Caterpillar Regression Example: Conjugate Priors, Conditional & Marginal Posteriors, Predictive Distribution, Variable Selection

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C. P. Robert, *The Bayesian Core*, *Springer, 2nd edition,* chapter 3 (full text available)

Regression

- Regression refers to statistical analysis that deals with the representation of dependencies between several variables.

- In particular, we want to find a representation of the distribution $f(y|\theta,\boldsymbol{x})$ of an observable variable y given a vector of observables $\boldsymbol{x}$, using samples of $(\boldsymbol{x}_i,y_i)$, $i=1,\ldots,n$.

- Here, we will consider a particular example of modeling dependencies in pine processionary caterpillar colony size.

Linear Regression Models

- In linear regression, we analyze the linear influence of some variables on others.

- In our particular example of pine processionary caterpillar colonies,

  - Response Variable: y is the number of processionary caterpillar colonies

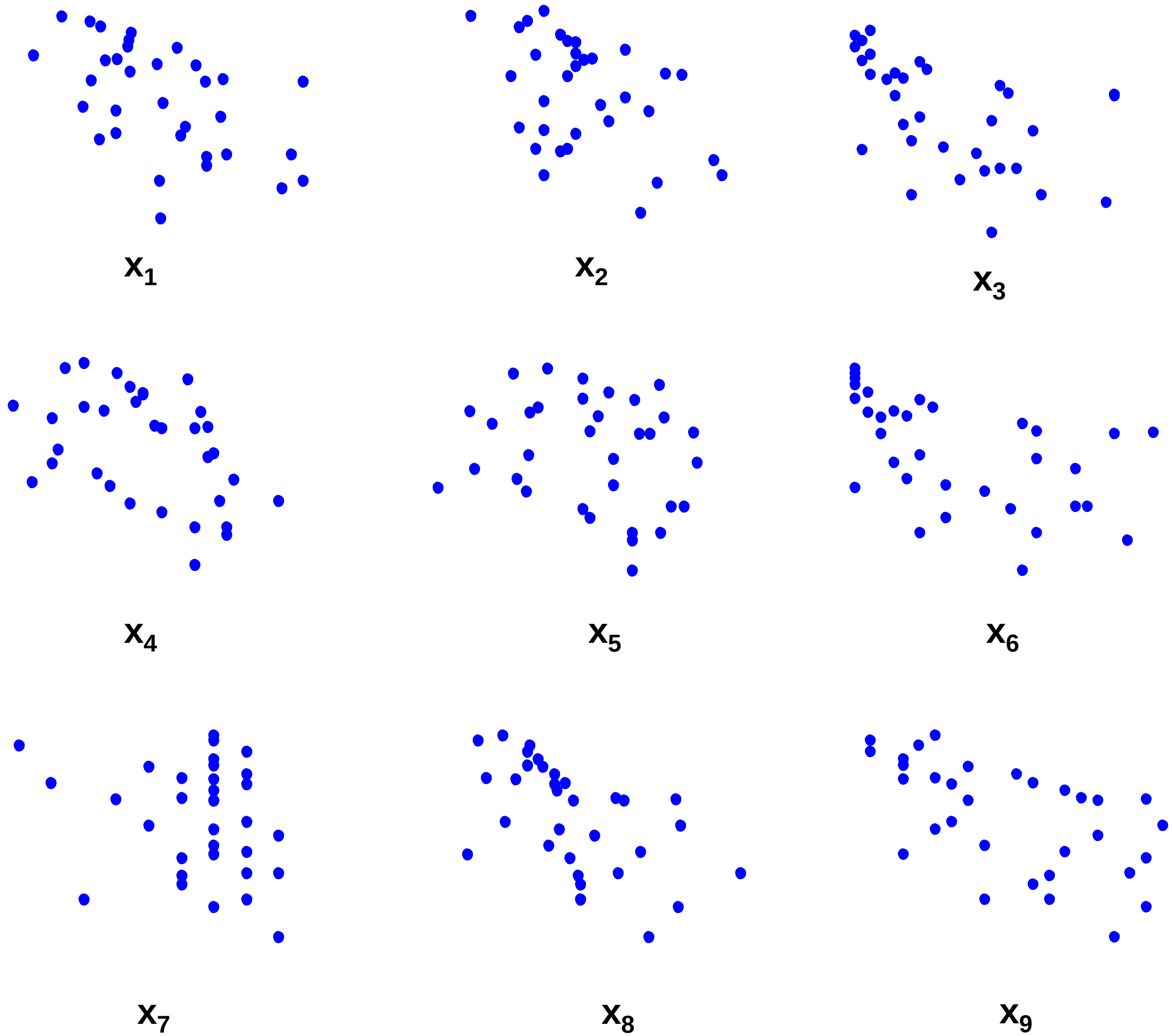
  - Explanatory Variables: Covariates $\boldsymbol{x}=(x_1,x_2,\ldots,x_k)$ as defined next (in general can be continuous, discrete or mixed type)

Caterpillar Regression Problem

The pine processionary caterpillar colony size (y=number of nests) is influenced by the following explanatory variables:

- x_1 is the altitude (in meters)
- x_2 is the slope (in degrees)
- x_3 is the number of pines in the square
- x_4 is the height (in meters) of the tree sampled at the center of the square
- x_5 is the diameter of the tree sampled at the center of the square
- x_6 is the index of the settlement density
- x_7 is the orientation of the square (from 1 if southbound to 2 otherwise)
- x_8 is the height (in meters) of the dominant tree
- x_9 is the number of vegetation strata
- x_{10} is the mix settlement index (from 1 if not mixed to 2 if mixed).

Caterpillar Regression Problem



Semilog-y plot of the data (x_i, y), $i=1,..,9$

An implementation is available
MatLab, C++

Regression

- The distribution of y given $\mathbf{x}$ is considered in the context of a set of experimental data, $i = 1, \ldots, n$, on which both y, and $x_{i1}, \ldots, x_{ik}$ are measured.

- The dataset is made from the outcomes $y = (y_1, \ldots, y_n)$ and of the $n \times (k+1)$ matrix of explanatory variables

$$\boldsymbol{X} = \begin{bmatrix} 1 & x_{11} & x_{12} & \ldots & x_{1k} \\ 1 & x_{21} & x_{22} & \ldots & x_{2k} \\ 1 & x_{31} & x_{32} & \ldots & x_{3k} \\ . & . & . & . & \\ . & . & . & . & . \\ . & . & . & . & . \\ 1 & x_{n1} & x_{n2} & \ldots & x_{nk} \end{bmatrix}$$

Linear Regression

- The most common linear regression model is of the form:

$$\boldsymbol{y} \mid \boldsymbol{\beta}, \sigma^2, \boldsymbol{X} \sim \mathcal{N}_n(\boldsymbol{X}\boldsymbol{\beta}, \sigma^2 \boldsymbol{I}_n)$$

- From this, we conclude that:

$$\mathbb{E}\left[y_i \mid \boldsymbol{\beta}, \boldsymbol{X} \right] = \beta_0 + \beta_1 x_{i1} + \ldots + \beta_k x_{ik}$$
$$Var\left[y_i \mid \sigma^2, \boldsymbol{X} \right] = \sigma^2$$

- Note the difference between finite valued regressors x_i (like in our caterpillar problem) and categorical variables which also take finite number of values but with range that has no numerical meaning.

- Makes no sense to involve x directly in the regression: replace the single regressor x [belonging in {1, . . . ,m}, say] with m indicator (or dummy) variables.

Categorical Variables

- Note the difference between finite valued regressors x_i (like in our caterpillar problem) and categorical variables which also take finite number of values but with range that has no numerical meaning.

- Makes no sense to use x directly in the regression: replace the single regressor x [in $\{1, . . . ,m\}$, say] with m indicator variables

$$x_1 = \mathbb{I}_1(x),\, x_2 = \mathbb{I}_2(x), \ldots, x_m = \mathbb{I}_m(x)$$

- Use different β_i for each class categorical variable value:

$$\mathbb{E}\left[y_i \mid \boldsymbol{\beta}, \boldsymbol{X}\right] = \ldots + \beta_1 \mathbb{I}_1(x) + \ldots + \beta_m \mathbb{I}_m(x) + \ldots$$

- *Identifiability* requires eliminating one of the classes (e.g. $\beta_1=0$) since $\sum_i \mathbb{I}_1(x) = 1$

Linear Regression

- Returning back to our linear regression model:

$$\boldsymbol{y} \mid \boldsymbol{\beta}, \sigma^2, \boldsymbol{X} \sim \mathcal{N}_n(\boldsymbol{X}\boldsymbol{\beta}, \sigma^2 \boldsymbol{I}_n)$$

- From this, we conclude that:

$$\mathbb{E}\left[y_i \mid \boldsymbol{\beta}, \boldsymbol{X} \right] = \beta_0 + \beta_1 x_{i1} + \ldots + \beta_k x_{ik}$$

$$Var\left[y_i \mid \sigma^2, \boldsymbol{X} \right] = \sigma^2$$

- Assume that $k+1<n$ and that $\boldsymbol{X}$ is of full rank: $rank(\boldsymbol{X}) = k + 1$
  - $\boldsymbol{X}$ is of full rank if and only if $\boldsymbol{X}^T \boldsymbol{X}$ is invertible

- The likelihood is then:

$$\ell(\boldsymbol{\beta}, \sigma^2 \mid \boldsymbol{y}, \boldsymbol{X}) = \frac{1}{\left(2\pi\sigma^2\right)^{n/2}} \exp\left(-\frac{1}{2\sigma^2} (\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta})^T (\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta}) \right)$$

MLE Estimator

- The MLE of β is the solution of the least squares minimization problem

$$\min_{\beta}(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta})^T(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta}) = \min_{\beta}\sum_{i=1}^{n}\left(y_i-\beta_0-\beta_1 x_{i1}-\ldots-\beta_k x_{ik}\right)^2 \Rightarrow \boxed{\widehat{\boldsymbol{\beta}} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{y}}$$

- Since $\widehat{\boldsymbol{\beta}} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{y}$ is a linear transform of $\boldsymbol{y} \sim \mathcal{N}_n(\boldsymbol{X}\boldsymbol{\beta}, \sigma^2\boldsymbol{I}_n)$

$$\widehat{\boldsymbol{\beta}} \sim \mathcal{N}\left((\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{X}\boldsymbol{\beta}, (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\sigma^2\boldsymbol{I}_n\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\right) = \mathcal{N}(\boldsymbol{\beta}, \sigma^2(\boldsymbol{X}^T\boldsymbol{X})^{-1})$$

- Thus $\widehat{\boldsymbol{\beta}}$ is an unbiased estimator and $\boxed{Var\left(\widehat{\boldsymbol{\beta}}|\sigma^2, \boldsymbol{X}\right) = \sigma^2(\boldsymbol{X}^T\boldsymbol{X})^{-1}}$

- Also $\widehat{\boldsymbol{\beta}}$ is the best linear unbiased estimator of β:

$$\boldsymbol{\alpha} \in \mathbb{R}^{k+1}, Var\left(\boldsymbol{\alpha}^T\widehat{\boldsymbol{\beta}}|\sigma^2, \boldsymbol{X}\right) \leq Var\left(\boldsymbol{\alpha}^T\widetilde{\boldsymbol{\beta}}|\sigma^2, \boldsymbol{X}\right) \textit{ where } \widetilde{\boldsymbol{\beta}} \textit{ is any unbiased linear estimator of } \boldsymbol{\beta}$$

MLE Estimator

- The MLE of σ^2 is the solution of:

$$\min_{\sigma^2} \frac{1}{2\sigma^2}(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta})^T(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta}) - \frac{n}{2}\ln\sigma\Big|_{\boldsymbol{\beta}=\widehat{\boldsymbol{\beta}}} = 0 \Rightarrow \boxed{\sigma_{MLE}^2 = \frac{\left(\boldsymbol{y}-\boldsymbol{X}\widehat{\boldsymbol{\beta}}\right)^T\left(\boldsymbol{y}-\boldsymbol{X}\widehat{\boldsymbol{\beta}}\right)}{n}}$$

- Let $\boldsymbol{M} = \boldsymbol{X}\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-\mathbf{1}}\boldsymbol{X}^T$ be the projection $n \times n$ matrix of the data **y** on the column space of $\boldsymbol{X}$

$$\mathbb{E}(\sigma_{MLE}^2) = \frac{\mathbb{E}\left[\left(\boldsymbol{y}-\boldsymbol{X}\widehat{\boldsymbol{\beta}}\right)^T\left(\boldsymbol{y}-\boldsymbol{X}\widehat{\boldsymbol{\beta}}\right)\right]}{n} = \frac{\mathbb{E}[(\boldsymbol{y}-\boldsymbol{M}\boldsymbol{y})^T(\boldsymbol{y}-\boldsymbol{M}\boldsymbol{y})]}{n} =$$

$$= \frac{\mathbb{E}[\boldsymbol{y}^T(\boldsymbol{I}-\boldsymbol{M})^T(\boldsymbol{I}-\boldsymbol{M})\boldsymbol{y}]}{n} = \frac{\mathbb{E}[\boldsymbol{y}^T(\boldsymbol{I}-\boldsymbol{M})\boldsymbol{y}]}{n}$$

- You can easily show that (see this slide):

$$For\ \boldsymbol{Y}\ a\ n-dim\ random\ vector\ with\ \mathbb{E}\left[\boldsymbol{Y}\right] = \boldsymbol{\mu},\ Cov\left[\boldsymbol{Y}\right] = \boldsymbol{V},$$

$$and\ \boldsymbol{A}\ a\ n \times n\ matrix\ \ then: \mathbb{E}\left[\boldsymbol{Y}^T\boldsymbol{A}\boldsymbol{Y}\right] = tr\left(\boldsymbol{A}\boldsymbol{V}\right) + \boldsymbol{\mu}^T\boldsymbol{A}\boldsymbol{\mu}$$

MLE Estimator

$$\mathbb{E}\left(\sigma^2_{MLE}\right) = \frac{\mathbb{E}\left[\boldsymbol{y}^T(\boldsymbol{I}-\boldsymbol{M})\boldsymbol{y}\right]}{n}$$

- You can easily show that (see this slide):

$$For\ \boldsymbol{Y}\ a\ n-dim\ random\ vector\ with\ \mathbb{E}\left[\boldsymbol{Y}\right] = \boldsymbol{\mu}, Cov\left[\boldsymbol{Y}\right] = \boldsymbol{V},$$
$$and\ \boldsymbol{A}\ a\ n\times n\ matrix\ \ then: \mathbb{E}\left[\boldsymbol{Y}^T\boldsymbol{A}\boldsymbol{Y}\right] = tr\left(\boldsymbol{A}\boldsymbol{V}\right) + \boldsymbol{\mu}^T\boldsymbol{A}\boldsymbol{\mu}$$

- Applying this we have:

$$\mathbb{E}\left[\boldsymbol{y}^T\left(\boldsymbol{I}-\boldsymbol{M}\right)\boldsymbol{y}\right] = tr\left(\left(\boldsymbol{I}-\boldsymbol{M}\right)\overbrace{Cov(\boldsymbol{y})}^{\sigma^2\boldsymbol{I}_n}\right) + \mathbb{E}\left(\boldsymbol{y}\right)^T\left(\boldsymbol{I}-\boldsymbol{M}\right)\mathbb{E}\left(\boldsymbol{y}\right) =$$

$$= \sigma^2 rank\left(\boldsymbol{I}-\boldsymbol{M}\right) + \boldsymbol{\beta}^T\boldsymbol{X}^T\underbrace{\left(\boldsymbol{I}-\boldsymbol{M}\right)\boldsymbol{X}\boldsymbol{\beta}}_{0} = \sigma^2\left(n-k-1\right)$$

$Note: \boldsymbol{I}-\boldsymbol{M}$ *has* 1 *as its eigenvalue with multiplicity* $n-k-1$ *and*

$$tr\left(\boldsymbol{I}-\boldsymbol{M}\right) = tr\left(\boldsymbol{I}\right) - tr\left(\boldsymbol{M}\right) = n - tr\left(\boldsymbol{X}\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T\right) = n - tr\left(\left(\boldsymbol{X}^T\boldsymbol{X}\right)\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\right) = n - tr\left(\boldsymbol{I}_{k+1}\right) = n-k-1$$

MLE Estimator

- The expectation of the MLE of σ^2 is then given as:

$$\mathbb{E}\left(\sigma^2_{MLE}\right) = \frac{\sigma^2}{n}(n-k-1)$$

- The unbiased estimator of σ^2 is thus given:

$$\hat{\sigma}^2 = \frac{1}{n-k-1}\left(\boldsymbol{y}-\boldsymbol{X}\widehat{\boldsymbol{\beta}}\right)^T\left(\boldsymbol{y}-\boldsymbol{X}\widehat{\boldsymbol{\beta}}\right) = \frac{s^2}{n-k-1},\ where{:}\ s^2 \triangleq \left(\boldsymbol{y}-\boldsymbol{X}\widehat{\boldsymbol{\beta}}\right)^T\left(\boldsymbol{y}-\boldsymbol{X}\widehat{\boldsymbol{\beta}}\right)$$

- Indeed using the result from the earlier slide we have:

$$\mathbb{E}(\hat{\sigma}^2) = \frac{1}{n-k-1}\mathbb{E}\left[\left(\boldsymbol{y}-\boldsymbol{X}\widehat{\boldsymbol{\beta}}\right)^T\left(\boldsymbol{y}-\boldsymbol{X}\widehat{\boldsymbol{\beta}}\right)\right] = \frac{\sigma^2(n-k-1)}{n-k-1} = \sigma^2$$

- To approximate the covariance of $\widehat{\boldsymbol{\beta}}$, in $\widehat{\boldsymbol{\beta}} = \mathcal{N}\left(\boldsymbol{\beta}, \sigma^2(\boldsymbol{X}^T\boldsymbol{X})^{-1}\right)$ use:

$$Var\left(\widehat{\boldsymbol{\beta}}|\sigma^2,\boldsymbol{X}\right) = \hat{\sigma}^2(\boldsymbol{X}^T\boldsymbol{X})^{-1}$$

Appendix

$$\textit{For } \boldsymbol{Y} \textit{ a } n-\dim \textit{ random vector with } \mathbb{E}[\boldsymbol{Y}] = \boldsymbol{\mu}, Cov[\boldsymbol{Y}] = \boldsymbol{V},$$
$$\textit{and } \boldsymbol{A} \textit{ a } n \times n \textit{ matrix then: } \mathbb{E}\left[\boldsymbol{Y}^T \boldsymbol{A} \boldsymbol{Y}\right] = tr(\boldsymbol{A}\boldsymbol{V}) + \boldsymbol{\mu}^T \boldsymbol{A} \boldsymbol{\mu}$$

- We can prove this theorem quite easily as follows:

$$\mathbb{E}\left[(\boldsymbol{Y}-\boldsymbol{\mu})^T \boldsymbol{A}(\boldsymbol{Y}-\boldsymbol{\mu})\right] = \mathbb{E}\left[\boldsymbol{Y}^T \boldsymbol{A}\boldsymbol{Y} + \boldsymbol{\mu}^T \boldsymbol{A}\boldsymbol{\mu} - \boldsymbol{\mu}^T \boldsymbol{A}\boldsymbol{Y} - \boldsymbol{Y}^T \boldsymbol{A}\boldsymbol{\mu}\right] \Rightarrow$$

$$\mathbb{E}\left[(\boldsymbol{Y}-\boldsymbol{\mu})^T \boldsymbol{A}(\boldsymbol{Y}-\boldsymbol{\mu})\right] = \mathbb{E}\left[\boldsymbol{Y}^T \boldsymbol{A}\boldsymbol{Y}\right] + \boldsymbol{\mu}^T \boldsymbol{A}\boldsymbol{\mu} - \boldsymbol{\mu}^T \boldsymbol{A}\boldsymbol{\mu} - \boldsymbol{\mu}^T \boldsymbol{A}\boldsymbol{\mu} = \mathbb{E}\left[\boldsymbol{Y}^T \boldsymbol{A}\boldsymbol{Y}\right] - \boldsymbol{\mu}^T \boldsymbol{A}\boldsymbol{\mu}$$

- Using the identity $\mathbb{E}\left[tr(\boldsymbol{W})\right] = tr\left[\mathbb{E}(\boldsymbol{W})\right]$ for any random square matrix ***W***, we can write the l.h.s. of the equation above as:

$$\mathbb{E}\left[(\boldsymbol{Y}-\boldsymbol{\mu})^T \boldsymbol{A}(\boldsymbol{Y}-\boldsymbol{\mu})\right] = \mathbb{E}\left[tr\left((\boldsymbol{Y}-\boldsymbol{\mu})^T \boldsymbol{A}(\boldsymbol{Y}-\boldsymbol{\mu})\right)\right] = \mathbb{E}\left[tr\left(\boldsymbol{A}(\boldsymbol{Y}-\boldsymbol{\mu})(\boldsymbol{Y}-\boldsymbol{\mu})^T\right)\right] =$$
$$tr\left[\mathbb{E}\left(\boldsymbol{A}(\boldsymbol{Y}-\boldsymbol{\mu})(\boldsymbol{Y}-\boldsymbol{\mu})^T\right)\right] = tr\left[\boldsymbol{A}\mathbb{E}\left((\boldsymbol{Y}-\boldsymbol{\mu})(\boldsymbol{Y}-\boldsymbol{\mu})^T\right)\right] = tr(\boldsymbol{A}\boldsymbol{V})$$

This proves the theorem.

