Project 2.3

Extending Euler's Method

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This project explores extensions of Euler's Method to systems of first-order differential equations and higher order differential equations. We recall Euler's method (See §9.2 in the textbook) in this project.

(1) For the first problem, I go more in depth into the underlying process for deriving the recursive relations that are integral in Euler's calculation — as
Professor Paulin had done in the opening part of the project.
This is the first half of methods that I need to solve the third problem.

(2) For the second problem, I rewrite a second-order differential equation into a system of simpler, first-order equations.
This completes the methods that I need to solve the third problem.

(3) Finally, for the third problems, I go into less depth and rather explore the applications of the ideas that I explored more closely in the first and second problems.

1 Problem 1

Consider the system of differential equations:

$$u'(t) = u + y$$
$$y'(t) = tuy$$
$$u(0) = 1$$
$$y(0) = 0$$

Using Euler's (extended) method (with $\frac{1}{3}$) estimate $y(1)$ and $u(1)$.

1.1 Definitions from Problem Statement

From first glance, the following can be written:

$$u_0 = 1, y_0 = 0$$

By extension:

$$u(t_0) = u_0 = 1$$
$$y(t_0) = y_0 = 0$$

I'm also introducing the following definitions:

$$F(t_n, u_n, y_n) = u'$$
$$G(t_n, u_n, y_n) = y'$$

It follows:

$$F(t_0, u_0, y_0) = \frac{u_1 - u_0}{h}$$

$$G(t_0, u_0, y_0) = \frac{y_1 - y_0}{h}$$

Where F and G denote the slopes of u and y, respectively.

1.2 Determining the Slopes

The next step is to find the recursive equations. These are integral in Euler's method. In the same way I found $F(t_0, u_0, y_0)$ and $G(t_0, u_0, y_0)$, I can write the slopes for $F(t_1, u_1, y_1$ and $G(t_1, u_1, y_1)$, respectively:

$$F(t_1, u_1, y_1) = \frac{u_2 - u_1}{h}$$

$$G(t_1, u_1, y_1) = \frac{y_2 - y_1}{h}$$

Similarly, I can do the same for the slope for the succeeding set of points in Euler's calculation:

$$F(t_2, u_2, y_2) = \frac{u_3 - u_2}{h}$$
$$G(t_2, u_2, y_2) = \frac{y_3 - y_2}{h}$$

Continued...

Now, I could repeat this operation for $n+1$ and n terms, therefore finding the slopes for any term up to and including n:

$$F(t_n, u_n, y_n) = \frac{u_{n+1} - u_n}{h}$$

$$G(t_n, u_n, y_n) = \frac{y_{n+1} - y_n}{h}$$

1.3 Solving for the Recursive Relations

Now to solve for the recursive solutions, I simply need to use the slopes expressions that I wrote in 1.2 and use them in here.

In addition, for a correct implementation of Euler's method, I need to make sure that I calculate the slope from the previous point, not from a possibly shifted original starting point of y_0 or u_0.

For this reason, I need to use the idea of a point-slope formula. The method I'll use involves multiplying the slope expressions by the run, h, in order to get the rise. I get the following:

$$rise_{0u} = h\left(\frac{u_1 - u_0}{h}\right)$$

$$rise_{0y} = h\left(\frac{y_1 - y_0}{h}\right)$$

$$u_1 = rise_{0u} + u_0 \Rightarrow u_1 = h\left(\frac{u_1 - u_0}{h}\right) + u_0$$

$$y_1 = \text{rise}_{0y} + y_0 \Rightarrow y_1 = h\left(\frac{y_1 - y_0}{h}\right) + y_0$$

By the same reasoning, I could write the following for the next slope point:

$$rise_{1u} = h\left(\frac{u_2 - u_1}{h}\right)$$

$$rise_{1y} = h\left(\frac{y_2 - y_1}{h}\right)$$

$$u_2 = rise_{1u} + u_1 \Rightarrow u_2 = h\left(\frac{u_2 - u_1}{h}\right) + u_1$$

$$y_2 = rise_{1u} + y_1 \Rightarrow y_2 = h\left(\frac{y_1 - y_0}{h}\right) + y_1$$

Continued...

By extension, the slope of n terms can be expressed below. I will use n to determine the general recursive equations that I will use in my Euler's calculation.

$$rise_{nu} = h\left(\frac{y_{n+1} - y_n}{h}\right)$$

$$rise_{ny} = h\left(\frac{y_{n+1} - y_n}{h}\right)$$

Now that I have the rise, I need to also include the starting point (u_0 and y_1) in case the functions F and G do not start from the origin. I get the following:

$$u_{n+1} = rise_{nu} + u_n \Rightarrow u_{n+1} = h\left(\frac{u_n - u_{n-1}}{h}\right) + u_n$$

$$y_{n+1} = rise_{ny} + y_n \Rightarrow y_{n+1} = h\left(\frac{y_n - y_{n-1}}{h}\right) + y_n$$

And I now have my recursive relations.

