Project 4.1

What are the Complex Numbers?

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1 Problem 1

What is a complex number?

A complex number is a mixture of real and imaginary numbers. In fact, the complex set of numbers, $\mathbb{C}$, is *the* overarching set of numbers, encompassing real and irrational numbers, since it is a mixture of both. Nothing encompasses $\mathbb{C}$.

Any complex number can be expressed in the form $a + bi$, where a and b are both real numbers, and where $i = \sqrt{-1}$. The number derives its complexity from the imaginary part, bi, and the real part, a.

The signature of complex numbers is the value i. What truly separates real numbers from the domain of complex numbers is this incalculable value. Thinking about all values in the domain $\mathbb{R}$ and operations we can do on them, besides the division by zero exception, we can describe practically any solution besides perhaps the most prominent one, $\sqrt{-1}$. In other words, there are physical or mental equivalents involving numbers in $\mathbb{R}$ except the square root of -1.

Despite their imaginary components, complex numbers are remarkably tangible. For many polynomials, complex numbers are irreducible roots.

For a common polynomial:

$$x^2 + 10x + 169$$

The roots are $x = 5 + 12i$ and $x = 512i$.

Another way (or domain) to unearth imaginary numbers is the polar plane. Unlike the Cartesian coordinate system which is limited to the $\mathbb{R}$ domain, the polar coordinate system can visually represent complex numbers.

The power of the polar coordinate system that is nonexistent in the Cartesian system is the ability to represent values as vectors.

I will go into this in more detail as I explain the complex number form and how we can utilize it to perform multiplication and division of complex numbers.

In short, like the Cartesian coordinate system, the polar form takes in two values, a real and imaginary value — the real axis is the horizontal one (the x) and the imaginary axis is the vertical one (the y). Recalling the form, any complex number in the form $a + bi$ effectively has a horizontal value of a and a vertical value of b since a is real and b is the coefficient of i.

However, the polar form goes beyond the Cartesian coordinate system by representing values as vectors. Thus, the polar system takes in an angle θ as well as a magnitude r. Therefore, where a Cartesian coordinate system has the following property:

$$(x_1, y_1) = (x_2, y_2) \Rightarrow x_1 = x_2, y_1 = y_2$$

A polar system coordinate can be mapped like so:

$$(r, \theta) = (r, \theta + 2\pi k)$$

What is interesting is that the polar form could equally represent real numbers and complex numbers alike. While real numbers they will be simply plotted on the real axis, complex numbers become vectors. Because it could represent values in $\mathbb{C}$, the polar system can thus represent any value in any set in number theory.

The distribution of complex numbers and the ability to visualize goes to show that complex numbers are not only all around us but also quite tangible — despite their imaginary parts throwing them into the abstract, uncountable worlds.

Nevertheless, imaginary numbers are integral to the field of mathematics. They are essential in algebraic calculations where they intermediately appear in electrical engineering and are spread through many mathematical calculations.

2 Problem 2

Explain addition and multiplication of complex numbers. This should be done using both Cartesian and polar coordinates. Explain, in your own words, why addition and multiplication are defined the way they are.

In order to explain both, I will show how the polar form for complex numbers was derived.

As I touched on the on the polar form in the introduction, the polar form represents complex numbers as vectors, with a magnitude r and an angle θ.

Only by ways of a magnitude and direction can a complex number be represented in terms of understandable numbers in the domain of R.

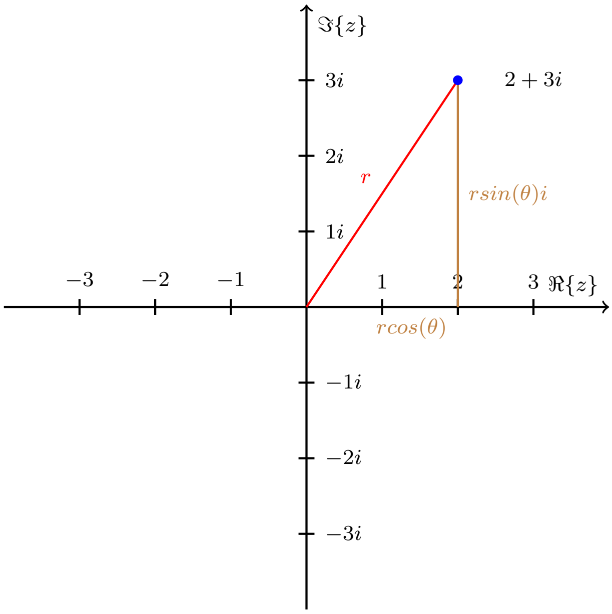
Everything stems from the following representation. Below, I drew a graph of a complex number $2 + 3i$ graphed below:

As you can see, the vector in this plot has a magnitude of r and the angle is θ.