T-Statistic

- We can define the standard t-statistic as follows:

$$T_i = \frac{\hat{\beta}_i - \beta_i}{\sqrt{\hat{\sigma}^2 \omega_{ii}}} \sim \mathcal{T}(n-k-1,0,1),\ where\text{:}\ \omega_{(i,i)} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}|_{(i,i)}$$

- This statistic can be used for hypothesis testing, e.g.

$$accept\ H_0 : \beta_i = 0\ versus\ H_1 : \beta_i \neq 0\ at\ the\ level\ \alpha\ if$$

$$\frac{|\hat{\beta}_i|}{\sqrt{\hat{\sigma}^2 \omega_{ii}}} \leq F_{n-k-1}^{-1}(1-a/2), \text{the } (1-a/2)\text{nd quantile of } \mathcal{T}_{n-k-1}$$

T-Statistic

- The frequentist argument in using this bound is that there is significant evidence against H_0 if the p-value is smaller than α.

$$p_i = P_{H_0}\left(|T_i| > |t_i| = \frac{|\hat{\beta}_i|}{\sqrt{\hat{\sigma}^2\omega_{ii}}}\right) =$$

$$P_{H_0}(T_i < -|t_i|) + P_{H_0}(T_i > |t_i|) = F_{n-k-1}(-|t_i|) + \left(1 - F_{n-k-1}(|t_i|)\right) < \alpha$$

- Finally the statistic T_i can be used to derive the frequentist marginal confidence interval is:

$$\left\{\beta_i \colon |\beta_i - \hat{\beta}_i| \leq \sqrt{\hat{\sigma}^2\omega_{ii}}\, F_{n-k-1}^{-1}(1 - a/2)\right\}$$

MLE (Least Squares) Estimates

	$\hat{\beta}_i$	$\sqrt{\hat{\sigma}^2 \omega_{ii}}$	$t_i = \frac{\hat{\beta}_i}{\sqrt{\hat{\sigma}^2 \omega_{ii}}}$	$p_i = P_{H_0}\left(\lvert T_i \rvert > \frac{\lvert\hat{\beta}_i\rvert}{\sqrt{\hat{\sigma}^2 \omega_{ii}}}\right)$	
Coefficients	Estimate	Std.Error	t-value	Pr(>|t|)	
intercept	10.998412	3.060892	3.594	0.00161	**
XV1	-0.004431	0.001557	-2.846	0.00939	**
XV2	-0.053830	0.021904	-2.458	0.02232	*
XV3	0.067939	0.099492	0.683	0.50174	
XV4	-1.293636	0.563925	-2.294	0.03168	*
XV5	0.231637	0.104399	2.219	0.03709	*
XV6	-0.356800	1.566782	-0.228	0.82193	
XV7	-0.237469	1.006210	-0.236	0.81558	
XV8	0.181060	0.236772	0.765	0.45248	
XV9	-1.285316	0.865023	-1.486	0.15142	
XV10	-0.433106	0.735018	-0.589	0.56162	

An implementation is available MatLab, C++

Likelihood

$$\ell(\boldsymbol{\beta},\sigma^2 \mid \boldsymbol{y},\boldsymbol{X}) = \frac{1}{\left(2\pi\sigma^2\right)^{n/2}}\exp\left(-\frac{1}{2\sigma^2}(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta})^T(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta})\right)$$

- Note that the likelihood above can now be written in the following very useful form:

$$\ell(\boldsymbol{\beta},\sigma^2|\boldsymbol{y},\boldsymbol{X}) = \frac{1}{(2\pi\sigma^2)^{n/2}}\exp\left(-\frac{1}{2\sigma^2}\left(\boldsymbol{y}-\boldsymbol{X}\left((\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}})+\widehat{\boldsymbol{\beta}}\right)\right)^T\left(\boldsymbol{y}-\boldsymbol{X}\left((\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}})+\widehat{\boldsymbol{\beta}}\right)\right)\right) \Rightarrow$$

$$\ell(\boldsymbol{\beta},\sigma^2|\boldsymbol{y},\boldsymbol{X}) = \frac{1}{(2\pi\sigma^2)^{n/2}}\exp\left(-\frac{1}{2\sigma^2}\left(\boldsymbol{y}-\boldsymbol{X}\widehat{\boldsymbol{\beta}}\right)^T\left(\boldsymbol{y}-\boldsymbol{X}\widehat{\boldsymbol{\beta}}\right)-\frac{1}{2\sigma^2}\left(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}\right)^T\boldsymbol{X}^T\boldsymbol{X}\left(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}\right)\right) =$$

$$= \frac{1}{(2\pi\sigma^2)^{n/2}}\exp\left(-\frac{1}{2\sigma^2}s^2-\frac{1}{2\sigma^2}\left(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}\right)^T\boldsymbol{X}^T\boldsymbol{X}\left(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}\right)\right)$$

$$where:\ s^2 \triangleq \left(\boldsymbol{y}-\boldsymbol{X}\widehat{\boldsymbol{\beta}}\right)^T\left(\boldsymbol{y}-\boldsymbol{X}\widehat{\boldsymbol{\beta}}\right),$$
$$\widehat{\boldsymbol{\beta}} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{y}$$

Conjugate Priors

- Observing the form of the likelihood suggests the following conjugate prior:

$$\boldsymbol{\beta}|\sigma^2,\boldsymbol{X}\sim\mathcal{N}_{k+1}\left(\widetilde{\boldsymbol{\beta}},\sigma^2\boldsymbol{M}^{-1}\right),\ \boldsymbol{M}\ a\ (k+1)\times(k+1)\ pos.\ def.\ symm.\ matrix$$
$$\sigma^2|\boldsymbol{X}\sim InvGamma(a,b),\ a,b>0$$

- The posterior is then

$$\pi\left(\boldsymbol{\beta},\sigma^2|\widehat{\boldsymbol{\beta}},s^2,\boldsymbol{X}\right)\propto\frac{1}{(2\pi\sigma^2)^{n/2}}\exp\left(-\frac{1}{2\sigma^2}s^2-\frac{1}{2\sigma^2}\left(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}\right)^T\boldsymbol{X}^T\boldsymbol{X}\left(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}\right)\right)$$
$$\times\frac{1}{(2\pi\sigma^2)^{(k+1)/2}}\exp\left(-\frac{1}{2\sigma^2}\left(\boldsymbol{\beta}-\widetilde{\boldsymbol{\beta}}\right)^T\boldsymbol{M}\left(\boldsymbol{\beta}-\widetilde{\boldsymbol{\beta}}\right)\right)\times\left(\frac{1}{\sigma^2}\right)^{a+1}\exp\left(-\frac{1}{\sigma^2}b\right)=$$
$$=\sigma^{-k-1-2a-2-n}\exp\left(-\frac{1}{2\sigma^2}\left\{s^2+2b+\left(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}\right)^T\boldsymbol{X}^T\boldsymbol{X}(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}})+\left(\boldsymbol{\beta}-\widetilde{\boldsymbol{\beta}}\right)^T\boldsymbol{M}\left(\boldsymbol{\beta}-\widetilde{\boldsymbol{\beta}}\right)\right\}\right)=$$
$$=\sigma^{-k-n-2a-3}\exp\left(-\frac{1}{2\sigma^2}\left\{s^2+2b+\widehat{\boldsymbol{\beta}}^T(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})\boldsymbol{\beta}-2\boldsymbol{\beta}^T\left(\boldsymbol{M}\widetilde{\boldsymbol{\beta}}+\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}\right)+\widetilde{\boldsymbol{\beta}}^T\boldsymbol{M}\widetilde{\boldsymbol{\beta}}+\widehat{\boldsymbol{\beta}}^T(\boldsymbol{X}^T\boldsymbol{X})\widehat{\boldsymbol{\beta}}\right\}\right)$$

Conjugate Priors: Posterior of β given σ^2

$$\pi\left(\boldsymbol{\beta},\sigma^2|\widehat{\boldsymbol{\beta}},s^2,\boldsymbol{X}\right) \propto \sigma^{-k-n-2a-3}\exp\left(-\frac{1}{2\sigma^2}\left\{s^2+2b+\boldsymbol{\beta}^T(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})\boldsymbol{\beta}-2\boldsymbol{\beta}^T\left(\boldsymbol{M}\widetilde{\boldsymbol{\beta}}+\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}\right)+\widetilde{\boldsymbol{\beta}}^T\boldsymbol{M}\widetilde{\boldsymbol{\beta}}+\widehat{\boldsymbol{\beta}}^T(\boldsymbol{X}^T\boldsymbol{X})\widehat{\boldsymbol{\beta}}\right\}\right)=$$

$$\sigma^{-k-n-2a-3}\exp\left(-\frac{1}{2\sigma^2}\left\{\begin{array}{c} s^2+2b+\left(\boldsymbol{\beta}-\left(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\left(\boldsymbol{M}\widetilde{\boldsymbol{\beta}}+\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}\right)\right)^T(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})\left(\boldsymbol{\beta}-(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-1}\left(\boldsymbol{M}\widetilde{\boldsymbol{\beta}}+\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}\right)\right) \\ +\widetilde{\boldsymbol{\beta}}^T\boldsymbol{M}\widetilde{\boldsymbol{\beta}}+\widehat{\boldsymbol{\beta}}^T\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}} \\ -\left(M\widetilde{\boldsymbol{\beta}}+\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}\right)^T(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-T}(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-1}\left(\boldsymbol{M}\widetilde{\boldsymbol{\beta}}+\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}\right) \end{array}\right\}\right)$$

- Based on this, the posterior of β given σ^2 is:

$$\pi(\boldsymbol{\beta}|\sigma^2,\boldsymbol{y},\boldsymbol{X}) \propto \exp\left(-\frac{1}{2\sigma^2}(\boldsymbol{\beta}-\mathbb{E}[\boldsymbol{\beta}|\boldsymbol{y},\boldsymbol{X}])^T(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})(\boldsymbol{\beta}-\mathbb{E}[\boldsymbol{\beta}|\boldsymbol{y},\boldsymbol{X}])\right)$$

$$\boldsymbol{\beta}|\sigma^2,\boldsymbol{y},\boldsymbol{X}\sim\mathcal{N}_{k+1}\left((\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-1}\left\{\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}+\boldsymbol{M}\widetilde{\boldsymbol{\beta}}\right\},\sigma^2(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-1}\right)$$

$$where$$

$$\mathbb{E}[\boldsymbol{\beta}|\sigma^2,\boldsymbol{y},\boldsymbol{X}]=(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-1}\left(\boldsymbol{M}\widetilde{\boldsymbol{\beta}}+\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}\right)$$

$$and$$

$$Var[\boldsymbol{\beta}|\sigma^2,\boldsymbol{y},\boldsymbol{X}]=\sigma^2(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-1}$$

Conjugate Priors: Marginal posterior of σ²

$$\pi\left(\boldsymbol{\beta},\sigma^2|\widehat{\boldsymbol{\beta}},s^2,\boldsymbol{X}\right)\propto$$

$$\sigma^{-k-n-2a-3}\exp\left(-\frac{1}{2\sigma^2}\begin{Bmatrix} s^2+2b+(\boldsymbol{\beta}-\mathbb{E}[\boldsymbol{\beta}|\sigma^2,\boldsymbol{y},\boldsymbol{X}])^T(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})(\boldsymbol{\beta}-\mathbb{E}[\boldsymbol{\beta}|\sigma^2,\boldsymbol{y},\boldsymbol{X}]) \\ +\widetilde{\boldsymbol{\beta}}^T\boldsymbol{M}\widetilde{\boldsymbol{\beta}}+\widehat{\boldsymbol{\beta}}^T\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}} \\ -\left(\boldsymbol{M}\widetilde{\boldsymbol{\beta}}+\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}\right)^T(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-1}\left(\boldsymbol{M}\widetilde{\boldsymbol{\beta}}+\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}\right)\end{Bmatrix}\right)$$

- Integrating out β gives the following marginal for σ^2:

$$\pi\left(\sigma^2|\widehat{\boldsymbol{\beta}},s^2,\boldsymbol{X}\right)$$

$$\propto\sigma^{-n-2a-2}\exp\left(-\frac{1}{2\sigma^2}\begin{Bmatrix}\widetilde{\boldsymbol{\beta}}^T\boldsymbol{M}\widetilde{\boldsymbol{\beta}}+\widehat{\boldsymbol{\beta}}^T\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}+s^2 \\ +2b-\left(\boldsymbol{M}\widetilde{\boldsymbol{\beta}}+\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}\right)^T(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-1}\left(\boldsymbol{M}\widetilde{\boldsymbol{\beta}}+\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}\right)\end{Bmatrix}\right)$$

- To simplify, we will use the following two identities

$$A:\ \left(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}=\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}-\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\left(\boldsymbol{M}^{-1}+\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\right)^{-1}\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}$$

$$B:\ \boldsymbol{X}^T\boldsymbol{X}\left(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{M}=\left(\boldsymbol{M}^{-1}+\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\right)^{-1}$$

Conjugate Priors: Marginal Posterior of σ^2

$$\pi\left(\sigma^2|\widehat{\boldsymbol{\beta}},s^2,\boldsymbol{X}\right)\propto\sigma^{-n-2a-2}\exp\left(-\frac{1}{2\sigma^2}\left\{\widetilde{\boldsymbol{\beta}}^T\boldsymbol{M}\widetilde{\boldsymbol{\beta}}+\widehat{\boldsymbol{\beta}}^T\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}+s^2+2b-\left(\boldsymbol{M}\widetilde{\boldsymbol{\beta}}+\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}\right)^T(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-1}\left(\boldsymbol{M}\widetilde{\boldsymbol{\beta}}+\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}\right)\right\}\right)$$

- We can simplify the last term above using identities A & B:

$$\left(\boldsymbol{M}\widetilde{\boldsymbol{\beta}}+\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}\right)^T(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-1}\left(\boldsymbol{M}\widetilde{\boldsymbol{\beta}}+\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}\right)=\left(expand\right)$$

$$2\widehat{\boldsymbol{\beta}}^T\boldsymbol{X}^T\boldsymbol{X}(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{M}\widetilde{\boldsymbol{\beta}}+\widetilde{\boldsymbol{\beta}}^T\boldsymbol{M}^T(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{M}\widetilde{\boldsymbol{\beta}}+\widehat{\boldsymbol{\beta}}^T\boldsymbol{X}^T\boldsymbol{X}(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}=$$

$$(\boldsymbol{X}^T\boldsymbol{X})^{-1}-(\boldsymbol{X}^T\boldsymbol{X})^{-1}(\boldsymbol{M}^{-1}+(\boldsymbol{X}^T\boldsymbol{X})^{-1})^{-1}(\boldsymbol{X}^T\boldsymbol{X})^{-1}$$

$$2\widehat{\boldsymbol{\beta}}^T\boldsymbol{X}^T\boldsymbol{X}(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{M}\widetilde{\boldsymbol{\beta}}+\widetilde{\boldsymbol{\beta}}^T\boldsymbol{M}^T(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{M}\widetilde{\boldsymbol{\beta}}+\widehat{\boldsymbol{\beta}}^T\boldsymbol{X}^T\boldsymbol{X}\left((\boldsymbol{X}^T\boldsymbol{X})^{-1}-(\boldsymbol{X}^T\boldsymbol{X})^{-1}(\boldsymbol{M}^{-1}+(\boldsymbol{X}^T\boldsymbol{X})^{-1})^{-1}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\right)\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}=$$

$$2\widehat{\boldsymbol{\beta}}^T\underbrace{\boldsymbol{X}^T\boldsymbol{X}(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{M}}_{(\boldsymbol{M}^{-1}+(\boldsymbol{X}^T\boldsymbol{X})^{-1})^{-1}}\widetilde{\boldsymbol{\beta}}+\widetilde{\boldsymbol{\beta}}^T\underbrace{\boldsymbol{M}^T(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{M}}_{\boldsymbol{M}-\boldsymbol{X}^T\boldsymbol{X}(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{M}=\boldsymbol{M}-(\boldsymbol{M}^{-1}+(\boldsymbol{X}^T\boldsymbol{X})^{-1})^{-1}}\widetilde{\boldsymbol{\beta}}+\widehat{\boldsymbol{\beta}}^T\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}^T(\boldsymbol{M}^{-1}+(\boldsymbol{X}^T\boldsymbol{X})^{-1})^{-1}\widehat{\boldsymbol{\beta}}=$$

$$2\widehat{\boldsymbol{\beta}}^T(\boldsymbol{M}^{-1}+(\boldsymbol{X}^T\boldsymbol{X})^{-1})^{-1}\widetilde{\boldsymbol{\beta}}+\widetilde{\boldsymbol{\beta}}^T\boldsymbol{M}\widetilde{\boldsymbol{\beta}}-\widetilde{\boldsymbol{\beta}}^T(\boldsymbol{M}^{-1}+(\boldsymbol{X}^T\boldsymbol{X})^{-1})^{-1}\widetilde{\boldsymbol{\beta}}+\widehat{\boldsymbol{\beta}}^T\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}^T(\boldsymbol{M}^{-1}+(\boldsymbol{X}^T\boldsymbol{X})^{-1})^{-1}\widehat{\boldsymbol{\beta}}=$$

$$\widehat{\boldsymbol{\beta}}^T\boldsymbol{M}\widetilde{\boldsymbol{\beta}}+\widehat{\boldsymbol{\beta}}^T\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}-\left(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}\right)^T(\boldsymbol{M}^{-1}+(\boldsymbol{X}^T\boldsymbol{X})^{-1})^{-1}\left(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}\right)$$

- Finally:

$$\pi\left(\sigma^2|\widehat{\boldsymbol{\beta}},s^2,\boldsymbol{X}\right)\propto\sigma^{-n-2a-2}\exp\left\{-\frac{1}{2\sigma^2}\left(\left(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}\right)^T(\boldsymbol{M}^{-1}+(\boldsymbol{X}^T\boldsymbol{X})^{-1})^{-1}\left(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}\right)+s^2+2b\right)\right\}$$

Conjugate Priors: Marginal Posterior of σ^2

$$\pi\left(\sigma^2|\widehat{\boldsymbol{\beta}},s^2,\boldsymbol{X}\right) \propto \sigma^{-n-2a-2}\exp\left\{-\frac{1}{2\sigma^2}\left(\left(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}\right)^T(\boldsymbol{M}^{-1}+(\boldsymbol{X}^T\boldsymbol{X})^{-1})^{-1}\left(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}\right)+s^2+2b\right)\right\}$$

- The marginal posterior is:

$$\sigma^2|\boldsymbol{y},\boldsymbol{X}\sim InvGamma\left(\frac{n}{2}+a, b+\frac{s^2}{2}+\frac{\left(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}\right)^T(\boldsymbol{M}^{-1}+(\boldsymbol{X}^T\boldsymbol{X})^{-1})^{-1}\left(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}\right)}{2}\right)$$

- The marginal posterior mean for $n \geq 2$ is then:

$$\mathbb{E}^{\pi}[\sigma^2|\boldsymbol{y},\boldsymbol{X}] = \frac{2b+s^2+\left(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}\right)^T(\boldsymbol{M}^{-1}+(X^T\boldsymbol{X})^{-1})^{-1}\left(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}\right)}{n+2a-2}$$

Inverse-gamma	$\theta \sim \text{Inv-gamma}(\alpha,\beta)$ $p(\theta) = \text{Inv-gamma}(\theta\|\alpha,\beta)$	shape $\alpha > 0$ scale $\beta > 0$

$p(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{-(\alpha+1)}e^{-\beta/\theta}, \quad \theta > 0$	$\text{E}(\theta) = \frac{\beta}{\alpha-1}$, for $\alpha > 1$ $\text{var}(\theta) = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)}, \alpha > 2$ $\text{mode}(\theta) = \frac{\beta}{\alpha+1}$

Conjugate Priors: MLE, Posterior Mean and Ridge Estimator

- Note that setting $\boldsymbol{M} = \boldsymbol{I}_{k+1}/c\,,\ c > 0\ and\ \widetilde{\boldsymbol{\beta}} = \boldsymbol{0}_{k+1}$ in the conditional posterior mean

$$\mathbb{E}[\boldsymbol{\beta}|\sigma^2, \boldsymbol{y}, \boldsymbol{X}] = (\boldsymbol{M} + \boldsymbol{X}^T\boldsymbol{X})^{-1}\left(\boldsymbol{M}\widetilde{\boldsymbol{\beta}} + \boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}\right)$$

we obtain the classical Ridge Regression estimate:

$$\mathbb{E}[\beta|\sigma^2, \boldsymbol{y}, \boldsymbol{X}] = \left(\frac{1}{c}\boldsymbol{I}_{k+1} + \boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}} = \left(\frac{1}{c}\boldsymbol{I}_{k+1} + \boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T\boldsymbol{y}$$

- The general estimator $\mathbb{E}\left[\boldsymbol{\beta}\,|\,\sigma^2, \boldsymbol{y}, \boldsymbol{X}\right]$ can be seen as a weighted average of the prior mean and the MLE.