Finally, I now have to relate t to h, as the *run* is incremented by h with each successive slope calculation.

By the same reasoning of the above recursive equations, I could extend the relationship of t_0 with t_1 to t_{n+1} to t_n:

$$t_1 = t_0 + h$$

By the same logic, I can write:

$$t_2 = t_1 + h \Rightarrow t_3 = t_2 + h$$

And extending this to the nth term brings me to the following:

$$t_{n+1} = t_n + h$$

Therefore, I have the recursive relations defined.

$$t_{n+1} = t_n + h \tag{1}$$

$$u_{n+1} = hF(t_n, u_n, y_n) + u_n \tag{2}$$

$$y_{n+1} = hG(t_n, u_n, y_n) + y_n \tag{3}$$

1.4 Estimating Values with Euler's Method

We are now in a position to solve the problem.
The following is a summary of formulas: For easy reference, the following Initially, we were given the following system of differential equations (A and B):

$$u'(t) = u + y$$

$$y'(t) = tuy$$

And the following initial conditions:

$$u(0) = 1$$

$$y(0) = 0$$

With step size: $h = \frac{1}{3}$

The following are the formulas and definitions I derived:

$$F(t_n, u_n, y_n) = u_n + y_n \tag{A}$$

$$G(t_n, u_n, y_n) = t_n u_n y_n \tag{B}$$

$$t_{n+1} = t_n + h \tag{1}$$

$$u_{n+1} = hF(t_n, u_n, y_n) + u_n \tag{2}$$

$$y_{n+1} = hG(t_n, u_n, y_n) + y_n \tag{3}$$

Applying Euler's Method (Extended):

There are three iterations of calculation involved for a step size of $\frac{1}{3}$ ([$h = \frac{1}{3}$), from $t = 0$ to $t = 1$ for we need to estimate the values of $y(1)$ and $u(1)$, where $t = 1$.

For convenient calculation, I will use the following table. Each row represents an iteration (n value) of the approximation. In the columns, t_n, u_n, y_n represent the variables at the nth iteration while $F(t_n, u_n, y_n)$ and $G(t_n, u_n, y_n)$ represent the slopes at the nth iteration.

n	t_n	u_n	y_n	$F(t_n, u_n, y_n)$	$G(t_n, u_n, y_n)$	u_{n+1}	y_{n+1}
0	0	1	0	1	0	4/3	0
1	1/3	4/3	0	4/3	0	7/9	4/3
2	2/3	7/9	4/3	19/9	56/81	40/27	380/243
3	1	40/27	380/243				

Table 1: The first row is directly taken from the initial conditions. I use 1 for t_n, 2 for u_n, 3 for y_n.
For the slopes, I use A for F and I use B for G.

Continued...

Therefore, as a result of the calculation, I approximate $y(1)$ and $u(1)$ to be:

$$\boxed{y(1) = \frac{380}{243} = 1.563786...}$$

$$\boxed{u(1) = \frac{40}{27} = 1.\overline{481}}$$

2 Problem 2

Consider the second-order initial-value problem:

$$a(t)y'' + b(t) + c(t)y' + y = 0, \;\; y(t_0) = y_0, \;\; y'(t_0) = u_0. \tag{2}$$

Express this as a system of equations similar to question 1. Hint: Think about y' as a new variable.

2.1 Splitting the Equation

The hint is correct — we must tackle this second-order differential equation by separating the y'' term and the y' term from each other in the same way that x^2 is distinct from x.

I introduce two definitions:

$$f = y$$
$$g = y'$$

Differentiating them:

$$f' = y' \tag{A}$$
$$g' = y'' \tag{B}$$

By matching the differentiated terms in the second-order differential equation, I'm doing this step in order to then solve for f' and g' and thus come up with equations similar to Problem 1 (u' and y').

2.2 Solving for f and g

I have differentiated f and g and thus matching them to the differential equation in the problem statement. Now, I can now solve for f' and g', my two new, split differential equations:

Taking equation 2, I solve for y' (f') and y'' (g'):

Equation 2, for reference:

$$a(t)y'' + b(t) + c(t)y' + y = 0$$

Solving for y':

$$y' = g$$
$$g = y'$$

Using the definition of g.

Continued...

