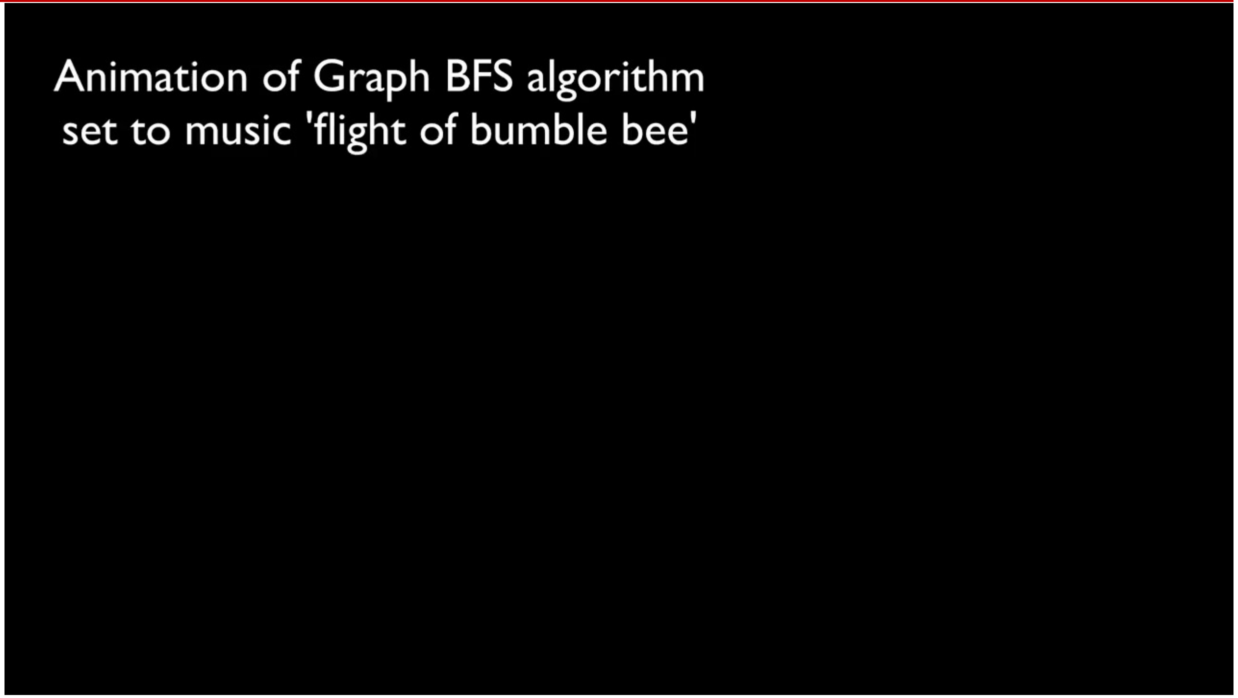
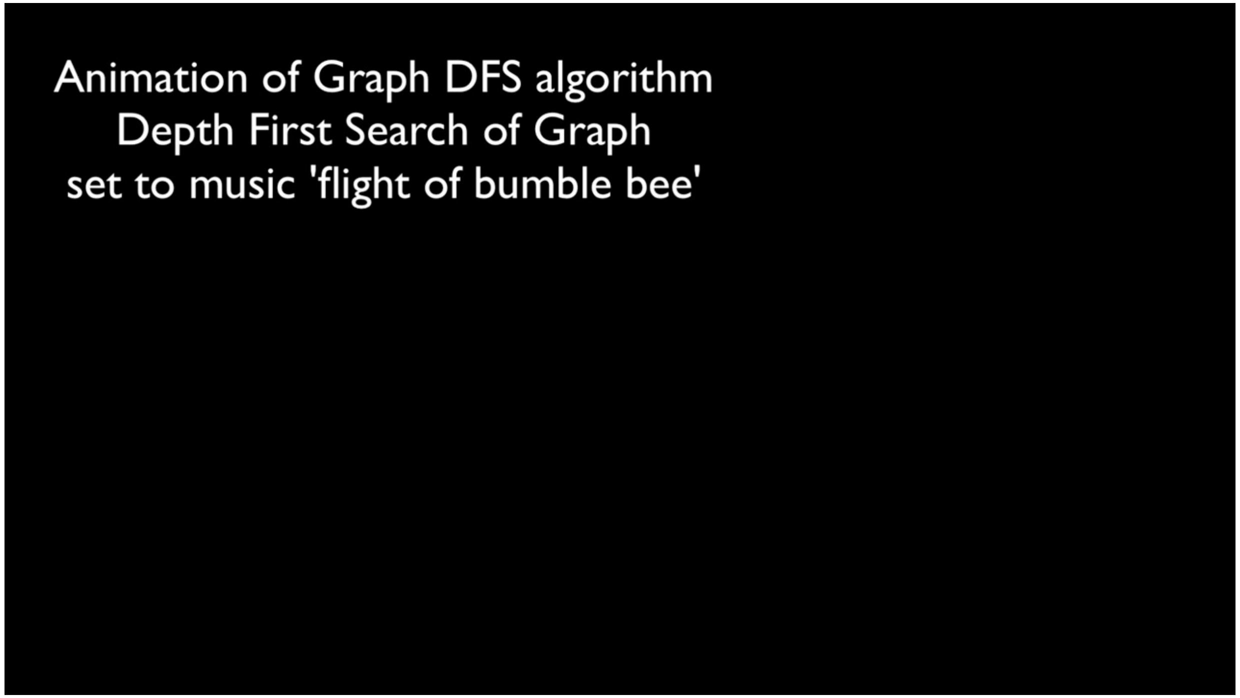
BFS vs. DFS

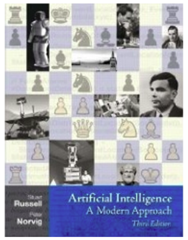
Gerry Jenkins
https://www.youtube.com/watch?v=x-VTfcmrLEQ
https://www.youtube.com/watch?v=oDqjPvD54Ss





Outline

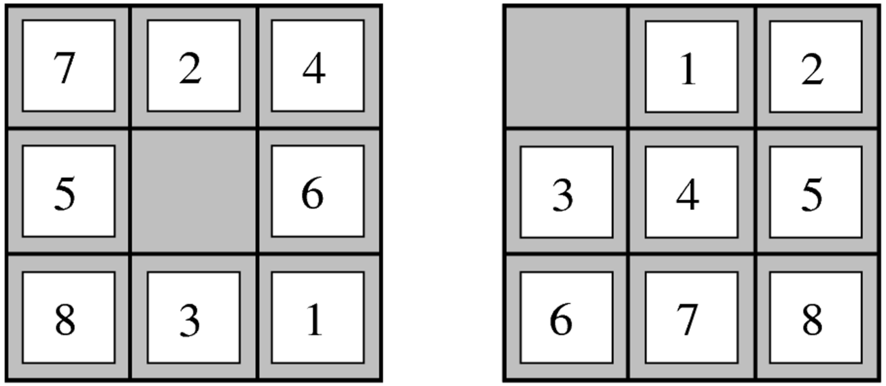
- Best-first search
  - Greedy best-first search
  - A* search
  - Heuristics
  - Admissible Heuristics
  - Graph Search
  - Consistent Heuristics
- Local search algorithms
  - Hill-climbing search
  - Beam search
  - Simulated annealing search
  - Genetic algorithms
- Constraint Satisfaction Problems



Many slides based on Russell & Norvig's slides
[Artificial Intelligence: A Modern Approach]

Motivation

- Uninformed search algorithms are too inefficient
  - they expand far too many unpromising paths
- Example:
  - 8-puzzle



Start State Goal State

  - Average solution depth = 22
  - Breadth-first search to depth 22 has to expand about 3.1 x 10^{10} nodes