Conjugate Priors: Marginal Posterior of β

$$\pi(\boldsymbol{\beta}|\boldsymbol{y},\boldsymbol{X}) = \int \pi(\boldsymbol{\beta}|\sigma^2,\boldsymbol{y},\boldsymbol{X})\pi(\sigma^2|\boldsymbol{y},\boldsymbol{X})d\sigma^2$$

$$\pi(\boldsymbol{\beta}|\sigma^2,\boldsymbol{y},\boldsymbol{X}) \propto \mathcal{N}_{k+1}\left((\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-1}\{\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}+\boldsymbol{M}\widetilde{\boldsymbol{\beta}}\},\sigma^2(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-1}\right)$$

$$\pi(\sigma^2|\boldsymbol{y},\boldsymbol{X}) \propto \mathcal{IG}\left(\frac{n}{2}+a, b+\frac{s^2}{2}+\frac{\left(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}\right)^T(\boldsymbol{M}^{-1}+(\boldsymbol{X}^T\boldsymbol{X})^{-1})^{-1}\left(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}\right)}{2}\right)$$

- To integrate, we use the following transformations:

$$\tau = 1/\sigma^2, \quad d\tau = -\tau^2 d\sigma^2, \quad \widehat{\boldsymbol{\mu}} = \{\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X}\}^{-1}\left[\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}+\boldsymbol{M}\widetilde{\boldsymbol{\beta}}\right]$$

$$Z = \tau\frac{1}{2}\frac{\left[(\boldsymbol{\beta}-\widehat{\boldsymbol{\mu}})^T(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})(\boldsymbol{\beta}-\widehat{\boldsymbol{\mu}})+2b+s^2+\left(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}\right)^T(\boldsymbol{M}^{-1}+(\boldsymbol{X}^T\boldsymbol{X})^{-1})^{-1}\left(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}\right)\right]}{G}$$

$$d\tau = \frac{2}{G}dZ$$

Inverse-gamma	$\theta \sim \text{Inv-gamma}(\alpha,\beta)$ $p(\theta) = \text{Inv-gamma}(\theta\|\alpha,\beta)$	shape $\alpha > 0$ scale $\beta > 0$

$$\tau = 1/\sigma^2, \quad d\tau = -\tau^2 d\sigma^2, \quad \widehat{\boldsymbol{\mu}} = \{\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X}\}^{-1}\left[\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}+\boldsymbol{M}\widetilde{\boldsymbol{\beta}}\right]$$

$$Z = \tau\frac{1}{2}\frac{\left[(\boldsymbol{\beta}-\widehat{\boldsymbol{\mu}})^T(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})(\boldsymbol{\beta}-\widehat{\boldsymbol{\mu}})+2b+s^2+\left(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}\right)^T(\boldsymbol{M}^{-1}+(\boldsymbol{X}^T\boldsymbol{X})^{-1})^{-1}\left(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}\right)\right]}{G}$$

$$d\tau = \frac{2}{G}dZ$$

Conjugate Priors: Marginal Posterior of β

$$\tau = 1/\sigma^2, \quad d\tau = -\tau^2 d\sigma^2, \quad \widehat{\boldsymbol{\mu}} = \{\boldsymbol{M} + \boldsymbol{X}^T\boldsymbol{X}\}^{-1}\left[\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}} + \boldsymbol{M}\widetilde{\boldsymbol{\beta}}\right]$$

$$Z = \tau\frac{1}{2}\left[(\boldsymbol{\beta} - \widehat{\boldsymbol{\mu}})^T(\boldsymbol{M} + \boldsymbol{X}^T\boldsymbol{X})(\boldsymbol{\beta} - \widehat{\boldsymbol{\mu}}) + 2b + s^2 + \left(\widetilde{\boldsymbol{\beta}} - \widehat{\boldsymbol{\beta}}\right)^T(\boldsymbol{M}^{-1} + (\boldsymbol{X}^T\boldsymbol{X})^{-1})^{-1}\left(\widetilde{\boldsymbol{\beta}} - \widehat{\boldsymbol{\beta}}\right)\right]$$

$$d\tau = \frac{2}{G}dZ$$

- The marginal now takes the form:

$$\pi(\boldsymbol{\beta}|\boldsymbol{y},\boldsymbol{X}) \propto \int_0^\infty e^{-Z^2}\tau^{\left(\frac{k+1}{2}\right)+\left(\frac{n}{2}+a+1\right)}d\sigma^2 \sim \int_0^\infty e^{-Z^2}\tau^{\frac{k}{2}+\frac{n}{2}+a-\frac{1}{2}}d\tau$$

$$\sim G^{-\frac{k}{2}-\frac{n}{2}-a-\frac{1}{2}}\int_0^\infty e^{-Z^2}Z^{\frac{k}{2}+\frac{n}{2}+a-\frac{1}{2}}dZ \sim G^{-\frac{k}{2}-\frac{n}{2}-a-\frac{1}{2}}$$

where

$$G = (\boldsymbol{\beta} - \widehat{\boldsymbol{\mu}})^T(\boldsymbol{M} + \boldsymbol{X}^T\boldsymbol{X})(\boldsymbol{\beta} - \widehat{\boldsymbol{\mu}}) + 2b + s^2 + \left(\widetilde{\boldsymbol{\beta}} - \widehat{\boldsymbol{\beta}}\right)^T(\boldsymbol{M}^{-1} + (\boldsymbol{X}^T\boldsymbol{X})^{-1})^{-1}\left(\widetilde{\boldsymbol{\beta}} - \widehat{\boldsymbol{\beta}}\right)$$

- The last integral above is a scalar quantity.

Conjugate Priors: Marginal Posterior of β

- Thus integrating out σ^2 leads to a multivariate Student's t marginal posterior on β :

$$\pi(\boldsymbol{\beta}|\boldsymbol{y},\boldsymbol{X}) \propto \left[(\boldsymbol{\beta}-\widehat{\boldsymbol{\mu}})^T(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})(\boldsymbol{\beta}-\widehat{\boldsymbol{\mu}})+2b+s^2+\left(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}\right)^T(\boldsymbol{M}^{-1}+(\boldsymbol{X}^T\boldsymbol{X})^{-1})^{-1}\left(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}\right)\right]^{-\frac{k}{2}-\frac{n}{2}-a-\frac{1}{2}}$$

where:

$$\widehat{\boldsymbol{\mu}} = \{\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X}\}^{-1}\left[\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}+\boldsymbol{M}\widetilde{\boldsymbol{\beta}}\right]$$

- Recall that the density of the multivariate $\mathscr{T}_p(\nu,\theta,\boldsymbol{\Sigma})$ is:

$$\mathscr{T}_p(\nu,\theta,\boldsymbol{\Sigma}) = \frac{\Gamma\left((\nu+p)/2\right)/\Gamma(\nu/2)}{\sqrt{\det(\boldsymbol{\Sigma})\nu\pi}}\left[1+\frac{(t-\theta)^T\boldsymbol{\Sigma}^{-1}(t-\theta)}{\nu}\right]^{-\frac{\nu+p}{2}}$$

Conjugate Priors: Marginal Posterior of β

- We can now see that our marginal

$$\pi(\boldsymbol{\beta}|\boldsymbol{y},\boldsymbol{X}) \propto \left[(\boldsymbol{\beta}-\widehat{\boldsymbol{\mu}})^T(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})(\boldsymbol{\beta}-\widehat{\boldsymbol{\mu}})+2b+s^2+\left(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}\right)^T(\boldsymbol{M}^{-1}+(\boldsymbol{X}^T\boldsymbol{X})^{-1})^{-1}\left(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}\right)\right]^{-\frac{k}{2}-\frac{n}{2}-a-\frac{1}{2}}$$

can be written (can check with substitution of the Eqs below) in the form of

$$\mathscr{T}_p(\nu,\theta,\boldsymbol{\Sigma})=\frac{\Gamma\left((\nu+p)/2\right)/\Gamma(\nu/2)}{\sqrt{\det(\boldsymbol{\Sigma})\nu\pi}}\left[1+\frac{(t-\theta)^T\boldsymbol{\Sigma}^{-1}(t-\theta)}{\nu}\right]^{-\frac{\nu+p}{2}}$$

as follows ($p=k+1$, $v=n+2a$):

$$\boldsymbol{\beta}|\boldsymbol{y},\boldsymbol{X}\sim\mathcal{T}_{k+1}\left(n+2a,\widehat{\boldsymbol{\mu}},\widehat{\boldsymbol{\Sigma}}\right)$$

$$\widehat{\boldsymbol{\mu}}=(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-1}\left((\boldsymbol{X}^T\boldsymbol{X})\widehat{\boldsymbol{\beta}}+\boldsymbol{M}\widetilde{\boldsymbol{\beta}}\right)$$

$$\widehat{\boldsymbol{\Sigma}}=\frac{2b+s^2+\left(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}\right)^T(\boldsymbol{M}^{-1}+(\boldsymbol{X}^T\boldsymbol{X})^{-1})^{-1}\left(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}\right)}{n+2a}(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-1}$$

Multivariate Student's-$\mathcal{T}$ Distribution

- Recall the properties of the multivariate Student's-$\mathcal{T}$ distribution:*

Multivariate Student-t	$\theta \sim t_\nu(\mu, \Sigma)$ $p(\theta) = t_\nu(\theta\|\mu, \Sigma)$ (implicit dimension d)	degrees of freedom $\nu > 0$ location $\mu = (\mu_1, .., \mu_d)$ symmetric, pos. definite $d \times d$ scale matrix Σ

$$p(\theta) = \frac{\Gamma((\nu+d)/2)}{\Gamma(\nu/2)\nu^{d/2}\pi^{d/2}}|\Sigma|^{-1/2} \times \left(1 + \frac{1}{\nu}(\theta-\mu)^T\Sigma^{-1}(\theta-\mu)\right)^{-(\nu+d)/2}$$

$$\mathrm{E}(\theta) = \mu, \text{ for } \nu > 1$$
$$\mathrm{var}(\theta) = \frac{\nu}{\nu-2}\Sigma, \text{ for } \nu > 2$$
$$\mathrm{mode}(\theta) = \mu$$

*Note that in the notation of the tables above the degrees of freedom of the distribution rather than the dimensionality are shown as subscripts, e,g, t_v vs t_d in the earlier slide. Hopefully the notation will be clear from the discussion.

Bayesian Data Analysis, A. Gelman, J. Carlin, H. Stern and D. Rubin, 2004

Conjugate Priors: Predictive Distribution

- For a given *(m,k+1)* explanatory matrix $\tilde{\boldsymbol{X}}$, the outcome $\tilde{\boldsymbol{y}}$ can be inferred through the predictive distribution*

$$\pi\left(\tilde{\boldsymbol{y}}|\sigma^2,\boldsymbol{y},\boldsymbol{X},\tilde{\boldsymbol{X}}\right)$$

- Since $\pi\left(\tilde{\boldsymbol{y}}|\tilde{\boldsymbol{X}},\boldsymbol{\beta},\sigma^2\right)\propto\mathcal{N}\left(\tilde{\boldsymbol{X}}\boldsymbol{\beta},\sigma^2\boldsymbol{I}_m\right),$

  and since the posterior of $\boldsymbol{\beta}$ conditional on σ^2 is given as

$$\boldsymbol{\beta}|\sigma^2,\boldsymbol{y},\boldsymbol{X}\sim\mathcal{N}_{k+1}\left((\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-1}\{\boldsymbol{X}^T\boldsymbol{X}\hat{\boldsymbol{\beta}}+\boldsymbol{M}\tilde{\boldsymbol{\beta}}\},\sigma^2(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-1}\right)$$

  we can see that:

$$\pi\left(\tilde{\boldsymbol{y}}|\sigma^2,\boldsymbol{y},\boldsymbol{X},\tilde{\boldsymbol{X}}\right)\propto\int\pi\left(\tilde{\boldsymbol{y}}|\boldsymbol{\beta},\sigma^2,\boldsymbol{y},\boldsymbol{X},\tilde{\boldsymbol{X}}\right)\pi(\boldsymbol{\beta}|\sigma^2,\boldsymbol{y},\boldsymbol{X})d\boldsymbol{\beta}$$

  is a Gaussian.

* We will later on integrate σ^2 out to compute: $\pi\left(\tilde{\boldsymbol{y}}|\boldsymbol{y},\boldsymbol{X},\tilde{\boldsymbol{X}}\right)$

Appendix

- We will make use next of the following conditional expectation equalities:

$$\mathbb{E}[\boldsymbol{X}|\boldsymbol{Z}] = \mathbb{E}[\mathbb{E}[\boldsymbol{X}|\boldsymbol{Y},\boldsymbol{Z}]|\boldsymbol{Z}]$$

$$Var[\boldsymbol{X}|\boldsymbol{Z}] = Var[\mathbb{E}[\boldsymbol{X}|\boldsymbol{Y},\boldsymbol{Z}]|\boldsymbol{Z}]+\mathbb{E}[Var[\boldsymbol{X}|\boldsymbol{Y},\boldsymbol{Z}]|\boldsymbol{Z}]$$

- The proof of these is quite straightforward, e.g.

$$\mathbb{E}[\boldsymbol{X}|\boldsymbol{Z}] = \int_{\boldsymbol{x}} \boldsymbol{x}p(\boldsymbol{x}|\boldsymbol{z})d\boldsymbol{x} = \int_{\boldsymbol{x}} \boldsymbol{x}\int_{\boldsymbol{y}} p(\boldsymbol{x},\boldsymbol{y}|\boldsymbol{z})\boldsymbol{dy}\,d\boldsymbol{x}$$

$$\int_{\boldsymbol{x}} \boldsymbol{x}\int_{\boldsymbol{y}} p(\boldsymbol{x}|\boldsymbol{y},\boldsymbol{z})\boldsymbol{p}(\boldsymbol{y}|\boldsymbol{z})\boldsymbol{dy}\,d\boldsymbol{x} = \int_{\boldsymbol{y}}\left(\int_{\boldsymbol{x}} \boldsymbol{x}p(\boldsymbol{x}|\boldsymbol{y},\boldsymbol{z})\boldsymbol{dx}\right)\boldsymbol{p}(\boldsymbol{y}|\boldsymbol{z})d\boldsymbol{y} = \mathbb{E}[\mathbb{E}[\boldsymbol{X}|\boldsymbol{Y},\boldsymbol{Z}]|\boldsymbol{Z}]$$

Conjugate Priors: Predictive Distribution

- We can compute the mean and the variance of this predictive distribution as follows:

$$\mathbb{E}_\pi\left[\widetilde{\boldsymbol{y}}|\sigma^2,\boldsymbol{y},\boldsymbol{X},\widetilde{\boldsymbol{X}}\right]=\mathbb{E}_\pi\left(\mathbb{E}_\pi\left(\widetilde{\boldsymbol{y}}|\boldsymbol{\beta},\sigma^2,\boldsymbol{y},\boldsymbol{X},\widetilde{\boldsymbol{X}}\right)|\sigma^2,\boldsymbol{y},\boldsymbol{X},\widetilde{\boldsymbol{X}}\right)=$$
$$\mathbb{E}_\pi\left(\widetilde{\boldsymbol{X}}\boldsymbol{\beta}|\sigma^2,\boldsymbol{y},\boldsymbol{X},\widetilde{\boldsymbol{X}}\right)=\widetilde{\boldsymbol{X}}(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-1}\left(\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}+\boldsymbol{M}\widetilde{\boldsymbol{\beta}}\right)$$

and

$$Var_\pi\left[\widetilde{\boldsymbol{y}}|\sigma^2,\boldsymbol{y},\boldsymbol{X},\widetilde{\boldsymbol{X}}\right]=$$
$$\mathbb{E}_\pi\left(Var_\pi\left(\widetilde{\boldsymbol{y}}|\boldsymbol{\beta},\sigma^2,\boldsymbol{y},\boldsymbol{X},\widetilde{\boldsymbol{X}}\right)|\sigma^2,\boldsymbol{y},\boldsymbol{X},\widetilde{\boldsymbol{X}}\right)+Var_\pi\left(\mathbb{E}_\pi\left(\widetilde{\boldsymbol{y}}|\boldsymbol{\beta},\sigma^2,\boldsymbol{y},\boldsymbol{X},\widetilde{\boldsymbol{X}}\right)|\sigma^2,\boldsymbol{y},\boldsymbol{X},\widetilde{\boldsymbol{X}}\right)=$$
$$\mathbb{E}_\pi\left(\sigma^2\boldsymbol{I}_m|\sigma^2,\boldsymbol{y},\boldsymbol{X},\widetilde{\boldsymbol{X}}\right)+Var_\pi\left(\widetilde{\boldsymbol{X}}\boldsymbol{\beta}|\sigma^2,\boldsymbol{y},\boldsymbol{X},\widetilde{\boldsymbol{X}}\right)=$$
$$\sigma^2\boldsymbol{I}_m+\widetilde{\boldsymbol{X}}\sigma^2(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-1}\widetilde{\boldsymbol{X}}^T=\sigma^2\left(\boldsymbol{I}_m+\widetilde{\boldsymbol{X}}(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-1}\widetilde{\boldsymbol{X}}^T\right)$$

