Mixed Integer Linear Programming

Which is sort of famous, for valid reasons

Mixed Integer Linear Programming

A Mixed Integer Linear Program is a problem in the form

$$\text{argmin}_x \left\{ c^T x \mid Ax \geq b, x \geq 0, x_I \in \mathbb{Z} \right\}$$

- The cost function and all constraints are linear
- All variables are non-negative
- Some variables (those with index in I) are integer

MILP is an extremely powerful formalism

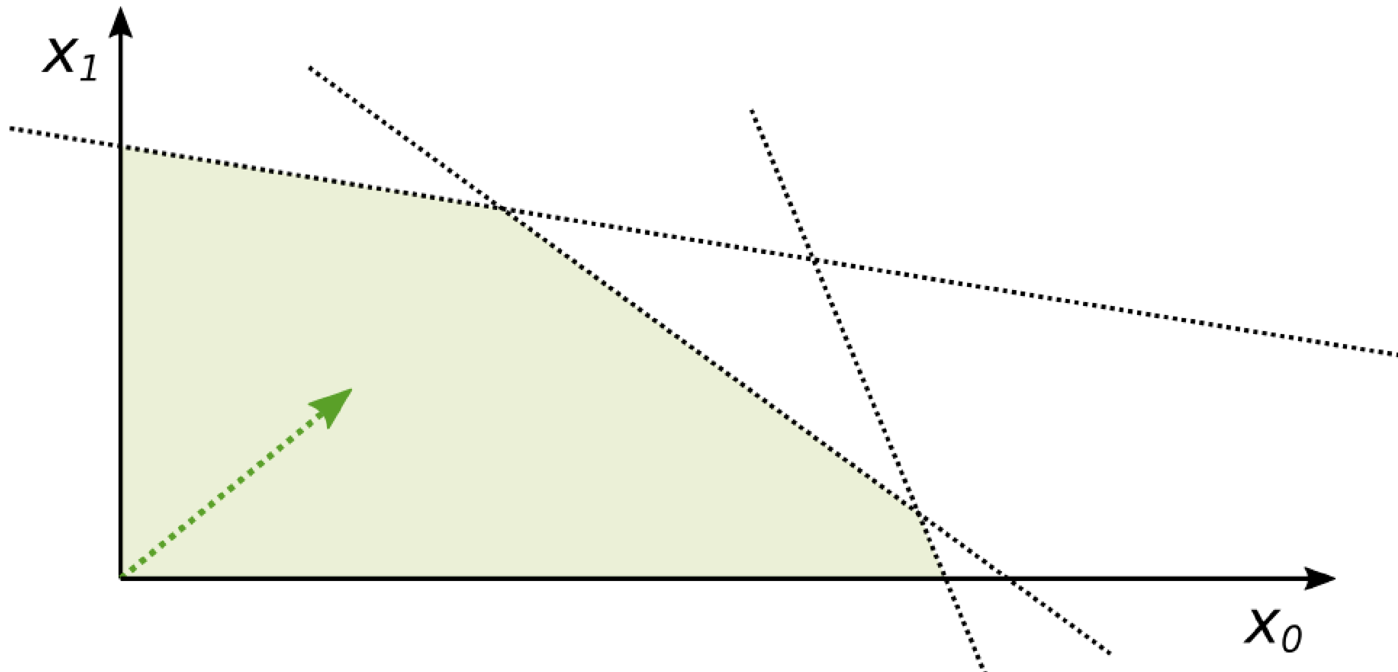
Thanks to the presence of integer variables

- ...Any combinatorial element can be modeled
- ...And non-linearity can be approximated

MILP solvers classically rely on three main techniques

Linear Relaxation

If we remove the integrality constraints from a MILP we obtain an LP



$$\text{argmin}_x \left\{ c^T x \mid Ax \geq b, x \geq 0 \right\}$$

This is called the linear (or LP) relaxation of the MILP

- The feasible space is defined via linear constraints $\Rightarrow$ is is a polytope
- The cost vector $\boldsymbol{c}$ is also the gradient and determines an optimization direction

