A Model for Our Constrained Subproblem

Let's put to work what we learned

The Model Variables

Our pricing problem requires to build paths

We will model this by introducing a variable for each time step:

$$x_0, x_1, \ldots x_{eoh-1}$$

In the domain of each variables, we include:

- One value for each node in the original graph
  - If $x_t = i$, then we visit node i at time t
- One special value to specify that the path has not yet started:
  - If $x_t = -1$, then the path has not yet started at time t
- One special value to specify that the path has finished early
  - If $x_t = -2$, then the path is already over at time t

Overall, we have $D_t = \{-2, -1, \ldots, n_v - 1\}$

The Model Variables

We also need to track the path weight

We will introducing again a variable for each time step:

$$y_0, y_1, \dots y_{eoh-1}$$

Where $y_t \in \{-M, \dots, M\}$, with M being a vary large number

- Using a large number here is not a problems
- ...Since propagation will reduce the domains already at the root node

The total cost of a path can be obtained by summation

$$z = \sum_{t=0}^{eoh-1} y_i + \alpha$$

If we want paths with negative weight, we can just add the constraint $z < 0$

Allowed Transitions

We now need to model transitions:

- We can move only along arcs in the original graph
  - I.g. we can move from i to j iff $(i, j) \in E$
  - ...Where E refers here to the set of arcs in the original graph
- ...But the special values make for an exception
  - We can always move from -1 to i
  - We can always move from i to -2

Overall, the allowed transitions are:

$$\{(i, j)\ \forall (i, j) \in E\} \cup \{(-1, i)\ \forall i \in V\} \cup \{(i, -2)\ \forall i \in V\}$$

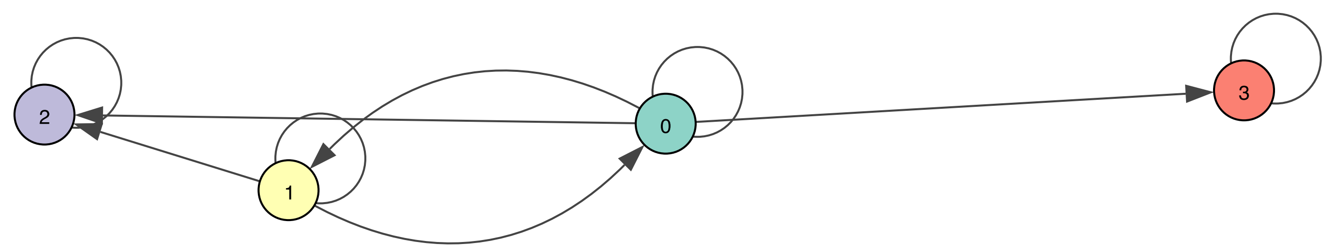
Where V refers here to the set of nodes in the original graph

Allowed Transitions

Let's use our graph as an example

In [2]:
```
ig.plot(g, **util.get_visual_style(g), bbox=(700, 150), margin=50)
```

Out[2]:



The allowed transitions are:

$$(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(2,2),(3,3),$$
$$(-1,0),(-1,1),(-1,2),(-1,2),$$
$$(0,-2),(1,-2),(2,-2),(3,-2)$$

Transition Weights

When we move, we accumulate weight

Let $n(t, i)$ and $e(t, i, j)$ be the TUG indexes for pair (t, i) and triple (t, i, j)

- When we move towards node i at time t, we accumulate $r^v_{n(t,i)} + \lambda_{n(t,i)}$
  - As an exception, moving towards -2 accumulates 0 weight
- When we move from node i at time 0, we also accumulate $r^v_{n(0,i)} + \lambda_{n(0,i)}$
- When we move from i to j at time t, we accumulate $r^e_{e(t,i,j)}$

In detail:

- If we move from i to j at time $t > 0$, we accumulate:
  - $r^v_{n(t,j)} + \lambda_{n(t,j)}$ for the destination node
  - $r^e_{n(t,i,j)}$ for the arc

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In detail:

- If we move from i to j at time $t = 0$, we accumulate:
  - $r^v_{n(t,i)} + \lambda_{n(t,i)}$ for the source node
  - $r^v_{n(t,j)} + \lambda_{n(t,j)}$ for the destination node
  - $r^e_{n(t,i,j)}$ for the arc