→ try to be more clever with what nodes to expand

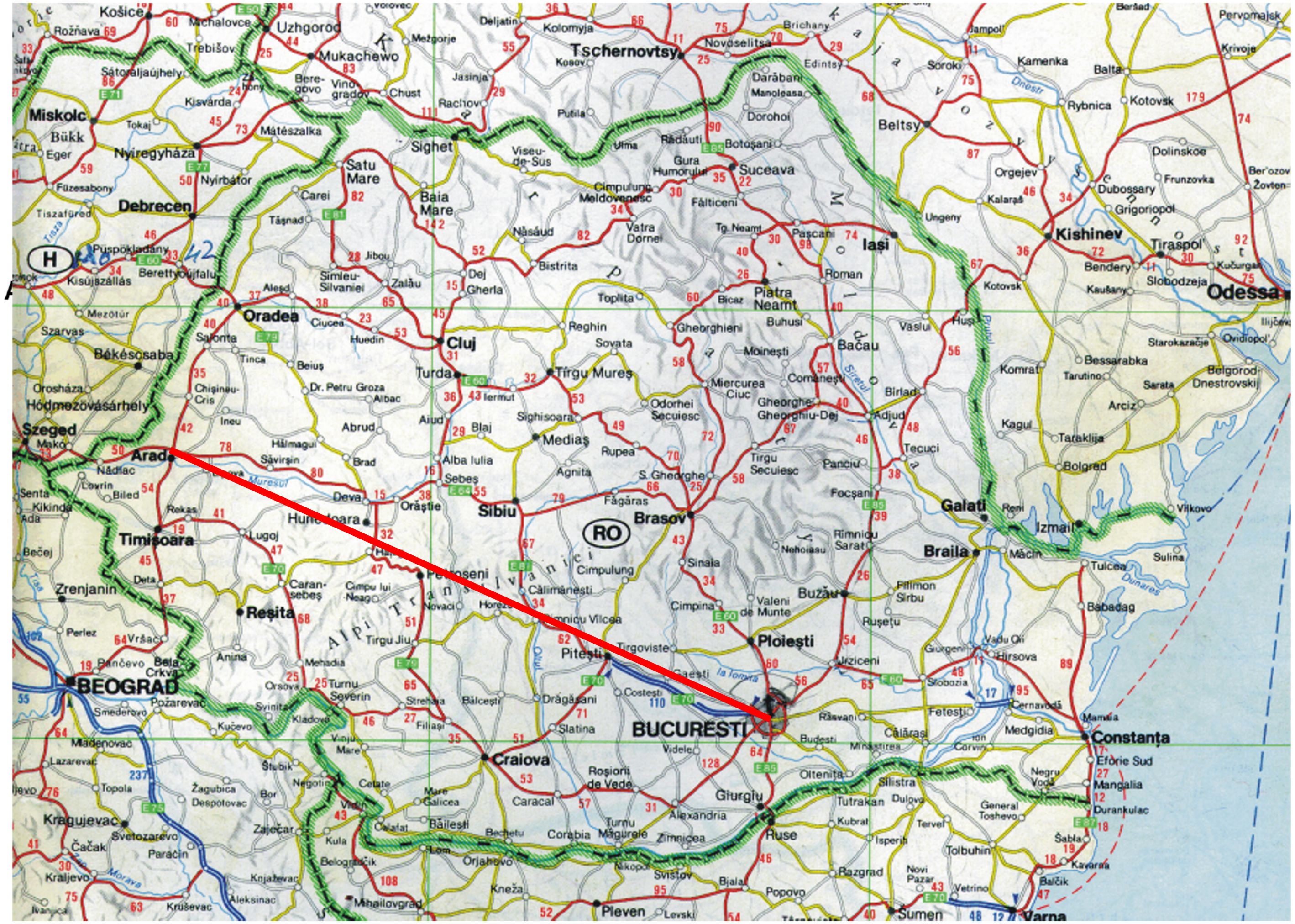
Best-First Search

- Recall
  - Search strategies are characterized by the order in which they expand the nodes of the search tree
  - Uninformed tree-search algorithms sort the nodes by problem-independent methods (e.g., recency)
- Basic Idea of Best-First Search
  - use a heuristic evaluation function $f(n)$ for each node
    - estimate of the "desirability" of the node's state
  - expand most desirable unexpanded node
- Implementation
  - use Tree Search algorithm
  - order the nodes in fringe in decreasing order of desirability
- Algorithms
  - Greedy best-first search
  - A* search

Heuristic

- Greek "heurisko" (εὑρίσκω) → "I find"
  - cf. also „Eureka!“
- informally denotes a „rule of thumb“
  - i.e., knowledge that may be helpful in solving a problem
  - note that heuristics may also go wrong!
- In tree-search algorithms, a heuristic denotes a function that estimates the remaining costs until the goal is reached
- Example:
  - straight-line distances may be a good approximation for the true distances on a map of Romania
  - and are easy to obtain (ruler on the map)
    - but cannot be obtained directly from the distances on the map

Romania Example: Straight-line Distances



Straight–line distance to Bucharest	
Arad	366
Bucharest	0
Craiova	160
Dobreta	242
Eforie	161
Fagaras	176
Giurgiu	77
Hirsova	151
Iasi	226
Lugoj	244
Mehadia	241
Neamt	234
Oradea	380
Pitesti	98
Rimnicu Vilcea	193
Sibiu	253
Timisoara	329
Urziceni	80
Vaslui	199
Zerind	374

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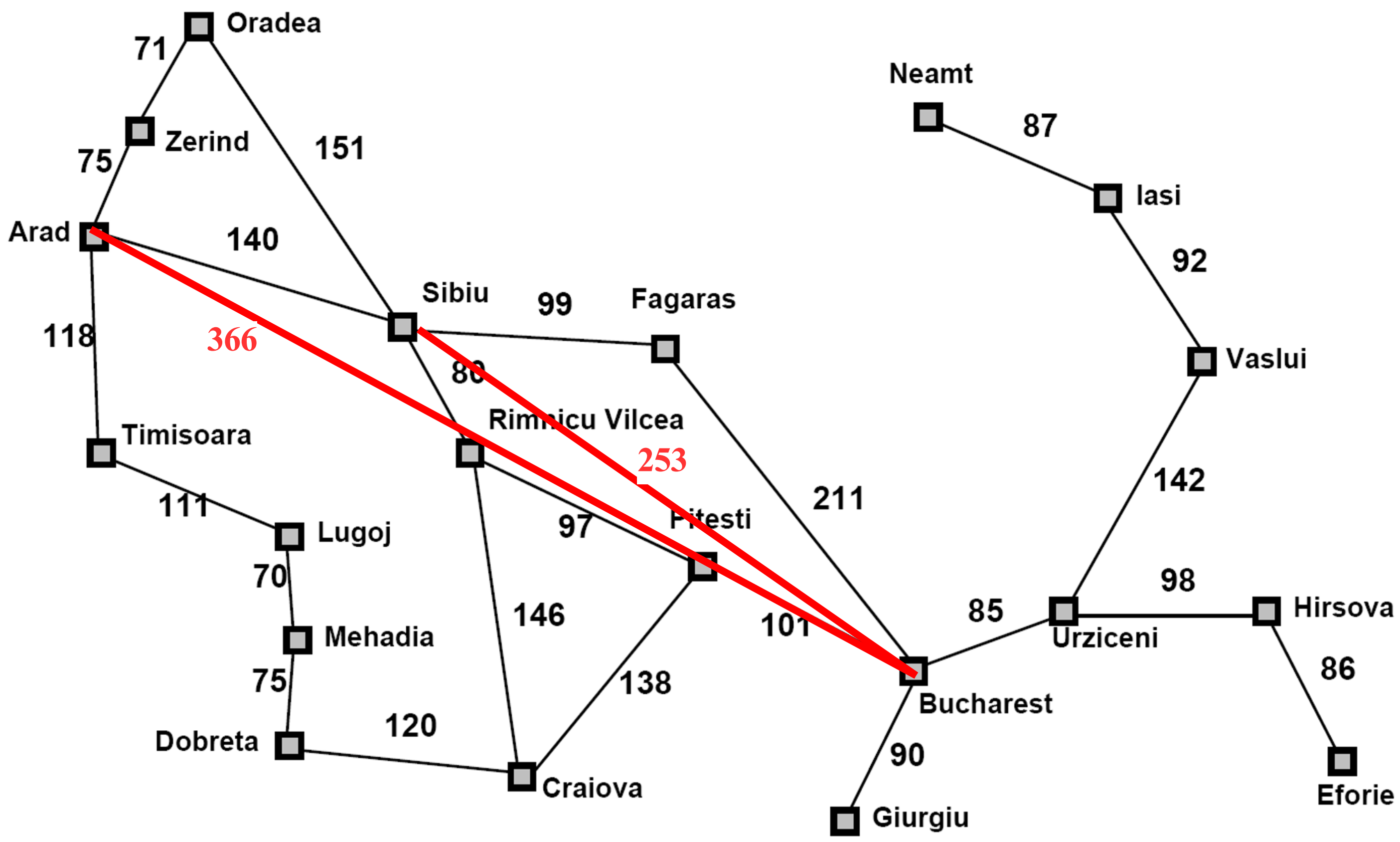

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Greedy Best-First Search

- Evaluation function $f(n) = h(n)$ ($h$euristic)
  - estimates the cost from node n to $goal$
  - e.g., $h_{SLD}(n)$ = straight-line distance from n to Bucharest
- Greedy best-first search expands the node that appears to be closest to goal
  - according to evaluation function
- Example:

Greedy Best-First Search

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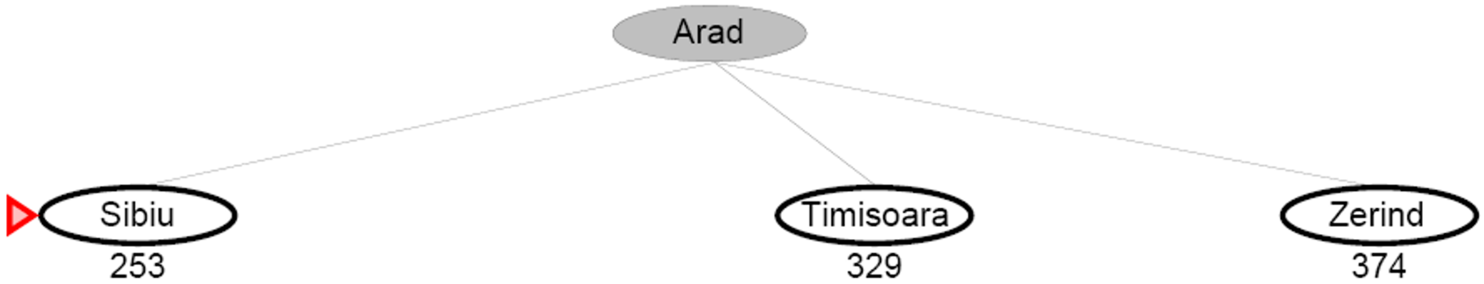

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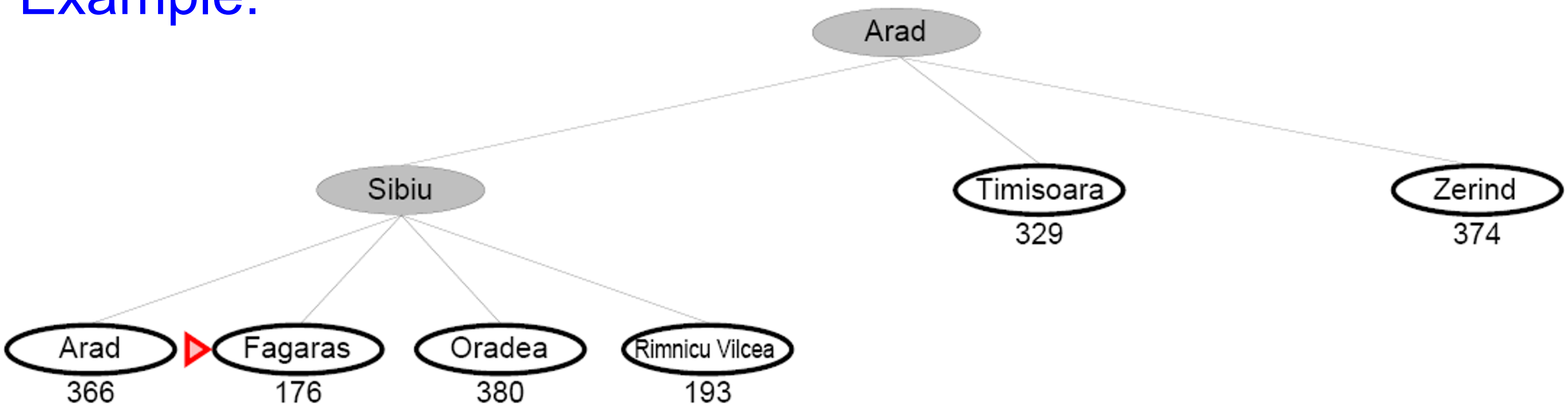

