Final Project

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Abstract

In this project, I explore the **Integration by Parts** method;, the three special types of convergent series that we can calculate the result, the **Arithmetic Series**, the **Geometric Series** and the **Telescoping Series** and using them in Calculus to our advantage; and, finally, finding challenging **series solutions** to special differential equations that can't be solved explicitly. Going beyond the projects in which I simply analyze some of the more advanced methods, I trace what I've learned since the beginning of the semester and analyze the *significance* in the field of mathematics, rather than simply the usage, of the methods. For all the above methods that I will cover in this project, I will outline not only describe them and use them to find solutions but I will also provide some examples that highlight their significance in mathematics. From the Abel's and Dirichlet's Tests for determining the convergence of more complex series to the more advanced method of Variation of Parameters which is a step beyond simple guessing and is a surefire method for determining the form of the solution by converting differential equations into a system of equations, I highlight why each method mentioned above has it's unique place in mathematics. As we've seen in the projects, I described methods such as undetermined coefficients fall short because not only does the algebra become long and complex but also the method is limited to a certain class of equations. For this reason and many others, the whole point of this next class in Calculus is to teach us the true complexity of mathematics and learning these sophisticated methods than simply assuming that a "plug-and-chug" equation. Unlike other lower math classes, Calculus 1B in UC Berkeley teaches students that they must problem-solve and come up with unique approaches to a varied problem set. As a Computer Science major with a deep appreciation in mathematics, I've not only mastered many of the methods covered in class and even going beyond in the projects as well as I could, but, most importantly, I've learned to embrace the true freedom of mathematics. Though I know that there's much more in mathematics and a multitude of branches of mathematics that I've never experienced, I appreciate the opportunity to gain a taste of the type of exploratory, and oftentimes creative, work that mathematicians around the world employ at the highest levels. In essence, with it's emphasis on complex understanding of the methods and equal emphasis on mastery of each of the methods we've learned in the course, Calculus 1B has broadened the students' and my horizons in not only mathematics but also how the world functions in terms of logic and numbers. As I set my sights on future math courses, such as Discrete Mathematics, CS 70, algorithm optimization courses, as well as artificial intelligence and machine learning classes, I hope to gain more knowledge of this wonderful field and its applications, applying the problem-solving skills and conceptual understanding that is rooted at the most basic ideas, to advance my understanding and intellectual experience.

1 Integration By Parts

1.1 Introduction

Since the first lectures of Calculus I, we've integrated simple expressions. Problems like these can be integrated with just a few rules in mind:

$$\int 2x dx$$
$$\int 5x^2 dx$$
$$\int -\frac{8}{x} dx$$
$$...$$

The most advanced integration covered in almost all Calculus I courses are those that involve u-substitution and the most challenging integrals involve two or more u-substitution or a "smart" u-substitution.

The following could be solved with one u-substitution:

$$\int \frac{1}{x^2} dx$$
$$\int 2x \sin 4x^2 dx$$
$$\int 24x^2 log(8x^3) dx$$

The following are best solved with multiple u-substitutions (double substitution):

$$\int e^{x+e^x} dx$$
$$\int \frac{1}{2} sec^2(2t) tan^2(2t) 2dt$$

There are the *tricky* integrals involving a "smart" substitution — they are pretty challenging even in Calculus II:

$$\int e^{sqrt(x)} dx, \quad x = u^2 \Rightarrow u = \sqrt{x}$$

These are certainly in scope for Calculus II, but there is a larger breadth of problems that cannot be solved this way. Even for the problem above, one cannot solve the integral using solely substitution.

$$\int e^{sqrt(x)} dx, \quad u = \sqrt{x}$$
$$du = \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{x}}\right) dx$$
$$du = \frac{1}{2\sqrt{x}} dx$$
$$dx = 2\sqrt{x}\, du$$
$$\int e^u \left(2\sqrt{x}\, du\right)$$
$$...$$

Clearly, there is no logical follow-up using basic Calculus I integration techniques. There are two functions present and basic integration only covers standard functions such as logarithms, sine, cosine or simply x that are functions of one variable such as t or x.