- In conclusion:

$$\widetilde{\boldsymbol{y}}|\sigma^2,\boldsymbol{y},\boldsymbol{X},\widetilde{\boldsymbol{X}}\sim\mathcal{N}_m\left(\widetilde{\boldsymbol{X}}\mathbb{E}[\boldsymbol{\beta}|\sigma^2,\boldsymbol{y},\boldsymbol{X}],\sigma^2\left(\boldsymbol{I}_m+\widetilde{\boldsymbol{X}}Var[\boldsymbol{\beta}|\sigma^2,\boldsymbol{y},\boldsymbol{X}]\widetilde{\boldsymbol{X}}^T\right)\right)$$
$$\mathbb{E}[\boldsymbol{\beta}|\sigma^2,\boldsymbol{y},\boldsymbol{X}]=(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-1}\left\{\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}+\boldsymbol{M}\widetilde{\boldsymbol{\beta}}\right\}$$

Conjugate Priors: Predictive Distribution

$$\widetilde{\boldsymbol{y}}|\sigma^2,\boldsymbol{y},\boldsymbol{X},\widetilde{\boldsymbol{X}}\sim\mathcal{N}\left(\widetilde{\boldsymbol{X}}\mathbb{E}[\boldsymbol{\beta}|\sigma^2,\boldsymbol{y},\boldsymbol{X}],\sigma^2\boldsymbol{I}_m+\widetilde{\boldsymbol{X}}Var[\boldsymbol{\beta}|\sigma^2,\boldsymbol{y},\boldsymbol{X}]\widetilde{\boldsymbol{X}}^T\right)$$

$$\mathbb{E}\left[\widetilde{\boldsymbol{y}}|\sigma^2,\boldsymbol{y},\boldsymbol{X},\widetilde{\boldsymbol{X}}\right]=\widetilde{\boldsymbol{X}}\mathbb{E}[\boldsymbol{\beta}|\sigma^2,\boldsymbol{y},\boldsymbol{X}]=\widetilde{\boldsymbol{X}}(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-1}\{\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}+\boldsymbol{M}\widetilde{\boldsymbol{\beta}}\},$$

$$\mathrm{Var}\left[\widetilde{\boldsymbol{y}}|\sigma^2,\boldsymbol{y},\boldsymbol{X},\widetilde{\boldsymbol{X}}\right]=\widetilde{\boldsymbol{X}}Var[\boldsymbol{\beta}|\sigma^2,\boldsymbol{y},\boldsymbol{X}]\widetilde{\boldsymbol{X}}^T=\sigma^2\widetilde{\boldsymbol{X}}(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-1}\widetilde{\boldsymbol{X}}^T$$

- We now integrate σ^2 against the posterior distribution

$$\sigma^2|\boldsymbol{y},\boldsymbol{X}\sim\mathcal{IG}\left(\frac{n}{2}+a,b+\frac{s^2}{2}+\frac{\left(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}\right)^T(\boldsymbol{M}^{-1}+(\boldsymbol{X}^T\boldsymbol{X})^{-1})^{-1}\left(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}\right)}{2}\right)$$

to obtain:

$$\pi\left(\widetilde{\boldsymbol{y}}|\boldsymbol{y},\boldsymbol{X},\widetilde{\boldsymbol{X}}\right)\propto\int_0^\infty\mathcal{N}\left(\widetilde{\boldsymbol{X}}\mathbb{E}[\boldsymbol{\beta}|\sigma^2,\boldsymbol{y},\boldsymbol{X}],\sigma^2\boldsymbol{I}_m+\widetilde{\boldsymbol{X}}Var[\boldsymbol{\beta}|\sigma^2,\boldsymbol{y},\boldsymbol{X}]\widetilde{\boldsymbol{X}}^T\right)$$

$$\times\,\mathcal{IG}\left(\frac{n}{2}+a,b+\frac{s^2}{2}+\frac{\left(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}\right)^T(\boldsymbol{M}^{-1}+(\boldsymbol{X}^T\boldsymbol{X})^{-1})^{-1}\left(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}\right)}{2}\right)d\sigma^2$$

Conjugate Priors: Predictive Distribution

$$\pi(\tilde{\boldsymbol{y}}|\boldsymbol{y},\boldsymbol{X},\tilde{\boldsymbol{X}}) \propto \int_0^\infty \mathcal{N}\left(\tilde{\boldsymbol{X}}\mathbb{E}[\boldsymbol{\beta}|\sigma^2,\boldsymbol{y},\boldsymbol{X}], \sigma^2\boldsymbol{I}_m + \tilde{\boldsymbol{X}}Var[\boldsymbol{\beta}|\sigma^2,\boldsymbol{y},\boldsymbol{X}]\tilde{\boldsymbol{X}}^T\right)$$
$$\times \mathcal{IG}\left(\frac{n}{2}+a, b+\frac{s^2}{2}+\frac{(\tilde{\boldsymbol{\beta}}-\hat{\boldsymbol{\beta}})^T(\boldsymbol{M}^{-1}+(\boldsymbol{X}^T\boldsymbol{X})^{-1})^{-1}(\tilde{\boldsymbol{\beta}}-\hat{\boldsymbol{\beta}})}{2}\right)d\sigma^2$$

- We substitute:

$$\sigma^2\boldsymbol{I}_m + \tilde{\boldsymbol{X}}Var[\boldsymbol{\beta}|\sigma^2,\boldsymbol{y},\boldsymbol{X}]\tilde{\boldsymbol{X}}^T = \sigma^2\left(\boldsymbol{I}_m + \tilde{\boldsymbol{X}}(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-1}\tilde{\boldsymbol{X}}^T\right)$$

$$\pi(\tilde{\boldsymbol{y}}|\boldsymbol{y},\boldsymbol{X},\tilde{\boldsymbol{X}}) \propto \int_0^\infty \mathcal{N}\left(\tilde{\boldsymbol{X}}\mathbb{E}[\boldsymbol{\beta}|\sigma^2,\boldsymbol{y},\boldsymbol{X}], \sigma^2\left(\boldsymbol{I}_m + \tilde{\boldsymbol{X}}(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-1}\tilde{\boldsymbol{X}}^T\right)\right)$$
$$\times \mathcal{IG}\left(\frac{n}{2}+a, b+\frac{s^2}{2}+\frac{(\tilde{\boldsymbol{\beta}}-\hat{\boldsymbol{\beta}})^T(\boldsymbol{M}^{-1}+(\boldsymbol{X}^T\boldsymbol{X})^{-1})^{-1}(\tilde{\boldsymbol{\beta}}-\hat{\boldsymbol{\beta}})}{2}\right)d\sigma^2$$
$$\propto \int_0^\infty \sigma^{-m-n-2a-2}\exp\begin{pmatrix} -\frac{1}{2\sigma^2}\{2b+s^2+(\tilde{\boldsymbol{\beta}}-\hat{\boldsymbol{\beta}})^T(\boldsymbol{M}^{-1}+(\boldsymbol{X}^T\boldsymbol{X})^{-1})^{-1}(\tilde{\boldsymbol{\beta}}-\hat{\boldsymbol{\beta}})+ \\ (\tilde{\boldsymbol{y}}-\mathbb{E}[\tilde{\boldsymbol{y}}|\sigma^2,\boldsymbol{y},\boldsymbol{X},\tilde{\boldsymbol{X}}])^T\left(\boldsymbol{I}_m+\tilde{\boldsymbol{X}}\left(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\tilde{\boldsymbol{X}}^T\right)^{-1}(\tilde{\boldsymbol{y}}-\mathbb{E}[\tilde{\boldsymbol{y}}|\sigma^2,\boldsymbol{y},\boldsymbol{X},\tilde{\boldsymbol{X}}])\}\end{pmatrix}d\sigma^2$$

Conjugate Priors: Predictive Distribution

$$\pi(\tilde{\boldsymbol{y}}|\boldsymbol{y},\boldsymbol{X},\tilde{\boldsymbol{X}}) \propto$$

$$\propto \int_0^\infty \sigma^{-m-n-2a-2} \exp\begin{pmatrix} -\frac{1}{2\sigma^2}\{2b+s^2+(\tilde{\boldsymbol{\beta}}-\hat{\boldsymbol{\beta}})^T(\boldsymbol{M}^{-1}+(\boldsymbol{X}^T\boldsymbol{X})^{-1})^{-1}(\tilde{\boldsymbol{\beta}}-\hat{\boldsymbol{\beta}})+ \\ (\tilde{\boldsymbol{y}}-\mathbb{E}[\tilde{\boldsymbol{y}}|\sigma^2,\boldsymbol{y},\boldsymbol{X},\tilde{\boldsymbol{X}}])^T\left(\boldsymbol{I}_m+\tilde{\boldsymbol{X}}\left(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\tilde{\boldsymbol{X}}^T\right)^{-1}(\tilde{\boldsymbol{y}}-\mathbb{E}[\tilde{\boldsymbol{y}}|\sigma^2,\boldsymbol{y},\boldsymbol{X},\tilde{\boldsymbol{X}}])\} \end{pmatrix} d\sigma^2$$

- If we call the term inside {.} as G, then:

$$\pi(\tilde{\boldsymbol{y}}|\boldsymbol{y},\boldsymbol{X},\tilde{\boldsymbol{X}}) \propto \int_0^\infty \sigma^{-m-n-2a-2}\, e^{-\frac{1}{2\sigma^2}G} d\sigma^2 \Rightarrow$$

$$\pi(\tilde{\boldsymbol{y}}|\boldsymbol{y},\boldsymbol{X},\tilde{\boldsymbol{X}}) \propto \int_0^\infty (\sigma^2)^{-\frac{m+n+2a+2}{2}} e^{-\frac{1}{2\sigma^2}G} d\sigma^2 \Rightarrow \pi(\tilde{\boldsymbol{y}}|\boldsymbol{y},\boldsymbol{X},\tilde{\boldsymbol{X}}) \propto G^{-\frac{m+n+2a+2}{2}} \int_0^\infty Z^{\frac{m+n+2a+2}{2}} e^{-Z}\left(-\frac{1}{2}G\frac{1}{Z^2}dZ\right) \sim G^{-\frac{m+n+2a}{2}}$$

$$Z=\frac{1}{2\sigma^2}G$$

$$\pi(\tilde{\boldsymbol{y}}|\boldsymbol{y},\boldsymbol{X},\tilde{\boldsymbol{X}}) \propto \{2b+s^2+(\tilde{\boldsymbol{\beta}}-\hat{\boldsymbol{\beta}})^T\left(\boldsymbol{M}^{-1}+\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\right)^{-1}(\tilde{\boldsymbol{\beta}}-\hat{\boldsymbol{\beta}})+$$

$$(\tilde{\boldsymbol{y}}-\mathbb{E}[\tilde{\boldsymbol{y}}|\sigma^2,\boldsymbol{y},\boldsymbol{X},\tilde{\boldsymbol{X}}])^T(\boldsymbol{I}_m+\tilde{\boldsymbol{X}}(\boldsymbol{M}+\boldsymbol{X}^T\boldsymbol{X})^{-1}\tilde{\boldsymbol{X}}^T)^{-1}(\tilde{\boldsymbol{y}}-\mathbb{E}[\tilde{\boldsymbol{y}}|\sigma^2,\boldsymbol{y},\boldsymbol{X},\tilde{\boldsymbol{X}}])\}^{-(m+n+2a)/2}$$

Conjugate Priors: Predictive Distribution

$$\pi(\tilde{\boldsymbol{y}}|\boldsymbol{y},\boldsymbol{X},\tilde{\boldsymbol{X}}) \propto \{2b + s^2 + (\tilde{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}})^T \left(\boldsymbol{M}^{-1} + (\boldsymbol{X}^T\boldsymbol{X})^{-1}\right)^{-1} (\tilde{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}}) +$$
$$(\tilde{\boldsymbol{y}} - \mathbb{E}[\tilde{\boldsymbol{y}}|\sigma^2,\boldsymbol{y},\boldsymbol{X},\tilde{\boldsymbol{X}}])^T \left(\boldsymbol{I}_m + \tilde{\boldsymbol{X}}(\boldsymbol{M} + \boldsymbol{X}^T\boldsymbol{X})^{-1}\tilde{\boldsymbol{X}}^T\right)^{-1} (\tilde{\boldsymbol{y}} - \mathbb{E}[\tilde{\boldsymbol{y}}|\sigma^2,\boldsymbol{y},\boldsymbol{X},\tilde{\boldsymbol{X}}])\}^{-(m+n+2a)/2}$$

- This corresponds to a Student's t distribution:

$$\mathcal{T}_p(\nu,\theta,\boldsymbol{\Sigma}) = \frac{\Gamma\left((\nu+p)/2\right)/\Gamma(\nu/2)}{\sqrt{\det(\boldsymbol{\Sigma})\nu\pi}}\left[1 + \frac{(t-\theta)^T\boldsymbol{\Sigma}^{-1}(t-\theta)}{\nu}\right]^{-\frac{\nu+p}{2}}$$

with

$$\tilde{\boldsymbol{y}}|\boldsymbol{y},\boldsymbol{X},\tilde{\boldsymbol{X}} \sim \mathcal{T}_m(n+2a,\hat{\boldsymbol{\mu}},\hat{\boldsymbol{\Sigma}})$$
$$\hat{\boldsymbol{\mu}} = \mathbb{E}[\tilde{\boldsymbol{y}}|\sigma^2,\boldsymbol{y},\boldsymbol{X},\tilde{\boldsymbol{X}}] = \tilde{\boldsymbol{X}}(\boldsymbol{M} + \boldsymbol{X}^T\boldsymbol{X})^{-1}(\boldsymbol{X}^T\boldsymbol{X}\hat{\boldsymbol{\beta}} + \boldsymbol{M}\tilde{\boldsymbol{\beta}})$$
$$\hat{\boldsymbol{\Sigma}} = \frac{\left\{2b + s^2 + (\tilde{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}})^T(\boldsymbol{M}^{-1} + (\boldsymbol{X}^T\boldsymbol{X})^{-1})^{-1}(\tilde{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}})\right\}\left(\boldsymbol{I}_m + \tilde{\boldsymbol{X}}(\boldsymbol{M} + \boldsymbol{X}^T\boldsymbol{X})^{-1}\tilde{\boldsymbol{X}}^T\right)}{n+2a}$$

Implementation of the Conjugate Prior

Our Prior:

$$\boldsymbol{\beta}|\sigma^2,\boldsymbol{X}\sim\mathcal{N}_{k+1}\left(\tilde{\boldsymbol{\beta}},\sigma^2\boldsymbol{M}^{-1}\right),\ \boldsymbol{M}\ a\ (k+1)\times(k+1)\ pos.\ def.\ symm.\ matrix$$

$$\sigma^2|\boldsymbol{X}\sim\mathcal{IG}(a,b),\ a,b>0$$

- We assume that there is no precise information available about $\tilde{\boldsymbol{\beta}},\boldsymbol{M},a,b$.

- Let us choose as $\boldsymbol{M}=\boldsymbol{I}_{k+1}/c$ and $\tilde{\boldsymbol{\beta}}=\mathbf{0}_{k+1}$.

$$\boldsymbol{\beta}\,|\,\sigma^2,\boldsymbol{X}\sim\mathcal{N}_{k+1}\left(\boldsymbol{0}_{k+1},c\sigma^2\boldsymbol{I}_{k+1}\right)$$

- Let us take a = 2.1 and b = 2, i.e. prior mean and prior variance of σ^2 equal to 1.82 and 33.06.

$$p(\theta)=\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{-(\alpha+1)}e^{-\beta/\theta},\quad\theta>0 \qquad \begin{aligned}&\mathrm{E}(\theta)=\frac{\beta}{\alpha-1},\ \text{for}\ \alpha>1\\&\mathrm{var}(\theta)=\frac{\beta^2}{(\alpha-1)^2(\alpha-2)},\alpha>2\\&\mathrm{mode}(\theta)=\frac{\beta}{\alpha+1}\end{aligned}$$

- The intuition is that if c is large, the prior on β should be more diffuse and have less bearing on the outcome.

- However, this turns out not to be the case. There is lasting influence of c on the posterior means of σ^2 and β_0.

Influence of the Prior Scale c on Bayesian Estimates of β

**************** Conjugate Priors *******************

	$\mathbb{E}_\pi\left(\sigma^2 \mid \boldsymbol{y}, \boldsymbol{X}\right)$	$\mathbb{E}_\pi\left(\boldsymbol{\beta}_0 \mid \boldsymbol{y}, \boldsymbol{X}\right)$	$V_\pi\left(\boldsymbol{\beta}_0 \mid \boldsymbol{y}, \boldsymbol{X}\right)$
c	𝔼[sigma^2 | y, X]	𝔼[beta_0 | y, X]	V[beta_0 | y, X]
0.1	1.0044	0.1251	0.0988
1.0	0.8541	0.9031	0.7733
10.0	0.6976	4.7299	3.8991
100.0	0.5746	9.6626	6.8355
1000.0	0.5470	10.8476	7.3419

An implementation is available MatLab, C++

$$\mathbb{E}^\pi[\sigma^2|\boldsymbol{y},\boldsymbol{X}] = \frac{2b + s^2 + \left(\widetilde{\boldsymbol{\beta}} - \widehat{\boldsymbol{\beta}}\right)^T (\boldsymbol{M}^{-1} + (X^T\boldsymbol{X})^{-1})^{-1}\left(\widetilde{\boldsymbol{\beta}} - \widehat{\boldsymbol{\beta}}\right)}{n + 2a - 2}$$

$$\mathbb{E}^\pi[\boldsymbol{\beta}|\boldsymbol{y},\boldsymbol{X}] = (\boldsymbol{M} + \boldsymbol{X}^T\boldsymbol{X})^{-1}\left((\boldsymbol{X}^T\boldsymbol{X})\widehat{\boldsymbol{\beta}} + \boldsymbol{M}\widetilde{\boldsymbol{\beta}}\right)$$

$$\mathrm{Var}[\boldsymbol{\beta}|\boldsymbol{y},\boldsymbol{X}] = \frac{2b+s^2+(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}})^T\left(\boldsymbol{M}^{-1}+\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\right)^{-1}(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}})}{n+2a}(\boldsymbol{M} + \boldsymbol{X}^T\boldsymbol{X})^{-1}$$

Bayesian Estimates of β for c=100

************** Conjugate Priors**************

Bayes estimates of beta for c=100

beta_i	$\mathbb{E}_\pi(\beta_i \mid y, X)$ $\mathbb{E}$[beta_i | y, X]	$V_\pi(\beta_i \mid y, X)$ V[beta_i | y, X]
beta_0	9.6626	6.8355
beta_1	-0.0040	0.0000
beta_2	-0.0516	0.0004
beta_3	0.0418	0.0077
beta_4	-1.2633	0.2615
beta_5	0.2307	0.0090
beta_6	-0.0832	1.9310
beta_7	-0.1917	0.8254
beta_8	0.1608	0.0462
beta_9	-1.2069	0.6127
beta_10	-0.2567	0.4267

An implementation is available MatLab, C++

Influence of the Conjugate Prior

- The value of *c* thus has a significant influence on the estimators and even more on the posterior variance.
- The Bayes estimates stabilize for very large values of *c*.

  Thus the prior associated with a particular c should not be considered as a weak or pseudo-noninformative prior but, on the opposite, associated with a specific proper prior information.

- The dependence on *(a, b)* is equally strong.

  Considering these limitations of conjugate priors on at least the posterior variance, **a more sophisticated noninformative strategy is needed**.

- We first look a middle-ground perspective which settles the problem of the choice of ***M***.

Zellner's Informative G-Prior

- We start with a middle ground solution by introducing only information about the location parameter of the regression but bypassing the selection of the prior correlation structure.

$$\boldsymbol{\beta}|\sigma^2,\boldsymbol{X}\sim\mathcal{N}_{k+1}\left(\widetilde{\boldsymbol{\beta}},c\sigma^2(\boldsymbol{X}^T\boldsymbol{X})^{-1}\right)$$

$$\sigma^2\sim\pi(\sigma^2|\boldsymbol{X})\propto\sigma^{-2}\text{ improper Jeffreys prior}$$

- Zellner, A. (1971). An Introduction to Bayesian Econometrics. John Wiley, New York.
- Zellner, A. (1984). Basic Issues in Econometrics. University of Chicago Press, Chicago.
- Carlin, B. and Louis, T. (1996). Bayes and Empirical Bayes Methods for Data Analysis. Chapman and Hall, New York.