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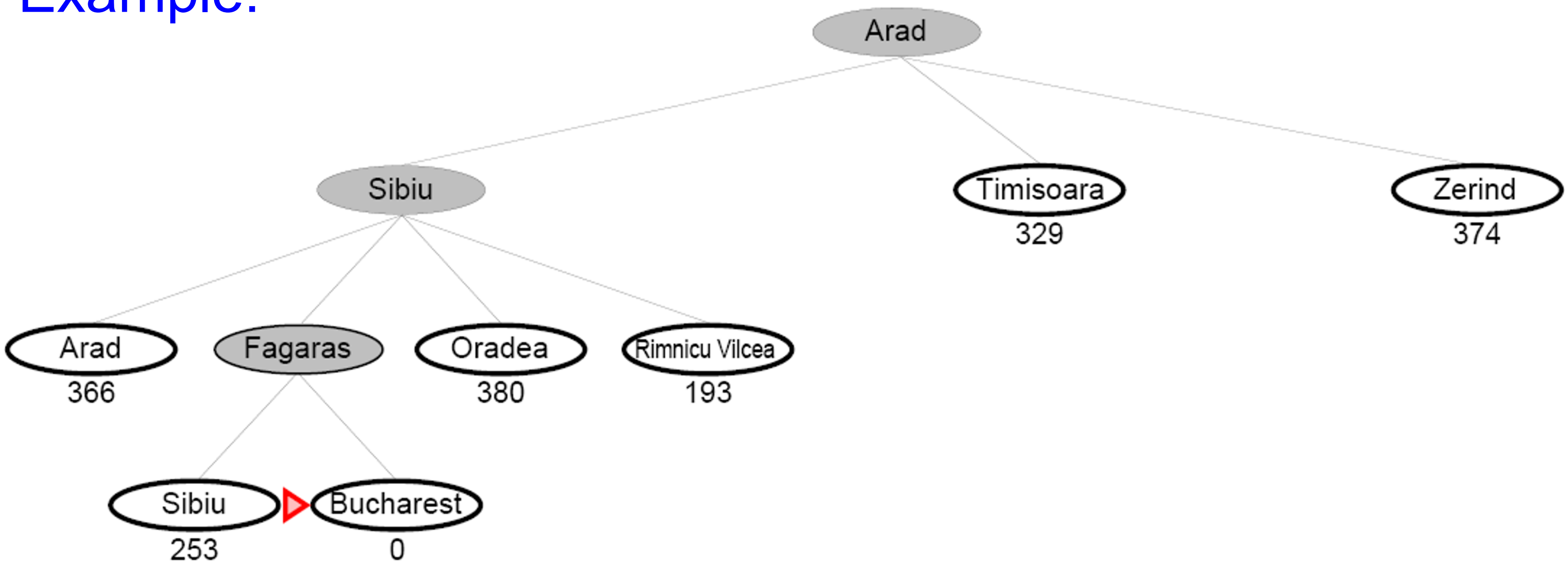
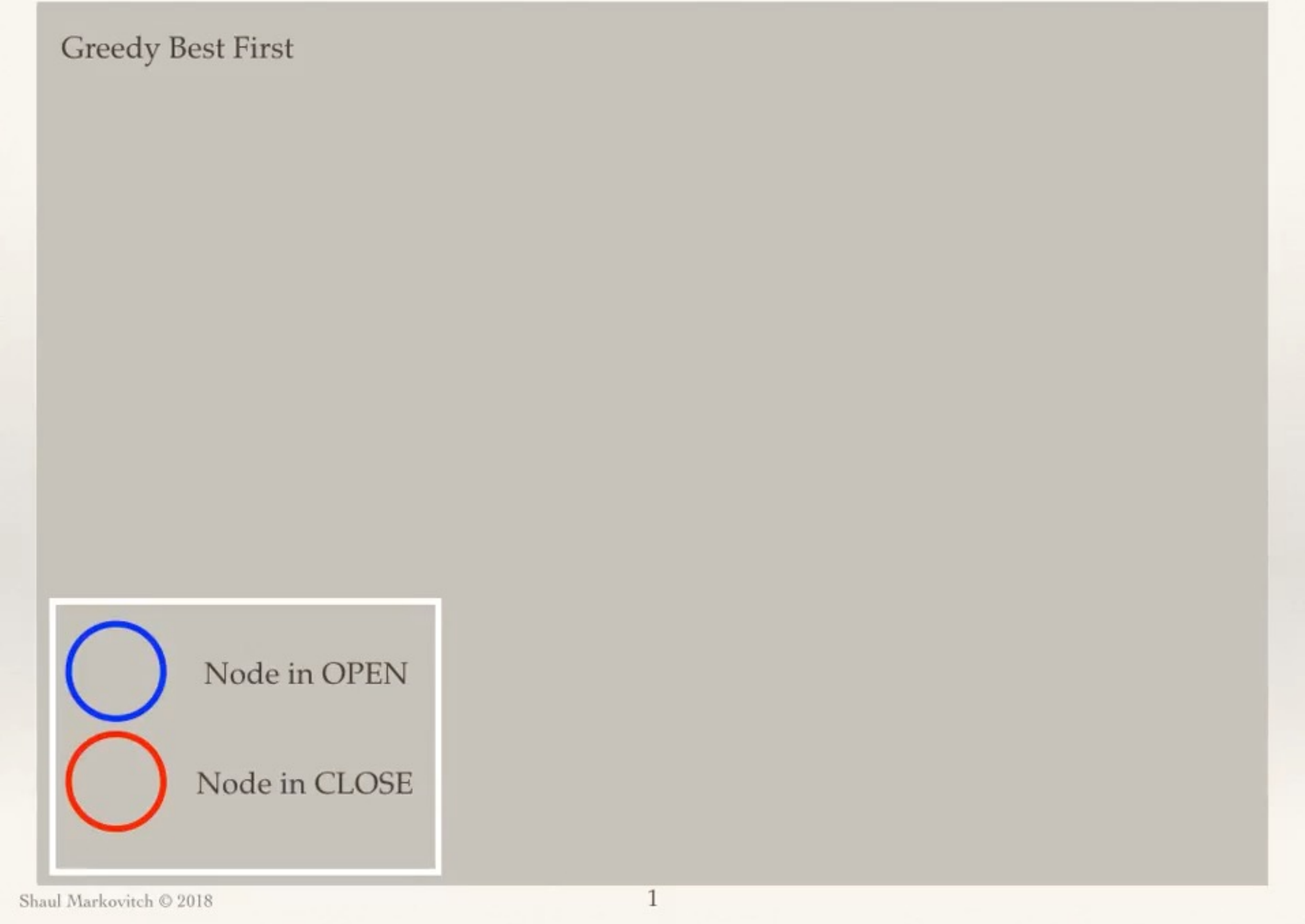


Greedy Best-First Search

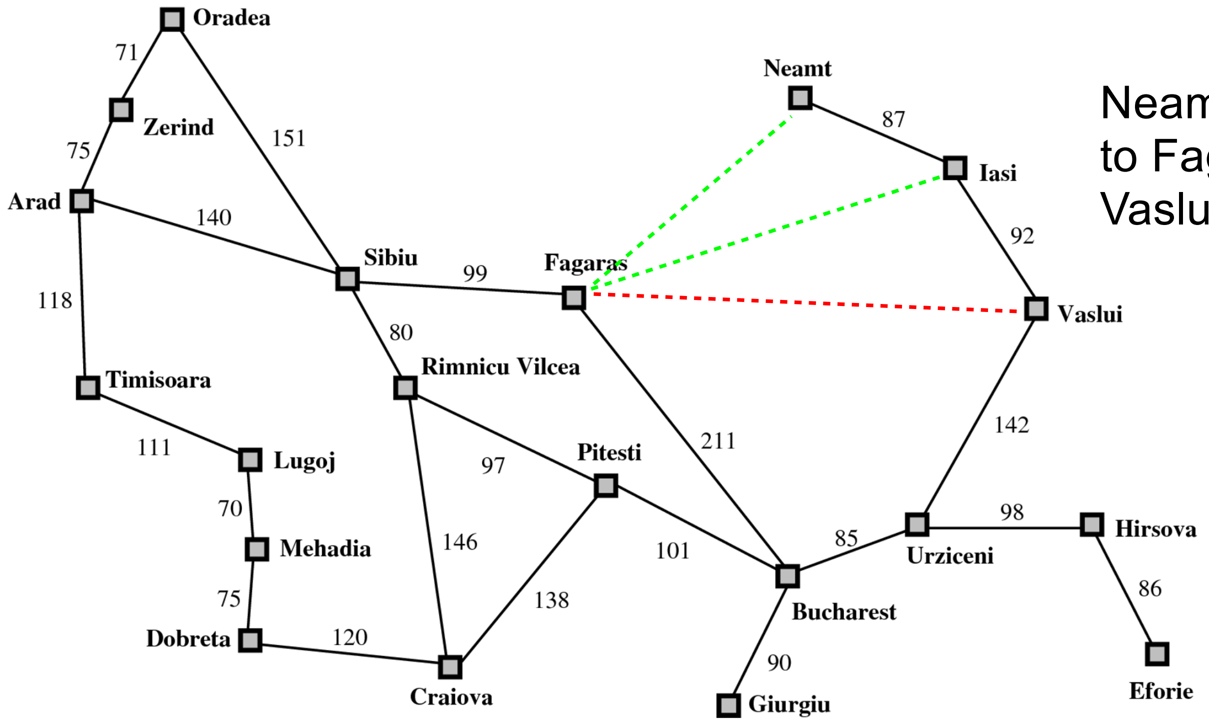
Shaul Markovitch, https://www.youtube.com/watch?v=A8pmud1Uh0Q&t=1s



Properties of Greedy Best-First Search

- Completeness
  - No – can get stuck in loops
  - Example: We want to get from Iasi to Fagaras
    - Iasi → Neamt → Iasi → Neamt → ...

Note:
These two are different search nodes referring to the same state!