From the above graph, it is clear that the complex number form $a + bi$ can be broken down into the following relations:

$$a + bi \Rightarrow a = rcos(\theta), bi = rsin(\theta)i$$

By extension, the polar form provides a convenient framework to operate on complex numbers with an interface that involves understandable and verifiable operations on real numbers. What makes it convenient is mainly the properties of trigonometric functions such as *cosine* and *sine*; their properties will be integral in demonstrating how multiplication and division work for complex numbers.



Knowing what we know from the plot, any complex number z can be represented like so:

$$z = r\left(cos(\theta) + isin(\theta)\right)$$

By taking the common coefficient r out from both parts of the complex number a and bi.

By extension, if we have two numbers z_1 and z_2:

$$z_1 = r_1\left(cos(\theta_1) + isin(\theta_1)\right) z_2 = r_2\left(cos(\theta_2) + isin(\theta_2)\right)$$

Where the magnitudes r's and the θ's are unique to each of the complex numbers.

Now, this definition is not too useful if I don't know what r and θ are. I can write as many complex numbers in polar form as I want, but it would all go to waste if I don't know what the coefficients I'm dealing with.

The beauty of the polar form is that the polar system provides us with the tools to find just that.

For r: Recalling the Pythagorean theorem:

$$c^2 = a^2 + b^2$$

I can simply apply this theorem to calculate my r value. In the polar form, I have a and bi in $a + bi$. Surprisingly, those two are conveniently correlated. Recalling that the horizontal axis of the polar system is a and the vertical axis is the value of b. The only variable that is different in name is r, and it is the hypotenuse length denoted by c.

Therefore, to find r I simply use:

$$r^2 = a^2 + b^2$$

For θ: The process is a bit more complicated, but the idea is the same — use trigonometric formulas to relate a and b in the complex number in the form $a + bi$.

To find θ, one must recall the trigonometric function that allows us to calculate the angle given the opposite and adjacent side. Due to the polar system, we have access to sides b (opposite) and a (adjacent). In order to relate the two, *tangent* is the appropriate choice.

We have:

$$tan(\theta) = \frac{b}{a}$$

However, I see that θ is trapped inside the *tangent*. Therefore, to get θ, I take the *arctan* of both sides.

Therefore, to find θ I use the formula:

$$\theta = arctan\left(\frac{b}{b}\right)$$

Now, for multiplication, I simply use the interface the polar form provides us and multiply the trigonometric equivalents:

$$z_1z_2 = r_1\left(cos(\theta_1) + isin(\theta_1)\right) * r_2\left(cos(\theta_2) + isin(\theta_2)\right)$$

$$z_1z_2 = r_1r_2(cos(\theta_1)cos(\theta_2) + icos(\theta_1)sin(\theta_2) + isin(\theta_1)cos(\theta_2) + i^2sin(\theta_1)sin(\theta_2))$$

Using the following trigonometric identities:

$$sin(a+b) = cos(a)sin(b) + sin(a)cos(b)$$
$$cos(a+b) = cos(a)cos(b) - sin(a)sin(b)$$

I get the following:

$$z_1z_2 = r_1r_2\left(cos(\theta_1+\theta_2) + isin(\theta_1+\theta_2)\right)$$

In the Cartesian system, multiplication is harder than simply using the formulas that I derived above — yet, it is more intuitive.

Similarly to how we perform addition in the Cartesian coordinate system, multiplying complex numbers in the Cartesian system involves treating the real and imaginary parts as separate terms — given complex numbers in the $a + bi$ form, and treat a and bi as separate terms.

Suppose I'm given the following numbers in the Cartesian form of $a_1 + b_1i$ and $a_2 + b_2i$:

$$z_1 = a_1 + b_1i, z_2 = a_2 + b_2i$$

Multiplying them, therefore, yields:

$$z_1z_2 = (a_1 + b_1i)(a_2 + b_2i)$$

Using the Distributive Property and, as I said, treating the a's and bi's as separate terms, it follows:

$$z_1z_2 = (a_1a_2 + a_1b_2i + b_1ia_2 - b_1b_2)$$

Therefore, I get the following:

$$z_1z_2 = a_1a_2 + a_1b_2i + b_1ia_2 - b_1b_2$$

From here, one typically simplifies the i terms as they are common and the real parts on both sides of the left-hand expression. It is evident that the Cartesian coordinate system for multiplying complex numbers is not the best approach and it is best to use the polar system. Also, if one is dealing with fractions, then he or she may use the complex conjugate to multiply the numerator and the denominator in order to turn it into a usable form, if needed.