The basic pattern for these problems is that they follow a *reverse chain rule* pattern. In other words, only problems that include the anti-derivative in the expression that is being integrated can be solved through u-substitution.

$$\int u * u' du$$

However, problems that follow this pattern make-up a small subset of solvable integration problems. Many more integration problems make-up the subset of problems that include two functions.

1.2 The Method

Thus, Calculus II begins with covering the technique that allows students to tackle on expressions that include two or more functions. Therefore, the technique of Integration by Parts opens up the more advanced course of Calculus.

As a result, now students can tackle expressions such as these:

$$\int e^x x^2$$

$$\int sin^3(x) cos^2(x) dx$$

The method takes in two functions, named by convention as u and v and has a corresponding formula.

$$\int u dv = uv - \int v du$$

Here's a proof proving the formula:

$$\begin{aligned}
f(x)g(x) &\quad \text{For integrals involving two functions} \\
\frac{d}{dx}(f(x)g(x)) &= f'(x)g(x) + f(x)g'(x) \quad \text{Product Rule} \\
\int \frac{d}{dx}(f(x)g(x)) &= \int f'(x)g(x)dx + \int f(x)g'(x)dx \quad \text{Integrating with respect to } x \\
f(x)g(x) &= \int f'(x)g(x)dx + \int f(x)g'(x)dx \\
-\int f(x)g'(x)dx &= -f(x)g(x) + \int f'(x)g(x)dx \quad \text{Rearranging to fit given integral} \\
\int f(x)g'(x)dx &= f(x)g(x) - \int f'(x)g(x)dx
\end{aligned}$$

$$\boxed{\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx}$$

Judging from the short proof, Integration by Parts is not used as a formula as it is a strategy for integration. In short, the point of Integration by Parts is to reduce the given integral into an expression involving a simpler integral. It's often necessary to perform many iterations of Integration by Parts in order to simplify the current integral further and further until the integral is solvable using Calculus I techniques. In effect, the true power of this method derives from this power to simplify integrals beyond the reach of basic integration techniques so that it could be solved using these same techniques. The basic techniques from previous Calculus I courses building on onto this Calculus II strategy gives rise to a synergistic blend of macro-simplification — breaking a given integral into smaller integrals — and micro solutions — using Calculus I techniques to solve the simpler problems. In essence, this problem-solving technique — of taking something bigger and breaking it down into smaller pieces — is something that has drastically improved my performance in my other classes, especially technical classes such as the CS 61 series, and has generally opened me to a strategy-based manner of solving problems instead of relying on formulas learned by heart.

2 Special Types of Series

Now, skipping forward to the more abstract and conceptual topics of Calculus are series, specifically infinite series. Now, there are many applications and tangents on series that we covered in class, but what most resonates with me are the series we could calculate the result of. The three special infinite series that we could calculate (given certain conditions) **Arithmetic Series**, **Geometric Series**, and, finally, the **Telescoping Series**.

Because these infinite series are calculable, they have many and serious applications in not only our class but also in the real-world. After all, what is most remarkable about these special series types is that they take the infinite and condense it into the finite. Thus, they are used in Calculus II to reduce the more abstract. The true beauty of these calculable series types is that power series could represent complex differential equations (as we'll see in §), then power series could be broken down into functions. Thus we've went full-circle: we've taken the infinite and we've broken it down into something understandable. If there's one thing that fascinates me about Calculus, it is this idea that infinity could be comprehensible and that it exists in the real-world equally with finite expressions.

The Geometric Series:

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$$

This series converges if $r < 1$, which stands for the rate. The formula is very common in converting functions to power series and vice versa.

The Arithmetic Series:

$$\sum_{k=1}^{n} k^2 = \frac{n\,(n+1)\,(2n+1)}{6}$$

This series converges when n is finite. I've come across this series multiple times in my CS 61B course, Data Structures and Algorithms, as it is highly applicable in complexity analysis, the study of the run time of functions. For instance, a function that runs in arithmetic time is considered θn^2 as the lower orders drop in the above expression — they are not considered in complexity analysis as they are treated as negligible. In other words, the lower-ordered terms do not affect the quadratic growth of a function that runs in arithmetic times. Seeing that connection was very worthwhile for me.

