On Kahn's basis conjecture

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Abstract

In 1991, Kahn made the following conjecture. For any n-dimensional vector space V and any $n \times n$ array of n^2 bases of V, it is possible to choose a representative vector from each of these bases in such a way that the representatives from each row form a basis and the representatives from each column also form a basis. Rota's basis conjecture can be viewed as a special case of Kahn's conjecture, where for each column, all the bases in that column are the same. Recently the authors showed that in the setting of Rota's basis conjecture it is possible to find suitable representatives in $(1/2 - o(1))n$ of the rows. In this companion note we give a slight modification of our arguments which generalises this result to the setting of Kahn's conjecture. Our results also apply to the more general setting of matroids.

1 Introduction

This note should be considered as a companion note to the paper [1]. Here we fill in the details of how to modify our proof given in [1] to also apply to the setting of Kahn's conjecture [2]. The proof presented here is mostly self-contained but we refer the interested reader to [1] for motivation and a more detailed take on the proof.

Matroids are objects that abstract the combinatorial properties of linear independence in vector spaces. Specifically, a finite matroid $M = (E, \mathcal{I})$ consists of a finite ground set E (whose elements may be thought of as vectors in a vector space), and a collection $\mathcal{I}$ of subsets of E, called independent sets. The defining properties of a matroid are that:

- the empty set is independent (that is, $\emptyset \in \mathcal{I}$);

- subsets of independent sets are independent (that is, if $A' \subseteq A \subseteq E$ and $A \in \mathcal{I}$, then $A' \in \mathcal{I}'$);

- if A and B are independent sets, and $|A| > |B|$, then an independent set can be constructed by adding an element of A to B (that is, there is $a \in A \backslash B$ such that $B \cup \{a\} \in \mathcal{I}$). This final property is called the *augmentation property*.

Observe that any finite set of elements in a vector space (over any field) naturally gives rise to a matroid, though not all matroids arise this way. A *basis* in a matroid M is a maximal independent set. By the augmentation property, all bases have the same size, and this common size is called the *rank* of M. The following conjecture is the natural matroid generalisation of Kahn's original conjecture.

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Conjecture 1.1. *Given a rank-n matroid and bases $B_{i,j}$ for each $1 \leq i,j \leq n$, there exist representatives $b_{i,j} \in B_{i,j}$ such that each $\{b_{1,j}, \ldots, b_{n,j}\}$ and each $\{b_{i,1}, \ldots, b_{i,n}\}$ are bases.*

Theorem 1.2. *For any $\varepsilon > 0$ the following holds for sufficiently large n. Given a rank-n matroid and bases $B_{i,j}$ for each $1 \leq i \leq n$ and $1 \leq j \leq f = (1-\varepsilon)n/2$, there exist representatives $b_{i,j} \in B_{i,j}$ and $L \subseteq \{1, \ldots, f\}$ such that each $\{b_{i,j} : i \in L\}$ is independent, and such that $\{b_{i,1}, \ldots, b_{i,n}\}$ is a basis for any $i \in L$ and $|L| \geq (1/2 - \varepsilon)n$.*

Note that if we are in the setting of Conjecture 1.1 where bases are given for all $1 \leq i,j \leq n$ then the above theorem allows us to choose roughly which rows we would like to find our bases in.
Notation. We will frequently want to denote the result of adding and removing single elements from a set. For a set S and some $x \notin S$, $y \in S$, we write $S + x$ to mean $S \cup \{x\}$, and we write $S - y$ to mean $S \setminus \{y\}$.

2 Finding many row bases

In this section we prove Theorem 1.2.

Let $t = f - \varepsilon n/2 = (1/2 - \varepsilon)n$. Let T be an $f \times n$ table partially filled with matroid elements coming from the associated bases. We denote by $T(i,j) \in B_{i,j}$ the element in cell (i,j), with $T(i,j) = \emptyset$ denoting that it is empty. We start with T empty and in each iteration increase the number of values entered in T while preserving the independence of the elements in each row and column. We denote by S_i the set of all current entries of T in row i, and by C_j the set of all current entries in column j.

