Project 5.3

Variation of Parameters

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In this project, I explore the Variations of Parameters method for solving differential equations. As we have seen in § 17.2, methods such as undetermined coefficients fall short on many types of differential equations because not only does the algebra become long and complex but also the method only works on a certain small class of equations. It is a slightly more sophisticated method than simply assuming that a solution will be proportional to some constant $e^{\alpha x}$ and thus has broader applications. In §1 and §2, I will lay down the foundation from which the method emerges. In §3, I will prove the method and finally, in §4, I will use the method to solve a non-homogeneous, second-order differential equation.

Consider the following homogeneous differential equation

$$ay'' + by' + cy = 0,$$

where a, b, c are constant and $a \neq 0$. Suppose that $y_1(x)$ and $y_2(x)$ are linearly independent solutions to this equation. Now imagine we have a non-homogeneous problem:

$$ay'' + by' + cy = g(x).$$

We are going to search for a particular solution of the form

$$y_p = c_1(x)y_1 + c_2(x)y_2,$$

where $c_1(x)$ and $c_2(x)$ are functions (if they were constants we would just get a homogeneous solution).

1 Problem 1

Calculate y_p' and y_p''.
I simply use the product rule for both.

For y_p':

$$y_p = c_1(x)y_1 + c_2(x)y_2$$
$$\boxed{y_p' = \left(c_1'(x)y_1 + c_1(x)y_1'\right) + \left(c_2'(x)y_2 + c_2(x)y_2'\right)}$$

For y_p'':

$$\begin{aligned}
y_p'' &= \left(\left(c_1''(x)y_1 + c_1'(x)y_1'\right) + \left(c_1'(x)y_1' + c_1(x)y_1''\right)\right) + \left(\left(c_2''(x)y_2 + c_2'(x)y_2'\right) + \left(c_2'(x)y_2' + c_2(x)y_2''\right)\right) \\
&= \left(c_1''(x)y_1 + c_1'(x)y_1'\right) + \left(c_1'(x)y_1' + c_1(x)y_1''\right) \left(c_2''(x)y_2 + c_2'(x)y_2'\right) + \left(c_2'(x)y_2' + c_2(x)y_2''\right) \\
&= c_1''(x)y_1 + c_1'(x)y_1' + c_1'(x)y_1' + c_1(x)y_1'' + c_2''(x)y_2 + c_2'(x)y_2' + c_2'(x)y_2' + c_2(x)y_2'' \\
&= c_1''(x)y_1 + 2c_1'(x)y_1' + c_1(x)y_1'' + c_2''(x)y_2 + 2c_2'(x)y_2' + c_2(x)y_2''
\end{aligned}$$
$$\boxed{y_p'' = c_1''(x)y_1 + c_2''(x)y_2 + 2c_1'(x)y_1' + 2c_2'(x)y_2' + c_1(x)y_1'' + c_2(x)y_2''} \qquad \blacksquare$$

2 Problem 2

Under the assumption that

$$c_1'(x)y_1 + c_2'(x)y_2 = 0,$$

simplify your answers in question 1.

For y_p':

$$\begin{aligned}
y_p &= c_1(x)y_1 + c_2(x)y_2 \\
y_p' &= \left(c_1'(x)y_1 + c_1(x)y_1'\right) + \left(c_2'(x)y_2 + c_2(x)y_2'\right) \\
y_p' &= c_1'(x)y_1 + c_1(x)y_1' + c_2'(x)y_2 + c_2(x)y_2' \\
y_p' &= c_1'(x)y_1 + c_2'(x)y_2 + c_1(x)y_1' + c_2(x)y_2' \\
y_p' &= (0) + c_1(x)y_1' + c_2(x)y_2' \quad \text{Substituting in } c_1'(x)y_1 + c_2'(x)y_2 = 0
\end{aligned}$$
$$\boxed{y_p' = c_1(x)y_1' + c_2(x)y_2'}$$

Thus, for y_p'':

$$\begin{aligned}
y_p' &= c_1(x)y_1' + c_2(x)y_2' \\
y_p'' &= \left(c_1'(x)y_1' + c_1(x)y_1''\right) + \left(c_2'(x)y_2' + c_2(x)y_2''\right)
\end{aligned}$$
$$\boxed{y_p'' = c_1'(x)y_1' + c_2'(x)y_2' + c_1(x)y_1'' + c_2(x)y_2''}$$

Continued...