Linear Relaxation

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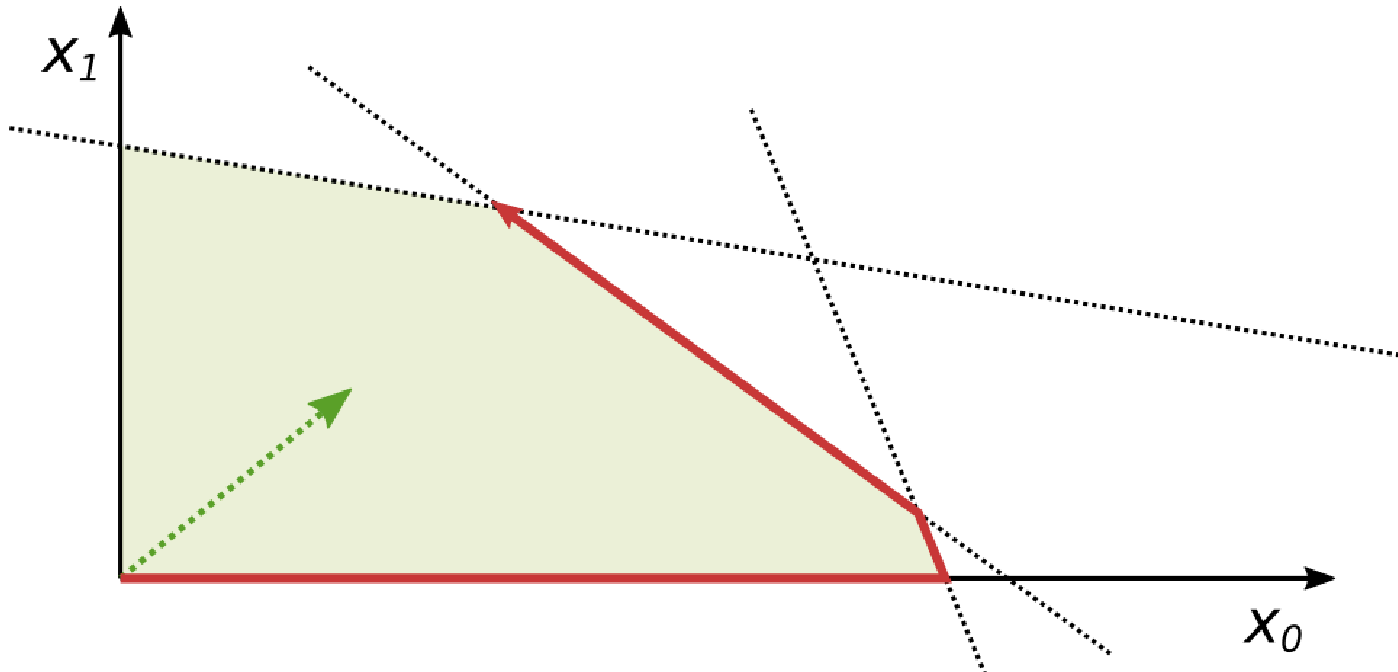

$$\text{argmin}_x \left\{ c^T x \mid Ax \geq b, x \geq 0 \right\}$$

LPs can be solved in pseudo-polynomial time via the [Simplex method]

- The method start from a polytope vertex
- ...And then moves between adjiacent vertexes until the optimum is reached