For addition, the process is intuitive in the Cartesian system and is harder in polar form.
To add in the Cartesian form, I simply use the $a + bi$ form and treat a and bi as separate terms.
So, suppose I have the following two complex numbers z_1 and z_2:

$$z_1 = a_1 + b_1i, z_2 = a_2 + b_2i$$

Adding them, therefore, I get the following:

$$z_1 + z_2 = a_1 + a_2 + i(b_1 + b_2)$$

Subtraction involves the same process.

Now, for addition in polar coordinates, I have to convert the complex number into polar form.

Suppose I have the following polar coordinates representing complex numbers in the form (r_1, θ_1) and (r_2, θ_2):

To add these two polar coordinates, I use the trigonometric relations derived from the plot to get the x and y components:

$$y_n = r_n(sin(\theta_n))x_n = r_n(cos(\theta_n))$$

Calculating the components:

$$y_1 = r_1(sin(\theta_1))x_1 = r_1(cos(\theta_1))$$
$$y_2 = r_2(sin(\theta_2))x_2 = r_2(cos(\theta_2))$$

Finally, I simply add the components together to get the new x and y components:

$$y = y_1 + y_2 = r_1(sin(\theta_1)) + r_2(sin(\theta_2))$$

$$x = r_1(cos(\theta_1)) + r_1(sin(\theta_1))$$

The polar system for addition is much less effective and thus the Cartesian system is best for addition and subtraction. Note, for fractions, the complex conjugate must be used on the fraction by multiplying the numerator and denominator by it and turn the fraction into a usable form for addition or subtraction.

For division, the process is almost identical, but simply using other trigonometric identities. The same formulas for find r and θ applies for both operations.

$$\frac{z_1}{z_2} = \frac{r_1\left(cos(\theta_1) + isin(\theta_1)\right)}{r_2\left(cos(\theta_2) + isin(\theta_2)\right)}$$

Then, as we approach any complex fraction, multiplying by the complex conjugate in order to get a usable fraction.

It follows:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\frac{(cos(\theta_1) + isin(\theta_1))}{(cos(\theta_2) + isin(\theta_2))} * \frac{(cos(\theta_2) - isin(\theta_2))}{(cos(\theta_2) - isin(\theta_2))}\right)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\frac{cos(\theta_1)cos(\theta_2) + sin(\theta_1 sin(\theta_2)}{cos^2(\theta_2) + sin^2(\theta_2)} + i\left(\frac{sin(\theta_1)cos(\theta_2) + cos(\theta_1)sin(\theta_2)}{cos^2(\theta_2) + sin^2(\theta_2)}\right)\right)$$

Therefore, for division:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(cos(\theta_1 - \theta_2) + isin(\theta_1 - \theta_2)\right)$$

3 Problem 3

Explain why Euler's formula $e^{i\theta} = cos(\theta) + isin(\theta)$ is true.

To explain why Euler's formula is true, I'll use the concepts we have learned in class, specifically the common McLaurin Series for *sine*, *cosine*, and e.

Here are the McLaurin Series equivalents for these functions:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

I chose these three because they make up the components of Euler's formula. To turn these equations into Euler's formula, I will perform a few substitutions that will get these components closer to how they appear in Euler's formula.

First, I substitute $x = i\theta$ for x, it follows:

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots$$

Now, an interesting pattern appears — it seems that I can break down $e^{i\theta}$ into *sine* and *cosine* equivalents. This is logical not only because of the McLaurin Series for *sine* and *cosine*, each taking on the odd and even terms of e^x but also because it follows the structure of Euler's formula — $e^{i\theta}$ defined in terms of *sine* and *cosine*.

Therefore:

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots$$

Becomes:

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = 1 + i\theta - \frac{(\theta)^2}{2!} - i\frac{(\theta)^3}{3!} + \frac{(\theta)^4}{4!} + i\frac{(\theta)^5}{5!} - \cdots$$

Rearranging, it follows:

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ix - i\frac{x^3}{3!} + i\frac{x^5}{5!} - \cdots$$

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

Therefore, replacing these partial sums with Taylor Series equivalents of *sine* and *cosine*, I get Euler's formula:

$$e^{i\theta} = cos(\theta) + isin(\theta)$$