3 Problem 3

3. Under the same assumption as question 2, prove that if

$$ay_p'' + by_p' + cy_p = g(x)$$

then

$$a\left(c_1'(x)y_1' + c_2'(x)y_2'\right) = g(x).$$

Hint: y_1 and y_2 are homogeneous solutions.

Under the same assumption as §2, one could simply substitute the equations derived in §1 and further simplified in §2, into the equation in the problem statement.

With the following equations:

$$\begin{aligned}
&ay_p'' + by_p' + cy_p = g(x)\\
&y_p = c_1(x)y_1 + c_2(x)y_2\\
&y_p' = c_1(x)y_1' + c_2(x)y_2'\\
&y_p'' = c_1'(x)y_1' + c_2'(x)y_2' + c_1(x)y_1'' + c_2(x)y_2''
\end{aligned}$$

Therefore, by substitution:

$$\begin{aligned}
g(x) = a&\left(c_1'(x)y_1' + c_2'(x)y_2' + c_1(x)y_1'' + c_2(x)y_2''\right)\\
&+ b\left(c_1(x)y_1' + c_2(x)y_2'\right) + c\left(c_1(x)y_1 + c_2(x)y_2\right)
\end{aligned}$$

Which can be rewritten into the following with simple variable manipulation:

$$\begin{aligned}
g(x) &= a(c_1'(x)y_1' + ac_2'(x)y_2' + ac_1(x)y_1'' + ac_2(x)y_2'' + bc_1(x)y_1' + bc_2(x)y_2' + cc_1(x)y_1 + cc_2(x)y_2\\
g(x) &= ac_1(x)y_1'' + bc_1(x)y_1' + cc_1(x)y_1 + ac_2(x)y_2'' + bc_2(x)y_2' + cc_2(x)y_2 + ac_1'(x)y_1' + ac_2'(x)y_2'\\
g(x) &= c_1\left(a(x)y_1'' + b(x)y_1' + c(x)y_1\right) + c_2\left(a(x)y_2'' + b(x)y_2' + c(x)y_2\right) + a\left(c_1'(x)y_1' + c_2'(x)y_2'\right)
\end{aligned}$$

Hence the name of the technique we're about to uncover — "Variation of Parameters."
Further simplification can be done since y_1 and y_2 are homogeneous solutions, which means that the right-hand side of the original equation is set to 0. Thus this means the following:

$$\begin{aligned}
ay_1'' + by_1' + cy_1 = 0\\
ay_2'' + by_2' + cy_2 = 0
\end{aligned}$$

With substitution, it follows:

$$\begin{aligned}
&c_1\,(0) + c_2\,(0) + a\left(c_1'(x)y_1' + c_2'(x)y_2'\right) = g(x)\\
&\boxed{a\left(c_1'(x)y_1' + c_2'(x)y_2'\right) = g(x)}
\end{aligned}$$

The resulting equation matches the one in the problem statement — hence proved.

4 Problem 4

In conclusion, if y_1 and y_2 are linearly independent solutions to $ay'' + by' + cy = 0$ and $c_1(x)$, $c_2(x)$ are functions such that

$$c_1'(x)y_1 + c_2'(x)y_2 = 0 \quad \text{and} \quad c_1'(x)y_1' + c_2'(x)y_2' = \frac{g(x)}{a} \tag{1}$$

then $y_p = c_1(x)y_1 + c_2(x)y_2$ is a solution to

$$ay'' + by' + cy = g(x)$$

Notice that we can simultaneously solve the two equations in (1) to express $c_1'(x)$ and $c_2'(x)$ in terms of a, $g(x)$, y_1, y_2, y_1' and y_2'. We could then integrate to get explicit formulae for $c_1(x)$ and $c_2(x)$, and thus y_p. This is called the method of **variation of parameters**.

