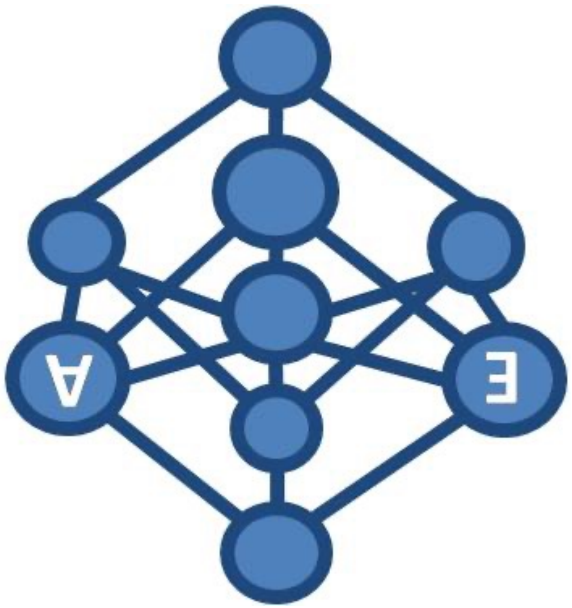
Probabilistic Graphical Models*

Bayesian Networks



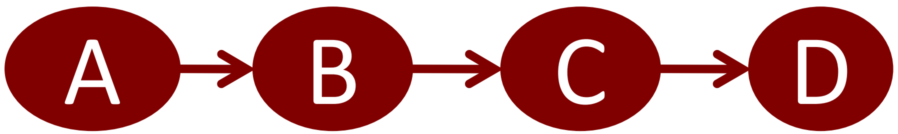
*Thanks to Carlos Guestrin, Pedro Domingos and many others for making their slides publically available

What you need to know thus far

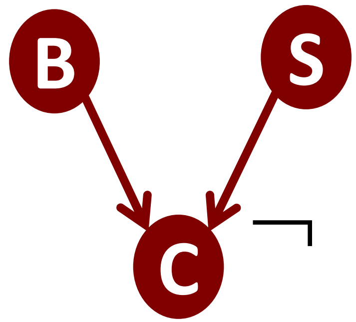
- Independence and conditional independence
- Definition of a Bayesian network
- Local Markov assumption
- The representation theorem
  - G is I-map for P iff P factorizes according to G
  - Interpretation

Independencies encoded in BN

- We said: all you need is the local Markov assumption **($X_i \perp$ $NonDescendants_{X_i}$ | Pa_{X_i})**
- But then we talked about other (in)dependencies such as explaining away



$$A \perp D \mid C$$



$$B \perp S$$

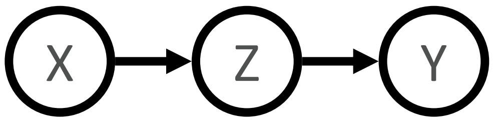
$$\neg \; B \perp S \mid C$$

- **So, what are the independencies encoded by a BN?**
  - Only assumption is local Markov but many other independencies can be derived using the algebra of conditional independencies!

Understanding independencies in BNs (with 3 nodes)

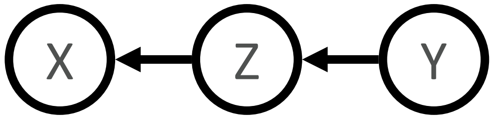
Local Markov Assumption: A variable X is independent of ist non-descendants given its parents and only ist parents:
(Xi ⊥ NonDescendants$_{Xi}$ | Pa$_{Xi}$)

Indirect causal effect:



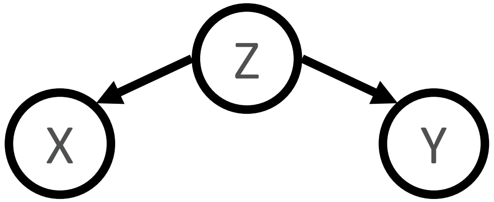
$$X \perp Y \mid Z$$
$$\neg X \perp Y$$

Indirect evidential effect:



$$X \perp Y \mid Z$$
$$\neg X \perp Y$$

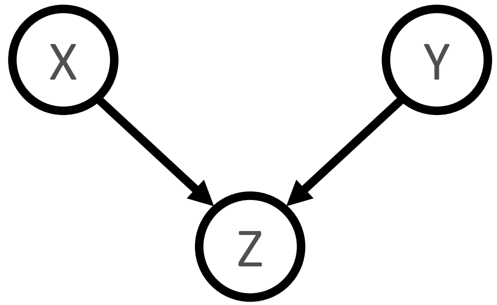
Common cause:



$$X \perp Y \mid Z$$
$$\neg X \perp Y$$

Represent all the same distributions

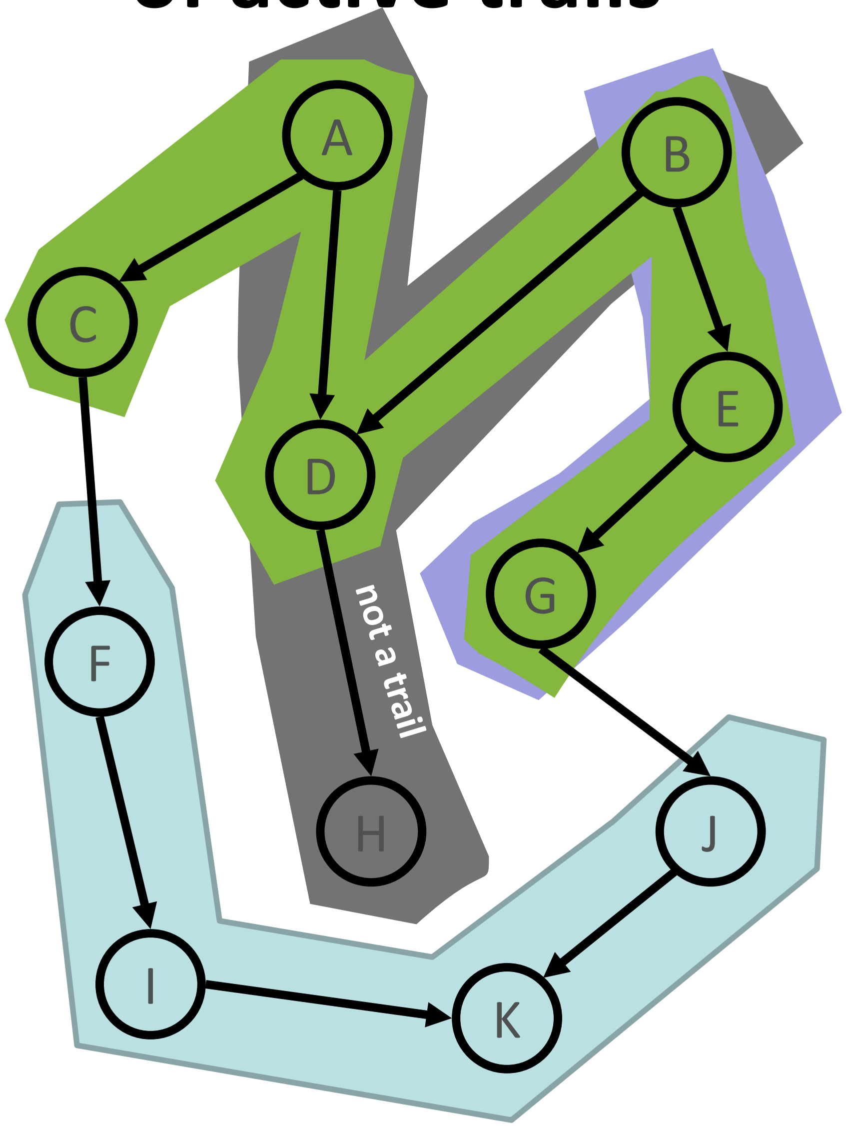
(v-structure)
Common effect:



inverted

$$X \perp Y$$
$$\neg X \perp Y \mid Z$$

This can be generlized using the notion of active trails



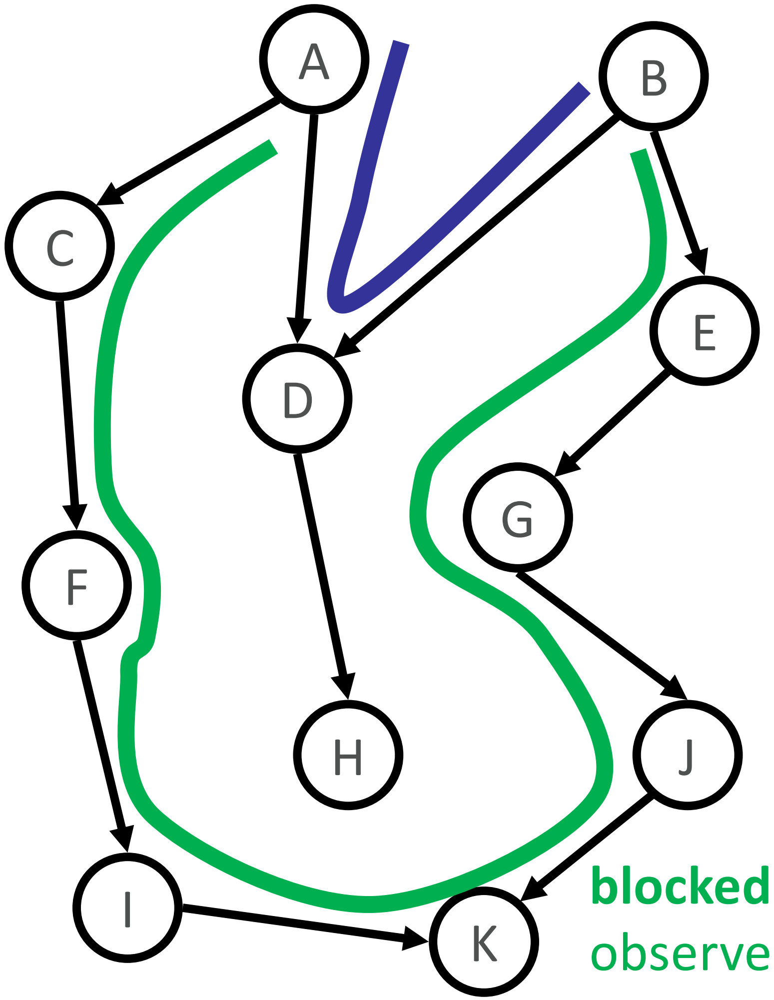
A trail is an undirected path that never visits a node twice

This can be generlized using the notion of active trails

- A trail $X'_1 - X'_2 - \cdots - X'_k$ is **active** (when some variables $\boldsymbol{O} \subseteq \{X_1,\ldots,X_m\}$ are observed) **if** for each consecutive triplet in the trail it holds:
  - $X_{i-1} \rightarrow X_i \rightarrow X_{i+1}$, and X_i is **not observed** ($X_i \notin \boldsymbol{O}$)
  - $X_{i-1} \leftarrow X_i \leftarrow X_{i+1}$, and X_i is **not observed** ($X_i \notin \boldsymbol{O}$)
  - $X_{i-1} \leftarrow X_i \rightarrow X_{i+1}$, and X_i is **not observed** ($X_i \notin \boldsymbol{O}$)
  - $X_{i-1} \rightarrow X_i \leftarrow X_{i+1}$, and X_i **is observed** ($X_i \in \boldsymbol{O}$), or **one of its descendents** **(v-structure)**

Intuitively, information flows a long active trials!

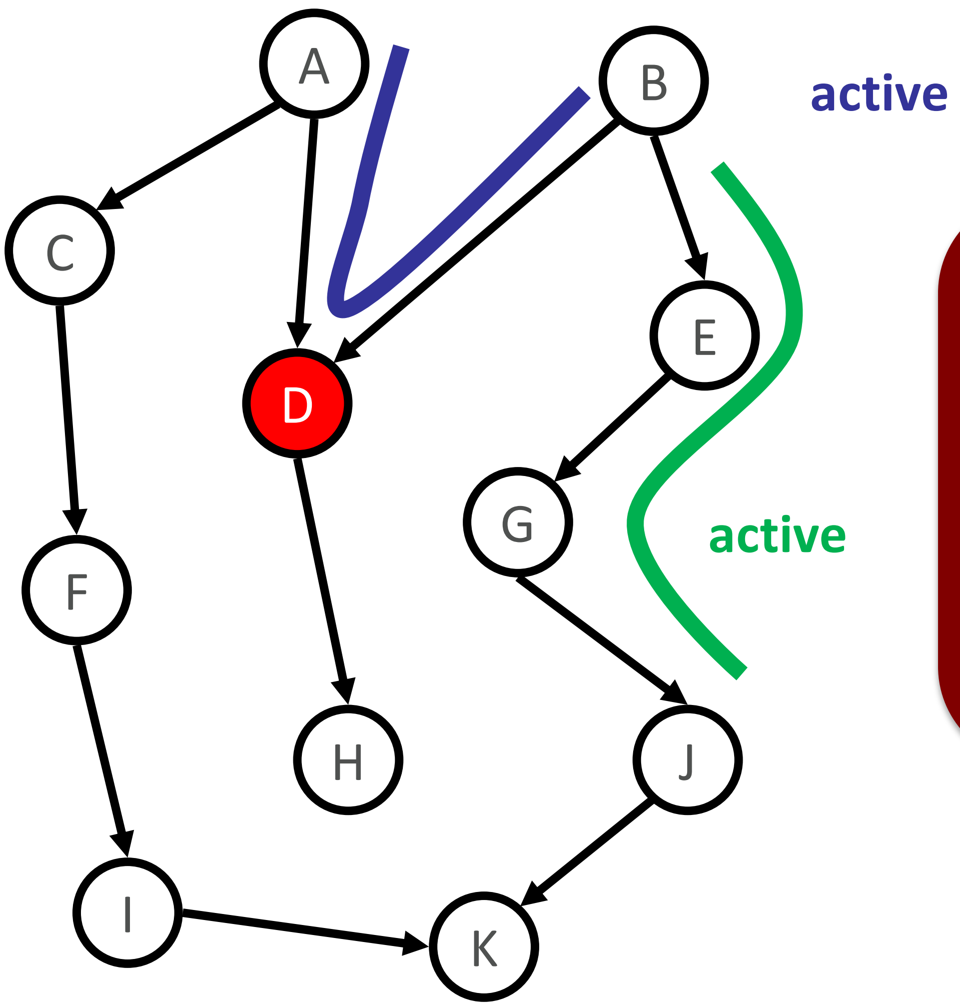
Active and Blocked Trails in BNs – Some Examples –