2.1 Simple swaps

Let U denote the set of currently non-empty positions in T. Our objective is to increase the size of U for as long as we don't have t full rows. If $T(i,j) = \emptyset$ and there is an $x \in B_{i,j}$ independent to S_i and to C_j then we can let $T(i,j) = x$.

We will want much more freedom than this: we also want to consider those elements that can be added to T after making a small change to the current entries of T. This motivates the following definition.

Definition 2.1. Let $T(i,b) = \emptyset$. Say an element $x \in B_{i,c}$ is (i,b)-*addable* if either

- $T(i,c) = \emptyset$ and both $S_i + x$ and $C_c + x$ are independent, or;

- $T(i,c) = x'$ and there is $y \in B_{i,b}$ such that $C_b + y$ and $S_i - x' + y + x$ are independent.

In the second case, we say that $S_i - x' + y$ is the result of applying a *simple swap* to S_i, and we say y is a *witness* for the (i,b)-addability of x. In the first case, there is no witness for the addability of x.

Note that if for some S_i missing an element in column b there is an (i,b)-addable element $x \in B_{i,c}$ which is also independent from C_c, then we can increase the size of U by setting $T(i,c) = x$, possibly after applying a simple swap to S_i. This unfortunately might not always be possible, but a simple swap still allows us to change which column of T is missing an entry in row i.

With this in mind, we study which elements of S can be used in a simple swap.

Definition 2.2. Let $T(i,b) = \emptyset$. We say that a column c with $T(i,c) \neq \emptyset$ is (i,b)-*swappable* if there is a simple swap making $T(i,c) = \emptyset$ and $T(i,b) = y$ in such a way that $C_b + y$ and $S_i + y - x$ are independent. We say that y is a witness for the (i,b)-swappability of c.

(Basically, a column is (i, b)-swappable if in row i we can add an element to column b and remove one from column c while preserving independence of C_b and S_i.)

Claim 2.3. *Suppose $T(i, b) = \emptyset$. We can either directly increase the size of U or there are at least $n - |C_b|$ columns which are (i, b)-swappable.*

Proof. By the augmentation property there is a set $I \subseteq B_{i,b}$ of at least $n - |C_b|$ elements which are independent to C_b. If $|S_i| < n - |C_b|$ then again by the augmentation property applied to I and S_i there is an element $x \in B_{i,b}$ which is also independent to S_i. Therefore, setting $T(i, b) = x$ we increase the size of U. Suppose now that $|S_i| \geq n - |C_b|$. Let $S \subseteq S_i$ be the set of all elements of S_i which are in an (i, b)-swappable column. Let us assume for the sake of contradiction that $|S| < n - |C_b|$. This implies $|S| < |S_i|$. As $|I| \geq n - |C_b|$ there is a $y \in I$ such that $S + y$ is independent. Using the augmentation property, we can add $|S_i - S| - 1$ elements of $S_i - S$ to $S + y$ to obtain an independent set $S_i + y - x'$ for some $x' \in S - S'$ with $T(i, c) = x'$. But this means c is (i, b)-swappable, which is a contradiction. $\qquad\square$

Now we show that if c is (i, b)-swappable, then all elements of $B_{i,c}$ which are independent to S_i are (i, b)-addable, unless there is an (i, b)-addable element not in F.

Claim 2.4. *Suppose $T(i, b) = \emptyset$, and let column c be (i, b)-swappable with witness y. Either $S_i + y$ is independent, or for any $x \in B_{i,c}$ independent of S_i, x is (i, b)-addable.*

Proof. Let $T(i, c) = x'$. Consider some $x \in B_{i,c}$ independent to S_i. Let $I = S_i + x$ and $J = S_i + y - x'$. Note that J is independent by the (i, b)-swappability of c so by the augmentation property, there is an element of $I \backslash J$ that is independent of J; this element is either x' or x. In the former case $S_i + y$ is independent. In the latter case, $S_i + y - x' + x$ is independent, showing that x is (i, b)-addable. $\quad\square$

Note that if the first case of Claim 2.4 occurs, it means we can set $T(i, b) = y$ and increase the size of U, while preserving the independence conditions. The above lemma is telling us that either we can increase the size of U or there are many elements which we can choose as $T(i, c)$ while preserving independence of S_i. Potentially these choices could cause C_c to be dependent, however this would mean that we can remove some other entry in column c. While this will not increase the number of entries in U it offers us a way to "move" an empty cell from (i, b) to a position in column c. This motivates the following definition, which was essentially implicit in [1] (in the general case of Kahn's basis conjecture it is more convenient to make this definition explicit).