4. Using the method of variation of parameters, find a general solution to

$$y'' + 2y' + y = x^{-4}e^{-x}.$$

4.1 Finding a Homogeneous Framework:

To start, let's determine the homogeneous solution — this will provide an idea of what our particular solution and thus what the final, general solution will look like. It follows:

$$y'' + 2y' + y = 0$$

Substituting the y's with r's with an exponent matching each order and solving for the roots:

$$\begin{aligned}
&r^2 + 2r + 1 = 0 \\
&(r+1)^2 = 0 \\
&r + 1 = 0 \\
&r = -1
\end{aligned}$$

The form of homogeneous solutions for the original differential equation is the following:

$$y_p(x) = c_1(x)y_1 + c_2(x)y_2 \tag{2}$$

Thus $r_{1,2} = -1$, so the homogeneous solution y_h for original differential equation takes the form of the following equation. This is due to the repeating roots:

$$\boxed{y_h = c_1e^{-x} + c_2xe^{-x}} \tag{3}$$

4.2 Determining a System of Equations:

Now, based on the homogeneous solution, the particular solution y_p will take the form of the following. Because we're implementing **variation of parameters**, the constants c_1 and c_2 are upgraded to be variable, thus becoming functions $c_1(x)$ and $c_2(x)$.

$$y_h \Rightarrow y_p = c_1(x)e^{-x} + c_2(x)xe^{-x} \tag{4}$$

Now, to arrive at the general solution, we must determine the variable parameters, functions $c_1(x)$ and $c_2(x)$. In order to solve for them, we take the equations in (1) and form a system of equations:

$$\begin{aligned}
c_1'(x)y_1 + c_2'(x)y_2 &= 0 \\
c_1'(x)y_1' + c_2'(x)y_2' &= \frac{g(x)}{a}
\end{aligned}$$

Continued...

Because $y_1 = e^{-x}$ and $y_2 = xe^{-x}$ determined from the standard particular solution form given in (3) and the particular solution form based on the original differential equation at (4), we can substitute the values for y_1 and y_2. It follows:

$$\begin{aligned} c_1'(x)\left(e^{-x}\right) + c_2'(x)\left(xe^{-x}\right) &= 0 \\ c_1'(x)y_1' + c_2'(x)y_2' &= \frac{g(x)}{a} \end{aligned}$$

Finding y_1' and y_2':

$$\begin{aligned} y_1' &= -e^{-x} \\ y_2' &= e^{-x} - xe^{-x} \quad \textit{Product Rule} \end{aligned}$$

we can substitute the values in the second equation in (1):

$$\begin{aligned} c_1'(x)e^{-x} + c_2'(x)xe^{-x} &= 0 \\ c_1'(x)\left(-e^{-x}\right) + c_2'(x)\left(e^{-x} - xe^{-x}\right) &= \frac{g(x)}{a} \end{aligned}$$

Finally, we know $g(x)$ and the constant a from the original differential equation because of the form $ay'' + by' + c = g(x)$.

$$\begin{aligned} &y'' + 2y' + y = x^{-4}e^{-x} \quad \text{Original differential equation.} \\ &g(x) = x^{-4}e^{-x} \qquad\qquad a = 1 \end{aligned}$$

Substituting these values into our system, we get the following:

$$\begin{aligned} c_1'(x)e^{-x} + c_2'(x)xe^{-x} &= 0 \\ -c_1'(x)e^{-x} + c_2'(x)\left(e^{-x} - xe^{-x}\right) &= \frac{\left(x^{-4}e^{-x}\right)}{(1)} \end{aligned}$$

Notice that $c_1'(x)$ and $c_2'(x)$ in the system of equations are not the functions $c_1(x)$ and $c_2(x)$, but once the derivatives are found, we can integrate them in order to determine the parameters $c_1(x)$ and $c_2(x)$ in y_p. Since $c_1'(x)$ and $c_2'(x)$ are common in both equations, we could treat them as two variables and solve for them.