Neamt is closer to Fagaras than Vaslui

Properties of Greedy Best-First Search

- Completeness
  - No – can get stuck in loops
  - can be fixed with careful checking for duplicate states

  → complete in finite state space with repeated-state checking
- Time Complexity
  - $O(b^m)$, like depth-first search
  - but a good heuristic can give dramatic improvement
    - optimal case: best choice in each step → only d steps
    - a good heuristic improves chances for encountering optimal case
- Space Complexity
  - has to keep all nodes in memory → same as time complexity
- Optimality
  - No
  - Example:
    - solution Arad → Sibiu → Fagaras → Bucharest is not optimal

A* Search

- Best-known form of best-first search

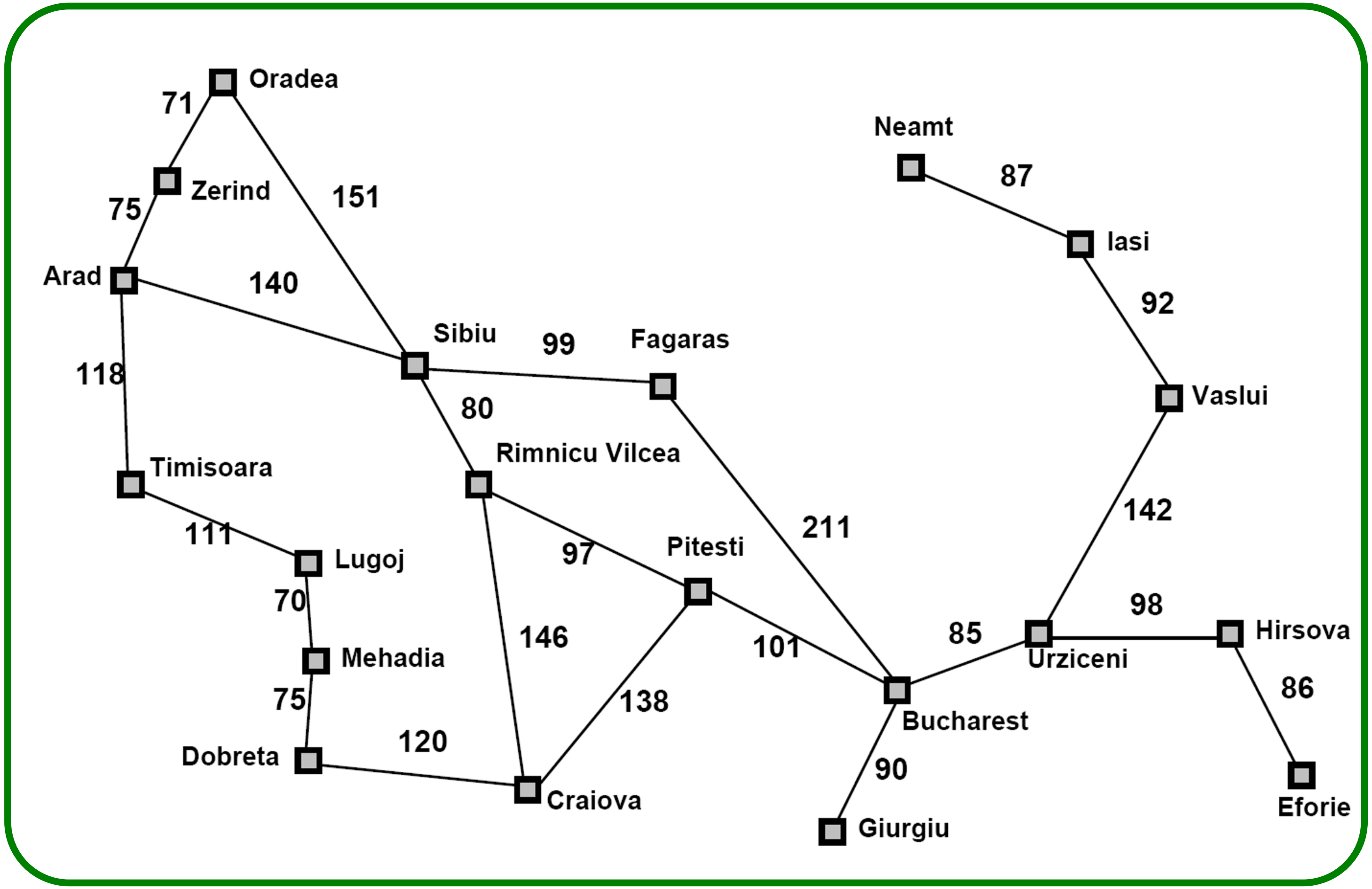
- Basic idea:
  - avoid expanding paths that are already expensive

  $\rightarrow$ evaluate complete path cost not only remaining costs

- Evaluation function: $f(n) = g(n) + h(n)$
  - $g(n)$ = cost so far to reach node n
  - $h(n)$ = estimated cost to get from n to goal
  - $f(n)$ = estimated cost of path to goal via n

Beispiel

$g(n)$



$h(n)$

Straight−line distance to Bucharest

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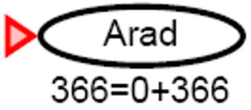
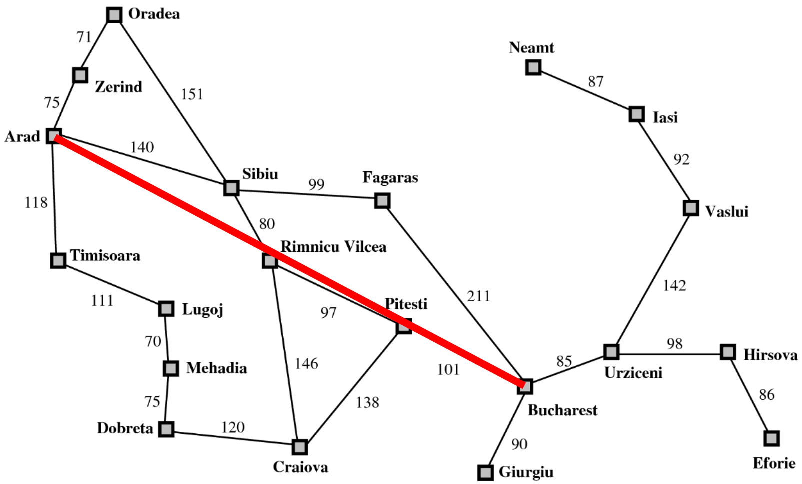
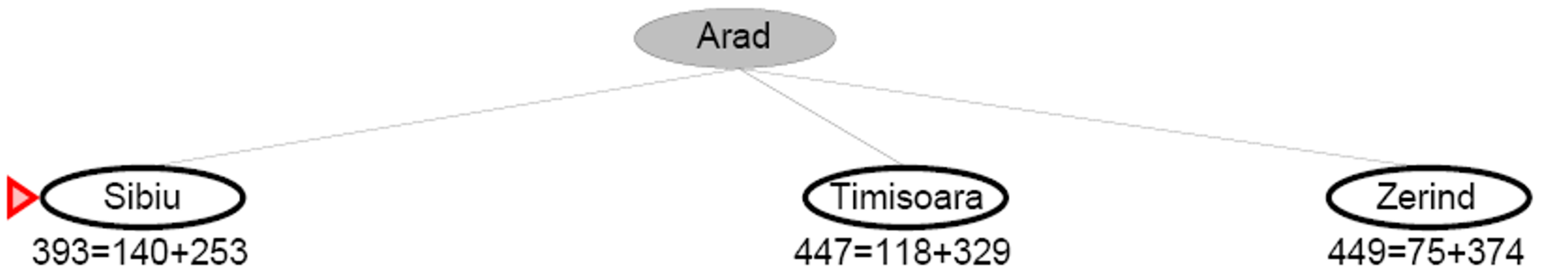
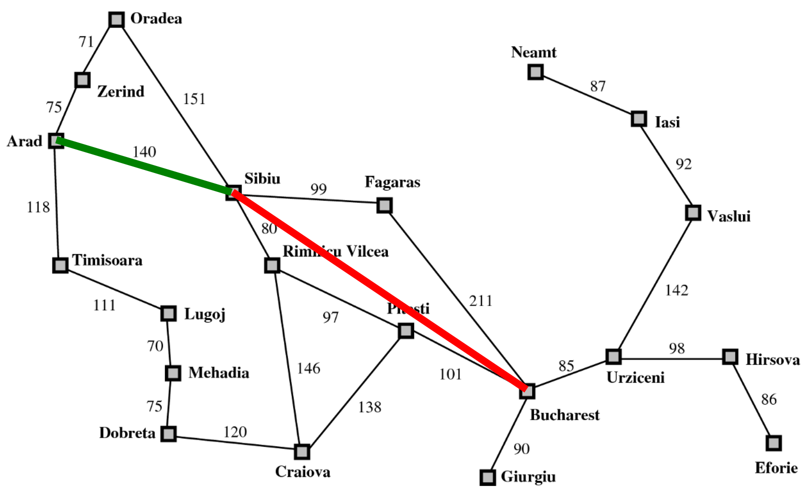
A* Search Example