Definition 2.5. Let $T(i, b) = \emptyset$ and $T(j, c) = y' \neq \emptyset$. We say that position (j, c) is (i, b)-*removable* if there is an (i, b)-addable element $x \in B_{i,c}$ such that $C_c + x - y'$ is independent.

Claim 2.6. *Let $T(i, b) = \emptyset$. If there are r distinct (i, b)-addable elements in $B_{i,c}$ then either one such element is independent of C_c or there are at least r positions in column c which are (i, b)-removable.*

Proof. Let X denote the set of (i, b)-addable elements in $B_{i,c}$, so that $|X| = r$. If $|S| = r > |C_c|$ then by the augmentation property there is an element $x \in S$ which is independent of C_c, so let us assume $r \leq |C_c|$. Once again the augmentation property implies that there is a set $Y \subseteq C_c$ such that $|Y| = r$ and $I = X + C_c - Y$ is independent. For any $y \in Y$, taking $J = C_c - y$ and using the augmentation property, there is an element $x \in I \setminus J$ such that $C_c - y + x$ is independent, showing that the position of y is removable. $\qquad\square$

Note again that if the first case of Claim 2.6 occurs, meaning that there is an (i, b)-addable element $x \in B_{i,c}$ which is independent of C_c, then we can increase the size of U by letting $T(i, c) = x$, possibly after a simple swap.

The following claim gives a good illustration of how to use the ideas developed in this section to find many addable elements. It will be very useful later on.

Claim 2.7. *Let $T(i,b) = \emptyset$. Then either we can increase the size of U or there are at least $(n - |S_i|)(n - |C_b|)$ elements which are (i,b)-removable.*

Proof. By Claim 2.3 there are at least $n - |C_b|$ columns that are (i,b)-swappable. For each such column c, by the augmentation property, there are at least $n - |S_i|$ elements $x \in B_{i,c}$ independent to all the elements of S_i. Each of these elements is (i,b)-addable by Claim 2.4, or else we can increase the size of U directly. For each such addable element, by Claim 2.6 we can again either increase the size of U directly or there are $n - |S_i|$ positions which are (i,b)-removable. That is to say, there are at least $(n - |S_i|)(n - |C_b|)$ positions which are (i,b)-removable, as claimed. $\square$

In our proof of Theorem 1.2 we will need the following lemma. It will allow us to ensure that the new removable positions we find which are in the same column are still distinct.

Lemma 2.8. *Then for each $B_{i,c}$, we can find an injection $\phi_{i,c} : S_i \to B_{i,c}$ such that for all $x \in S_i$, $\phi_{i,c}(x)$ is independent of $S_i - x$.*

Proof. Consider the bipartite graph G where the first part consists of the elements of S_i and the second part consists of the elements of $B_{i,c}$, with an edge between $x \in S_i$ and $y \in B_{i,c}$ if y is independent of $S_i - x$. We use Hall's theorem to show that there is a matching in this bipartite graph covering S_i. Indeed, consider some $W \subseteq S_i$. By the augmentation property, there are at least $|W|$ elements $y \in B_{i,c}$ such that $S_i - W + y$ is an independent set, and again using the augmentation property, each of these can be extended to an independent set of the form $S_i + y - x$ for some $x \in W$. That is to say, W has at least $|W|$ neighbours in G. $\square$

2.2 Cascading swaps

Informally speaking, for any i_0 for which S_{i_0} is not a basis, we have showed that either we can increase the size of U, or there are many positions $(i_1, c_1) \in U$ which we can free up after performing a simple swap. In the latter case, we find a row with as many removable elements as possible and then repeat the argument starting from this row, either finding a way to increase the size of U or finding more positions that we can free up after a sequence of two swaps. We then iterate this argument, continually increasing the number of positions we can free up. This cannot continue for too long, and eventually we will find a way to increase the size of U via a cascading sequence of swaps, as desired.

