Honours Readings Summaries

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1 Combinatorial estimates by the switching method

M. Hasheminezhad and B. D. McKay, Combinatorial estimates by the switching method, Contemporary Mathematics, 531 (2010) 209-221.

1.1 Summary

Consider a finite set Ω of objects. A switching is a (nondeterministic) operation that transforms one object into another (or more generally, a switching is a relation $R \subseteq \Omega \times \Omega$). We can partition Ω into subsets $\{C(v)\}_{v \in V}$ and put a directed graph structure (possibly with loops) on the index set V: if an element in $C(v)$ can switch to an element in $C(w)$ then there is an arc between v and w.

Hopefully, for each v we can find a good lower bound $a(v)$ on the number of ways an $\omega \in C(v)$ can be switched, and a good upper bound $b(v)$ on the number of switchings an $\omega \in C(v)$ can be produced by. If we imagine that all switchings are performed at once, a and b give bounds on the inflow and outflow at each vertex in terms of the size of each $C(v)$. (So, $a(v)$ and $b(v)$ can more generally be bounds on the *average* number of switchings per element in $C(v)$).

This gives some information about the relative sizes of the classes $C(v)$. Given a set of vertices X, let $N(X)$ be the total amount of elements in all $C(x)$, where $x \in X$. The objective of the paper is to bound $N(Y)/N(X)$ for given vertex sets X and Y.

In order to obtain bounds on $N(Y)/N(X)$, we write the constraints as a system of linear inequalities. Let $s'(v,w)$ be the (unknown) amount of switchings going from $C(v)$ to $C(w)$. For instance, for any vertex v, we have $\sum s'(vw) \geq a(v)N(v)$. A nonzero assignment of s and N satisfying the full set of inequalities is called a *feasible solution*, and a feasible solution which maximizes $N(Y)/N(X)$ is called an *optimal solution*.

The key result of the paper is to show that optimal solutions always take one of six standard forms. This is proved by a reduction of the inequalities to a linear program, and the fact that an optimal solution of a linear program always occurs at a vertex of the corresponding convex polytope. The analysis is amenable to a general bound α on the arcs of the digraph instead of the functions a, b (in that case we have $\alpha(v,w) = b(w)/a(v)$). Althoughα doesn't have a clear combinatorial interpretation, the paper suggests that different α may arise from richer information bounding the behaviour of the flow of switchings.

For a number of commonly-satisfied assumptions, the paper proves some alternative bounds for $N(Y)/N(X)$ that are slightly looser but easier to apply. In particular, a common use case is that we have some statistic on the objects in Ω and we believe that objects with a high value of that statistic make up a negligible fraction of all the objects in Ω. We would then partition the objects according to our statistic, and design a switching that tends to decrease the statistic, but by no more than some fixed amount. We would choose X to be the set of all vertices (partititions) and Y would be the set of all vertices (partitions) with a statistic value higher than some M.

1.2 Remarks

- I'm not sure why $i_{k-1} = \max\{M - (k-1)K, N+1\}$ is required for cases (a) and (b) of Lemma 3, instead of just $M - (k-1)K$ as in case (c).
- In Corollary 1, it isn't spelled out what assumptions X should satisfy, but it would seem A1 still has to hold.
- In Corollary 1, the somewhat inscrutable expression $k = \left\lceil \frac{M+\min\{0,K-\rho-1\}}{K} \right\rceil$ can be more easily seen to be $k = \left\lceil \frac{M-\lfloor\rho\rfloor}{K} \right\rceil$.