Linear Relaxation

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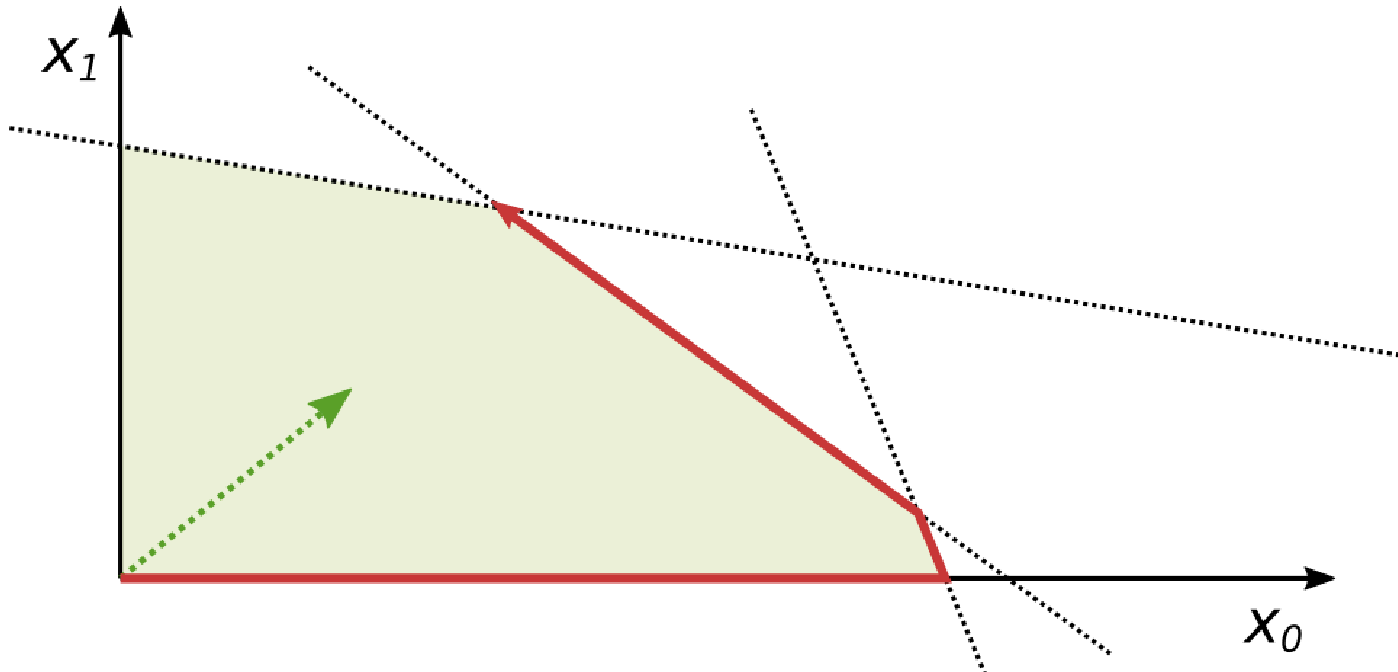


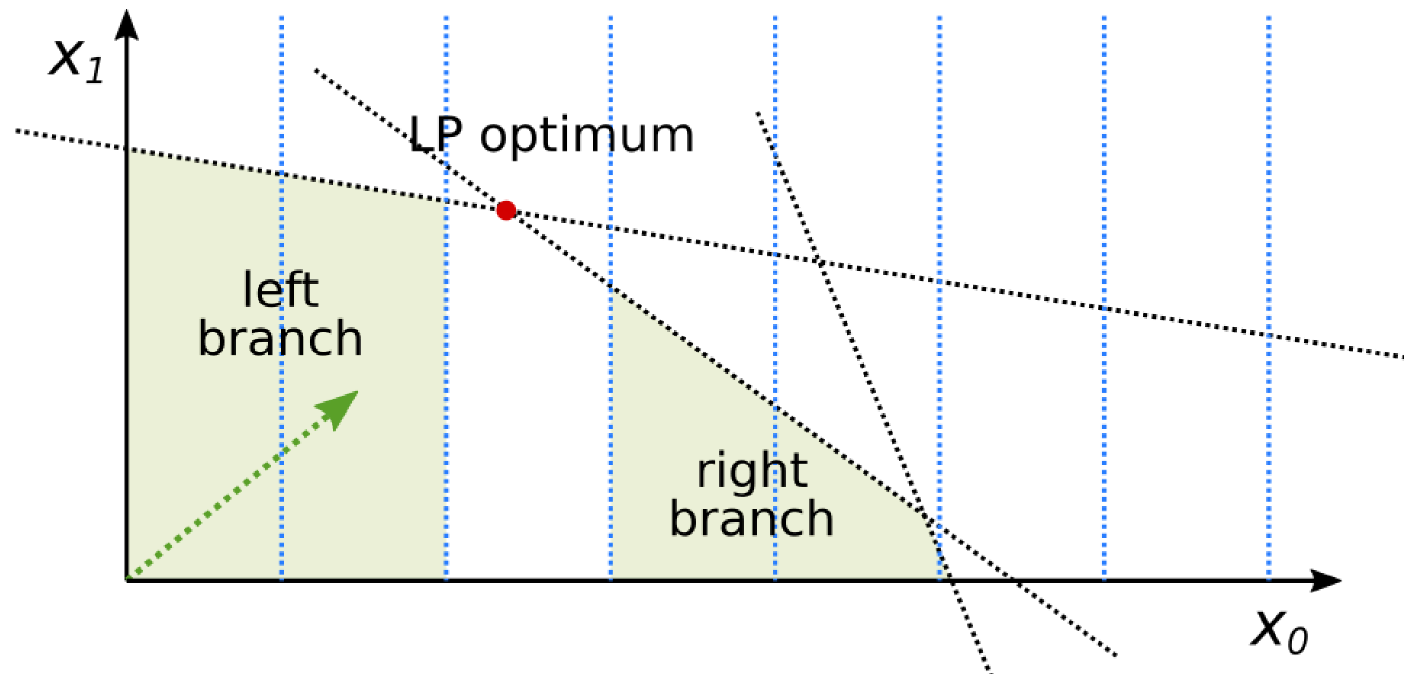
$$\text{argmin}_x \left\{ c^T x \mid Ax \geq b, x \geq 0 \right\}$$