blocked as D and H are NOT observed

blocked as K is NOT observed

Active and Blocked Trails in BNs – Some Examples –

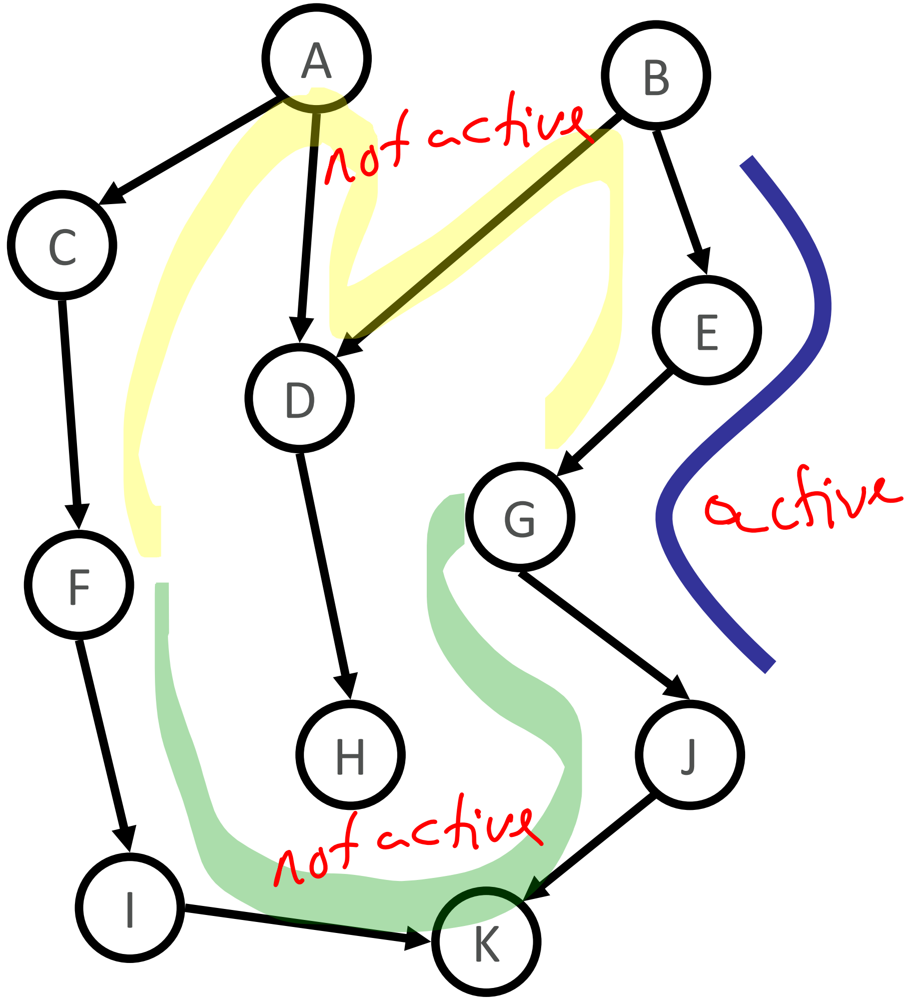


Information can flow if there is a trail x…y that is not blocked by z <u>(active trail)</u>

Active trail and independence?

