# Conjugate Distributions for Bayesian Inference

## Beta, Bernoulli and Related Distributions

| **Beta** | $Y \sim \mathsf{Beta}(a, b)$ |
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| pdf | $p(y) = \frac{1}{B(a,b)} y^{a-1}(1-y)^{b-1}, 0 \leq y \leq 1, a > 0, b > 0$ |
| nc | $\int_0^1 y^{a-1}(1-y)^{b-1} dy \equiv B(a, b)$ |
| mean | $\mathsf{E}[Y] = \frac{a}{a+b}$ |
| variance | $\mathsf{Var}[Y] = \frac{ab}{(a+b)^2(a+b+1)}$ |
| `R> help(Beta)` | `rbeta(k, a, b)` # generate $k$ Beta rv's |
| **Bernoulli** | $Y \sim \mathsf{Ber}(\theta)$ |
| pmf | $p(Y = y \mid \theta) = \theta^y(1-\theta)^{1-y}; y = 0, 1; 0 \leq \theta \leq 1$ |
| mean | $\mathsf{E}[Y] = \theta = P(Y = 1)$ |
| variance | $\mathsf{Var}[Y] = \theta(1 = \theta)$ |
| `R> help(Binomial)` | `rbinom(k, size=1, p=.5) # generate` $k$ `Ber(.5) rv's` |
| **Updating** | |
| Data Model | $Y_i \mid \theta \overset{\text{iid}}{\sim} \mathsf{Ber}(\theta)$ for $i = 1, \ldots n$, $\mathbf{Y} \equiv (Y_1, \ldots Y_n)$ |
| Prior $p(\theta)$ | $\theta \sim \mathsf{Beta}(a_0, b_0)$ |
| Likelihood | $L(\theta; y_1, y_n) = \theta^{\sum y_i}(1-\theta)^{n-\sum y_i}$ |
| Posterior $p(\theta \mid \mathbf{Y})$ | $\theta \mid \mathbf{Y} \sim \mathsf{Beta}(a_n, b_n), \quad a_n = a_0 + \sum y_i,\ b_n = b_o + n - \sum y_i$ |
| Marginal of $\mathbf{Y}$ | $p(y_1, \ldots, y_n) = \frac{B(a_n, b_n)}{B(a_0, b_0)}$ |
| Notes | Reference prior $\mathsf{Beta}(1/2, 1/2)$ |

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| **Binomial** | $Y \sim \mathsf{Bin}(n, \theta)$ |
| pmf | $p(Y = y \mid \theta) = \binom{n}{y} \theta^y (1-\theta)^{n-y}; y = 0, 1 \ldots, n; 0 \le \theta \le 1$ |
| mean | $\mathsf{E}[Y] = n\theta$ |
| variance | $\mathsf{Var}[Y] = n\theta(1-\theta)$ |
| `R> help(Binomial)` | `rbinom(k, size=n, p=.5) # generate` $k$ `Bin(n,.5) rv's` |
| **Updating** | |
| Data Model | $Y \mid \theta \sim \mathsf{Bin}(n, \theta)$ |
| Prior $p(\theta)$ | $\theta \sim \mathsf{Beta}(a_0, b_0)$ |
| Likelihood | $L(\theta; y, n) = \binom{n}{y} \theta^y (1-\theta)^{n-y}$ |
| Posterior $p(\theta \mid y)$ | $\theta \mid y \sim \mathsf{Beta}(a_n, b_n), \quad a_n = a_0 + y,\ b_n = b_o + n - y$ |
| Marginal of $Y$ | $p(y) = \frac{\binom{n}{y}\ B(a_n, b_n)}{B(a_0, b_0)}$ |
| **Negative Binomial** | $Y \sim \mathsf{NegBin}(r, \theta)$ |
| pmf | $p(Y = y \mid \theta) = \binom{r+y-1}{y} \theta^r (1-\theta)^y; r = 0, 1 \ldots; 0 \le \theta \le 1$ |
| mean | $\mathsf{E}[Y] = \frac{r(1-\theta)}{\theta}$ |
| variance | $\mathsf{Var}[Y] = \frac{r(1-\theta)}{\theta^2}$ |
| `R> help(NegBinomial)` | `rnbinom(k,r, p=.5) # generate` $k$ `NegBin(r,.5) rv's` |
| Notes | Y = Number of Bernoulli failures before the $r$th success |
| **Updating** | |
| Data Model | $X \mid \theta \sim \mathsf{NegBin}(y, \theta)\ n \equiv X + y$ |
| Prior $p(\theta)$ | $\theta \sim \mathsf{Beta}(a_0, b_0)$ |
| Likelihood | $L(\theta; x, y) \propto \theta^y (1-\theta)^x$ |
| Posterior $p(\theta \mid x, y)$ | $\theta \mid x \sim \mathsf{Beta}(a_n, b_n), \quad a_n = a_0 + y,\ b_n = b_o + x$ |
| Notes | Same posterior for $\theta$ as with Binomial sampling $x = n - y$ |