Finally, there is the Telescoping Series. A telescoping series "eats itself." Like a telescope, the larger and larger components fold in. If the telescope is stretched out to infinity, this can visually represent the telescoping series. These are highly important in simplifying complex mathematical expressions and I've come across many instances where I noticed a telescoping series. These series are always convergent and their values are almost always represented by the first and last values. Here's an example.

$$\sum_{k=1}^{\infty} \frac{-2}{(n+1)(n+2)}$$

Now it's not obvious that this series is in fact a telescoping series. But, it must be noted that after doing some algebra, specifically after breaking down the expression into partial fractions, then one can truly expand the sum and see that this is, in fact, a telescoping series.

$$\sum_{k=1}^{\infty} \frac{-2}{(n+1)(n+2)} \Rightarrow \frac{-2}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2}$$
$$\sum_{k=1}^{\infty} \frac{-2}{(n+1)(n+2)} \Rightarrow \frac{-2}{(n+1)(n+2)} = \frac{A(n+2)}{(n+1)(n+2)} + \frac{B(n+1)}{(n+1)(n+2)}$$
$$= \frac{-2}{(n+1)(n+2)} \frac{Bn+B+An+2A)}{(n+1)(n+2)}$$

$$An + Bn = 0$$
$$2A + B = -2$$

$$A + B = 0$$
$$B = -2 - 2A$$

$$A + (-2 - 2A) = 0$$
$$B = -2 - 2A$$

$$A + -2 - 2A = 0$$
$$B = -2 - 2A$$

$$-2 - A = 0$$
$$B = -2 - 2A$$

$$-A = 2$$
$$B = -2 - 2A$$

$$A = -2$$
$$B = -2 - 2(-2)$$

$$A = -2$$
$$B = -2 + 4$$

$$A = -2$$
$$B = 2$$

Therefore:

$$\sum_{k=1}^{\infty} \frac{-2}{(n+1)(n+2)} = \sum_{k=1}^{\infty} \left[-\frac{2}{n+1} + \frac{2}{n+2} \right]$$
$$= \left(-\frac{2}{3} + \frac{2}{4} \right) +$$
$$\left(-\frac{2}{4} + \frac{2}{5} \right) + ... + \left(-\frac{2}{n-1} + \frac{2}{n} \right) +$$
$$\left(-\frac{2}{n} + \frac{2}{n+1} \right) + \left(-\frac{2}{n+1} + \frac{2}{n+2} \right)$$

Clearly, the *intermediate* terms cancel and the only terms remaining are the first and last terms. Thus the telescoping series converges to the following value. We can then take the limit as n approaches infinity if we want to find the exact value, which simply converges to the first value: $-\frac{2}{3}$.

3 Finding Series Solutions

Skipping toward the end of the course, another method — and perhaps the most diverse method that we've learned in the course — that condenses the complex and oftentimes incomprehensible differential equations that fall outside through a workaround using power series. An equation such as $y'' - 2xy' + y = 0$ mentioned by the textbook does not have an explicit solution so methods such as using an integration constant or trying to solve the equation using Variation of Parameters that produce explicit and finite solutions will not work. This method particularly excites me because the differential equations we could solve using series representations have direct applications in advanced fields humanity has only started grasping. As the book mentions, differential equations that have series solutions arise from physics problems and quantum mechanics in relation to the Schrödinger equation. This method with no formal name is covered in §17.4, the last chapter covered in Math 1B. As always with more advanced methods, this method could be also used to solve series but it is much less efficient.

The basic strategy involves parsing through the given differential equation, and representing each instance of the variable (including any coefficients) that you're solving for such as y. This is run under the assumption that a power series could represent the variable you're solving for. Stemming from a theorem that any value could be expressed as a power series, this power series representation for a variable is logical. Then, these power series representations, written in the form: $c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 ...$ can effectively represent each term (with modulation); regardless of how differentiated or how many or what type of coefficients a term has, it could be represented by a power series that takes into account these changes. For example, the first derivative of y': $c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4$ and the second derivative y'': $2c_2 + 6c_3x + 12c_4x^3$ and on and on. Any coefficients, even if they are variable such as x, are taken into account.