Transition Weights

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- When we move from i to j at time t, we accumulate $r^e_{e(t,i,j)}$

Let's see some examples:

- If we move from -1 to j at time t, we accumulate:
  - $r^v_{n(t,j)} + \lambda_{n(t,j)}$ for the destination node
- If we move from i to -2 at time $t = 0$, we accumulate:
  - $r^v_{n(t,i)} + \lambda_{n(t,i)}$ for the source node
- If we move from i to -2 at time $t > 0$, we accumulate 0

Allowed Transitions

We can use this information to populate tables

...And use them within a set of ALLOWED constraints:

$$
\begin{aligned}
&\text{ALLOWED}([x_0, x_1, y_0], T_0) && \text{for time } 0 \\
&\text{ALLOWED}([x_1, x_2, y_1], T_1) && \text{for time } 1 \\
&\ldots \\
&\text{ALLOWED}([x_{eoh-2}, x_{eoh-1}, y_{eoh-1}], T_{eoh-1}) && \text{for time } eoh - 1
\end{aligned}
$$

- The constraints allow only feasible transitions
- ...And compute the corresponding cost

As a result of propagation

...A restriction on the cost may result in pruned values

- This prevents us from considering many useless paths

Forbidden Transitions

We can handle the maximum wait restriction via forbidden transitions

...Using of course the $\textbf{FORBIDDEN}$ constraint

- Let n_w be the maximum number of allowed waits
- ...Then the forbidden transitions are:

$$\bar{T} = \{\{i\}_{h=0..n_w} \forall i \in V\}$$

I.e. any repetition of a node index for $n_w + 1$ times

Since we have $n_w = 2$ in our case, we forbid:

$$\{(0,0,0),(1,1,1),(2,2,2),(3,3,3)\}$$

I.e. we cannot spend 3 time steps on any node

Forbidden Transitions

We need to add $eoh - n_w$ constraints using this table

...So as to prevent excessive waiting over all the time horizon

$$
\begin{aligned}
&\text{FORBIDDEN}([x_0, \ldots, x_{n_w}], \bar{T}) && \text{for time } n_w \\
&\text{FORBIDDEN}([x_1, \ldots, x_{n_w+1}], \bar{T}) && \text{for time } n_w + 1 \\
&\ldots \\
&\text{FORBIDDEN}([x_{eoh-1-n_w}, \ldots, x_{eoh-1}], \bar{T}) && \text{for time } eoh - 1
\end{aligned}
$$

Both in this and in the previous case:

- The number of constraints grows linearly with $\boldsymbol{eoh}$
- The table size is relatively limited

Model Code

The code for this model is in the `solve_pricing_problem_maxwaits` function

We start by building a model using the Google Or-tools CP-SAT solver:

```
mdl = cp_model.CpModel()
```

Then we build the variables:

```
x = {i: mdl.NewIntVar(-2, mni, f'x_{i}') for i in range(eoh)}
c = {i: mdl.NewIntVar(minwgt, maxwgt, f'c_{i}') for i in range(1, eoh)}
z = mdl.NewIntVar(minwgt * eoh, maxwgt * eoh, 'z')
```

We are using integer variables even if have real weights:

- The trick is to rely on finite precision
- Given a weight w, we transform it as $round(w * p)$
- So that we obtain an integer, at the expense of some precision

Model Code

The code for this model is in the `solve_pricing_problem_maxwaits` function

We add all ALLOWED constraints

```
for t in range(1, eoh):
    # Build the table
    ...
    mdl.AddAllowedAssignments([x[t-1], x[t], c[t]], alw)
```

Then the FORBIDDEN constraints

```
if max_waits is not None:
    for t in range(max_waits, eoh):
        # Build the table
        ...
        mdl.AddForbiddenAssignments(scope, frb)
```

Model Code

The code for this model is in the `solve_pricing_problem_maxwaits` function

Finally, we define the total path weight:

```
mdl.Add(z == sum(c[i] for i in range(1, eoh)))
```

...And we define a constraint on the z variable:

```
mdl.Add(z < -round(alpha / prec))
```