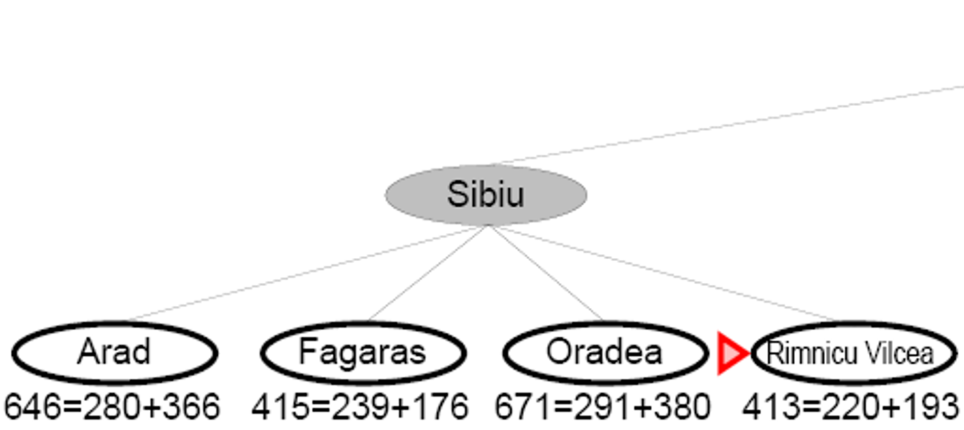


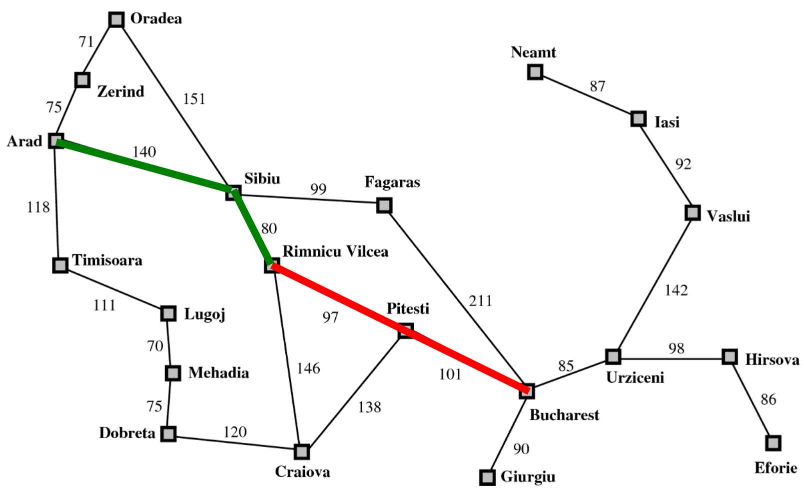
A* Search Example



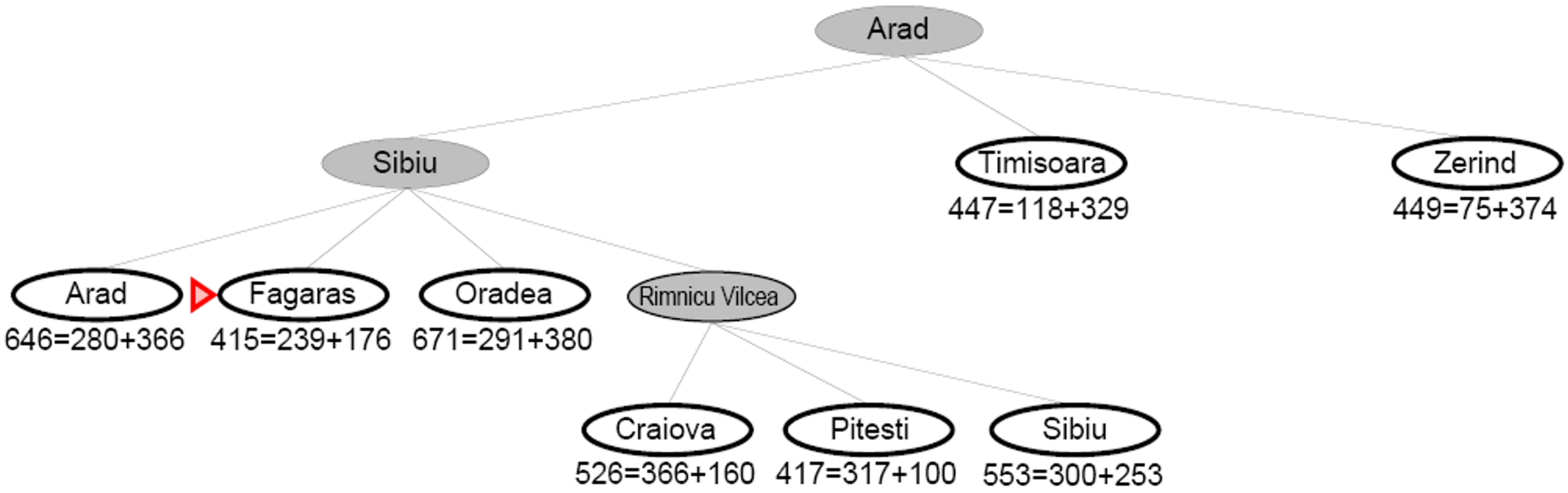


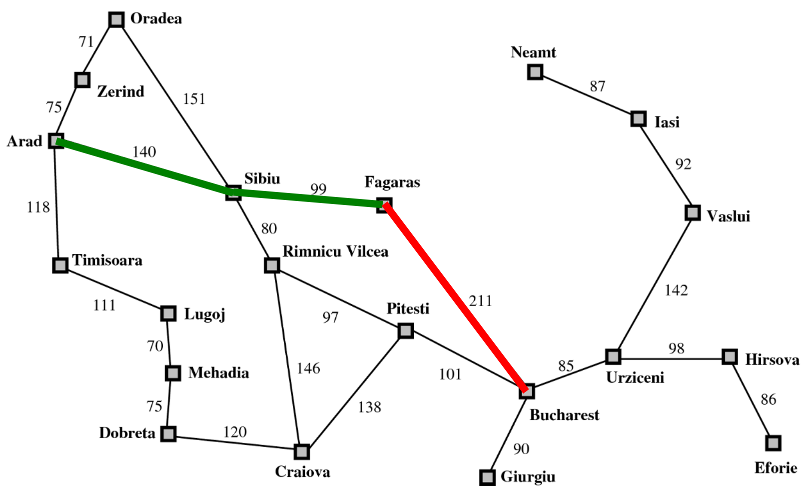
A* Search Example



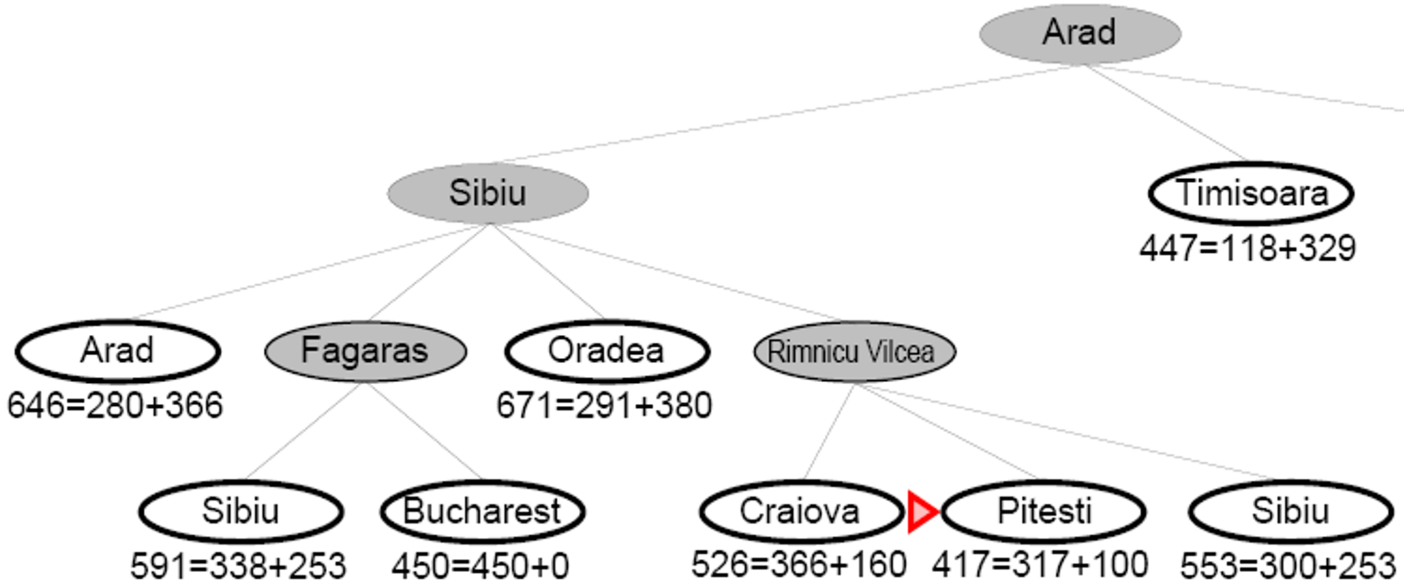


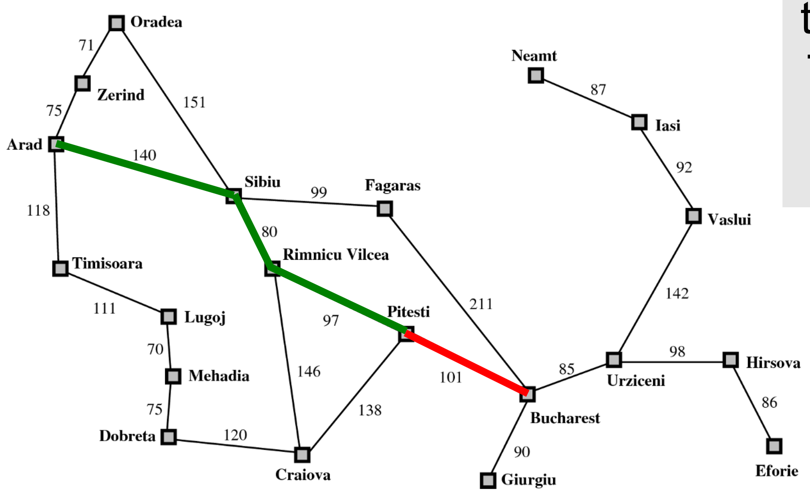
A* Search Example





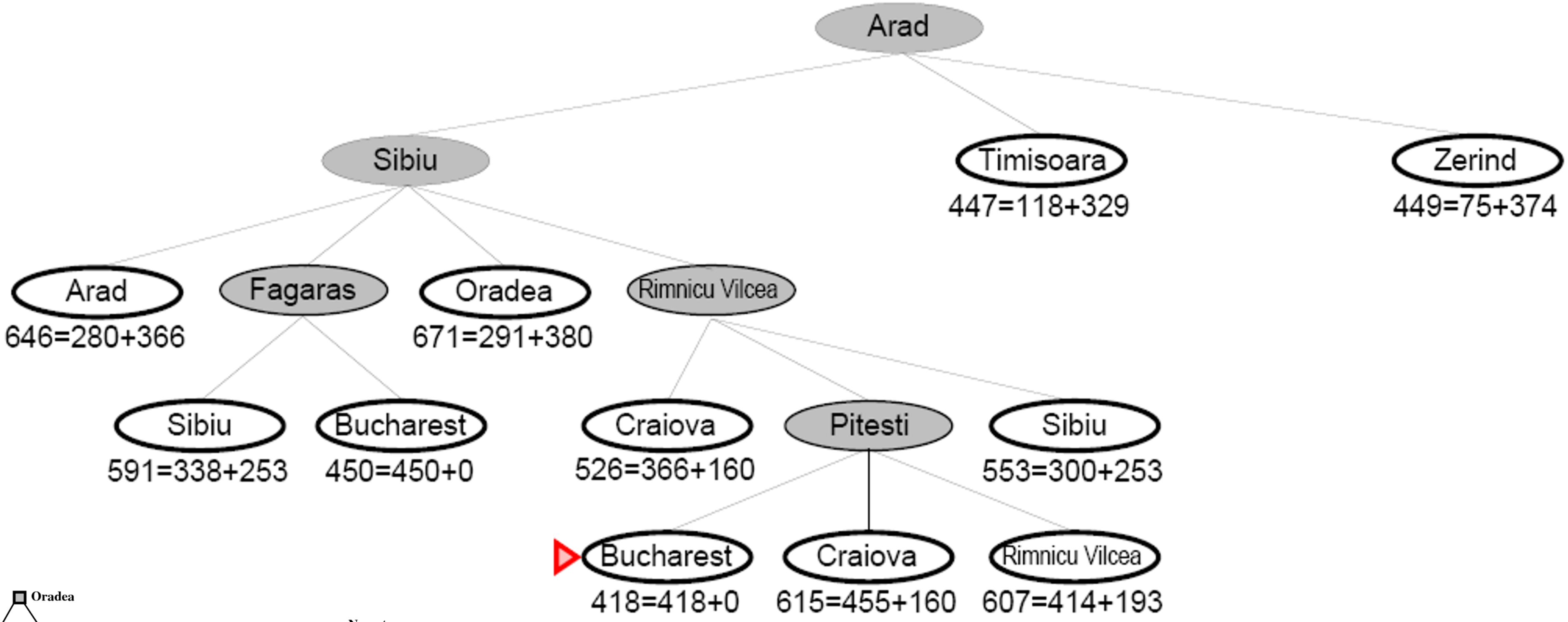
A* Search Example

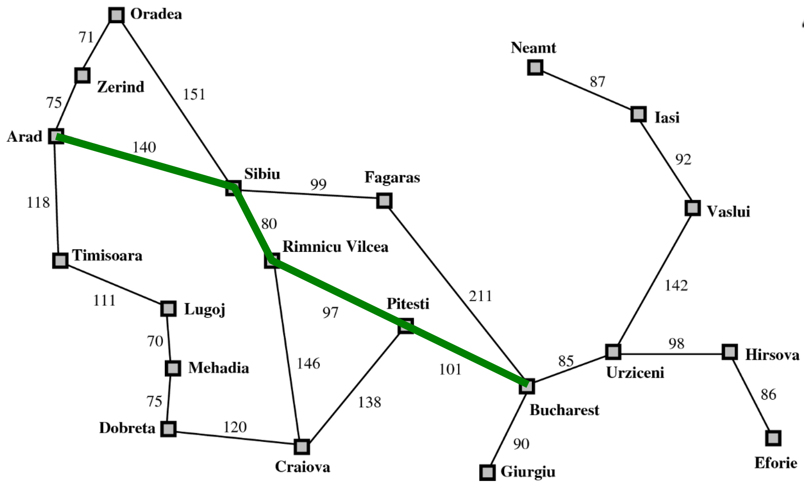




Note that Pitesti will be expanded even though Bucharest is already in the fringe! This is good, because we may still find a shorter way to Budapest. Greedy Search would not do that.

A* Search Example





Properties of A*

- Completeness
  - Yes
  - unless there are infinitely many nodes with $f(n) \leq f(G)$
- Time Complexity
  - it can be shown that the number of nodes grows exponentially unless the error of the heuristic $h(n)$ is bounded by the logarithm of the value of the actual path cost $h^*(n)$, i.e.

$$| h(n) - h^*(n) | \leq O\left(\log h^*(n)\right)$$

- Space Complexity
  - keeps all nodes in memory
  - typically the main problem with A*
- Optimality
  - ???

→ following pages

Admissible Heuristics

A heuristic is admissible if it *never* overestimates the cost to reach the goal

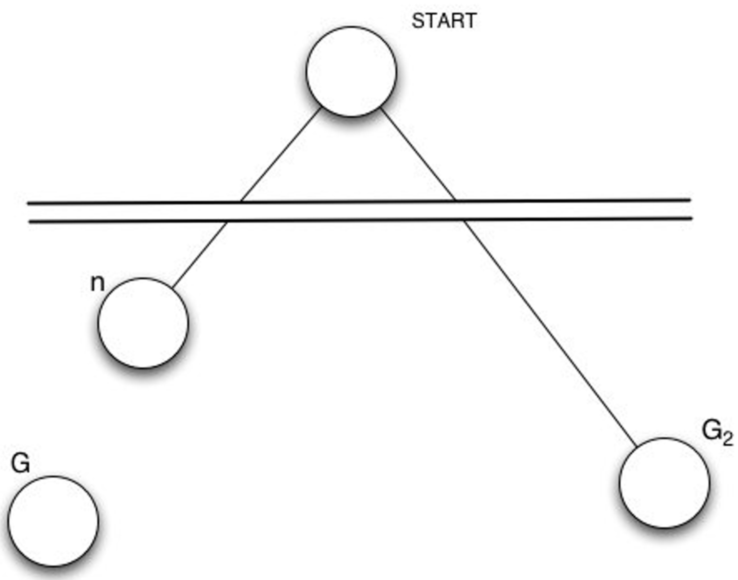
- Formally:
  - $h(n) \leq h^*(n)$ if $\boldsymbol{h^*(n)}$ **are the true cost** from n to goal
- Example:
  - Straight-Line Distances h_{SLD} are an admissible heuristics for actual road distances h^*
- Note:
  - $h(n) \geq 0$ must also hold, so that $h(goal) = 0$

Theorem

If $h(n)$ is admissible, A* using `TREE-SEARCH` is optimal.