LPs can be solved in polynomial time via Interior Point methods

- These used to be slower in practice than the Simplex, but not anymore
- In a MILP complex, the Simplex might still be preferred (later we will see why)

Technique #1: Branching

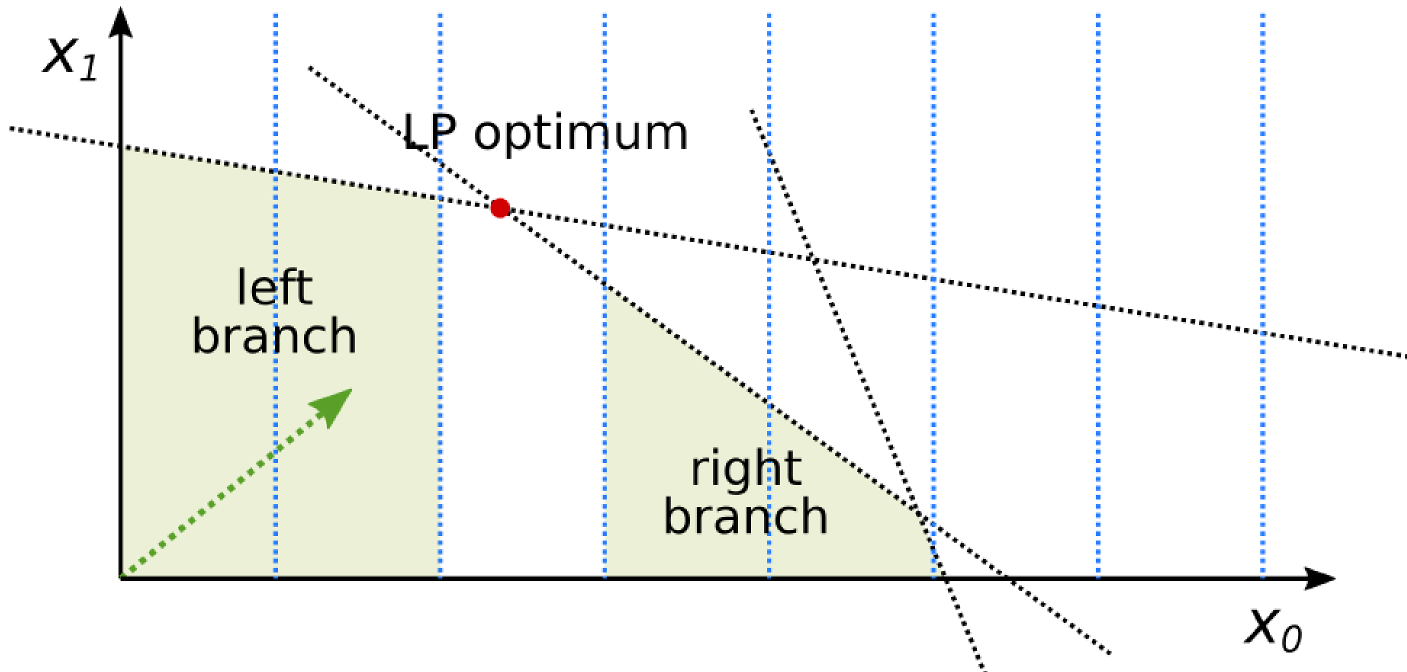
When tackling a MILP, we start by solving its LP relaxation