- We do not need to minimize z (although we may)
- ...Since it is enough to search for paths with negative weight

Model Code

The code for this model is in the `solve_pricing_problem_maxwaits` function

We build a solver and set a time limit:

```
slv = cp_model.CpSolver()
slv.parameters.max_time_in_seconds = time_limit
```

We tell the solver not to stop after the first solution:

```
slv.parameters.enumerate_all_solutions = True
```

We define a callback to store all solutions:

```
class Collector(cp_model.CpSolverSolutionCallback):
```

...And the we solve the problem:

```
status = slv.SolveWithSolutionCallback(mdl, collector)
```

Maximum Wait Pricing in Action

Let's test our new code in an enumeration task

In [3]:
```
ncosts_n, npaths_n = util.solve_pricing_problem_maxwaits(tug, rflows_n, rpaths_n,
                                                         node_counts_n, arc_counts_n, max_waits=2,
                                                         cover_duals=mvc_duals,
                                                         alpha=alpha, filter_paths=False, max_paths=10)
print('COST: PATH')
util.print_solution(tug, ncosts_n, npaths_n, sort='ascending')
```

```
COST: PATH
0.20: 0,0 > 1,0 > 2,1 > 3,0
0.20: 2,1 > 3,0
0.20: 2,1 > 3,2
0.20: 0,0 > 1,0 > 2,1 > 3,2
0.28: 1,1 > 2,1 > 3,2
0.28: 1,1 > 2,1 > 3,0
0.34: 0,0 > 1,1 > 2,1 > 3,0
0.34: 0,0 > 1,1 > 2,1 > 3,2
0.48: 1,0 > 2,1 > 3,2
0.76: 0,1 > 1,0 > 2,1 > 3,2
0.76: 0,1 > 1,0 > 2,1 > 3,0
```

- Paths with more than 2 consecutive visits to the same node are not built

Maximum Wait Pricing in Action

Let's test our new code in an enumeration task

```
In [4]: ncosts_n, npaths_n = util.solve_pricing_problem_maxwaits(tug, rflows_n, rpaths_n,
                                    node_counts_n, arc_counts_n, max_waits=2,
                                    cover_duals=mvc_duals,
                                    alpha=alpha, filter_paths=True, max_paths=10)
        print('FLOW: PATH')
        util.print_solution(tug, ncosts_n, npaths_n, sort='ascending')
```

```
FLOW: PATH
-0.00: 1,0 > 2,0 > 3,2
-0.00: 1,0 > 2,0 > 3,3
-0.00: 2,0 > 3,0
-0.00: 2,0 > 3,2
-0.00: 2,3 > 3,3
-0.00: 2,0 > 3,3
-0.00: 0,0 > 1,0 > 2,3 > 3,3
```

- Some paths (erroneously) have negative waits due to the use of finite precision
- Our column generation code can handle this issue

Column Generation with Maximum Waits

Finally, we can test the column generation code itself

In [5]:
```
rflows_cg, rpaths_cg = util.trajectory_extraction_cg(tug, node_counts_n, arc_counts_n,
                                    alpha=alpha, min_vertex_cover=mvc, max_iter=30,
                                    verbose=1, max_paths_per_iter=10, max_waits=2, solver='
print('FLOW: PATH')
util.print_solution(tug, rflows_cg, rpaths_cg, sort='descending', max_paths=6)
sse = util.get_reconstruction_error(tug, rflows_cg, rpaths_cg, node_counts_n, arc_counts_n)
print(f'RSSE: {np.sqrt(sse):.2f}')
```

```
It.0, sse: 209.13, #paths: 27, new: 11
It.1, sse: 202.55, #paths: 38, new: 11
It.2, sse: 141.07, #paths: 45, new: 7
It.3, sse: 131.99, #paths: 51, new: 6
It.4, sse: 103.51, #paths: 59, new: 8
It.5, sse: 83.07, #paths: 59, new: 0
FLOW: PATH
8.29: 3,3
5.76: 0,2
5.39: 2,3
4.56: 3,2
3.29: 0,1 > 1,1 > 2,0 > 3,0
3.05: 1,2
...
RSSE: 9.11
```