Proof:

Let n be an unexpanded node in the fringe such that n is on a shortest path to an optimal goal G with path cost $C^* = g(G)$.



Suppose some suboptimal goal G_2 has been generated and is in the fringe.

$$C^* = g(n) + h^*(n)$$

$$f(n) = g(n) + h(n)$$

$$h(n) \leq h^*(n)$$

because h admissible

$$f(n) \leq C^* < f(G_2)$$

G_2 will never be expanded, and G will be returned

$$g(G_2) > C^*$$

because G_2 suboptimal

$$f(G_2) = g(G_2)$$

because $h(G_2) = 0$

(holds for all goal nodes)

A* and Graph Search

- So far we only had tree search where each node in the search tree represents one possible path to the associated domain state
  - e.g., one can go directly from Arad to Sibiu or via Oradea
- Problem:
  - In some cases the path that is detected later may be the better path
  - so that a previously found solution starting from Sibiu has to be re-investigated with the new, cheaper path
- Two solutions
  - Add ability to detect repeated states
    - → graph search
    - general solution that also works for BFS, DFS, ...
  - Or ensure that the cheaper path is always taken first
    - → consistent heuristics
    - specific solution for A*

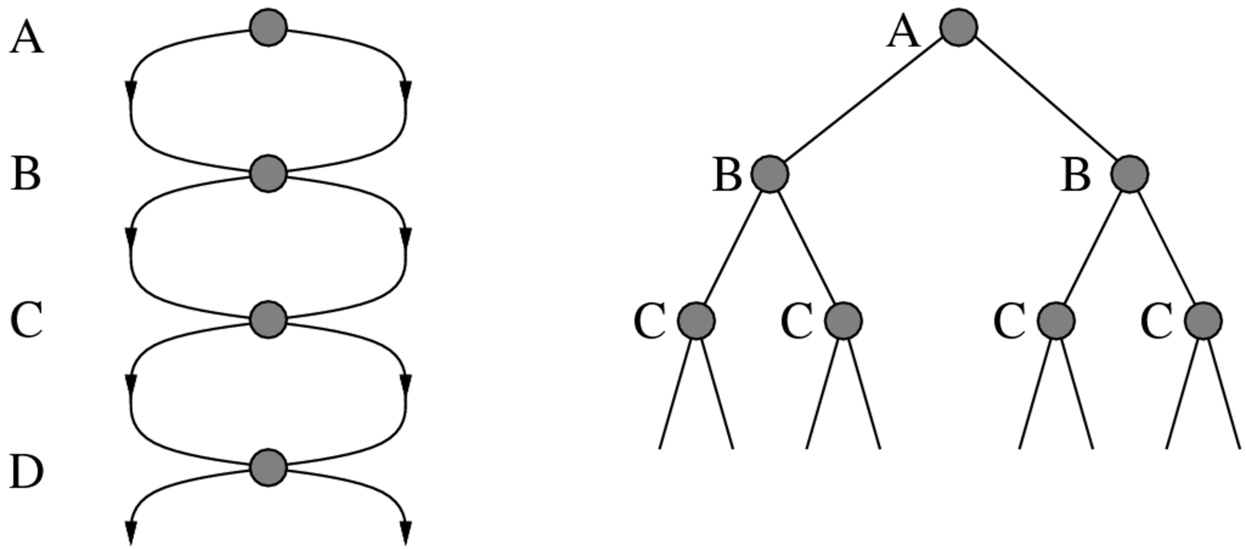
Repeated States

- Failure to detect repeated states can turn a linear problem into an exponential one!

Ribbon Example

- two connections from each state to the next

d states but state space is 2^d



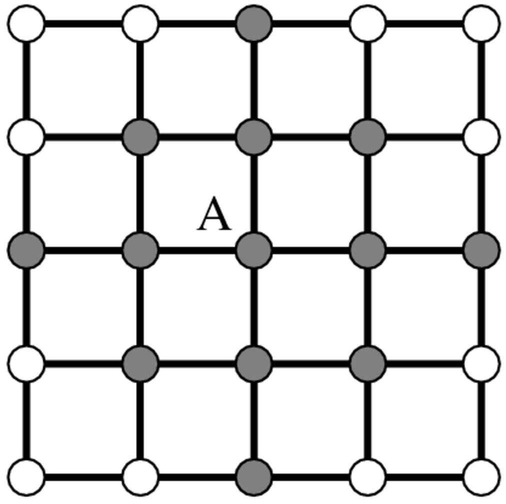
(a) (b)

Repeated States

- Failure to detect repeated states can turn a linear problem into an exponential one!