Solving for y'':

$$\begin{aligned} a(t)y'' + b(t) + c(t)y' + y &= 0 \\ a(t)y'' &= -c(t)y' - b(t) - y \\ y'' &= \frac{-c(t)y' - b(t) - y}{a(t)} \end{aligned}$$

Therefore:

$$f' = y' = g \qquad (By\ A\ in\ 2.1)$$

$$g' = y'' = -\frac{c(t)y' + b(t) + y}{a(t)} \qquad (By\ B\ in\ 2.1)$$

Replacing y' and y" with our new functions f and g yields:

$$f' = g \quad \textit{(Already replaced)}$$

$$g' = -\frac{c(t)g + b(t) + f}{a(t)}$$

2.3 Converting Initial Conditions to New Equations

Finally, I now have to convert the initial conditions into the two new differential equations. Here are the initial conditions:

$$y(t_0) = y_0, \;\; y'(t_0) = u_0$$

Now, I take these conditions and feed them through the new differential equations f and g:

$$\begin{aligned} y(t_0) = y_0 &\Rightarrow f(t_0) = y_0 \\ y'(t_0) = u_0 &\Rightarrow g(t_0) = y_0 \end{aligned}$$

Since:

$$\begin{aligned} f(t_0) &= y(t_0) \\ g(t_0) &= y'(t_0) \end{aligned}$$

By equations A and B (respectively) in 2.1

Therefore, the new system of differential equations is:

$$f(t_0) = y_0$$

$$g(t_0) = u_0$$

$$f'(t) = g$$

$$g'(t) = \frac{c(t)g + b(t) + f}{a(t)}$$

3 Problem 3

3. Consider the following initial-value problem:

$$2y'' - 5y' + ty = 0, \;\; y(0) = 1, \;\; y'(0) = 2. \tag{3}$$

In light of question 2, apply Euler's (extended) method (with $h = 1$) to this problem to estimate $y(3)$.

3.1 Splitting the Equation into a System of Eqs.

The initial-value problem is a second-order equation that needs to be split into two differential equations. Going by the logic of Problem 2, I perform the following steps to split the equation into two new differential equations:

I name two new functions f and g:

$$f = y$$

$$g = y'$$

Differentiating them:

$$f' = y'$$

$$g' = y''$$

In the spirit of Problem 2, I'm treating the y' and the y'' variables as new terms.

Now, I solve for f' and g', (y' and y'' respectively). In this way, I generate two differential equations from one.

Solving for y':

$$y' = g$$
$$g = y'$$

Using the definition of g.

Solving for y'':

$$2y'' - 5y' + ty = 0$$
$$2y'' = 5y' - ty$$
$$y'' = \frac{5y' - ty}{2}$$

Therefore:

$$f' = y' = g$$
$$g' = y'' = \frac{5y' - ty}{2}$$

Continued...

Replacing y' and y'' with our new functions f and g yields:

$$f' = g \quad \textit{(Already replaced)}$$

$$g' = \frac{5g - tf}{2}$$

Finally, converting the initial conditions:

$$y(0) = 1, \;\; y'(0) = 2$$

I feed these initial conditions defined in terms of their old function through my new functions, f and g:

$$y(0) = 1 \Rightarrow f(0) = 1$$

$$y'(0) = 2 \Rightarrow g(0) = 2$$

Hence, I have the following system:

$$f(0) = 1 \tag{A}$$

$$g(0) = 2 \tag{B}$$

$$f' = g \tag{1}$$

$$g' = \frac{5g - tf}{2} \tag{2}$$

3.2 Writing Recursive Formulas for Euler's Method

In light of Problem 1, here are the steps I took to write the recursive equations for Euler's calculation.

As in Problem 1, I will use the variable n to signify the current term
(nth iteration in the calculation.

I also use the variable h which signifies the step size,
(introduced in the problem statement).

For reference, the system I derived in §3.1 is below:

$$f(0) = 1 \tag{A}$$

$$g(0) = 2 \tag{B}$$

$$f' = g \tag{1}$$

$$g' = \frac{5g - tf}{2} \tag{2}$$

Continued...

First, I define my slope formulas, I'll call them F and G:

$$F(t_n, f_n, g_n) = f'$$

$$G(t_n, f_n, g_n) = g'$$

It follows:

$$F(t_0, f_0, g_0) = \frac{f_1 - f_0}{h} \tag{4}$$

$$G(t_0, f_0, g_0) = \frac{g_1 - g_0}{h} \tag{5}$$

As in Problem 1, I can logically expand these slope formulas for any nth term:

$$F(t_n, f_n, g_n) = \frac{f_{n+1} - f_n}{h} \tag{6}$$

$$G(t_n, f_n, g_n) = \frac{g_{n+1} - gn}{h} \tag{7}$$

Having found the slope formulas, I now incorporate them into the recursive relations. In other words, I need to calculate each next point from its previous point to ensure recursion.

