.

2 Asymptotic enumeration of sparse multigraphs with given degrees

C. Greenhill and B. D. McKay, Asymptotic enumeration of sparse multigraphs with given degrees, arXiv preprint arXiv:1303.4218 (2013)

2.1 Summary

The paper introduces a generalized version of the switching method, where there are multiple different "colours" of switchings that operate on the same set of objects. By slightly loosening the bound, this generalized model can be reduced to an instance of the single-switching model.

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3 Asymptotic enumeration by degree sequence of graphs with degrees $o\left(n^{1/2}\right)$

B. D. McKay and N. C. Wormald, Asymptotic enumeration by degree sequence of graphs with degrees $o\left(n^{1/2}\right)$, Combinatorica, 11 (1991) 369-382.

3.1 Summary

(unfinished)

3.2 Remarks