But it is important to be able to solve equations such as Equation 1 because they arise from physical problems and, in particular, in connection with the Schrödinger equation in quantum mechanics.

To demonstrate the true power of this method, I will solve the following equation using series representation to arrive at a solution.

$$y'' - xy' + 2y = 0$$

Let y:

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4...$$

Therefore:

$$\begin{aligned} y' &= c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3... \\ y'' &= 2c_2 + 6c_3x + 12c_4x^2... \end{aligned}$$

It follows for every term in the differential equation:

$$\begin{aligned} 2y &= -2\left(c_0 + c_1x + c_2x^2 + c_3x^3...\right) \\ -xy' &= -x\left(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3...\right) \\ y'' &= 2c_2 + 6c_3x + 12c_4x^2... \end{aligned}$$

Combining power series yields power series:

$$\begin{aligned} 2y - xy' &= 2\left(c_0 + c_1x + c_2x^2 + c_3x^3...\right) - x\left(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3\right) \\ y'' + 2y - xy' &= \left(2c_2 + 6c_3x + 12c_4x^2...\right) + 2\left(c_0 + c_1x + c_2x^2 + c_3x^3...\right) - x\left(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3\right) \\ y'' - xy' + 2y &= 2c_2 + 6c_3x + 12c_4x^2... + 2c_0 + 2c_1x + 2c_2x^2 + 2c_3x^3 \\ &\quad ... - c_1x - 2c_2x^2 - 3c_3x^3 - 4c_4x^4... \end{aligned}$$

Now, the patterns that we are looking for in our series solution are beginning to emerge.

I can continue on finding the pattern this way, but I'll simplify the beginning approach by replacing the power series expansions as sums. I will then apply the same ideas to find patterns that emerge out of this differential equation. When the algebra becomes complicated, this is the preferred way to find patterns and ultimately find solutions to the differential equation.

It follows:

Let y:

$$y = \sum_{n=0}^{\infty} c_n x^n$$

Therefore:

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} c_n n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} c_n n\left(n-1\right) x^{n-2} \end{aligned}$$

We adjust the n value by 1 for each differentiation. This is done to account for the loss of terms and the resulting *shift* that occurs with differentiation.

It follows for every term in the differential equation:

$$2y = 2\sum_{n=0}^{\infty} c_n x^n$$

$$-xy' = -x\sum_{n=1}^{\infty} c_n n x^{n-1} \Rightarrow -\sum_{n=1}^{\infty} c_n n x^n$$

$$y'' = \sum_{n=2}^{\infty} c_n n\left(n-1\right)x^{n-2}$$

Therefore:

$$\sum_{n=2}^{\infty} c_n n\left(n-1\right)x^{n-2} - x\sum_{n=1}^{\infty} c_n n x^{n-1} + 2\sum_{n=0}^{\infty} c_n x^n = 0$$

Now, we perform what is called re-indexing the sums. We take each sum expression and change the exponent on the x term — this is done so we can combine the sum expressions eventually and we can factor out the x term. Thus, I'm introducing a new variable k to equal our new value and I will set k to different values to make sure all the sum expressions have the same exponent on the x.

Let k:

$$k = n - 2$$
$$n = k + 2$$

$$\sum_{n=2}^{\infty} c_n n\left(n-1\right)x^{n-2} \Rightarrow \sum_{k=0}^{\infty} c_{k+2}\left(k+2\right)\left(k+1\right)x^k$$

$$k = n$$
$$n = k$$

$$-\sum_{n=1}^{\infty} c_n n x^n \Rightarrow -\sum_{k=1}^{\infty} c_k x^k$$

$$k = 0$$

$$2\sum_{n=0}^{\infty} c_n x^n \Rightarrow \sum_{k=0}^{\infty} 2c_k x^k$$

Therefore:

$$\sum_{k=0}^{\infty} c_{k+2}\left(k+2\right)\left(k+1\right)x^k - \sum_{k=1}^{\infty} c_k x^k + \sum_{k=0}^{\infty} 2c_k x^k = 0$$

Having accounted for equalizing the x term throughout my series terms, now we can proceed to equalizing the starting bounds. We can do this easily by distributing the appropriate number of terms out of the sum. In this case $k = 0$ must be changed to $k = 1$ by the k term of the middle term. Therefore, I take out the $k = 0$ terms of the first and third terms of my new equation.