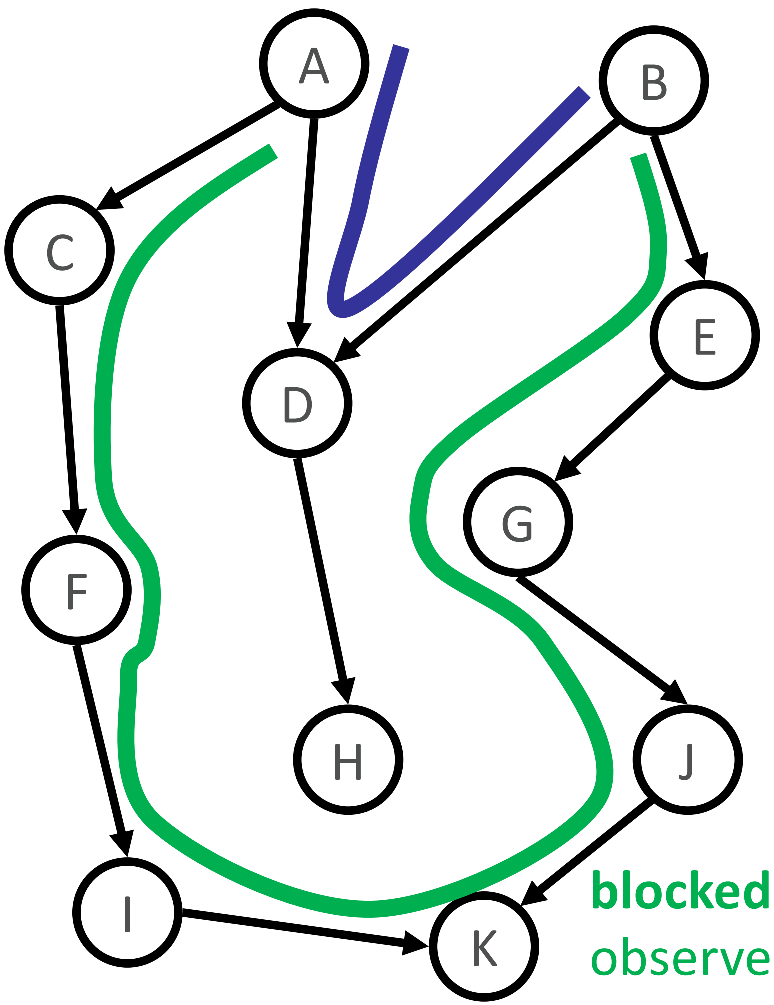
- **Theorem**: Sets of variables $\mathbf{X_i}$ **and $\mathbf{X_j}$ are independent given** $Z \in \{X_1,\ldots,X_n\}$ if there is **no active trail** between $\mathbf{X_i}$ and $\mathbf{X_j}$ when variables $Z \in \{X_1,\ldots,X_n\}$ are observed

- We say that $\mathbf{X_i}$ and $\mathbf{X_j}$ are **d-separated** given Z (dependence separation)