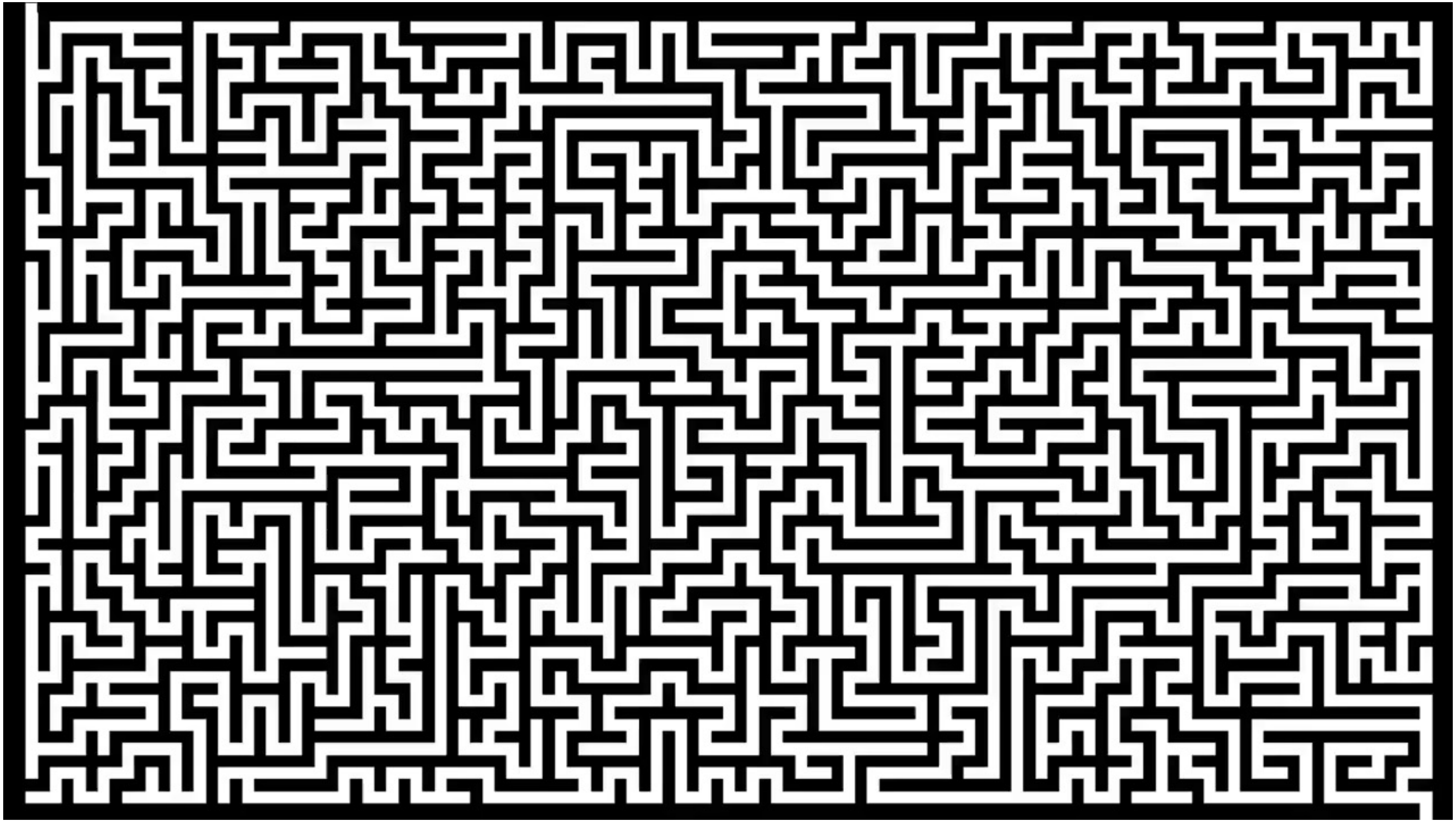
(more realistic) Grid Example

- each square on grid has 4 neighboring states in
- thus, game tree w/o repetitions has 4^d nodes
- but only about $2d^2$ different states are reachable in d steps



A* Search Example

Sebastian Lague, https://www.youtube.com/watch?v=-L-WgKMFuhE



Graph Search

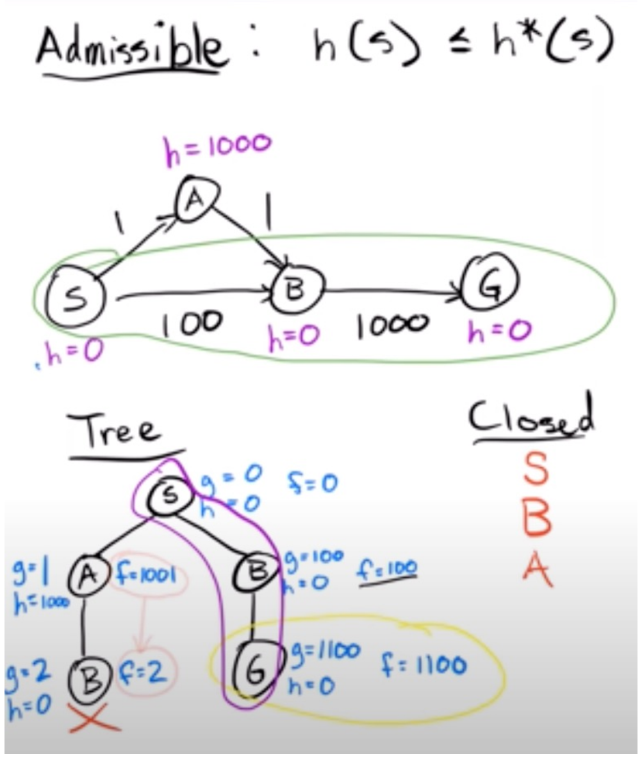
- remembers the states that have been visited in a list *closed*
  - Note: the fringe list is often also called the open list

```
function GRAPH-SEARCH( problem, fringe) returns a solution, or failure
   closed ← an empty set
   fringe ← INSERT(MAKE-NODE(INITIAL-STATE[problem]), fringe)
   loop do
      if fringe is empty then return failure
      node ← REMOVE-FRONT(fringe)
      if GOAL-TEST(problem, STATE[node]) then return node
      if STATE[node] is not in closed then
         add STATE[node] to closed
         fringe ← INSERTALL(EXPAND(node, problem), fringe)
   end
```

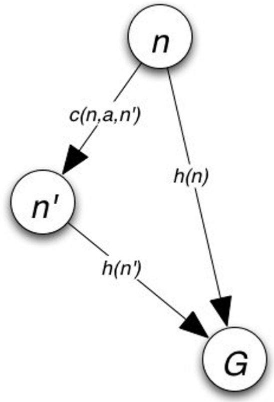
- Example:
  - Dijkstra's algorithm is the graph-search variant of uniform cost search

Consistent Heuristics

https://www.youtube.com/watch?v=AVKPExS4TBY



- Graph-Search discards new paths to repeated state even though the new path may be cheaper
  → Previous proof breaks down
- 2 Solutions
  1. Add extra bookkeeping to remove the more expensive path
  2. Ensure that optimal path to any repeated state is always followed first
- Requirement for Solution 2:



A heuristic is consistent if for every node n and every successor n' generated by any action a it holds that $h(n) \leq c(n,a,n') + h(n')$

Lemma 1

Every consistent heuristic is admissible.

Proof Sketch by induction

Base Case: for all nodes n, in which an action a leads to goal G

$$h(n) \leq c(n, a, G) + h(G) = h^*(n)$$

By *induction on the path length from goal*, this argument can be extended to all nodes, so that eventually

$$\forall n: h(n) \leq h^*(n)$$

- Note:
  - not every admissible heuristic is consistent
  - but most of them are
    - it is hard to find non-consistent admissible heuristics

Lemma 2

If $h(n)$ is consistent, then the values of $f(n)$ along any path are non-decreasing.

Proof:

$$f(n) = g(n) + h(n) \leq g(n) + c(n, a, n') + h(n') =$$

$$g(n) + c(n, a, n') + h(n') = g(n') + h(n') = f(n')$$

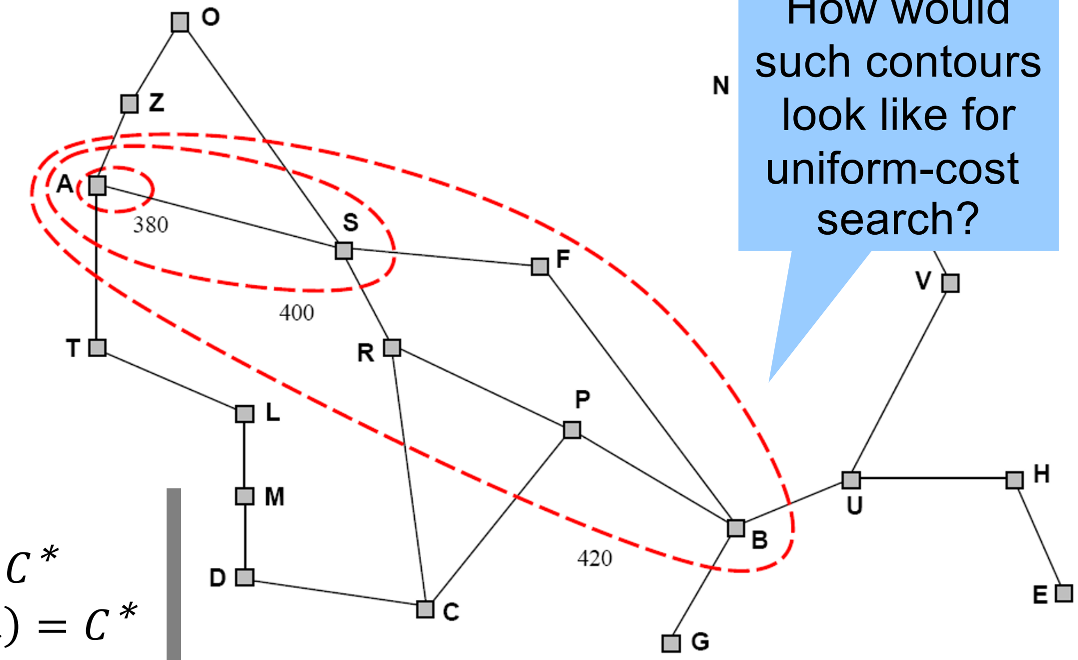
Theorem

If $h(n)$ is consistent, graph-search A* is optimal.