Zellner's Informative G-Prior

- We start with a middle ground solution by introducing only information about the location parameter of the regression but bypassing the selection of the prior correlation structure.

$$\boldsymbol{\beta}|\sigma^2, \boldsymbol{X} \sim \mathcal{N}_{k+1}\left(\tilde{\boldsymbol{\beta}}, c\sigma^2(\boldsymbol{X}^T\boldsymbol{X})^{-1}\right)$$
$$\sigma^2 \sim \pi(\sigma^2|\boldsymbol{X}) \propto \sigma^{-2} \; improper\, Jeffreys\, prior$$

- The prior determination is restricted to the choices of $\tilde{\boldsymbol{\beta}}$ and constant *c*. *c* can be interpreted as a measure of the information available in the prior relative to the sample.
  - E.g., we will see that setting 1/c = 0.5 gives the prior the same weight as 50% of the sample.
- There is still strong influence of c.

Zellner's Informative G-Prior: Conditional Posterior of β

- The joint posterior now takes the form (note $\boldsymbol{X}^T\boldsymbol{X}$ is used in both likelihood and prior):

$$\pi(\boldsymbol{\beta},\sigma^2|\boldsymbol{y},\boldsymbol{X}) \propto (\sigma^2)^{-(n/2)}\exp\left(-\frac{1}{2\sigma^2}\left(\boldsymbol{y}-\boldsymbol{X}\widehat{\boldsymbol{\beta}}\right)^T\left(\boldsymbol{y}-\boldsymbol{X}\widehat{\boldsymbol{\beta}}\right)-\frac{1}{2\sigma^2}\left(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}\right)^T\boldsymbol{X}^T\boldsymbol{X}\left(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}\right)\right)(\sigma^2)^{-(k+1)/2}$$
$$\times\exp\left(-\frac{1}{2c\sigma^2}\left(\boldsymbol{\beta}-\widetilde{\boldsymbol{\beta}}\right)^T\boldsymbol{X}^T\boldsymbol{X}\left(\boldsymbol{\beta}-\widetilde{\boldsymbol{\beta}}\right)\right)(\sigma^2)^{-1}$$

- We can now compute the following:

$$\boldsymbol{\beta}|\sigma^2,\boldsymbol{y},\boldsymbol{X}\sim\exp\left(-\frac{1}{2\sigma^2}\left(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}\right)^T\boldsymbol{X}^T\boldsymbol{X}\left(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}\right)-\frac{1}{2c\sigma^2}\left(\boldsymbol{\beta}-\widetilde{\boldsymbol{\beta}}\right)^T\boldsymbol{X}^T\boldsymbol{X}\left(\boldsymbol{\beta}-\widetilde{\boldsymbol{\beta}}\right)\right)$$
$$\sim\exp\left(-\frac{1}{2}\left(\boldsymbol{\beta}-\frac{c}{c+1}\left(\frac{\widetilde{\boldsymbol{\beta}}}{c}+\widehat{\boldsymbol{\beta}}\right)\right)^T\frac{c+1}{c\sigma^2}\boldsymbol{X}^T\boldsymbol{X}\left(\boldsymbol{\beta}-\frac{c}{c+1}\left(\frac{\widetilde{\boldsymbol{\beta}}}{c}+\widehat{\boldsymbol{\beta}}\right)\right)\right)\Rightarrow$$

$$\boldsymbol{\beta}|\sigma^2,\boldsymbol{y},\boldsymbol{X}\sim\mathcal{N}_{k+1}\left(\frac{c}{c+1}\left(\frac{\widetilde{\boldsymbol{\beta}}}{c}+\widehat{\boldsymbol{\beta}}\right),\frac{c\sigma^2}{c+1}\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\right)$$

Zellner's Informative G-Prior: Posterior Marginal of σ^2

- Starting again with the posterior

$$\pi(\boldsymbol{\beta},\sigma^2|\boldsymbol{y},\boldsymbol{X}) \propto (\sigma^2)^{-(n+k+3/2)}\exp\left(-\frac{1}{2\sigma^2}s^2 - \frac{1}{2\sigma^2}(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}})^T\boldsymbol{X}^T\boldsymbol{X}(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}})\right)$$

$$\times\exp\left(-\frac{1}{2c\sigma^2}(\boldsymbol{\beta}-\widetilde{\boldsymbol{\beta}})^T\boldsymbol{X}^T\boldsymbol{X}(\boldsymbol{\beta}-\widetilde{\boldsymbol{\beta}})\right) =$$

$$\sim(\sigma^2)^{-(k+1)/2}\exp\left(-\frac{1}{2}\left(\boldsymbol{\beta}-\frac{c}{c+1}\left(\frac{\widetilde{\boldsymbol{\beta}}}{c}+\widehat{\boldsymbol{\beta}}\right)\right)^T\frac{c+1}{c\sigma^2}\boldsymbol{X}^T\boldsymbol{X}\left(\boldsymbol{\beta}-\frac{c}{c+1}\left(\frac{\widetilde{\boldsymbol{\beta}}}{c}+\widehat{\boldsymbol{\beta}}\right)\right)\right)$$

$$\times(\sigma^2)^{-(n/2+1)}\exp\left(-\frac{1}{2\sigma^2}\left(s^2+\frac{1}{c+1}(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}})^T\boldsymbol{X}^T\boldsymbol{X}(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}})\right)\right)$$

- We can now compute the following posterior marginal with integration in β:

$$\sigma^2|\boldsymbol{y},\boldsymbol{X}\sim InvGamma\left(\frac{n}{2},\frac{s^2}{2}+\frac{1}{2(c+1)}(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}})^T\boldsymbol{X}^T\boldsymbol{X}(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}})\right)$$

$$p(\theta)=\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{-(\alpha+1)}e^{-\beta/\theta},\ \theta>0 \qquad \mathrm{E}(\theta)=\frac{\beta}{\alpha-1},\ \text{for } \alpha>1 \quad \mathrm{var}(\theta)=\frac{\beta^2}{(\alpha-1)^2(\alpha-2)},\alpha>2 \quad \mathrm{mode}(\theta)=\frac{\beta}{\alpha+1}$$

$$\mathbb{E}[\sigma^2|\boldsymbol{y},\boldsymbol{X}] = \frac{s^2+\frac{1}{(c+1)}(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}})^T\boldsymbol{X}^T\boldsymbol{X}(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}})}{n-2}$$

Zellner's Informative G-Prior: Posterior Marginal of β

- Starting again with the posterior

$$\pi(\boldsymbol{\beta},\sigma^2|\boldsymbol{y},X) \propto (\sigma^2)^{-(n+k+3/2)}\exp\{-\frac{1}{2\sigma^2}[\left(\boldsymbol{\beta}-\frac{c}{c+1}\left(\frac{\widetilde{\boldsymbol{\beta}}}{c}+\widehat{\boldsymbol{\beta}}\right)\right)^T\frac{c+1}{c}\boldsymbol{X}^T\boldsymbol{X}\left(\boldsymbol{\beta}-\frac{c}{c+1}\left(\frac{\widetilde{\boldsymbol{\beta}}}{c}+\widehat{\boldsymbol{\beta}}\right)\right)+ s^2+\frac{1}{c+1}(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}})^T\boldsymbol{X}^T\boldsymbol{X}(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}})]\right\}$$

- We can now compute the following posterior marginal with integration in σ^2:

$$\boldsymbol{\beta}|\boldsymbol{y},\boldsymbol{X}\sim\left\{\left(\boldsymbol{\beta}-\frac{c}{c+1}\left(\frac{\widetilde{\boldsymbol{\beta}}}{c}+\widehat{\boldsymbol{\beta}}\right)\right)^T\frac{c+1}{c}\boldsymbol{X}^T\boldsymbol{X}\left(\boldsymbol{\beta}-\frac{c}{c+1}\left(\frac{\widetilde{\boldsymbol{\beta}}}{c}+\widehat{\boldsymbol{\beta}}\right)\right)+s^2+\frac{1}{c+1}(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}})^T\boldsymbol{X}^T\boldsymbol{X}(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}})\boldsymbol{\beta}\right\}^{-(n+k+1/2}$$

$$\boldsymbol{\beta}|\boldsymbol{y},\boldsymbol{X}\sim\mathcal{T}_{k+1}\left(n,\frac{c}{c+1}\left(\frac{\widetilde{\boldsymbol{\beta}}}{c}+\widehat{\boldsymbol{\beta}}\right),\frac{c\left(s^2+(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}})^T\boldsymbol{X}^T\boldsymbol{X}\left(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}\right)/(c+1)\right)}{n(c+1)}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\right)$$

Multivariate Student-t	$\theta\sim t_\nu(\mu,\Sigma)$ $p(\theta)=t_\nu(\theta\|\mu,\Sigma)$ (implicit dimension d)	degrees of freedom $\nu>0$ location $\mu=(\mu_1,\ldots,\mu_d)$ symmetric, pos. definite $d\times d$ scale matrix Σ	$p(\theta)=\frac{\Gamma((\nu+d)/2)}{\Gamma(\nu/2)\nu^{d/2}\pi^{d/2}}\|\Sigma\|^{-1/2}$ $\times(1+\frac{1}{\nu}(\theta-\mu)^T\Sigma^{-1}(\theta-\mu))^{-(\nu+d)/2}$	$\mathrm{E}(\theta)=\mu$, for $\nu>1$ $\mathrm{var}(\theta)=\frac{\nu}{\nu-2}\Sigma$, for $\nu>2$ $\mathrm{mode}(\theta)=\mu$

Zellner's Informative G-Prior: Bayes Estimates

- The Bayes estimate of $\boldsymbol{\beta}$ can be derived from:

$$\boldsymbol{\beta}|\boldsymbol{y},\boldsymbol{X} \sim \mathcal{T}_{k+1}\left(n, \frac{c}{c+1}\left(\frac{\tilde{\boldsymbol{\beta}}}{c}+\hat{\boldsymbol{\beta}}\right), \frac{c\left(s^2+(\tilde{\boldsymbol{\beta}}-\hat{\boldsymbol{\beta}})^T\boldsymbol{X}^T\boldsymbol{X}(\tilde{\boldsymbol{\beta}}-\hat{\boldsymbol{\beta}})/(c+1)\right)}{n(c+1)}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\right)$$

$$\mathbb{E}[\boldsymbol{\beta}|\boldsymbol{y},\boldsymbol{X}] = \frac{1}{c+1}\left(\tilde{\boldsymbol{\beta}}+c\hat{\boldsymbol{\beta}}\right)$$

- The posterior variance of $\boldsymbol{\beta}$ is also given from above as:

$$V_\pi[\boldsymbol{\beta}|\boldsymbol{y},\boldsymbol{X}] = \frac{c}{c+1}\frac{\left(s^2+(\tilde{\boldsymbol{\beta}}-\hat{\boldsymbol{\beta}})^T\boldsymbol{X}^T\boldsymbol{X}(\tilde{\boldsymbol{\beta}}-\hat{\boldsymbol{\beta}})/(c+1)\right)}{n-2}(\boldsymbol{X}^T\boldsymbol{X})^{-1}$$

$$p(\theta) = \frac{\Gamma((\nu+d)/2)}{\Gamma(\nu/2)\nu^{d/2}\pi^{d/2}}|\Sigma|^{-1/2} \times\left(1+\frac{1}{\nu}(\theta-\mu)^T\Sigma^{-1}(\theta-\mu)\right)^{-(\nu+d)/2}$$

$$\mathrm{E}(\theta) = \mu, \text{ for } \nu > 1$$
$$\mathrm{var}(\theta) = \frac{\nu}{\nu-2}\Sigma, \text{ for } \nu > 2$$
$$\mathrm{mode}(\theta) = \mu$$

Zellner's Informative G-Prior: Bayes Estimates

$$\mathbb{E}[\boldsymbol{\beta}|\boldsymbol{y},\boldsymbol{X}] = \frac{1}{c+1}\left(\tilde{\boldsymbol{\beta}} + c\hat{\boldsymbol{\beta}}\right)$$

- If c = 1, you can see from the above equation that it is like putting the same weight on the prior information and on the sample:

$$\mathbb{E}[\beta|\boldsymbol{y},\boldsymbol{X}] = \frac{1}{2}\left(\tilde{\boldsymbol{\beta}} + \hat{\boldsymbol{\beta}}\right)$$

  which is the average of the prior mean and the MLE estimator.

- If c=100, the prior weights 1% of the sample.

Zellner's Informative G-Prior: Bayes Estimates

- The Bayes estimate of σ^2 (use the mean of InvGamma) can be derived from

$$\sigma^2|\boldsymbol{y},\boldsymbol{X} \sim \mathcal{IG}\left(\frac{n}{2},\frac{s^2}{2}+\frac{1}{2(c+1)}\left(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}\right)^T\boldsymbol{X}^T\boldsymbol{X}\left(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}\right)\right) \Rightarrow$$

as

$$\mathbb{E}[\sigma^2|\boldsymbol{y},\boldsymbol{X}] = \frac{s^2+\left(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}\right)^T\boldsymbol{X}^T\boldsymbol{X}\left(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}\right)/(c+1)}{n-2}$$

- Note that only when c goes to infinity the influence of the prior vanishes.

$$p(\theta)=\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{-(\alpha+1)}e^{-\beta/\theta},\quad \theta>0$$

$$\mathrm{E}(\theta)=\frac{\beta}{\alpha-1},\ \text{for } \alpha>1$$
$$\mathrm{var}(\theta)=\frac{\beta^2}{(\alpha-1)^2(\alpha-2)},\ \alpha>2$$
$$\mathrm{mode}(\theta)=\frac{\beta}{\alpha+1}$$

Zellner's Informative G-Prior: Marginal Posterior Mean and Variance of β

$$\mathbb{E}[\boldsymbol{\beta}|\boldsymbol{y},\boldsymbol{X}] = \frac{1}{c+1}(\tilde{\boldsymbol{\beta}} + c\hat{\boldsymbol{\beta}})$$

$$V_\pi[\boldsymbol{\beta}|\boldsymbol{y},\boldsymbol{X}] = \frac{c}{c+1}\frac{\left(s^2 + (\tilde{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}})^T \boldsymbol{X}^T\boldsymbol{X}(\tilde{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}})/(c+1)\right)}{n-2}(\boldsymbol{X}^T\boldsymbol{X})^{-1}$$

**************** Zellner's G-Prior ****************

Posterior mean and variance of beta for c=100

beta_i	𝔼[beta_i|y, X] $\mathbb{E}_\pi(\beta_i \mid y, X)$	V[beta_i|y, X] $V_\pi(\beta_i \mid y, X)$	log10(BF) $\log_{10}(BF)$
(Intercept)	10.8895	6.8229	2.1873 (****)
X1	-0.0044	0.0000	1.1571 (***)
X2	-0.0533	0.0003	0.6667 (**)
X3	0.0673	0.0072	-0.8585
X4	-1.2808	0.2316	0.4726 (*)
X5	0.2293	0.0079	0.3861 (*)
X6	-0.3533	1.7877	-0.9860
X7	-0.2351	0.7373	-0.9849
X8	0.1793	0.0408	-0.8225
X9	-1.2726	0.5449	-0.3461
X10	-0.4288	0.3934	-0.8949

See here for BF calculation

An implementation is available MatLab, C++

Evidence against H_0:

(****) decisive
(***) strong
(**) substantial
(*) poor

Zellner's Informative G-Prior: Marginal Posterior Mean and Variance of β

$$\mathbb{E}[\boldsymbol{\beta}|\boldsymbol{y},\boldsymbol{X}] = \frac{1}{c+1}(\tilde{\boldsymbol{\beta}} + c\hat{\boldsymbol{\beta}})$$

$$V_\pi[\boldsymbol{\beta}|\boldsymbol{y},\boldsymbol{X}] = \frac{c}{c+1}\frac{\left(s^2 + (\tilde{\boldsymbol{\beta}}-\hat{\boldsymbol{\beta}})^T\boldsymbol{X}^T\boldsymbol{X}(\tilde{\boldsymbol{\beta}}-\hat{\boldsymbol{\beta}})/(c+1)\right)}{n-2}(\boldsymbol{X}^T\boldsymbol{X})^{-1}$$

*************** Zellner's G-Prior ****************

Posterior mean and variance of beta for c=1000

$\mathbb{E}_\pi(\boldsymbol{\beta}_i \mid \boldsymbol{y},\boldsymbol{X})$ $V_\pi(\boldsymbol{\beta}_i \mid \boldsymbol{y},\boldsymbol{X})$ $\log_{10}(BF)$

See here for BF calculation

beta_i	𝔼[beta_i|y, X]	V[beta_i|y, X]	log10(BF)
(Intercept)	10.9874	6.6644	1.7973 (***)
X1	-0.0044	0.0000	0.7375 (**)
X2	-0.0538	0.0003	0.2313 (*)
X3	0.0679	0.0070	-1.3506
X4	-1.2923	0.2262	0.0307
X5	0.2314	0.0078	-0.0588
X6	-0.3564	1.7461	-1.4834
X7	-0.2372	0.7202	-1.4822
X8	0.1809	0.0399	-1.3130
X9	-1.2840	0.5323	-0.8177
X10	-0.4327	0.3843	-1.3885

An implementation is available MatLab, C++

Evidence against H_0:

(****) decisive
(***) strong
(**) substantial
(*) poor

Zellner's Informative G-Prior: Predictive Modeling

- We want to predict (m ≥ 1) future observations when the explanatory variables $\tilde{X}$ but not the outcome variables

$$\tilde{\boldsymbol{y}} \sim \mathcal{N}_m\left(\tilde{\boldsymbol{X}}\boldsymbol{\beta}, \sigma^2 \boldsymbol{I}_m\right)$$

have been observed.

- Predictive distribution on $\tilde{y}$ defined as marginal of the joint posterior distribution on $(\tilde{\boldsymbol{y}}, \boldsymbol{\beta}, \sigma^2)$. Can be computed analytically by

$$\pi\left(\tilde{\boldsymbol{y}} | \boldsymbol{y}, \boldsymbol{X}, \tilde{\boldsymbol{X}}\right) \propto \int \pi\left(\tilde{\boldsymbol{y}} | \sigma^2, \boldsymbol{y}, \boldsymbol{X}, \tilde{\boldsymbol{X}}\right) \pi\left(\sigma^2 | \boldsymbol{y}, \boldsymbol{X}, \tilde{\boldsymbol{X}}\right) d\sigma^2$$

Zellner's Informative G-Prior: Predictive Modeling

$$\boldsymbol{\beta}|\sigma^2,\boldsymbol{y},\boldsymbol{X}\sim\mathcal{N}_{k+1}\left(\frac{c}{c+1}\left(\frac{\widetilde{\boldsymbol{\beta}}}{c}+\widehat{\boldsymbol{\beta}}\right),\frac{c\sigma^2}{c+1}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\right)$$

- Conditional on σ^2 the future vector of observations has a Gaussian distribution with

$$\mathbb{E}_\pi\left(\widetilde{\boldsymbol{y}}|\sigma^2,\boldsymbol{y},\boldsymbol{X},\widetilde{\boldsymbol{X}}\right)=\mathbb{E}_\pi\left[\mathbb{E}_\pi\left(\widetilde{\boldsymbol{y}}|\boldsymbol{\beta},\sigma^2,\boldsymbol{y},\boldsymbol{X},\widetilde{\boldsymbol{X}}\right)|\sigma^2,\boldsymbol{y},\boldsymbol{X},\widetilde{\boldsymbol{X}}\right]$$
$$=\mathbb{E}_\pi\left[\widetilde{\boldsymbol{X}}\boldsymbol{\beta}|\sigma^2,\boldsymbol{y},\boldsymbol{X},\widetilde{\boldsymbol{X}}\right]=\widetilde{\boldsymbol{X}}\mathbb{E}_\pi\left[\boldsymbol{\beta}|\sigma^2,\boldsymbol{y},\boldsymbol{X},\widetilde{\boldsymbol{X}}\right]\Rightarrow$$

$$\boxed{\mathbb{E}_\pi\left(\widetilde{\boldsymbol{y}}|\sigma^2,\boldsymbol{y},\boldsymbol{X},\widetilde{\boldsymbol{X}}\right)=\widetilde{\boldsymbol{X}}\frac{\widetilde{\boldsymbol{\beta}}+c\widehat{\boldsymbol{\beta}}}{c+1}\ (independent\,of\,\sigma^2)}$$

- This representation is quite intuitive, being the product of the matrix of explanatory variables $\widetilde{\boldsymbol{X}}$ by the Bayes estimate of β.