$F \perp G$

D-Separation – Some Examples

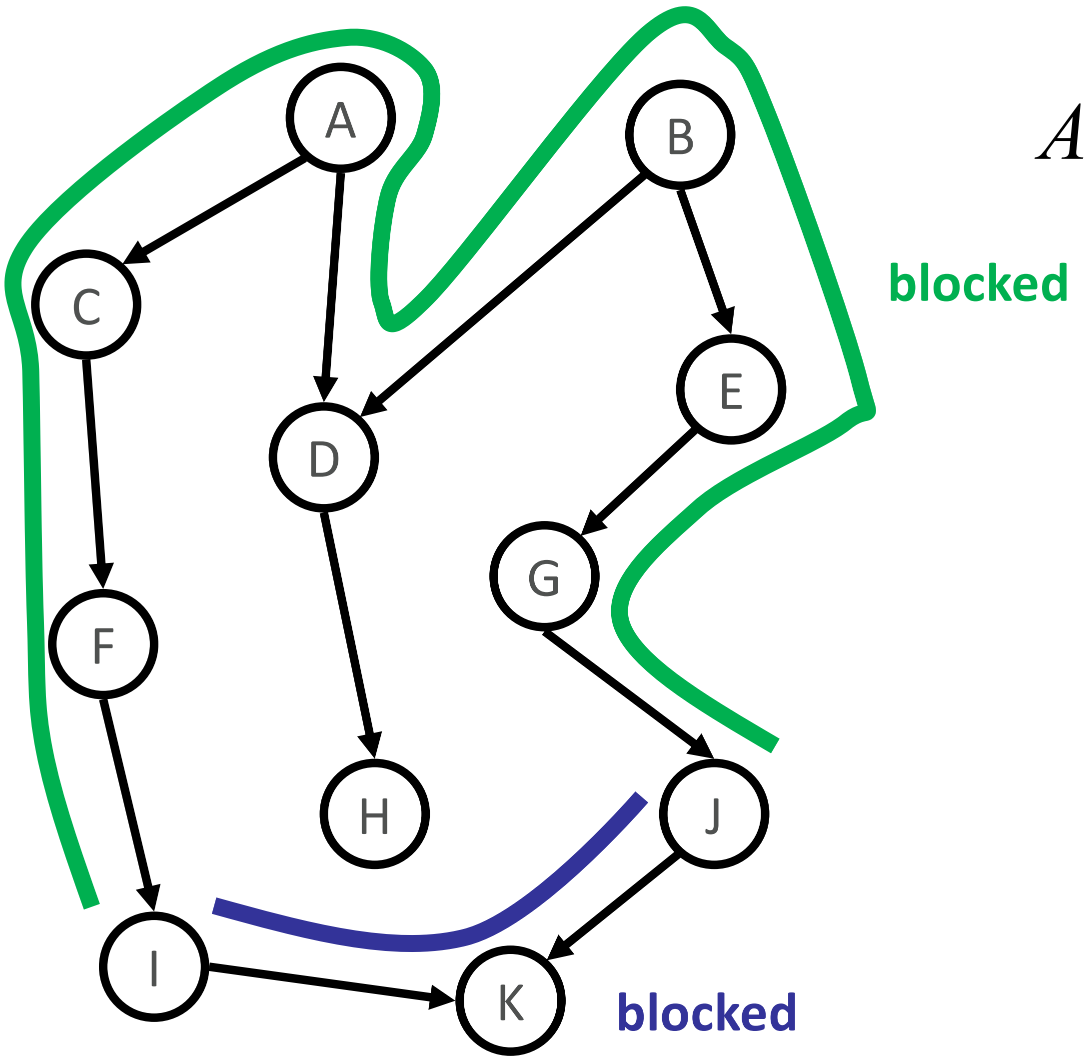


$$A \perp B$$

blocked as D and H are NOT observed

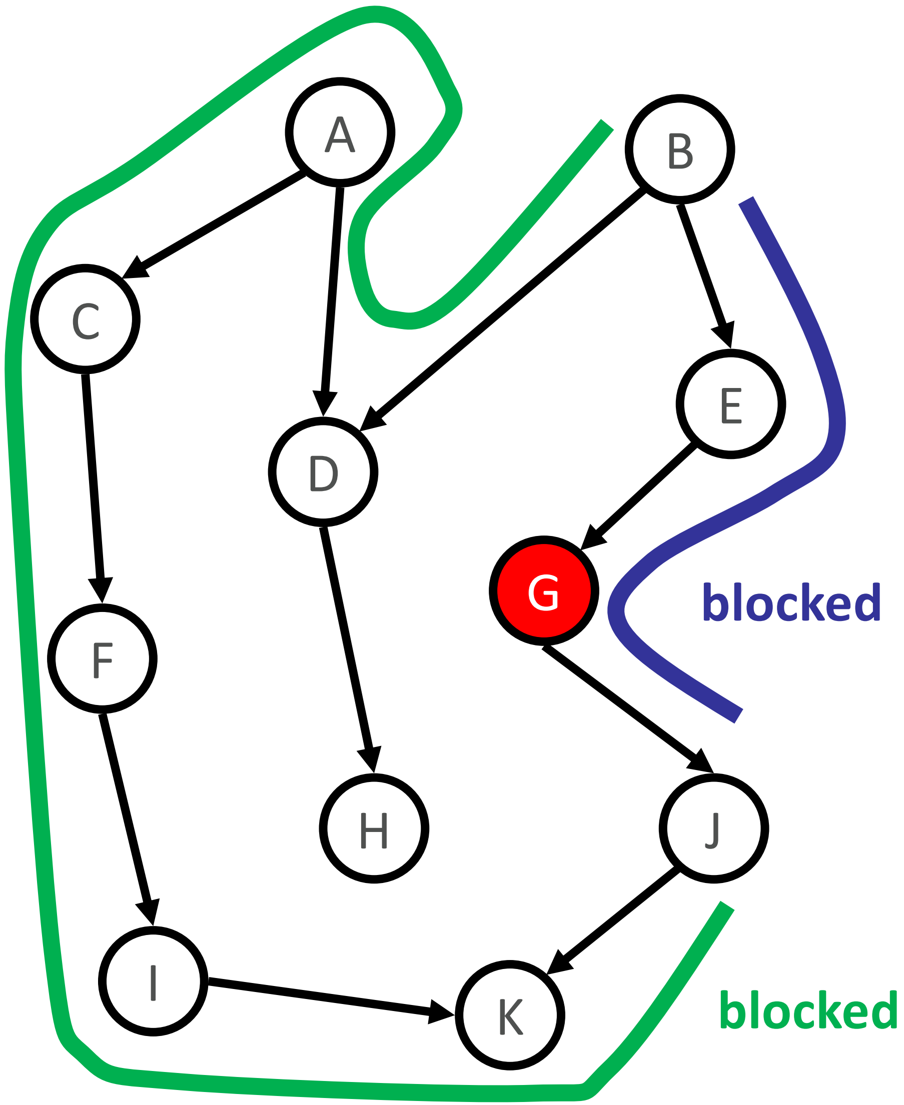