It follows:

$$2c_2 + 2c_0 + \sum_{k=1}^{\infty} c_{k+2}\,(k+2)\,(k+1)\,x^k - \sum_{k=1}^{\infty} c_k x^k + \sum_{k=1}^{\infty} 2c_k x^k = 0$$

Thus, now we can proceed with combining everything into one quite large but generalized sum expression.

$$\begin{aligned}
2c_2 + 2c_0 + \sum_{k=1}^{\infty} c_{k+2}\,(k+2)\,(k+1)\,x^k - \sum_{k=1}^{\infty} c_k x^k + \sum_{k=1}^{\infty} 2c_k x^k &= 0 \\
2(c_2 + c_0) + \sum_{k=1}^{\infty} \left[c_{k+2}\,(k+2)\,(k+1)\,x^k - c_k x^k + 2c_k x^k\right] &= 0 \\
2(c_2 + c_0) + \sum_{k=1}^{\infty} x^k \left[c_{k+2}\,(k+2)\,(k+1) - c_k + 2c_k\right] &= 0
\end{aligned}$$

Now, we can use the same idea as used in Partial Fraction Decomposition — we can use *Linear Independence* here to relate c_0 to c_2. Because 0 is a constant, we can simply ignore the sum and set 0 to c_0 and c_2:

$$\begin{aligned}
2(c_2 + c_0) &= 0 \\
2c_2 + 2c_0 &= 0 \\
2c_2 &= -2c_0 \\
c_2 &= -c_0
\end{aligned}$$

Now, we can set everything in the brackets of the sum to 0. We can ignore the x^k term just like when we solve for roots in algebra and we can ignore the sum because the coefficients are linearly independent from the sum. By being coefficients, they are simply constants, and they represent values completely independent from the sum.

$$\left[c_{k+2}\,(k+2)\,(k+1) - c_k + 2c_k\right] = 0$$

Now, we can solve for the recursive relation that will dictate our pattern for the ultimate conclusion — finding the solution to the given differential equation. You can solve for any of the c_k terms, but it's advised to solve for the larger coefficient: in this case, it is c_{k+2}.

It follows:

$$c_{k+2} = c_k \frac{k-2}{(k+1)\,(k+2)}, \quad k \geqslant 1$$

Notice, this is a recursive relation, also known as a *recurrence relation.* We can now use this relationship to find a **recurring pattern** between the coefficients in order to start building the sum solution. Notice also that the recursive relation does not make sense when $k < 1$, firstly because k was 1 in the combined sum and secondly, the recursive relation will clearly fail for $k < 1$

because we go into negatively numbered terms, which clearly violates the power series definitions in the beginning of the problem where we defined the power series to represent the y values starting from $n = 0$.

It follows:

$$\begin{aligned}
k = 1, &\quad c_3 = c_1 \frac{1-2}{2*3} \\
k = 2, &\quad c_4 = c_2 \frac{2-2}{3*4} \\
k = 3, &\quad c_5 = c_3 \frac{3-2}{4*5} \\
k = 5, &\quad c_7 = c_5 \frac{5-2}{6*7} \\
k = 7, &\quad c_9 = c_7 \frac{7-2}{8*9}
\end{aligned}$$

Now, we can go on, on and on, for quite a while. Without evaluating these expressions and seeing what numbers come out, this work would be useless. Moreover, we not only need to see what numbers come out but also relate them to one another to see any relations that we can integrate in our sum solution.