- If all integrality constraints are satisfied, we have found the true optimum
- If some x_j has a fractional value v_j, we split the problem in two:
  - In the first subproblem, we add the constraint $x_j \leq \lfloor v_j \rfloor$
  - In the second subproblem, we add $x_j \geq \lceil v_j \rceil$
- Then we can repeat the whole process

Technique #1: Branching

This approach is referred to as branching



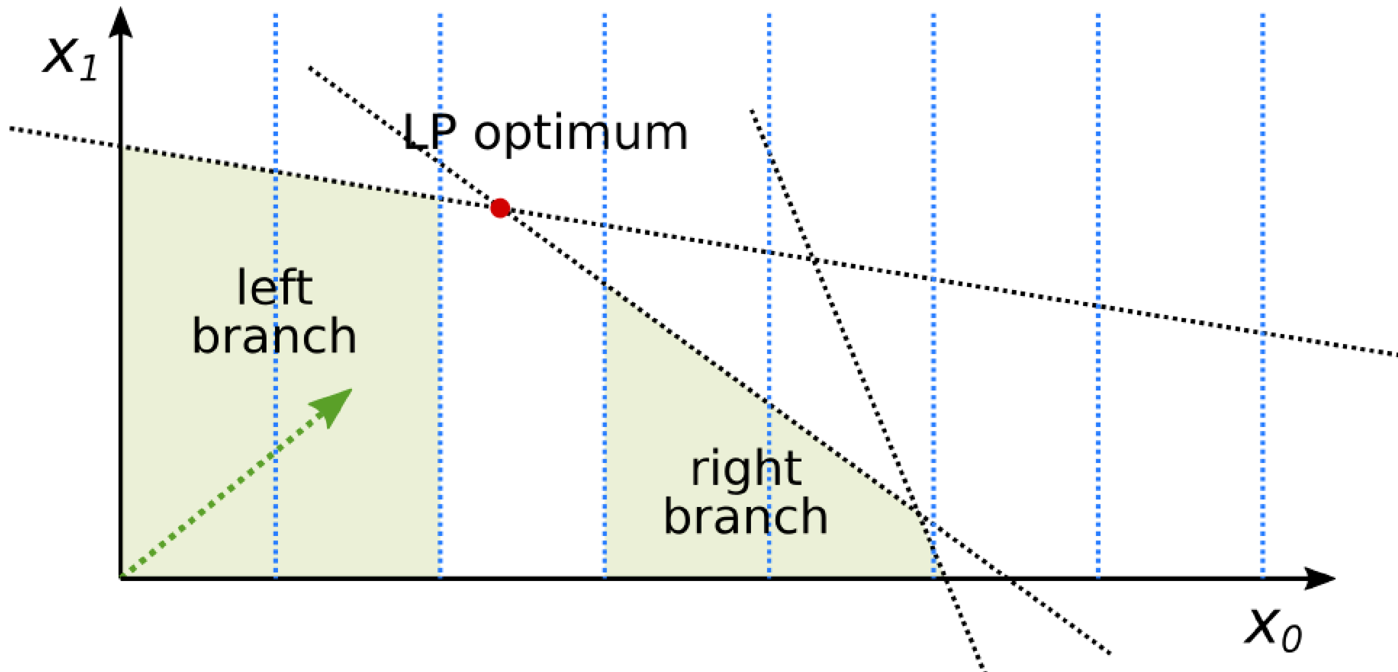
- The first subproblem is also known as the left branch
- The second as the right branch

Branching is the main method that makes MILP solvers complete

- In the worst case, we end up with a search tree having an exponential number of nodes
- ...But that's somewhat unavoidable, since solving MILP is NP-hard