blocked as K is NOT observed

D-Separation – Some Examples



$A \perp B \qquad I \perp J$

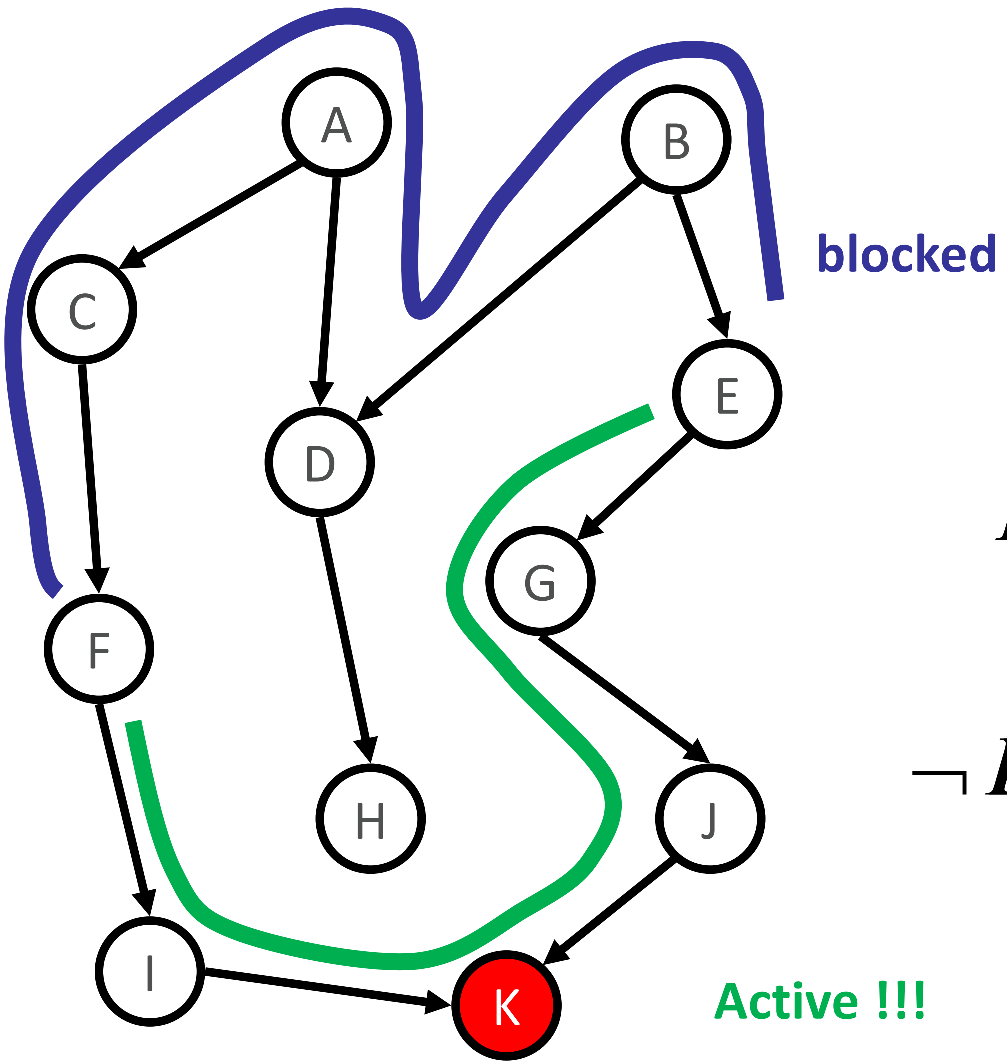
D-Separation – Some Examples