Zellner's Informative G-Prior: Predictive Modeling

$$\boldsymbol{\beta}|\sigma^2, \boldsymbol{y}, \boldsymbol{X} \sim \mathcal{N}_{k+1}\left(\frac{c}{c+1}\left(\frac{\widetilde{\boldsymbol{\beta}}}{c} + \widehat{\boldsymbol{\beta}}\right), \frac{c\sigma^2}{c+1}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\right)$$

- Similarly, we can compute:

$$\begin{aligned} V_\pi\left(\widetilde{\boldsymbol{y}}|\sigma^2, \boldsymbol{y}, \boldsymbol{X}, \widetilde{\boldsymbol{X}}\right) = \mathbb{E}_\pi\left[V\left(\widetilde{\boldsymbol{y}}|\boldsymbol{\beta}, \sigma^2, \boldsymbol{y}, \boldsymbol{X}, \widetilde{\boldsymbol{X}}\right)|\sigma^2, \boldsymbol{y}, \boldsymbol{X}, \widetilde{\boldsymbol{X}}\right] \\ + V_\pi\left[\mathbb{E}_\pi\left(\widetilde{\boldsymbol{y}}|\boldsymbol{\beta}, \sigma^2, \boldsymbol{y}, \boldsymbol{X}, \widetilde{\boldsymbol{X}}\right)|\sigma^2, \boldsymbol{y}, \boldsymbol{X}, \widetilde{\boldsymbol{X}}\right] = \\ = \mathbb{E}_\pi\left[\sigma^2\boldsymbol{I}_m|\sigma^2, \boldsymbol{y}, \boldsymbol{X}, \widetilde{\boldsymbol{X}}\right] + V_\pi\left[\widetilde{\boldsymbol{X}}\boldsymbol{\beta}|\sigma^2, \boldsymbol{y}, \boldsymbol{X}, \widetilde{\boldsymbol{X}}\right] \end{aligned}$$

$$\boxed{V_\pi\left(\widetilde{\boldsymbol{y}}|\sigma^2, \boldsymbol{y}, \boldsymbol{X}, \widetilde{\boldsymbol{X}}\right) = \sigma^2\left(\boldsymbol{I}_m + \frac{c}{c+1}\widetilde{\boldsymbol{X}}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\widetilde{\boldsymbol{X}}^T\right)}$$

- Here, we are interested on the highest posterior density (HPD) regions on subvectors of the parameter $\boldsymbol{\beta}$ derived from the marginal posterior distribution of $\boldsymbol{\beta}$.

- For a single parameter,

$$\beta_i|\boldsymbol{y},\boldsymbol{X}\sim\mathcal{T}_1\left(n,\frac{c}{c+1}\left(\frac{\widetilde{\boldsymbol{\beta}}_i}{c}+\widehat{\boldsymbol{\beta}}_i\right),\frac{c\left(s^2+(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}})^T\boldsymbol{X}^T\boldsymbol{X}(\widetilde{\boldsymbol{\beta}}-\hat{\beta})/(c+1)\right)}{n(c+1)}\omega_{ii}\right)$$

where ω_{ii} is the (i,i) element of $\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}$

- Let us define

$$\boldsymbol{\tau} = \frac{\widetilde{\boldsymbol{\beta}} + c\widehat{\boldsymbol{\beta}}}{c+1}$$

- Also

$$\boldsymbol{K} = \frac{c\left(s^2 + (\widetilde{\boldsymbol{\beta}} - \widehat{\boldsymbol{\beta}})^T \boldsymbol{X}^T \boldsymbol{X} (\widetilde{\boldsymbol{\beta}} - \widehat{\boldsymbol{\beta}})/(c+1)\right)}{n(c+1)} (\boldsymbol{X}^T \boldsymbol{X})^{-1} = (K_{ij})$$

- The variable

$$\zeta_i = \frac{\boldsymbol{\beta}_i - \tau_i}{\sqrt{K_{ii}}}$$

has a $\mathcal{T}$-distribution with n degrees of freedom.

- A $1-\alpha$ HPD interval on β_i is thus given by (see also here)

$$\left[\tau_i - \sqrt{K_{ii}} F_n^{-1}(1-\alpha/2), \tau_i + \sqrt{K_{ii}} F_n^{-1}(1-\alpha/2)\right]$$

where F_n is the CDF of $\mathcal{T}(\nu=n)$.

- Note that these HPD are different from the frequentist confidence intervals defined earlier as:

$$\beta_i|\boldsymbol{y},\boldsymbol{X} \sim \mathcal{T}\left(n-k-1, \hat{\beta}_i, \omega_{(i,i)}\, s^2/(n-k-1)\right)$$

$$\frac{\beta_i - \hat{\beta}_i}{\omega_{(i,i)}\, s^2/(n-k-1)} \sim \mathcal{T}(\nu = n-k-1, 0, 1),\ where\text{: } \omega_{(i,i)} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}|_{(i,i)}$$

$$\left\{\beta_i\text{: } |\beta_i - \hat{\beta}_i| \le F_{n-k-1}^{-1}(1-a/2)\sqrt{\omega_{(i,i)}\, s^2/(n-k-1)}\right\}$$

$$s^2 \triangleq \left(\boldsymbol{y} - \boldsymbol{X}\widehat{\boldsymbol{\beta}}\right)^T\left(\boldsymbol{y} - \boldsymbol{X}\widehat{\boldsymbol{\beta}}\right)$$

Zellner's Informative G-Prior: 95% High Posterior Density (HPD) Intervals for β_i's

beta_i	HPD Interval
beta_0	[4.6518, 17.3450]
beta_1	[-0.0077, -0.0012]
beta_2	[-0.0992, -0.0084]
beta_3	[-0.1384, 0.2742]
beta_4	[-2.4629, -0.1244]
beta_5	[0.0152, 0.4481]
beta_6	[-3.6054, 2.8918]
beta_7	[-2.3238, 1.8489]
beta_8	[-0.3099, 0.6720]
beta_9	[-3.0789, 0.5083]
beta_10	[-1.9571, 1.0909]

c=100

An implementation is available MatLab, C++

$$\left\{\beta_i \colon |\beta_i - \hat{\beta}_i| \leq F^{-1}_{n-k-1}(1-a/2)\sqrt{\omega_{(i,i)}\, s^2/(n-k-1)}\right\}$$

beta_i	HPD Interval
beta_0	[5.7435, 16.2533]
beta_1	[-0.0071, -0.0018]
beta_2	[-0.0914, -0.0162]
beta_3	[-0.1029, 0.2387]
beta_4	[-2.2618, -0.3255]
beta_5	[0.0524, 0.4109]
beta_6	[-3.0466, 2.3330]
beta_7	[-1.9649, 1.4900]
beta_8	[-0.2254, 0.5875]
beta_9	[-2.7704, 0.1998]
beta_10	[-1.6950, 0.8288]

$\{\beta_i : |\beta_i - \hat{\beta}_i|$

c=100

An implementation is available MatLab, C++

Note: The results given in ``C. P. Robert, *The Bayesian Core*, *Springer, 2nd edition,* chapter 3 '' refer to 90% HPD intervals rather than 95% as posted. These results agree with what is shown above.

Zellner's Informative G-Prior: Marginal Distribution of y

- The marginal distribution of $\boldsymbol{y}$ (evidence) is a multivariate $\mathcal{T}$.
- Since $\boldsymbol{\beta}|\sigma^2, \boldsymbol{X} \sim \mathcal{N}_{k+1}(\widetilde{\boldsymbol{\beta}}, c\sigma^2(\boldsymbol{X}^T\boldsymbol{X})^{-1})$, the linear transform of $\boldsymbol{\beta}$ satisfies:

$$\boldsymbol{X\beta}|\sigma^2, \boldsymbol{X} \sim \mathcal{N}\left(\boldsymbol{X}\widetilde{\boldsymbol{\beta}}, c\sigma^2\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\right)$$

which implies

$$\boldsymbol{y}|\sigma^2, \boldsymbol{X} \sim \mathcal{N}_n\left(\boldsymbol{X}\widetilde{\boldsymbol{\beta}}, \sigma^2(\boldsymbol{I}_n + c\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T)\right)$$

- Integration in σ^2 with $\pi(\sigma^2) = 1/\sigma^2$ gives (see Appendices):

$$f(\boldsymbol{y}|\boldsymbol{X}, c) = \int_0^\infty \frac{1}{(2\pi)^{n/2}} \frac{1}{(c+1)^{(k+1)/2}} (\sigma^2)^{-n/2-1} e^{-\frac{1}{2\sigma^2}(\boldsymbol{y}-\boldsymbol{X}\widetilde{\boldsymbol{\beta}})^T\left(\boldsymbol{I}_n + c\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\right)^{-1}(\boldsymbol{y}-\boldsymbol{X}\widetilde{\boldsymbol{\beta}})} \, d\sigma^2$$

$$f(\boldsymbol{y}|\boldsymbol{X}, c) = (c+1)^{-(k+1)/2}\pi^{-n/2}\Gamma(\frac{n}{2})\left[\boldsymbol{y}^T\boldsymbol{y} - \frac{c}{c+1}\boldsymbol{y}^T\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{y} + \frac{1}{c+1}\widetilde{\boldsymbol{\beta}}^T\boldsymbol{X}^T\boldsymbol{X}\widetilde{\boldsymbol{\beta}} - \frac{2}{c+1}\boldsymbol{y}^T\boldsymbol{X}\widetilde{\boldsymbol{\beta}}\right]^{-n/2}$$

Appendix

- It can be easily shown that

$$\left(\boldsymbol{I}_n + c\boldsymbol{X}\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T\right)^{-1} = \boldsymbol{I}_n - \frac{c}{c+1}\boldsymbol{X}\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T$$

- Indeed:

$$\left(\boldsymbol{I}_n - \frac{c}{c+1}\boldsymbol{X}\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T\right)\left(\boldsymbol{I}_n + c\boldsymbol{X}\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T\right) = \boldsymbol{I}_n - \frac{c}{c+1}\boldsymbol{X}\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T + c\boldsymbol{X}\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T$$

$$-\frac{c^2}{c+1}\boldsymbol{X}\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T\boldsymbol{X}\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T = \boldsymbol{I}_n + \left(c - \frac{c}{c+1} - \frac{c^2}{c+1}\right)\boldsymbol{X}\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T = \boldsymbol{I}_n$$

- This can simplify the term in the exponential in the earlier slide:

$$(\boldsymbol{y} - \boldsymbol{X}\widetilde{\boldsymbol{\beta}})^T\left(\boldsymbol{I}_n + c\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\right)^{-1}(\boldsymbol{y} - \boldsymbol{X}\widetilde{\boldsymbol{\beta}}) = (\boldsymbol{y} - \boldsymbol{X}\widetilde{\boldsymbol{\beta}})^T\left(\boldsymbol{I}_n - \frac{c}{c+1}\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\right)(\boldsymbol{y} - \boldsymbol{X}\widetilde{\boldsymbol{\beta}}) =$$

$$\boldsymbol{y}^T\boldsymbol{y} - \frac{c}{c+1}\boldsymbol{y}^T\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{y} - \boldsymbol{y}^T\boldsymbol{X}\widetilde{\boldsymbol{\beta}} + \frac{c}{c+1}\boldsymbol{y}^T\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{X}\widetilde{\boldsymbol{\beta}}$$

$$-\widetilde{\boldsymbol{\beta}}^T\boldsymbol{X}^T\boldsymbol{y} + \widetilde{\boldsymbol{\beta}}^T\boldsymbol{X}^T\boldsymbol{X}\widetilde{\boldsymbol{\beta}} + \frac{c}{c+1}\widetilde{\boldsymbol{\beta}}^T\boldsymbol{X}^T\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{y} - \frac{c}{c+1}\widetilde{\boldsymbol{\beta}}^T\boldsymbol{X}^T\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{X}\widetilde{\boldsymbol{\beta}} =$$

$$= \boldsymbol{y}^T\boldsymbol{y} - \frac{c}{c+1}\boldsymbol{y}^T\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{y} + \frac{1}{c+1}\widetilde{\boldsymbol{\beta}}^T\boldsymbol{X}^T\boldsymbol{X}\widetilde{\boldsymbol{\beta}} - \frac{2}{c+1}\boldsymbol{y}^T\boldsymbol{X}\widetilde{\boldsymbol{\beta}}$$

Appendix

- Let us revisit the matrix $\boldsymbol{I}_n + c\boldsymbol{X}\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T$
- Note that for any $\boldsymbol{\beta} \in \mathbb{R}^{k+1}$:

$$\left[\boldsymbol{I}_n + c\boldsymbol{X}\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T\right]\boldsymbol{X}\boldsymbol{\beta} = \boldsymbol{X}\boldsymbol{\beta} + c\boldsymbol{X}\boldsymbol{\beta} = (1+c)\boldsymbol{X}\boldsymbol{\beta}$$

which implies that $\boldsymbol{X}\boldsymbol{\beta}$ is an eigenvector with eigenvalue *(1+c).* There are *(k+1)* of those.

- Finally note that for any *z* in the null space of X^T, $\boldsymbol{X}^T z = 0,$

$$\left[\boldsymbol{I}_n + c\boldsymbol{X}\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T\right]z = z + c\boldsymbol{X}\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T z = z$$

which implies that these *z* (*n-k-1* in number) are eigenvectors of our matrix with *1* as the eigenvalues.

- The determinant of the matrix is then $(1+c)^{k+1}1^{n-k-1} = (1+c)^{k+1}$. This explains the derivation in the earlier slide.

Zellner's Informative G-Prior: Point Null Hypothesis

- If a null hypothesis is $H_0 : R\boldsymbol{\beta} = r$, R being a q x $(k+1)$ matrix, the model under H_0 can be rewritten as

$$\boldsymbol{y} \mid \boldsymbol{\beta}^0, \sigma^2, \boldsymbol{X}_0 \overset{H_0}{\sim} \mathcal{N}_n(\boldsymbol{X}_0\boldsymbol{\beta}^0, \sigma^2\boldsymbol{I}_n)$$

where $\boldsymbol{\beta}^0$ is $(k + 1 - q)$ dimensional.

- Under the prior

$$\boldsymbol{\beta}^0|\sigma^2, \boldsymbol{X}_0 \sim \mathcal{N}_{k+1-q}\left(\widetilde{\boldsymbol{\beta}}^0, c_0\sigma^2\left(\boldsymbol{X}_0^T\boldsymbol{X}_0\right)^{-1}\right)$$

- The marginal distribution of **y** under H_0 is:

$$\boldsymbol{f}(\boldsymbol{y}|\boldsymbol{X}_0, H_0) = (c_0+1)^{-(k+1-q)/2}\pi^{-n/2}\Gamma\left(\frac{n}{2}\right)$$

$$\times\left[\boldsymbol{y}^T\boldsymbol{y} - \frac{c_0}{c_0+1}\boldsymbol{y}^T\boldsymbol{X}_0(\boldsymbol{X}_0^T\boldsymbol{X}_0)^{-1}\boldsymbol{X}_0^T\boldsymbol{y} + \frac{1}{c_0+1}\widetilde{\boldsymbol{\beta}}_0^T\boldsymbol{X}_0^T\boldsymbol{X}_0\,\widetilde{\boldsymbol{\beta}}_0 - \frac{2}{c_0+1}\boldsymbol{y}^T\boldsymbol{X}_0\,\widetilde{\boldsymbol{\beta}}_0\right]^{-n/2}$$

Zellner's Informative G-Prior: Bayes Factor

- The Bayes factor is then given in analytical form as:

$$B_{10}^{\pi} = \frac{f(\boldsymbol{y}|\boldsymbol{X})}{f(\boldsymbol{y}|\boldsymbol{X}_0, H_0)} = \frac{(c_0+1)^{(k+1-q)/2}}{(c+1)^{(k+1)/2}} \times$$

$$\left[\frac{\boldsymbol{y}^T\boldsymbol{y} - \frac{c_0}{c_0+1}\boldsymbol{y}^T\boldsymbol{X}_0(\boldsymbol{X}_0^T\boldsymbol{X}_0)^{-1}\boldsymbol{X}_0^T\boldsymbol{y} + \frac{1}{c_0+1}\widetilde{\boldsymbol{\beta}}_0^T\boldsymbol{X}_0^T\boldsymbol{X}_0\,\widetilde{\boldsymbol{\beta}}_0 - \frac{2}{c_0+1}\boldsymbol{y}^T\boldsymbol{X}_0\,\widetilde{\boldsymbol{\beta}}_0}{\boldsymbol{y}^T\boldsymbol{y} - \frac{c}{c+1}\boldsymbol{y}^T\boldsymbol{X}\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T\boldsymbol{y} + \frac{1}{c+1}\widetilde{\boldsymbol{\beta}}^T\boldsymbol{X}^T\boldsymbol{X}\,\widetilde{\boldsymbol{\beta}} - \frac{2}{c+1}\boldsymbol{y}^T\boldsymbol{X}\,\widetilde{\boldsymbol{\beta}}}\right]^{n/2}$$

- Note that we use the same σ^2 in both models

- The Bayes factor depends on c_0 and c.

- This calculation can be used to evaluate the inclusion or not of each of the terms β_i, $i = 1, .., k+1$ in our regression model.

Note: $\boldsymbol{X}_0 = \boldsymbol{X}(:, setdiff(1{:}k+1, j))$

Noninformative Prior Analysis: Jeffreys' Prior

- Considering the robustness issues of the two priors examined earlier in the case of a complete lack of prior information, we consider now the non-informative Jeffreys' prior.