Vertical-axis value: I do this by adding the *rise* (of f_{n+1} or g_{n+1}) to the previous point's vertical-axis value (f_n or g_n).

Finding the *rise* of f and g, respectively:

By equation 4:

$$rise_0 = h\left(\frac{f_1 - f_0}{h}\right)$$

By equation 5:

$$rise_0 = h\left(\frac{g_1 - g_0}{h}\right)$$

By the slope formula, $\frac{rise}{run}$, I can find the rise by multiplying by the run which is h (the step size).

Therefore:

$$f_1 = rise_{f0} + f_0 \Rightarrow f_1 = h\left(\frac{f_1 - f_0}{h}\right) + f_0$$

$$g_1 = \text{rise}_{g0} + g_0 \Rightarrow g_1 = h\left(\frac{g_1 - g_0}{h}\right) + g_0$$

Continued...

By the same reasoning, I could write the following for the slope of the nth term:

$$rise_{nf} = h\left(\frac{f_{n+1} - f_n}{h}\right)$$

$$rise_{ng} = h\left(\frac{g_{n+1} - g_n}{h}\right)$$

$$f_{n+1} = rise_{nf} + f_n \Rightarrow f_{n+1} = h\left(\frac{f_{n+1} - f_n}{h}\right) + f_n$$

$$g_{n+1} = rise_{ng} + g_n \Rightarrow g_{n+1} = h\left(\frac{g_{n+1} - g_n}{h}\right) + g_n$$

Therefore, I have the recursive relations:

$$f_{n+1} = hF(t_n, f_n, g_n) + f_n$$

$$g_{n+1} = hG(t_n, f_n, g_n) + g_n$$

Horizontal-axis value: Finally, I need to ensure that the vertical-value of the next point is determined from the previous point I do this by adding h to the previous point's horizontal-axis value (t_n) to determine the next point's horizontal-axis value (t_{n+1}).

I can don't have to rely on a function value unlike the horizontal values of f and g because h is constant.

I also don't have to come up with two equations for t as it is an independent variable (unlike f and g which are dependent vars) and thus both f and g depend on the value of t.

Similarly to Problem 1, the above can be expressed by the following:

$$t_1 = t_0 + h$$

By extension:

$$t_{n+1} = t_n + h$$

Therefore we have the following recursive equations:

$$t_{n+1} = t_n + h \tag{1}$$

$$f_{n+1} = h\left(F(t_n, f_n, g_n)\right) + f_n \tag{2}$$

$$g_{n+1} = h\left(G(t_n, f_n, g_n)\right) + g_n \tag{3}$$

Continued...

3.3 Estimating Values with Euler's Method

Having found the recursive relations for calculating each point in Euler's method, we can now use Euler's method (Extended).

The following is the original system of equations:

$$f(0) = 1$$

$$g(0) = 2$$

$$f' = g$$

$$g' = \frac{5g - tf}{2}$$

And here are the equations I derived from the previous step for reference:

$$F(t_n, f_n, g_n) = g_n \tag{A}$$

$$G(t_n, f_n, g_n) = \frac{5g_n - t_n f_n}{2} \tag{B}$$

$$t_{n+1} = t_n + h \tag{1}$$

$$f_{n+1} = h\left(F(t_n, f_n, g_n)\right) + f_n \tag{2}$$

$$g_{n+1} = h\left(G(t_n, f_n, g_n)\right) + g_n \tag{3}$$

Applying Euler's Method (Extended): There are three iterations of calculation involved for a step size of 1 ([$h = 1$), from $t = 0$ to $t = 3$ for we need to estimate the values of $y(3)$, where $t = 3$.
As in Problem 1, for convenient calculation, I will use a table. Each row represents an iteration ($n$value) of the approximation. In the columns, t_n, f_n, g_n represent the variables at the nth iteration while $F(t_n, f_n, g_n)$ and $G(t_n, f_n, g_n)$ represent the slopes at the nth iteration.

n	t_n	f_n	g_n	$F(t_n, f_n, g_n)$	$G(t_n, f_n, g_n$	f_{n+1}	g_{n+1}
0	0	1	2	2	5	3	7
1	1	3	7	7	16	10	23
2	2	10	23	23	95/2	33	141/2
3	3	33	141/2				

Table 2: The first row is directly taken from the initial conditions. I use 1 for t_n, I use 2 for f_n, I use 3 for g_n. For the slopes, I use A for F and I use B for G.

Therefore the approximation for $y(3)$ is:

$$\boxed{y(3) = f(3) = 33}$$