Technique #1: Branching

This approach is referred to as branching



Branching is also the reason why the Simplex method is preferred to MILPs

- The Simplex method has a "dual" version
- ...Whose optimum can be updated efficiently when new constraints are added

...And you can guess that's a pretty common operation ;-)

Technique #2: Bounding

Let's look again at the LP relaxation

$$\text{argmin}_x \left\{ c^T x \mid Ax \geq b, x \geq 0 \right\}$$

- The problem has the same structure
- ...But a larger feasible space (that's why it is called a relaxation)

Hence, its optimal cost will be a lower bound (say lb) for the MILP

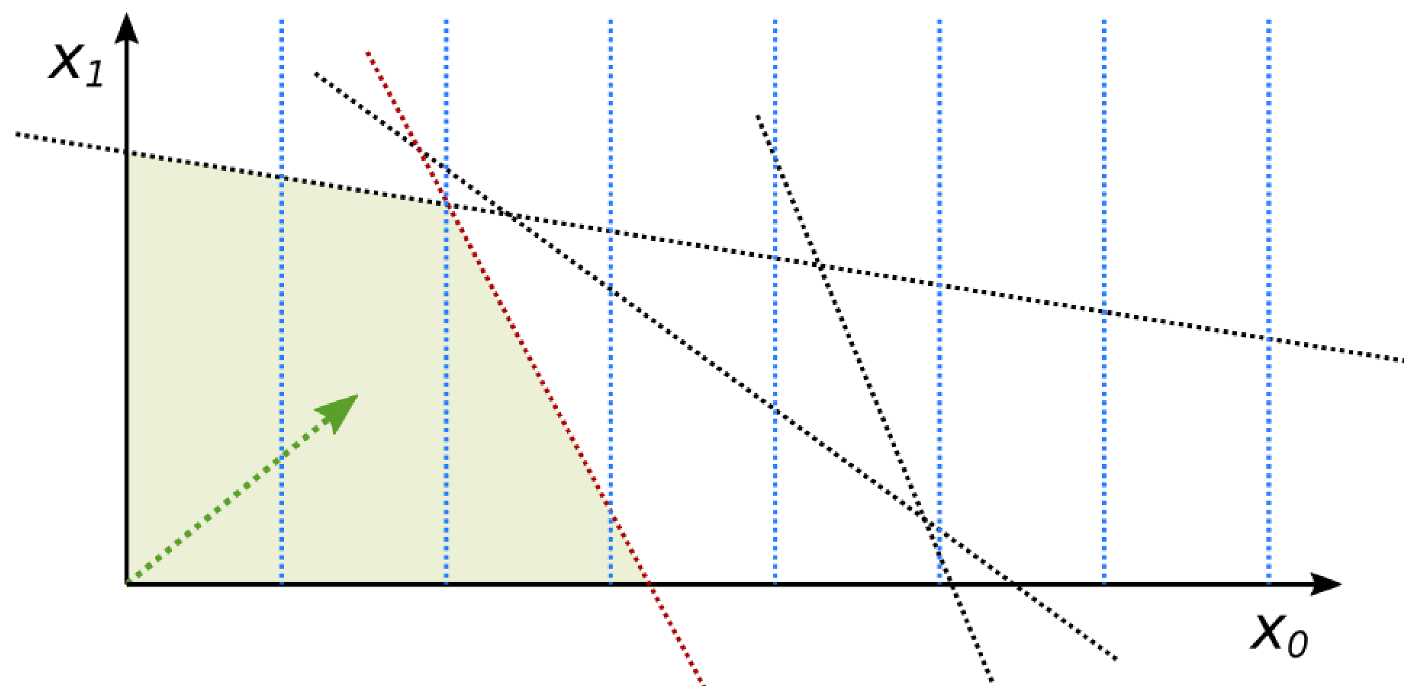
We can use this bound as an early stopping criterion

- Let x^* be the best (mixed-integer) solution we have found so far
- If for some node of the search tree we have $lb > c^T x^*$
- Then we have no hope of beating x^* and we can destroy (fathom) the node