- The Jeffreys' prior is a flat prior on $(\boldsymbol{\beta}, \log \sigma^2)$

$$\pi^J(\boldsymbol{\beta}, \sigma^2 \mid \boldsymbol{X}) \propto \sigma^{-2}$$

- The posterior is then given as:

$$\pi^J(\boldsymbol{\beta}, \sigma^2 | \boldsymbol{y}, \boldsymbol{X}) \propto (\sigma^{-2})^{n/2} \exp\left(-\frac{1}{2\sigma^2}\left(\boldsymbol{y} - \boldsymbol{X}\widehat{\boldsymbol{\beta}}\right)^T(\boldsymbol{y} - \boldsymbol{X}\widehat{\boldsymbol{\beta}}) - \frac{1}{2\sigma^2}\left(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}\right)^T \boldsymbol{X}^T\boldsymbol{X}(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}})\right) \times \sigma^{-2}$$

$$= (\sigma^{-2})^{(k+1)/2} \exp\left(-\frac{1}{2\sigma^2}\left(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}\right)^T \boldsymbol{X}^T\boldsymbol{X}(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}})\right) \times$$

$$(\sigma^{-2})^{(n-k-1)/2+1} \exp\left(-\frac{1}{2\sigma^2} s^2\right)$$

Noninformative Prior Analysis: Jeffreys' Prior

$$\pi^J(\boldsymbol{\beta}, \sigma^2 | \boldsymbol{y}, \boldsymbol{X}) \propto (\sigma^{-2})^{(k+1)/2} \exp\left(-\frac{1}{2\sigma^2}(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}})^T \boldsymbol{X}^T \boldsymbol{X}(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}})\right) \times$$
$$(\sigma^{-2})^{(n-k-1)/2+1} \exp\left(-\frac{1}{2\sigma^2} s^2\right)$$

- From this joint posterior, we can immediately evaluate the following conditional and marginal posteriors:

$$\pi^J(\sigma^2 | \boldsymbol{y}, \boldsymbol{X}) \propto \left(\sigma^2\right)^{-(n-k-1)/2-1} \exp\left(-\frac{1}{2\sigma^2} s^2\right) = \mathcal{IG}\left(\frac{(n-k-1)}{2}, \frac{s^2}{2}\right)$$

$$\pi^J(\boldsymbol{\beta} | \sigma^2, \boldsymbol{y}, \boldsymbol{X}) \propto (\sigma^{-2})^{(k+1)/2} \exp\left(-\frac{1}{2\sigma^2}(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}})^T \boldsymbol{X}^T \boldsymbol{X}(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}})\right)$$
$$= \mathcal{N}_{k+1}\left(\widehat{\boldsymbol{\beta}}, \sigma^2 (\boldsymbol{X}^T \boldsymbol{X})^{-1}\right)$$

- From the 1st of these eqs., the Bayes' estimate of σ^2 is:

$$\mathbb{E}^\pi(\sigma^2 | \boldsymbol{y}, \boldsymbol{X}) = \frac{s^2}{n-k-3}$$

$$p(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{-(\alpha+1)} e^{-\beta/\theta}, \quad \theta > 0 \qquad \begin{aligned} &\mathrm{E}(\theta) = \frac{\beta}{\alpha-1}, \text{ for } \alpha > 1 \\ &\mathrm{var}(\theta) = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)}, \alpha > 2 \\ &\mathrm{mode}(\theta) = \frac{\beta}{\alpha+1} \end{aligned}$$

- This estimate is larger (more pessimistic) than earlier estimates

Noninformative Prior Analysis: Jeffreys' Prior

$$\pi(\boldsymbol{\beta},\sigma^2|\boldsymbol{y},\boldsymbol{X}) \propto (\sigma^{-2})^{n/2+1}\exp\left(-\frac{s^2+\left(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}\right)^T\boldsymbol{X}^T\boldsymbol{X}\left(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}\right)}{2\sigma^2}\right)$$

- To compute the marginal posterior of β, we need to integrate in σ^2. Using symbolic integrator:

$$\pi(\boldsymbol{\beta}|\boldsymbol{y},\boldsymbol{X}) \propto \int (\sigma^2)^{-n/2-1}\exp\left(-\frac{s^2+\left(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}\right)^T\boldsymbol{X}^T\boldsymbol{X}\left(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}\right)}{2\sigma^2}\right)d\sigma^2$$

$$\propto \left(s^2+\left(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}\right)^T\boldsymbol{X}^T\boldsymbol{X}\left(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}\right)\right)^{-n/2} = \left(1+\frac{\left(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}\right)^T\left(\frac{s^2(\boldsymbol{X}^T\boldsymbol{X})^{-1}}{n-k-1}\right)^{-1}\left(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}\right)}{n-k-1}\right)^{-n/2}$$

- This is a Students' $\mathcal{T}$ distribution:

$$\pi(\boldsymbol{\beta}|\boldsymbol{y},\boldsymbol{X}) \propto \mathcal{T}_{k+1}(\nu=n-k-1,\boldsymbol{\theta},\boldsymbol{\Sigma}),\ with{:}\ \boldsymbol{\theta}=\widehat{\boldsymbol{\beta}},\ \boldsymbol{\Sigma}=\frac{s^2(\boldsymbol{X}^T\boldsymbol{X})^{-1}}{n-k-1}$$

- Thus the Bayes' estimate of β is: $\mathbb{E}^{\pi}(\boldsymbol{\beta}|\boldsymbol{y},\boldsymbol{X}) = \widehat{\boldsymbol{\beta}}$

- The HPD intervals coincide with the frequentist confidence intervals.

Zellner's Non-Informative G-Prior

- The main difference with informative G-prior setup is that **we now consider *c* as unknown**.
- We use the same G-prior distribution with $\tilde{\boldsymbol{\beta}} = \mathbf{0}_{k+1}$ conditional on c, and introduce a diffuse prior on c,

$$\pi(c) = c^{-1}\mathbb{I}_{N^*}(c)$$

- The corresponding marginal posterior is given as:

$$\pi(\boldsymbol{\beta},\sigma^2 \mid \boldsymbol{y},\boldsymbol{X}) = \int \pi(\boldsymbol{\beta},\sigma^2 \mid \boldsymbol{y},\boldsymbol{X},c)\pi(c \mid \boldsymbol{y},\boldsymbol{X})dc$$

$$\propto \sum_{c=1}^{\infty} \pi(\boldsymbol{\beta},\sigma^2 \mid \boldsymbol{y},\boldsymbol{X},c)\frac{f\left(\boldsymbol{y} \mid \boldsymbol{X},c\right)\pi(c)}{\pi\left(\boldsymbol{y} \mid \boldsymbol{X}\right)}$$

$$\propto \sum_{c=1}^{\infty} \pi(\boldsymbol{\beta},\sigma^2 \mid \boldsymbol{y},\boldsymbol{X},c) f\left(\boldsymbol{y} \mid \boldsymbol{X},c\right)c^{-1}$$

Zellner's Non-Informative G-Prior

$$\pi(\boldsymbol{\beta},\sigma^2 \mid \boldsymbol{y},\boldsymbol{X}) \propto \sum_{c=1}^{\infty} \pi(\boldsymbol{\beta},\sigma^2 \mid \boldsymbol{y},\boldsymbol{X},c)\, f\left(\boldsymbol{y} \mid \boldsymbol{X},c\right) c^{-1}$$

- We have proved earlier (for constant c) that:

$$f(\boldsymbol{y}|\boldsymbol{X},c) = (c+1)^{-(k+1)/2}\pi^{-n/2}\Gamma\left(\frac{n}{2}\right)$$
$$\left[\boldsymbol{y}^T\boldsymbol{y} - \frac{c}{c+1}\boldsymbol{y}^T\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{y} + \frac{1}{c+1}\tilde{\boldsymbol{\beta}}^T\boldsymbol{X}^T\boldsymbol{X}\tilde{\boldsymbol{\beta}} - \frac{2}{c+1}\boldsymbol{y}^T\boldsymbol{X}\tilde{\boldsymbol{\beta}}\right]^{-n/2}$$

- This for our problem with $\tilde{\boldsymbol{\beta}} = \boldsymbol{0}_{k+1}$

$$\boldsymbol{f}\left(\boldsymbol{y} \mid \boldsymbol{X},c\right) \propto \left(c+1\right)^{-(k+1)/2}\left[\boldsymbol{y}^T\boldsymbol{y} - \frac{c}{c+1}\boldsymbol{y}^T\boldsymbol{X}\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T\boldsymbol{y}\right]^{-n/2}$$

Zellner's Non-Informative G-Prior: Posterior Mean of β

- Recall that

$$\boldsymbol{y},\boldsymbol{X} \sim \mathcal{T}_{k+1}\left(n, \frac{c}{c+1}\left(\frac{\tilde{\boldsymbol{\beta}}}{c}+\hat{\boldsymbol{\beta}}\right), \frac{c\left(s^2+\left(\tilde{\boldsymbol{\beta}}-\hat{\boldsymbol{\beta}}\right)^T \boldsymbol{X}^T\boldsymbol{X}\left(\tilde{\boldsymbol{\beta}}-\hat{\boldsymbol{\beta}}\right)/(c+1)\right)}{n(c+1)}\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\right)$$

- The Bayes estimates of $\boldsymbol{\beta}$ for $\tilde{\boldsymbol{\beta}} = \mathbf{0}_{k+1}$ is now given by:

$$\mathbb{E}_\pi(\boldsymbol{\beta}|\boldsymbol{y},\boldsymbol{X}) = \mathbb{E}_\pi[\mathbb{E}_\pi(\boldsymbol{\beta}|\boldsymbol{y},\boldsymbol{X},c)|\boldsymbol{y},\boldsymbol{X}] = \mathbb{E}_\pi\left[\frac{c}{c+1}\hat{\boldsymbol{\beta}}|\boldsymbol{y},\boldsymbol{X}\right] = \frac{\sum_{c=1}^{\infty}\frac{c}{c+1}\pi(c|\boldsymbol{y},\boldsymbol{X})}{\sum_{c=1}^{\infty}\pi(c|\boldsymbol{y},\boldsymbol{X})}\hat{\boldsymbol{\beta}} \Rightarrow$$

$$\mathbb{E}_\pi(\boldsymbol{\beta}|\boldsymbol{y},\boldsymbol{X}) = \frac{\sum_{c=1}^{\infty}\frac{c}{c+1}f(\boldsymbol{y}|\boldsymbol{X},c)c^{-1}}{\sum_{c=1}^{\infty}f(\boldsymbol{y}|\boldsymbol{X},c)c^{-1}}\hat{\boldsymbol{\beta}}$$

Zellner's Non-Informative G-Prior: Posterior Mean of σ^2

- Recall that $\sigma^2|\boldsymbol{y},\boldsymbol{X}\sim\mathcal{IG}\left(\frac{n}{2},\frac{s^2}{2}+\frac{1}{2(c+1)}(\tilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}})^T\boldsymbol{X}^T\boldsymbol{X}(\tilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}})\right)\Rightarrow$

$$\mathbb{E}[\sigma^2|\boldsymbol{y},\boldsymbol{X}]=\frac{s^2+(\tilde{\beta}-\hat{\beta})^T\boldsymbol{X}^T\boldsymbol{X}(\tilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}})/(c+1)}{n-2}$$

- The Bayes estimate of σ^2 is given similarly by:

$$\mathbb{E}_{\pi}(\sigma^2|\boldsymbol{y},\boldsymbol{X})=\frac{\sum_{c=1}^{\infty}\frac{s^2+\widehat{\boldsymbol{\beta}}^T\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}/(c+1)}{n-2}f(\boldsymbol{y}|\boldsymbol{X},c)c^{-1}}{\sum_{c=1}^{\infty}f(\boldsymbol{y}|\boldsymbol{X},c)c^{-1}}$$

- Both Bayes estimates involve infinite summations on c. The denominator in both cases is the normalizing constant of the posterior $\sum_{c=1}^{\infty}f(y|X,c)c^{-1}$

Zellner's Non-Informative G-Prior: Posterior Variance of β

$$\boldsymbol{\beta}|\boldsymbol{y},\boldsymbol{X} \sim \mathcal{T}_{k+1}\left(n, \frac{c}{c+1}\left(\frac{\widetilde{\boldsymbol{\beta}}}{c}+\widehat{\boldsymbol{\beta}}\right), \frac{c\left(s^2+(\tilde{\beta}-\widehat{\boldsymbol{\beta}})^T \boldsymbol{X}^T\boldsymbol{X}(\widetilde{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}})/(c+1)\right)}{n(c+1)}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\right)$$

$$V_\pi(\boldsymbol{\beta}|\boldsymbol{y},\boldsymbol{X}) = \mathbb{E}_\pi[V_\pi(\boldsymbol{\beta}|\boldsymbol{y},\boldsymbol{X},c)|\boldsymbol{y},\boldsymbol{X}] + V_\pi[\mathbb{E}_\pi(\boldsymbol{\beta}|\boldsymbol{y},\boldsymbol{X},c)|\boldsymbol{y},\boldsymbol{X}] =$$

$$\mathbb{E}_\pi\left[\frac{c}{(n-2)(c+1)}\left(s^2+\widehat{\boldsymbol{\beta}}^T\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}/(c+1)\right)(\boldsymbol{X}^T\boldsymbol{X})^{-1}\right] + V_\pi\left[\frac{c}{(c+1)}\widehat{\boldsymbol{\beta}}|\boldsymbol{y},\boldsymbol{X}\right] =$$

$$\left[\frac{\sum_{c=1}^{\infty}\frac{f(\boldsymbol{y}|\boldsymbol{X},\boldsymbol{c})}{(n-2)(c+1)}\left(s^2+\widehat{\boldsymbol{\beta}}^T\boldsymbol{X}^T\boldsymbol{X}\widehat{\boldsymbol{\beta}}/(c+1)\right)}{\sum_{c=1}^{\infty} f(\boldsymbol{y}|\boldsymbol{X},c)c^{-1}}\right](\boldsymbol{X}^T\boldsymbol{X})^{-1} +$$

$$\widehat{\boldsymbol{\beta}}\left[\frac{\sum_{c=1}^{\infty}\left(\frac{c}{(c+1)}-\mathbb{E}_\pi\left(\frac{c}{(c+1)}|\boldsymbol{y},\boldsymbol{X}\right)\right)^2 f(\boldsymbol{y}|\boldsymbol{X},c)c^{-1}}{\sum_{c=1}^{\infty} f(\boldsymbol{y}|\boldsymbol{X},c)c^{-1}}\right]\widehat{\boldsymbol{\beta}}^T$$

See earlier slide for the term $f(y|X,c)$

Zellner's Non-Informative G-Prior: Marginal Distribution

- The marginal distribution of the dataset is available in closed form

$$f\left(\boldsymbol{y} \mid \boldsymbol{X}\right)=\sum_{c=1}^{\infty} f\left(\boldsymbol{y} \mid \boldsymbol{X}, c\right) c^{-1} \propto$$

$$\sum_{c=1}^{\infty} c^{-1}(c+1)^{-(k+1)/2}\left[\boldsymbol{y}^{T} \boldsymbol{y}-\frac{c}{c+1} \boldsymbol{y}^{T} \boldsymbol{X}\left(\boldsymbol{X}^{T} \boldsymbol{X}\right)^{-1} \boldsymbol{X}^{T} \boldsymbol{y}\right]^{-n/2}$$

- The $\mathcal{T}$-shape means that we can also compute the normalizing constant!

Zellner's Informative G-Prior: Posterior Mean and Variance

$$\mathbb{E}_\pi(\boldsymbol{\beta}|\boldsymbol{y},\boldsymbol{X}) = \frac{\sum_{c=1}^{\infty} \frac{c}{c+1} f(\boldsymbol{y}|\boldsymbol{X},c)c^{-1}}{\sum_{c=1}^{\infty} f(\boldsymbol{y}|\boldsymbol{X},c)c^{-1}} \widehat{\boldsymbol{\beta}}$$

beta_i	$\mathbb{E}_\pi(\boldsymbol{\beta}_i \mid \boldsymbol{y}, \boldsymbol{X})$ 𝔼[beta_i | y, X]	$V_\pi(\boldsymbol{\beta}_i \mid \boldsymbol{y}, \boldsymbol{X})$ V[beta_i | y, X]
(Intercept)	9.2714	9.6424
X1	-0.0037	0.0000
X2	-0.0454	0.0005
X3	0.0573	0.0092
X4	-1.0905	0.3079
X5	0.1953	0.0105
X6	-0.3008	2.2750
X7	-0.2002	0.9383
X8	0.1526	0.0522
X9	-1.0835	0.7063
X10	-0.3651	0.5020

$$V_\pi(\boldsymbol{\beta}|\boldsymbol{y},\boldsymbol{X}) = \left[\frac{\sum_{c=1}^{\infty} \frac{f(\boldsymbol{y}|\boldsymbol{X},\boldsymbol{c})}{(n-2)(c+1)}\left(s^2 + \widehat{\boldsymbol{\beta}}^T \boldsymbol{X}^T \boldsymbol{X} \widehat{\boldsymbol{\beta}}/(c+1)\right)}{\sum_{c=1}^{\infty} f(\boldsymbol{y}|\boldsymbol{X},c)c^{-1}}\right](\boldsymbol{X}^T\boldsymbol{X})^{-1} + \widehat{\boldsymbol{\beta}}\left[\frac{\sum_{c=1}^{\infty}\left(\frac{c}{(c+1)} - \mathbb{E}_\pi\left(\frac{c}{(c+1)}|\boldsymbol{y},\boldsymbol{X}\right)\right)^2 f(\boldsymbol{y}|\boldsymbol{X},c)c^{-1}}{\sum_{c=1}^{\infty} f(\boldsymbol{y}|\boldsymbol{X},c)c^{-1}}\right]\widehat{\boldsymbol{\beta}}^T$$

A Matlab implementation can be downloaded here

$$\left(We\ use\ \widetilde{\boldsymbol{\beta}} = \boldsymbol{0}_{11},\ c = 100\right)$$

Zellner's Non-Informative G-Prior: Point Null Hypothesis

- If a null hypothesis is $H_0 : R\beta = r$, the model under H_0 can be rewritten as

$$y \mid \boldsymbol{\beta}^0, \sigma^2, \boldsymbol{X}_0 \overset{H_0}{\sim} \mathcal{N}_n(\boldsymbol{X}_0 \boldsymbol{\beta}^0, \sigma^2 \boldsymbol{I}_n)$$

where $\boldsymbol{\beta}^0$ is $(k + 1 - q)$ dimensional.

- Under $\pi(c) = c^{-1}$ and the prior

$$\boldsymbol{\beta}^0 \mid \sigma^2, \boldsymbol{X}_0, c \sim \mathcal{N}_{k+1-q}\left(\boldsymbol{0}_{k+1-q}, c\sigma^2 \left(\boldsymbol{X}_0^T \boldsymbol{X}_0\right)^{-1}\right)$$

the marginal distribution of **y** under H_0 is:

$$\boldsymbol{f}\left(\boldsymbol{y} \mid \boldsymbol{X}_0, H_0\right) \propto \sum_{c=1}^{\infty} c^{-1}(c+1)^{-(k+1-q)/2} \left[\boldsymbol{y}^T \boldsymbol{y} - \frac{c}{c+1} \boldsymbol{y}^T \boldsymbol{X}_0 \left(\boldsymbol{X}_0^T \boldsymbol{X}_0\right)^{-1} \boldsymbol{X}_0^T \boldsymbol{y}\right]^{-n/2}$$

- The Bayes factor $\boldsymbol{B}_{10}^{\pi} = \boldsymbol{f}\left(\boldsymbol{y} \mid \boldsymbol{X}\right) / \boldsymbol{f}\left(\boldsymbol{y} \mid \boldsymbol{X}_0, H_0\right)$ can now be computed

Zellner's Non-Informative G-Prior: Posterior Mean and Variance

For $H_0 : \beta_7 = \beta_8 = 0, \log_{10}\left(B_{10}^{\pi}\right) = -0.7884$ (*We use* $\boldsymbol{\beta} = 0_{11}$)

beta_i	$\mathbb{E}_\pi(\boldsymbol{\beta}_i \mid \boldsymbol{y}, \boldsymbol{X})$ 𝔼[beta_i | y, X]	$V_\pi(\boldsymbol{\beta}_i \mid \boldsymbol{y}, \boldsymbol{X})$ V[beta_i | y, X]	$\log_{10}(BF)$ log10(BF)
(Intercept)	9.2714	9.6424	1.4205 (***)
X1	-0.0037	0.0000	0.8503 (**)
X2	-0.0454	0.0005	0.5664 (**)
X3	0.0573	0.0092	-0.3609
X4	-1.0905	0.3079	0.4520 (*)
X5	0.1953	0.0105	0.4007 (*)
X6	-0.3008	2.2750	-0.4411
X7	-0.2002	0.9383	-0.4404
X8	0.1526	0.0522	-0.3383
X9	-1.0835	0.7063	-0.0424
X10	-0.3651	0.5020	-0.3838

Evidence against H_0:

(****) decisive
(***) strong
(**) substantial
(*) poor

A Matlab implementation can be downloaded here

Variable Selection

- Let us return to our regression model with one dependent random variable y and a set of *k* $\{x_1, x_2, ..., x_k\}$ explanatory variables.

- Are all the x_i's needed in the regression?

- We assume that every q-subset $\{i_1, i_2, ..., i_q\}$, $0 \le q \le k$, of the explanatory variables,

$$\{1, x_{i_1}, x_{i_2}, ..., x_{i_q}\}$$

  is a proper set of explanatory variables for the regression of y (as before, the intercept is included in all models).

- We have a total of 2^k models to select from!

Variable Selection

- Following earlier notation, we denote: $\boldsymbol{X} = \left[\mathbf{1}_n \; x_1 \; x_2 \ldots x_k\right]$ as the matrix that contains $\mathbf{1}_n$ and the k potential predictor variables.