Proof:

A* expands nodes in order of increasing f value

Contour labelled f_i contains all nodes with $f(n) < f_i$



How would such contours look like for uniform-cost search?

Contours expand gradually
Cannot expand f_{i+1} until f_i is finished.

Eventually

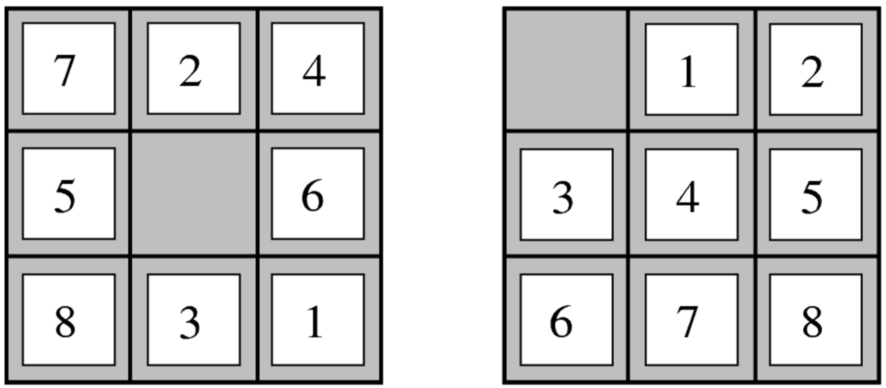
- A* expands all nodes with $f(n) < C^*$
- A* expands some nodes with $f(n) = C^*$
- A* expands no nodes with $f(n) > C^*$

Memory-Bounded Heuristic Search

- Some solutions to A* space problems
  (maintaining completeness and optimality)
  - Iterative-deepening A* (IDA*)
    - like iterative deepening
    - cutoff information is the f-cost $(g + h)$ instead of depth
  - Recursive best-first search (RBFS)
    - recursive algorithm that attempts to mimic
    - standard best-first search with linear space.
    - keeps track of the f-value of the best alternative
    - path available from any ancestor of current node
    - heuristic evaluations are updated with results of successors
  - (Simple) Memory-bounded A* ((S)MA*)
    - drop the worst leaf node when memory is full
    - its value will be updated to its parent
    - May need to be re-searched later

Admissible Heuristics: 8-Puzzle

- $h_{\text{MIS}}(n)$ = number of misplaced tiles
  - admissible because each misplaced tile must be moved at least once
- $h_{\text{MAN}}(n)$ = total Manhattan distance
  - i.e., no. of squares from desired location of each tile
  - admissible because this is the minimum distance of each tile to its target square
- Example:



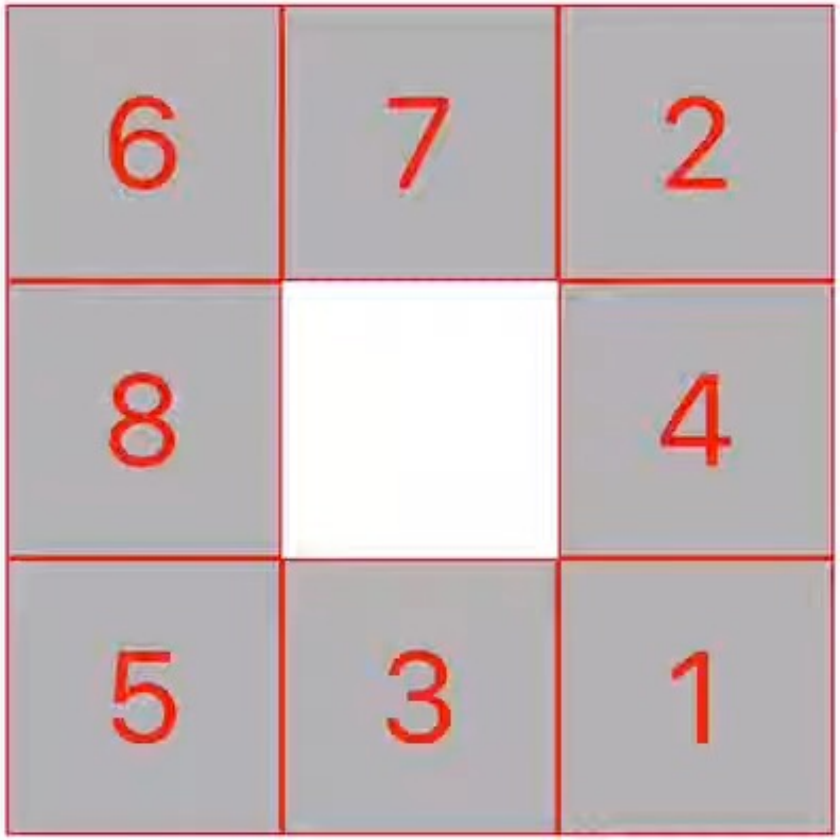
Start State Goal State

$h_{MIS}(start) = 8$

$h_{MAN}(start) = 18$

$h^{*}(start) = 26$

Solving 8-puzzle using A*



https://www.youtube.com/watch?v=_UyyJK3jeF4

Can we measure the quality of a heuristic? Effective Branching Factor

- Evaluation Measure for a search algorithm:
  - assume we searched N nodes and found solution in depth d
  - the effective branching factor b^* is the branching factor of a uniform tree of depth d with $N+1$ nodes, i.e.

$$1 + N = 1 + b^* + (b^*)^2 + \ldots + (b^*)^d$$

- A* find a solution at depth 5 using 52 nodes, then b* is 1.92

- **Measure is fairly constant for different instances of sufficiently hard problems**
  - Can thus provide a good empirical guide to the heuristic’s overall usefulness.
  - A good value of b^* is 1

Efficiency of A* Search / Comparing Heuristics

- Comparison of number of nodes searched by A* and Iterative Deepening Search (IDS)
  - average of 100 different 8-puzzles with different solutions
  - **Note:** heuristic $h_2 = h_{MAN}$ is always better than $h_1 = h_{MIS}$

	Suchkosten			Effektiver Verzweigungsfaktor		
d	IDS	$A^*(h_1)$	$A^*(h_2)$	IDS	$A^*(h_1)$	$A^*(h_2)$
2	10	6	6	2,45	1,79	1,79
4	112	13	12	2,87	1,48	1,45
6	680	20	18	2,73	1,34	1,30
8	6384	39	25	2,80	1,33	1,24
10	47127	93	39	2,79	1,38	1,22
12	3644035	227	73	2,78	1,42	1,24
14	–	539	113	–	1,44	1,23
16	–	1301	211	–	1,45	1,25
18	–	3056	363	–	1,46	1,26
20	–	7276	676	–	1,47	1,27
22	–	18094	1219	–	1,48	1,28
24	–	39135	1641	–	1,48	1,26

Constructing Heuristics: Dominance

If h_1 and h_2 are admissible, h_2 dominates h_1 if $\forall n: h_2(n) \geq h_1(n)$

- if h_2 dominates h_1 it will perform better because it will *always* be closer to the optimal heuristic h^*
- Example:
  - h_{MAN} dominates h_{MIS} because if a tile is misplaced, its Manhattan distance is ≥ 1

Theorem: (Combining admissible heuristics)

If h_1 and h_2 are two admissible heuristics than

$$h(n) = max\left(h_1(n), h_2(n)\right)$$

is also admissible and dominates h_1 and h_2

Relaxed Problems

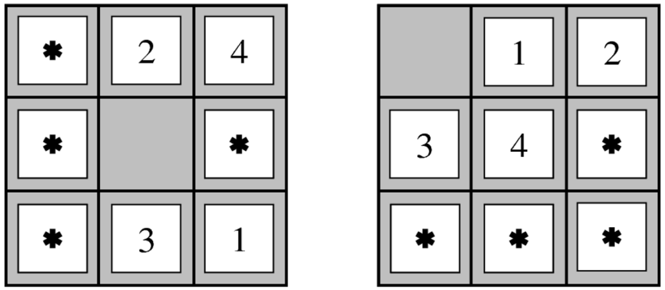
- A problem with fewer restrictions on the actions is called a relaxed problem
- **The cost of an optimal solution to a relaxed problem is an admissible heuristic for the original problem**

- Examples:
  - If the rules of the 8-puzzle are relaxed so that a tile can move anywhere, then h_{MIS} gives the shortest solution
  - If the rules are relaxed so that a tile can move to any adjacent square, then h_{MAN} gives the shortest solution

- Thus, looking for relaxed problems is a good strategy for inventing admissible heuristics.

Pattern Databases

- Admissible heuristics can also be derived from the solution cost of a subproblem of a given problem.
  - This cost is a lower bound on the cost of the real problem.
- Pattern databases store the exact solution (length) for every possible subproblem instance
  - constructed once for all by searching backwards from the goal and recording every possible pattern
- Example:
  - store exact solution costs for solving 4 tiles of the 8-puzzle
  - sample pattern:



Start State Goal State

Learning of Heuristics

- Another way to find a heuristic is through learning from experience
- Experience:
  - states encountered when solving lots of 8-puzzles
  - states are encoded using features, so that similarities between states can be recognized
- Features:
  - for the 8-puzzle, features could, e.g. be
    - the number of misplaced tiles
    - number of pairs of adjacent tiles that are also adjacent in goal
    - ...
- An inductive learning algorithm can then be used to predict costs for other states that arise during search.
- No guarantee that the learned function is admissible!

Summary

- Heuristic functions estimate the costs of shortest paths
- Good heuristics can dramatically reduce search costs
- Greedy best-first search expands node with lowest estimated remaining cost
  - incomplete and not always optimal
- A* search minimizes the path costs so far plus the estimated remaining cost
  - complete and optimal, also optimally efficient:
    - no other search algorithm can be more efficient, because they all have search the nodes with $f(n) < C^*$
    - otherwise it could miss a solution
- Admissible search heuristics can be derived from exact solutions of reduced problems
  - problems with less constraints
  - subproblems of the original problem