$$\begin{aligned}
k = 1, &\quad c_3 = c_1 \frac{1-2}{2*3} = -c_1 \frac{-1}{6} \\
k = 2, &\quad c_4 = c_2 \frac{2-2}{3*4} = 0 \\
k = 3, &\quad c_5 = c_3 \frac{3-2}{4*5} = c_3 \frac{1}{20} \\
k = 5, &\quad c_7 = c_5 \frac{5-2}{6*7} = -c_5 \frac{3}{42} \\
k = 7, &\quad c_9 = c_7 \frac{7-2}{8*9} = c_7 \frac{5}{72}
\end{aligned}$$

Now, this is not to helpful either. Simply simplifying the results yields pretty random results, at least on first glance it seems. However, there looks like to be some pattern before I simplified the fractional expressions. It seems that while the formula remains constant in subtracting 2 from the numerator, it does something quite interesting in the denominator. It looks like there is some pattern occurring that could be attributed to a mathematical function — the factorial:

$$\begin{aligned}
k = 1, &\quad c_3 = c_1 \frac{1-2}{2*3} = c_1 \frac{-1}{3!} \\
k = 2, &\quad c_4 = 0 \;\; \text{All } c_{2k} \text{ are all equal to 0 (due to recursion)} \\
k = 3, &\quad c_5 = c_1 \frac{1-2}{2*3} * \frac{3-2}{4*5} = -c_1 \frac{1}{5!} \\
k = 5, &\quad c_7 = c_1 \frac{1-2}{2*3} * \frac{3-2}{4*5} * \frac{5-2}{6*7} = -c_1 \frac{3}{7!} \\
k = 7, &\quad c_9 = c_1 \frac{1-2}{2*3} * \frac{3-2}{4*5} * \frac{5-2}{6*7} * \frac{7-2}{8*9} = -c_1 \frac{5*3}{9!}
\end{aligned}$$

Finally, we can now define the solution. Using the patterns that have emerged from the above expansion, we can now write a closed-form solution because there is no apparent pattern here; on the other hand, we know that there are no even terms (c_{2k}) because of the pattern we discovered — an even term effectively "nullifies" any term that recursively calls it.

Thus, we write:

$$y = \sum_{k=1}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + ...$$

Because of the linearly independent relation $c_2 = -c_0$ we produced on top, we write the following:

$$c_2 = -c_0 \Rightarrow y = c_0 + c_1 x - c_0 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + ...$$

Therefore:

$$y = c_0(1 - x^2) + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + ...$$
$$y = c_0(1 - x^2) + \big(c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + ...\big)$$

Now, we replace the terms of the equation with the closed-form solution that involves the simplified results we calculated using the recursive relation.

$$y = c_0(1 - x^2) + \left(-c_1 \frac{1}{3!} - c_1 \frac{1}{5!} - c_1 \frac{3}{7!} - c_1 \frac{5 * 3}{9!} + ...\right)$$

We can distribute out the c_1 common term.

$$y = c_0(1 - x^2) - c_1 \left(\frac{1}{3!} + \frac{1}{5!} + \frac{3}{7!} + \frac{5 * 3}{9!} + ...\right)$$

$$\boxed{y = c_0(1 - x^2) - c_1 \left(\frac{1}{3!} + \frac{1}{5!} + \frac{3}{7!} + \frac{5 * 3}{9!} + ...\right)}$$

And there we have our final solution.

Having shown some of the problems I most enjoyed in the course, Math 1B is one of the most important required classes I've taken. I know that the same problem-solving techniques will carry on to the mathematics classes that I will take in the near future. Most of all, I feel fortunate to have experienced this introductory course at one of the greatest institutions if not the greatest institution that specializes in technical fields such as mathematics. Chances are high that I wouldn't have learned so much of the material I've picked up as not only a result of taking UC Berkeley's Math 1B but also due to the unprecedented global pandemic, which shifted this course's evaluation to be focused more on holistic evaluation. Specifically, the advanced projects and the opportunity to reflect on our learning — an opportunity that is seldom given — more so, it is undoubtedly masked — due to students concentrating on the final exam. Whether for good or for the worse, doing the projects was one of the most valuable experiences I've had this semester and certainly, given the health situation, presented itself as an unprecedented opportunity to delve deeper than on an average semester and truly master, or at least try to master, mathematics.