The next definition makes precise the cascades that we consider. We remark that the definitions of addability and removability are with respect to a table T, and it makes sense to say that an element or position is addable or removable in a different table T'.

Definition 2.9. Consider a sequence of distinct rows $i_0, \ldots, i_{\ell-1}, i_\ell$. Say a position (i_ℓ, c_ℓ) is *cascade-removable* with state T_ℓ, with respect to $i_0, \ldots, i_{\ell-1}$, if

- $\ell = 0$ and $T(i_0, c_0) = \emptyset$, and $T_0 := T$, or;

- $\ell > 0$, and there is a position $(i_{\ell-1}, c_{\ell-1})$ which is cascade-removable with state $T_{\ell-1}$, with respect to $i_0, \ldots, i_{\ell-1}$, such that (i_ℓ, c_ℓ) is $(i_{\ell-1}, c_{\ell-1})$-removable in $T_{\ell-1}$. Moreover T_ℓ is the result of performing this removal in $T_{\ell-1}$.

We will call the sequence of operations resulting in the removal of (i_ℓ, c_ℓ) in T_ℓ a *cascade*, and we require that there is a cascade in which all the columns c_i are distinct. We write $Q(i_0, \ldots, i_{\ell-1})$ for the set of all elements outside $i_0, \ldots, i_{\ell-1}$ which are cascade-removable with respect to $i_0, \ldots, i_{\ell-1}$.

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We remark that this notion is almost the same as the notion of cascade-addability defined in [1]. The only difference is that the notion of cascade-addability essentially allows for the possibility that a new element can be directly added into T, increasing the size of U. In the general case of Kahn's basis conjecture, it is more convenient to first consider cascades that can free up positions (keeping the size of U constant), then separately consider the ways to add an element after this operation.

In the next claim, we show that given $i_0, \ldots, i_{\ell-1}$, we can either increase the size of U or it is possible to choose i_ℓ in such a way that the number of cascade-removable elements increases.

Claim 2.10. *Consider a sequence of distinct rows $i_0, \ldots, i_{\ell-1}$ with $1 \le \ell < f$ (recall that f denotes the number of rows of T). Then either we can increase the size of U, or we can choose $i_\ell \neq i_0, \ldots, i_{\ell-1}$ such that*

$$|Q(i_0, \ldots, i_\ell)| \ge \frac{|Q(i_0, \ldots, i_{\ell-1})|}{f - \ell} \cdot (n - f - \ell) - (\ell + 1)n. \tag{1}$$

Proof. Let us choose $i_\ell \in [f] \setminus \{i_0, \ldots i_{\ell-1}\}$ with the maximum number of elements of $Q(i_0, \ldots, i_{\ell-1})$. Let $Q = i_\ell \cap Q(i_0, \ldots, i_{\ell-1})$, so

$$|Q| \ge \frac{|Q(S_0, \ldots, S_{\ell-1})|}{f - \ell}.$$

Apply Lemma 2.8 to row i_ℓ to obtain an injection $\phi_{i_\ell, b}$ for every column b. We suppress the dependence on i_ℓ and just write ϕ_b.