Branching + Bounding = Branch & Bound

Technique #3: Cutting Planes

It is also common to speed-up MILP solution by using cutting planes



Cutting planes are linear inequalities inferred by relying on some property

- In MILP they are typically inferred based on integrality constraints
- They must be valid for any feasible solution
- They are useful if they force a fractional solution to become closer to integer

Technique #3: Cutting Planes

A common example is that of Gomory Cuts

While solving the simplex, we end up with many equalities in the form:

$$x_i + \sum_{j \in \bar{B}} \bar{a}_{ij} x_j = \bar{b}_i$$

- Where $x_i > 0$ with $i \in B$ and $x_j = 0, \forall j \in \bar{B}$
- $B =$ the set of indexes of non-zero variables in the current LP solution (base)
- $\bar{B} =$ the set of indexes of zero variables in the current LP solution
- We will assume all variables are integer, for simplicity

We can rewrite the equation as

$$x_i + \sum_{j \in \bar{B}} (\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor + \lfloor \bar{a}_{ij} \rfloor) x_j = \bar{b}_i - \lfloor \bar{b}_i \rfloor + \lfloor \bar{b}_i \rfloor$$

Technique #3: Cutting Planes

By simple algebraic manipulation we can then get:

$$x_i + \sum_{j \in \bar{B}} \lfloor \bar{a}_{ij} \rfloor x_j - \lfloor \bar{b}_i \rfloor = - \sum_{j \in \bar{B}} (\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor) x_j + (\bar{b}_i - \lfloor \bar{b}_i \rfloor)$$

We will build an inequality that is valid for any feasible, integer point:

- The right-most part is necessarily < 1, since:
  - $\bar{b}_i - \lfloor \bar{b}_i \rfloor$ is positive and fractional
  - Each $\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor$ is positive (and fractional)
  - Each x_j must be ≥ 0
- The left-most part is necessarily an integer, since:
  - $\lfloor \bar{b}_i \rfloor$ is integer and each $\lfloor \bar{a}_{ij} \rfloor$ is integer
  - Variables are integer as per our assumption

Technique #3: Cutting Planes

By simple algebraic manipulation we can then get:

$$x_i + \sum_{j \in \bar{B}} \lfloor \bar{a}_{ij} \rfloor x_j - \lfloor \bar{b}_i \rfloor = -\sum_{j \in \bar{B}} (\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor) x_j + (\bar{b}_i - \lfloor \bar{b}_i \rfloor)$$

- Hence, the right-most part should be < 1 and integer
- ...Meaning that it must be ≤ 0

$$-\sum_{j \in \bar{B}} (\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor) x_j + (\bar{b}_i - \lfloor \bar{b}_i \rfloor) \leq 0$$

And from here:

$$\sum_{j \in \bar{B}} (\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor) x_j \geq (\bar{b}_i - \lfloor \bar{b}_i \rfloor)$$

Technique #3: Cutting Planes

This inequality is the Gomory Cut

$$\sum_{j \in \bar{B}} (\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor) x_j \geq (\bar{b}_i - \lfloor \bar{b}_i \rfloor)$$

- Now, in the current solution we have $x_i = \bar{b}_i$
- If we target a x_i that should be integer, but it's fractional in the current solution

Then we have $\bar{b}_i - \lfloor \bar{b}_i \rfloor > 0$

- Combined with the fact that $x_j = 0, \forall j \in \bar{B}$ in the current solution
- We have that the cut is actually making the solution no longer feasible

Branching + Bounding + Cutting Planes = Branch & Cut

- Using cutting planes can speed up the solution process considerably
- But it's best not to overdo it, since subsequent cuts may become weaker

Some Considerations

We have just scratched the surface with MILP

Modern MILP solver do much more:

- Presolving
- Constraint propagations
- Symmetry breaking
- ...

MILP methods have a long history

- There is a huge gap between the solver performance
- The best solvers (Gurobi, Cplex, Mosek) are commercial (free for academics)
- Then you have a single semi-free solver (SCIP)
- ...A good free solver (CBC)
- ...And finally there is stuff you should not touch (glpk, lpsolve)

Some References

If you are interested in MILP, you might check one of the following:

- "Introduction to Linear Optimization", by Bertsimas and Tsitsiklis
- "Integer Programming" by Conforti, Cornuéjols, and Zambelli