- Each model M_γ is associated with binary indicator vector

$$\gamma \in \Gamma = \{0,1\}^k$$

  where $\gamma_i=1$ means that the variable x_i is included in the model M_γ and $\gamma_i=0$ that it is not.

- The number of variables included in the model M_γ is:

$$q_\gamma = \mathbf{1}^T \boldsymbol{\gamma}$$

- The indices of the variables included in the model and not included in the model are denoted, respectively, as: $t_1(\gamma), t_0(\gamma)$

Variable Selection – Models in Competition

- For $\boldsymbol{\beta} \in \mathbb{R}^{k+1}$ and $\boldsymbol{X}$, we define $\boldsymbol{\beta}_\gamma$ as the sub-vector

$$\boldsymbol{\beta}_\gamma = \left(\beta_0, \left(\beta_i\right)_{i \in t_1(\gamma)}\right)$$

- Let $\boldsymbol{X}_\gamma$ be the submatrix of $\boldsymbol{X}$ where only the column $\mathbf{1}_n$ and the columns in $t_1(\gamma)$ have been left.

- The model M_γ is then defined as:

$$\boldsymbol{y} \mid \boldsymbol{\gamma}, \boldsymbol{\beta}_\gamma, \sigma^2, X \sim \mathcal{N}_n\left(\boldsymbol{X}_\gamma \boldsymbol{\beta}_\gamma, \sigma^2 \boldsymbol{I}_n\right)$$

  where $\beta_\gamma \in \mathbb{R}^{q_\gamma + 1}, \sigma^2 \in \mathbb{R}_+^*$ are the unknown parameters.

- The σ^2 is common to all models and we use the same prior for all models.

Variable Selection – Models in Competition

- We have a high number 2^k of models in competition.

- We cannot specify a prior on every M_γ in a completely subjective and autonomous manner.

- We derive all priors from a single global prior **associated with the full model** that corresponds to $\gamma = (1, . . . , 1)$.

Zellner's Informative Prior: Variable Selection

- For the full model that corresponds to $\gamma = (1, \ldots, 1)$, we use the Zellner's informative G-prior:

$$\boldsymbol{\beta}|\sigma^2, \boldsymbol{X} \sim \mathcal{N}_{k+1}\left(\widetilde{\boldsymbol{\beta}}, c\sigma^2(\boldsymbol{X}^T\boldsymbol{X})^{-1}\right)$$
$$\sigma^2 \sim \pi(\sigma^2|\boldsymbol{X}) \propto \sigma^{-2} \ improper\,Jeffreys\,prior$$

- For each model M_γ, the prior distribution of $\boldsymbol{\beta}_\gamma$ conditional on σ^2 is fixed as:

$$\boldsymbol{\beta}_\gamma\,|\boldsymbol{\gamma}, \sigma^2 \sim \mathcal{N}_{q_\gamma+1}\left(\widetilde{\boldsymbol{\beta}}_\gamma, c\sigma^2\left(\boldsymbol{X}_\gamma^T\boldsymbol{X}_\gamma\right)^{-1}\right)$$

where $\widetilde{\boldsymbol{\beta}}_\gamma = \left(X_\gamma^T\boldsymbol{X}_\gamma\right)^{-1}\boldsymbol{X}_\gamma^T\widetilde{\boldsymbol{\beta}}$ and same prior on σ^2.

Zellner's Informative Prior: Variable Selection – Prior

- The joint prior for model M_γ is the improper prior

$$\pi(\boldsymbol{\beta}_\gamma, \sigma^2|\boldsymbol{\gamma}) \propto (\sigma^2)^{-(q_\gamma+1)/2-1} \exp\left[-\frac{1}{2(c\sigma^2)}(\boldsymbol{\beta}_\gamma - \tilde{\boldsymbol{\beta}}_\gamma)^T(\boldsymbol{X}_\gamma^T\boldsymbol{X}_\gamma)(\boldsymbol{\beta}_\gamma - \tilde{\boldsymbol{\beta}}_\gamma)\right]$$

- Infinitely many ways of defining a prior on the model index γ:

  Our choice is a uniform prior $p(\gamma|\boldsymbol{X}) = 2^{-k}$.

- Posterior distribution of γ is central to variable selection since it is proportional to marginal density of $\mathbf{y}$ on M_γ (or evidence of M_γ)

- Posterior distribution of γ is proportional to the marginal density of **y** on M_γ (so it can also be used to compute Bayes factors)

$$\pi(\gamma \mid \boldsymbol{y}, \boldsymbol{X}) \propto f(\boldsymbol{y} \mid \gamma, \boldsymbol{X})\pi(\gamma \mid \boldsymbol{X}) \propto f(\boldsymbol{y} \mid \gamma, \boldsymbol{X})$$
$$= \int\left(\int f(\boldsymbol{y} \mid \gamma, \boldsymbol{\beta}, \sigma^2, \boldsymbol{X})\pi(\boldsymbol{\beta} \mid \gamma, \sigma^2, \boldsymbol{X})d\boldsymbol{\beta}\right)\pi(\sigma^2 \mid \boldsymbol{X})d\sigma^2$$

where (see earlier derivation)

$$f(\boldsymbol{y}|\boldsymbol{\gamma},\sigma^2,\boldsymbol{X}) = \int f(\boldsymbol{y}|\boldsymbol{\gamma},\boldsymbol{\beta},\sigma^2)\pi(\boldsymbol{\beta}|\boldsymbol{\gamma},\sigma^2)d\boldsymbol{\beta} =$$
$$= (c+1)^{-(q_\gamma+1)/2}(2\pi)^{-n/2}(\sigma^2)^{-n/2} \times$$
$$\exp\left(-\frac{1}{2\sigma^2}\boldsymbol{y}^T\boldsymbol{y} + \frac{1}{2\sigma^2(c+1)}\left\{c\boldsymbol{y}^T\boldsymbol{X}_\gamma\left(\boldsymbol{X}_\gamma^T\boldsymbol{X}_\gamma\right)^{-1}\boldsymbol{X}_\gamma^T\boldsymbol{y} - \widetilde{\boldsymbol{\beta}}_\gamma^T\boldsymbol{X}_\gamma^T\boldsymbol{X}_\gamma\widetilde{\boldsymbol{\beta}}_\gamma + 2\boldsymbol{y}^T\boldsymbol{X}_\gamma\widetilde{\boldsymbol{\beta}}_\gamma\right\}\right)$$

Zellners Informative Prior: Variable Selection – Prior

$$\pi(\gamma \mid \boldsymbol{y}, \boldsymbol{X}) \propto \int f(\boldsymbol{y} \mid \gamma, \sigma^2, \boldsymbol{X}) \pi(\sigma^2 \mid \boldsymbol{X}) d\sigma^2 = \int f(\boldsymbol{y} \mid \gamma, \sigma^2, \boldsymbol{X}) \frac{1}{\sigma^2} d\sigma^2$$

- Posterior distribution of γ is then given as:

$$f(\gamma|\boldsymbol{y},\boldsymbol{X}) \propto (c+1)^{-(q_\gamma+1)/2} \times$$
$$\left(\boldsymbol{y}^T\boldsymbol{y} - \frac{c}{(c+1)}\boldsymbol{y}^T\boldsymbol{X}_\gamma\left(\boldsymbol{X}_\gamma^T\boldsymbol{X}_\gamma\right)^{-1}\boldsymbol{X}_\gamma^T\boldsymbol{y} + \frac{1}{c+1}\widetilde{\boldsymbol{\beta}}_\gamma^T\boldsymbol{X}_\gamma^T\boldsymbol{X}_\gamma\widetilde{\boldsymbol{\beta}}_\gamma - \frac{2}{c+1}\boldsymbol{y}^T\boldsymbol{X}_\gamma\widetilde{\boldsymbol{\beta}}_\gamma\right)^{-n/2}$$

- We already have seen this distribution earlier for a fixed c:

$$f(\boldsymbol{y}|\boldsymbol{X},c) = (c+1)^{-(k+1)/2}\pi^{-n/2}\Gamma(\frac{n}{2})\left[\boldsymbol{y}^T\boldsymbol{y} - \frac{c}{c+1}\boldsymbol{y}^T\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{y} + \frac{1}{c+1}\widetilde{\boldsymbol{\beta}}^T\boldsymbol{X}^T\boldsymbol{X}\widetilde{\boldsymbol{\beta}} - \frac{2}{c+1}\boldsymbol{y}^T\boldsymbol{X}\widetilde{\boldsymbol{\beta}}\right]^{-n/2}$$

Model Selection

- Most likely models ordered by decreasing posterior probabilities using Zellner's informative G-prior with *c=100*.

$t_1(\gamma)$	$\pi(\gamma \mid \boldsymbol{y}, \boldsymbol{X})$
t1_gamma	pi(gamma | y, X)
0 1 2 4 5	0.231543
0 1 2 4 5 9	0.037358
0 1 9	0.034435
0 1 2 4 5 10	0.032975
0 1 4 5	0.030606
0 1 2 9	0.025016
0 1 2 4 5 7	0.024144
0 1 2 4 5 8	0.023784
0 1 2 4 5 6	0.023735
0 1 2 3 4 5	0.023207
0 1 6 9	0.014587
0 1 2 3 9	0.014491
0 9	0.014281
0 1 2 6 9	0.013551
0 1 4 5 9	0.012761
0 1 3 9	0.011712
0 1 2 8	0.011477
0 1 8	0.009519
0 1 2 3 4 5 9	0.009036
0 1 2 4 5 6 9	0.009031

An implementation is available MatLab, C++

Model Selection

- Model M_γ with the highest posterior probability is $t_1(\gamma) = (1, 2, 4, 5)$, which corresponds to the variables
  - altitude,
  - slope,
  - height of the tree sampled in the center of the area, and
  - diameter of the tree sampled in the center of the area.

Model Selection

- For the Zellner's non-informative prior with $\pi(c)=1/c$, we have ($\tilde{\boldsymbol{\beta}} = \mathbf{0}_{k+1}$) :

$$\pi(\boldsymbol{\gamma} \mid \boldsymbol{y}, \boldsymbol{X}) = \sum_{c=1}^{\infty} c^{-1}(c+1)^{-(q_\gamma+1)/2}\left[\boldsymbol{y}^T\boldsymbol{y} - \frac{c}{c+1}\boldsymbol{y}^T\boldsymbol{X}_\gamma\left(\boldsymbol{X}_\gamma^T\boldsymbol{X}_\gamma\right)^{-1}\boldsymbol{X}_\gamma^T\boldsymbol{y}\right]^{-n/2}$$

- Again we have seen this before as

$$f(\boldsymbol{y} \mid \boldsymbol{X}) = \sum_{c=1}^{\infty} f(\boldsymbol{y} \mid \boldsymbol{X}, c)c^{-1} \propto$$

$$\sum_{c=1}^{\infty} c^{-1}(c+1)^{-(k+1)/2}\left[\boldsymbol{y}^T\boldsymbol{y} - \frac{c}{c+1}\boldsymbol{y}^T\boldsymbol{X}\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T\boldsymbol{y}\right]^{-n/2}$$

Model Selection

- Most likely models ordered by decreasing posterior probabilities using Zellner's non-informative G-prior.

$t_1(\gamma)$ $\pi(\gamma \mid \boldsymbol{y}, \boldsymbol{X})$

t1_gamma	pi(gamma | y, X)
0 1 2 4 5	0.092914
0 1 2 4 5 9	0.032553
0 1 2 4 5 10	0.029512
0 1 2 4 5 7	0.023114
0 1 2 4 5 8	0.022843
0 1 2 4 5 6	0.022807
0 1 2 3 4 5	0.022409
0 1 2 3 4 5 9	0.016733
0 1 2 4 5 6 9	0.016725
0 1 2 4 5 8 9	0.013726
0 1 4 5	0.011031
0 1 2 4 5 9 10	0.009933
0 1 2 3 9	0.009698
0 1 2 9	0.009316
0 1 2 4 5 7 9	0.009253
0 1 2 6 9	0.009189
0 1 4 5 9	0.008756
0 1 2 3 4 5 10	0.007933
0 1 2 4 5 8 10	0.007901
0 1 2 4 5 7 10	0.007896

An implementation is available MatLab, C++

Stochastic Search for the Most Likely Model

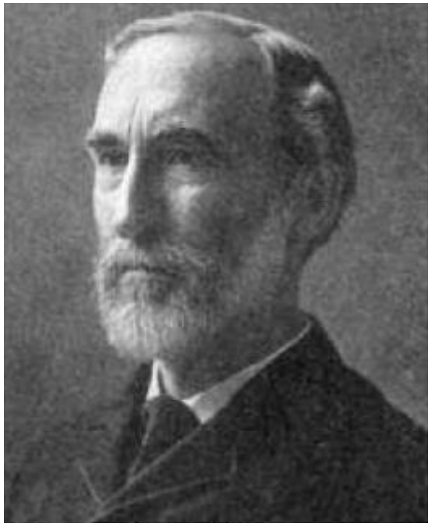
- When k is large, it becomes computationally intractable to compute the posterior probabilities of the 2^k models.
- Need of a tailored algorithm that samples from $\pi(\gamma|\mathbf{y},\mathbf{X})$ and selects the most likely models.
- Can be done by Gibbs sampling*, given the availability of the full conditional posterior probabilities of the γ_i's. If

$$\gamma_{-i} = (\gamma_1, \ldots, \gamma_{i-1}, \gamma_{i+1}, \ldots, \gamma_k) \ (1 \leq i \leq k)$$

then

$$\pi(\gamma_i \,|\mathbf{y}, \gamma_{-i}, \mathbf{X}) \propto \pi\,(\gamma\,|\mathbf{y},\mathbf{X})$$

(to be evaluated in both $\gamma_i = 0$ and $\gamma_i = 1$)



* Gibbs and other sampling algorithms will be introduced and discussed in detail in forthcoming lectures.

Gibbs Sampling for Variable Selection

Initialization: Draw γ^0 from the uniform distribution on Γ

Iteration t: Given $\left(\gamma_1^{(t-1)},\ldots,\gamma_k^{(t-1)}\right)$, generate

- $\gamma_1^{(t)}$ *according to* $\pi\left(\gamma_1 \mid \boldsymbol{y},\gamma_2^{(t-1)},\ldots,\gamma_k^{(t-1)},\boldsymbol{X}\right)$
- $\gamma_2^{(t)}$ *according to* $\pi\left(\gamma_2 \mid \boldsymbol{y},\gamma_1^{(t)},\gamma_3^{(t-1)},\ldots,\gamma_k^{(t-1)},\boldsymbol{X}\right)$
- ..
- ..
- $\gamma_k^{(t)}$ *according to* $\pi\left(\gamma_k \mid \boldsymbol{y},\gamma_1^{(t)},\gamma_2^{(t)},\ldots,\gamma_{k-1}^{(t)},\boldsymbol{X}\right)$

Gibbs Sampling for Variable Selection

Question: How to sample γ_1^t according to $\pi\left(\gamma_1 \mid \boldsymbol{y}, \gamma_2^{(t-1)}, \ldots, \gamma_k^{(t-1)}, \boldsymbol{X}\right)$

1. The conditional distribution $\pi\left(\gamma_1 \mid \boldsymbol{y}, \gamma_2^{(t-1)}, \ldots, \gamma_k^{(t-1)}, \boldsymbol{X}\right)$ is proportional to $\pi\left(\gamma \mid \boldsymbol{y}, \boldsymbol{X}\right)$

2. Since γ_1^t only has two possible values which are 0 and 1, we get

$$p_0 \propto \pi\left(\gamma_1 = 0, \gamma_2^{(t-1)}, \ldots, \gamma_k^{(t-1)} \mid \boldsymbol{y}, \boldsymbol{X}\right)$$

$$p_1 \propto \pi\left(\gamma_1 = 1, \gamma_2^{(t-1)}, \ldots, \gamma_k^{(t-1)} \mid \boldsymbol{y}, \boldsymbol{X}\right)$$

So the probability that $\gamma_1^t = 1$ is p_1, then we can use Gibbs Sampling to approximate the distribution of $\{\gamma_i^t\}$

Gibbs Sampling: Posterior Probabilities

- After $T \gg 1$ MCMC iterations, we approximate the posterior probabilities $p(\gamma|\mathbf{y},\mathbf{X})$ by empirical averages

$$\hat{\pi}(\gamma|\boldsymbol{y},\boldsymbol{X}) = \frac{1}{T - T_0 + 1}\sum_{t=T_0}^{T} \mathbb{I}_{\gamma^{(t)}=\gamma}$$

The T_0 first values (burn in) in the MCMC chain are eliminated.

Model Choice Comparison: Gibbs Estimates

First level Informative G-prior model with ($\tilde{\boldsymbol{\beta}} = 0_{11}$, c=100) compared with the Gibbs estimates of the top ten posterior probabilities

$t_1(\gamma)$	$\pi(\gamma \mid \boldsymbol{y}, \boldsymbol{X})$	$\hat{\pi}(\gamma \mid \boldsymbol{y}, \boldsymbol{X})$
t1_gamma	pi(gamma | y, X)	pi_hat(gamma | y, X)
0 1 2 4 5	0.231543	0.239276
0 1 2 4 5 9	0.037358	0.034397
0 1 9	0.034435	0.032397
0 1 2 4 5 10	0.032975	0.030097
0 1 4 5	0.030606	0.029397
0 1 2 9	0.025016	0.025297
0 1 2 4 5 7	0.024144	0.022498
0 1 2 4 5 8	0.023784	0.024898
0 1 2 4 5 6	0.023735	0.023598
0 1 2 3 4 5	0.023207	0.022998

A MatLab implementation is available

Model Choice Comparison: Gibbs Estimates

Non-informative G-prior variable model choice compared with the Gibbs estimates of the top ten posterior probabilities

$t_1(\gamma)$	$\pi(\gamma \mid \boldsymbol{y}, \boldsymbol{X})$	$\hat{\pi}(\gamma \mid \boldsymbol{y}, \boldsymbol{X})$
t1_gamma	pi(gamma | y, X)	pi_hat(gamma | y, X)
0 1 2 4 5	0.092914	0.093391
0 1 2 4 5 9	0.032553	0.033097
0 1 2 4 5 10	0.029512	0.032597
0 1 2 4 5 7	0.023114	0.025097
0 1 2 4 5 8	0.022843	0.023098
0 1 2 4 5 6	0.022807	0.022498
0 1 2 3 4 5	0.022409	0.021698
0 1 2 3 4 5 9	0.016733	0.015998
0 1 2 4 5 6 9	0.016725	0.014899
0 1 2 4 5 8 9	0.013726	0.013399

A MatLab implementation is available

Gibbs Sampling: Probabilities of Inclusion

- An approximation of the probability to include the i-th variable:

$$\hat{P}^{\pi}(\gamma_i = 1|\boldsymbol{y}, \boldsymbol{X}) = \frac{1}{T - T_0 + 1} \sum_{t=T_0}^{T} \mathbb{I}_{\gamma_i^{(t)}=1}$$

Probabilities of Inclusion Estimates

Informative ($\tilde{\boldsymbol{\beta}} = 0_{11}$, c=100) and non-informative G-prior variable inclusion estimates (based on the same Gibbs output as in the earlier two tables)

γ_i	$\hat{P}^{\pi}(\gamma_i = 1 \mid \boldsymbol{y}, \boldsymbol{X})$	$\hat{P}^{\pi}(\gamma_i = 1 \mid \boldsymbol{y}, \boldsymbol{X})$
gamma_i	P(gamma_i|y, X)	P(gamma_i|y, X)
gamma_1	0.8733	0.8806
gamma_2	0.7100	0.7789
gamma_3	0.1515	0.2958
gamma_4	0.6842	0.7422
gamma_5	0.6635	0.7234
gamma_6	0.1659	0.2992
gamma_7	0.1343	0.2812
gamma_8	0.1478	0.2740
gamma_9	0.3942	0.5015
gamma_10	0.1135	0.2556

A MatLab implementation is available