First we will show that unless we can increase the size of U there are at least $|Q|(n - \ell - f)$ cascade-removable positions (with respect to $i_0, \ldots, i_\ell$) which are not in any of the rows $i_0, \ldots, i_\ell$. We start by showing that for any $(i_\ell, c_\ell) \in Q$ (which is, by the definition of Q, cascade-removable with respect to $i_0, \ldots, i_{\ell-1}$, say with state $T_{\ell-1}$), there are $n - f$ columns c for which $\phi_c(i_\ell, c_\ell)$ is (i_ℓ, c_ℓ)-addable in $T_{\ell-1}$. This follows from Claim 2.3, which implies there are at least $n - |C_{c_\ell}| \ge n - f$ columns which are (i_ℓ, c_ℓ)-swappable, and Claim 2.4, which implies that for each such column c, $\phi_c((i_\ell, c_\ell))$ is indeed (i_ℓ, c_ℓ)-addable. If any of these elements does not appear in T we can add it (possibly after a simple swap) after performing a cascade to obtain $T_{\ell-1}$, thereby increasing the size of U. So, we can assume that all $(n - f)$ elements of the form $\phi_c((i_\ell, c_\ell))$ appear in T. Now, the cascade that removes (i_ℓ, c_ℓ) affects at most ℓ columns apart from c_ℓ, so the remaining columns are equal in T and T_ℓ. Ignoring the affected columns, we have found at least $(n - f - \ell)$ positions (at locations of) $\phi_c((i_\ell, c_\ell))$ which are cascade-removable with respect to $i_0, \ldots, i_\ell$, unless they are in the rows $i_0, \ldots, i_\ell$. Considering all $(i_\ell, c_\ell) \in Q$ gives $|Q|(n - f - \ell)$ such positions, at most $(\ell + 1)n$ of which are in the rows $i_0, \ldots, i_\ell$. $\square$

Now, we want to iteratively apply Claim 2.10 starting from some row i_0, to obtain a sequence $i_0, i_1, \ldots, i_h$. There are two ways this process can stop: either we find a way to increase the size of U, in which case we are done, or else we run out of distinct rows (that is, $h = f - 1$). We want to show that this latter possibility cannot occur by deducing from (1) that the $|Q(i_0, \ldots, i_\ell)|$ increase in size at an exponential rate: after logarithmically many steps there will be so many cascade-removable positions that they cannot all be contained in U, and it must be possible to increase the size of U.

A slight snag with this plan is that (1) only yields an exponentially growing recurrence if the "initial term" is rather large. To be precise, let C (depending on ε) be sufficiently large such that

$$C(1 + \varepsilon/2)^{\ell-1} \frac{1}{1 - \varepsilon} - \ell - 1 \ge C(1 + \varepsilon/2)^\ell \tag{2}$$

for all $\ell \ge 1$.

Claim 2.11. *For $S_0, \ldots, S_h$ as above, suppose that $|Q(S_0)| \geq Cn$ or $|Q(S_0, S_1)| \geq Cn$. Then, for $0 < \ell \leq \min\{h, \varepsilon n/4\}$, we have*

$$|Q(S_0, \ldots, S_\ell)| \geq C(1 + \varepsilon/2)^{\ell-1}n.$$

Proof. Let $Q_\ell = Q(i_0, \ldots, i_\ell)$. We proceed by induction. First observe that if $|Q_0| \geq Cn$ then (1) and (2) for $\ell = 1$ imply $|Q_1| \geq Cn$, giving us the base case. If $|Q_\ell| \geq C(1 + \varepsilon/2)^{\ell-1}n$ then

$$
\begin{aligned}
|Q_{\ell+1}| &\geq \frac{C(1 + \varepsilon/2)^{\ell-1}n}{f - \ell} \cdot (n - f - \ell) - (\ell + 1)n \\
&\geq \left(C(1 + \varepsilon/2)^{\ell-1} \frac{(n - f - \ell)}{f} - \ell - 1 \right) n \\
&\geq \left(C(1 + \varepsilon/2)^{\ell-1} \frac{1}{1 - \varepsilon} - \ell - 1 \right) n \\
&\geq C(1 + \varepsilon/2)^{\ell}n. \qquad \square
\end{aligned}
$$

If we could choose i_0, i_1 such that $|Q(i_0)| \geq Cn$ or $|Q(i_0, i_1)| \geq Cn$, then we would be done, as in [1]. There may not exist suitable starting rows $i_0, i_1 \in \mathcal{S}$, but in the next section we will show that if at least $\varepsilon n/2$ of the S_i in S are not bases, then it is possible to modify T without changing the size of U, in such a way that suitable i_0, i_1 exist.