## Normal/Gamma Models

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| **Normal** | $Y \sim \mathsf{N}(\mu, 1/\phi)$ |
| pdf | $p(y) = \sqrt{\frac{\phi}{2\pi}} \exp\{-\frac{1}{2}\phi(Y-\mu)^2\},\ y \in \mathbb{R}, \mu \in \mathbb{R},\ \phi > 0$ |
| mean | $\mathsf{E}[Y] = \mu$ |
| variance | $\mathsf{Var}[Y] = 1/\phi$ |
| `R> help(Normal)` | `rnorm(k, 0, 10)` # generate $k$ $N(0, 10^2)$ rv's |
| Notes | $\phi$ is the precision |
| **Gamma** | $Y \sim \mathsf{Gamma}(\alpha, \beta)$ |
| pdf | $p(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} \exp\{-y\beta\},\ y > 0,\ \alpha > 0,\ \beta > 0$ |
| mean | $\mathsf{E}[Y] = \alpha/\beta$ |
| variance | $\mathsf{Var}[Y] = \alpha/\beta^2$ |
| `R> help(rgamma)` | `rgamma(k, 2, rate=3)` # generate $k$ $\mathsf{Gamma}(2, 3)$ rv's |
| Notes: | Special cases: Exponential $\alpha = 1$ and |
| | Chi-Squared with $d$ degrees of freedon $\alpha = d/2$, $\beta = 1/2$ |
| **Student-t** | $Y \sim \mathsf{St}(d, m, s)$, for $d > 0, m \in \mathbb{R}, s > 0$ |
| pdf | $p(y) = \frac{\Gamma((d+1)/2)}{\Gamma(d/2)\Gamma(1/2)} (ds)^{-1/2} \left[1 + \frac{(y-m)^2}{ds}\right]^{-(d+1)/2}$ |
| mean | $\mathsf{E}[Y] = m$, if $d > 1$ |
| variance | $\mathsf{Var}[Y] = sd/(d-2)$, if $d > 2$ |
| `R> help(rt)` | `rt(k, 1, rate=3)` # generate $k$ $\mathsf{St}(1, 0, 1)$ rv's |
| Notes: | Special cases: Cauchy $d = 1$ |
| | if $\mu \sim \mathsf{St}(d, m, s)$ then $(\mu - m)/\sqrt{s} \sim \mathsf{St}(1, 0, 1)$ |
| | if $\mu \mid \phi \sim N(m, s/\phi)$, $\phi \sim \mathsf{Gamma}(d/2, 1/2)$ then $\mu \sim \mathsf{St}(d, m, s)$ |

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| **Congugate Updating** | $\mathbf{Y} \equiv (Y_1, \dots Y_n), \bar{Y} = \frac{1}{n}\sum_i^n Y_i, S^2 = \sum_i^n (Y_i - \bar{Y})^2/(n-1)$ |
| Data | $Y_i \mid \mu, \phi \overset{\text{iid}}{\sim} \mathsf{N}(\mu, 1/\phi)$ |
| Prior | $\mu \mid \phi \sim \mathsf{N}(m_0, 1/(n_0\phi))$ |
| | $\phi \sim \mathsf{Gamma}(v_0/2, v_0 S_0^2/2)$ |
| Likelihood | $L(\mu, \phi) = (2\pi)^{-n/2}\phi^{n/2} \exp\left(-\frac{\phi}{2}\{n(\bar{Y} - \mu)^2 + S^2(n-1)\}\right.$ |
| Posterior | $n_n = n + n_0,\ m_n = (n\bar{Y} + n_0 m_0)/n_n,$ |
| | $v_n = n + v_0,\ S_n^2 = (v_0 S_0^2 + (n-1)S^2 + \frac{nn_0}{n_n}(\bar{Y} - m_0)^2)/v_n$ |
| | $\mu \mid \phi, \mathbf{Y} \sim N(m_n, \frac{1}{\phi n_n}),$ |
| | $\phi \mid \mathbf{Y} \sim \mathsf{Gamma}(v_n/2, v_n S_n^2/2),$ |
| | $\mu \mid \mathbf{Y} \sim \mathsf{St}(v_n, m_n, S_n^2/n_n)$ |
| Reference Analysis | |
| Prior | $p(\mu, \phi) \propto 1/\phi$ |
| Posterior | |
| | $\mu \mid \phi, \mathbf{Y} \sim N(\bar{Y}, \frac{1}{\phi n}),$ |
| | $\phi \mid \mathbf{Y} \sim \mathsf{Gamma}((n-1)/2, (n-1)S^2/2),$ |
| | $\mu \mid \mathbf{Y} \sim \mathsf{St}(n-1, \bar{Y}, S^2/n)$ |