4.3 Solving for $c_1'(x)$ and $c_2'(x)$

Here's our system:

$$\begin{aligned} c_1'(x)e^{-x} + c_2'(x)xe^{-x} &= 0 \\ -c_1'(x)e^{-x} + c_2'(x)\left(e^{-x} - xe^{-x}\right) &= x^{-4}e^{-x} \end{aligned}$$

Adding them together yields:

$$\left(c_1'(x)e^{-x} + c_2'(x)xe^{-x}\right) - c_1'(x)e^{-x} + c_2'(x)\left(e^{-x} - xe^{-x}\right) = x^{-4}e^{-x}$$

Rearranging:

$$\begin{aligned} c_1'(x)e^{-x} + c_2'(x)xe^{-x} - c_1'(x)e^{-x} + c_2'(x)\left(e^{-x} - xe^{-x}\right) &= x^{-4}e^{-x} \\ c_1'(x)e^{-x} - c_1'(x)e^{-x} + c_2'(x)xe^{-x} + c_2'(x)e^{-x} - c_2'(x)xe^{-x} &= x^{-4}e^{-x} \\ c_1'(x)e^{-x} - c_1'(x)e^{-x} + c_2'(x)xe^{-x} - c_2'(x)xe^{-x} + c_2'(x)e^{-x} &= x^{-4}e^{-x} \end{aligned}$$

Continued...

Simplifying gives the following:

$$c_2'(x)e^{-x} = x^{-4}e^{-x}$$
$$\boxed{c_2'(x) = x^{-4}}$$

Thus, we found $c_2'(x)$.
But, before we solve for $c_2(x)$, let's use $c_2'(x)$ to find $c_1'(x)$. Thus, by substituting $c_2'(x) = x^{-4}$ in the first equation in (1), we get:

$$c_1'(x)e^{-x} + x^{-4}xe^{-x} = 0$$
$$c_1'(x)e^{-x} + x^{-3}e^{-x} = 0$$
$$c_1'(x)e^{-x} = -x^{-3}e^{-x}$$
$$\boxed{c_1'(x) = -x^{-3}}$$

Having found both $c_1'(x)$ and $c_2'(x)$, we can now integrate them to get $c_1(x)$ and $c_2(x)$. It follows:

For $c_1'(x)$

$$c_1'(x) = -x^{-3}$$
$$\int c_1'(x) = \int -x^{-3}\,dx$$
$$c_1(x) = \int -x^{-3}\,dx$$
$$\boxed{c_1(x) = \frac{x^{-2}}{2}}$$

For $c_2'(x)$:

$$c_2'(x) = x^{-4}$$
$$\int c_2'(x) = \int x^{-4}\,dx$$
$$c_2(x) = \int x^{-4}\,dx$$
$$\boxed{c_2(x) = -\frac{x^{-3}}{3}}$$

Notice that there are no constants for both here as they are covered by the variable parameters, $c_1(x)$ and $c_2(x)$.

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Having found the values of $c_1(x)$ and $c_2(x)$, we can now substitute them into the particular solution form we found at (4) to get the particular solution and thus the general solution. Therefore, our particular solution is the following:

$$\begin{aligned}
y_p &= \left(\frac{x^{-2}}{2}\right)e^{-x} + \left(-\frac{x^{-3}}{3}\right)xe^{-x} \\
&= \frac{x^{-2}}{2}e^{-x} - \frac{x^{-3}}{3}xe^{-x} \\
&= \frac{x^{-2}}{2}e^{-x} - \frac{x^{-2}}{3}e^{-x} \\
&= \frac{3x^{-2}}{6}e^{-x} - \frac{2x^{-2}}{6}e^{-x} \\
&= \frac{x^{-2}}{6}e^{-x}
\end{aligned}$$

$$\boxed{y_p = \frac{e^{-x}}{6x^2}}$$

Because we have found y_h at (3) and y_p at (4) the general solution for the original differential equation is the following:

$$\begin{aligned}
y_g &= y_h + y_p \\
y_g &= \left(c_1e^{-x} + c_2xe^{-x}\right) + \left(\frac{e^{-x}}{6x^2}\right)
\end{aligned}$$

Therefore:

$$\boxed{y = c_1e^{-x} + c_2xe^{-x} + \frac{e^{-x}}{6x^2}}$$