2.3 Increasing the amount of initial addable elements

Let us assume there are at least $\varepsilon n/2$ S_i which are not bases. Recall the choice of C from the previous section, and let $D = 2C + 4$, so that $D(n - f - 1) - 2n \geq Cn$ for large n. We prove the following (for large n).

Claim 2.12. *We can modify T in such a way that at least one of the following holds.*

(a) *The size of U increases;*

(b) *the size of U does not change, and there is a row i_0 missing entries in at least D columns;*

(c) *the size of U does not change, and there are now distinct rows i_0, i_1 such that i_1 contains at least D elements that are (i_0, b)-removable.*

This suffices for our proof of Theorem 1.2; indeed, if row i_0 is missing entries in at least D columns, then by Claim 2.10, either we can increase the size of U or there are at least $D(n - f) \geq Cn$ elements which are (i_0, b)-removable, meaning that $|Q(i_0)| \geq Cn$. If i_1 contains at least D elements that are (i_0, b)-addable, then in the proof of (1) with $\ell = 1$ we have $|Q| \geq D$ so either we can increase the size of U or $|Q(i_0, i_1)| \geq D(n - f - 1) - 2n \geq Cn$.

Before proceeding to the proof of Claim 2.12, we first observe that using Lemma 2.8 we can modify T to ensure that every row i that is not a basis can be assigned a distinct column b_i with $T(i, b_i) = \emptyset$. To show this, we prove the following lemma.

Lemma 2.13. *We can modify T in such a way that the size of each S_i remains the same, and in such a way that there is a choice of disjoint columns $\{b_1, \ldots, b_f\}$ for which any S_i that is not a basis has no element in column b_i.*

Proof. Suppose for some i that we found distinct columns $b_1, \ldots, b_{i-1}$ such that, for all S_j which are not bases, no element of S_j is in column b_j. If S_i is a basis we choose an arbitrary unused column as b_i. Otherwise there is a column, say c, such that $T(i, c)$ is empty. Then by Claim 2.3 there are at least $n - |C_c| \geq n - f \geq n/2$ columns which are are (i, c)-swappable. At least one of these columns does not appear in $\{b_1, \ldots, b_{i-1}\}$, since $i - 1 < f \leq n/2$. Let b be such a column and set $b_i = b$. By performing a simple swap we can modify S_i in such a way that the cell (i, b) becomes empty, while preserving the independence conditions. $\qquad\square$

Now we prove Claim 2.12.

Proof of Claim 2.12. Recall that we are assuming there are at least $\varepsilon n/2$ S_i that are not bases. Let E be the largest integer such that there are at least $M_E = \left(\varepsilon/\left(4D^2\right)\right)^E n$ S_i missing entries in at least E columns. We may assume $1 \leq E < D$. By Lemma 2.13 we may find distinct columns $b_1, \ldots, b_f$ such that each S_i which is not a basis has $T(i, b_i) = \emptyset$. We describe a procedure that modifies T to increase E.

We create an auxiliary digraph G on the vertex set $[f]$ as follows. For every i such that S_i is missing entries in at least E columns, put an arc to i from every j such that S_j contains at least $E + 1$ elements that are $(i, b(i))$-removable.

Say an $(E + 1)$-*out-star* in a digraph is a set of $E + 1$ arcs directed away from a single vertex. Exactly the same proof as in [1] shows that there are M_{E+1} vertex-disjoint $(E + 1)$-out-stars. Now, consider an $(E + 1)$-out-star (with centre S_j, say). We show how to transfer $E + 1$ elements from S_j to its out-neighbours, the end result of which is that S_j is then missing entries in $E + 1$ columns. We will then be able to repeat this process for each of our out-stars.

For each of the $E + 1$ out-neighbours S_i of S_j there are at least $E + 1$ positions of S_j which are (i, b_i)-removable. Therefore, for each such S_i we can make a specific choice of such a position, in such a way that each of these $E + 1$ choices are *distinct*. For each S_i we can then remove the chosen element from S_j by increasing the size of S_i. These modifications will not create any conflicts, because any addability witness for any element in S_0 is in a column unique to that S_i (by Lemma 2.13). After this operation, S_i is now missing entries in at least $E + 1$ columns. $\qquad\square$

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