$A \perp B$ $\quad$ $I \perp J$

$B \perp J \mid G$

D-Separation – Some Examples

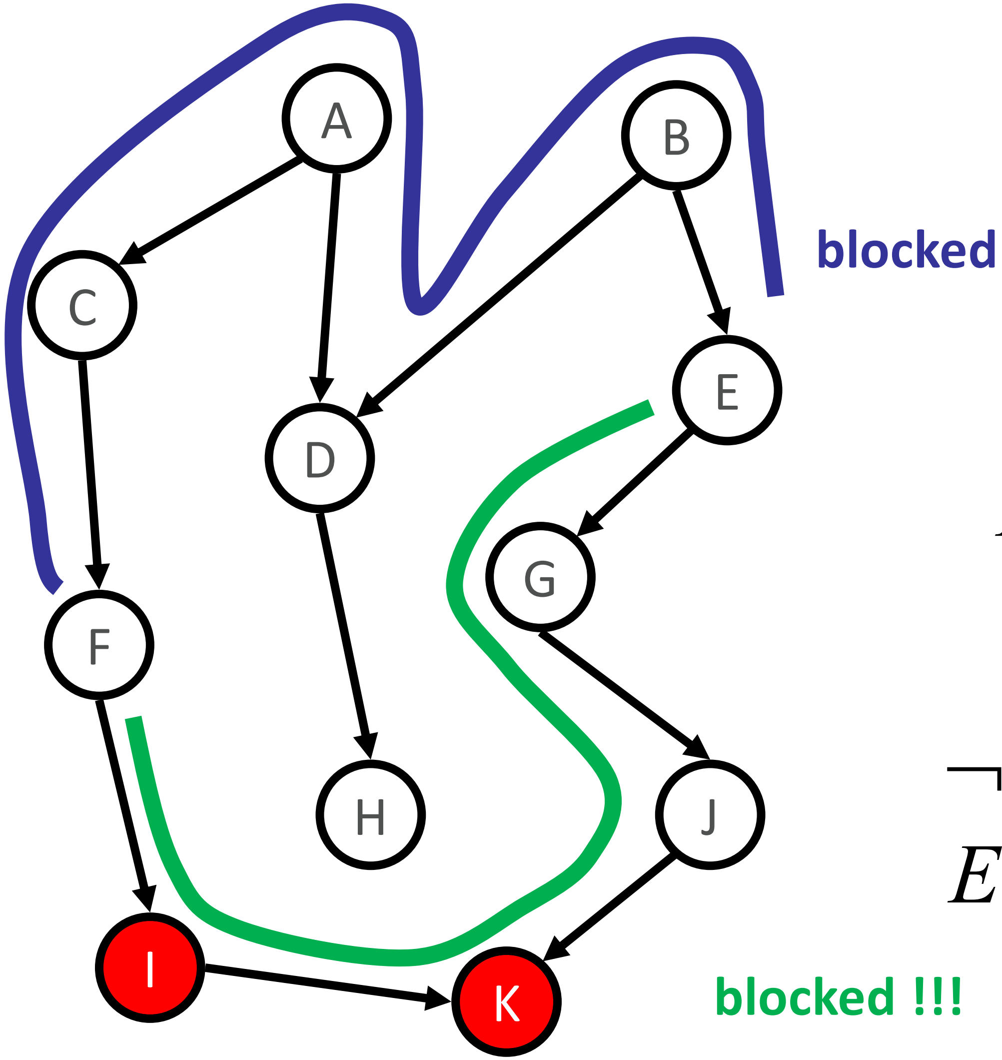


$$A \perp B \qquad I \perp J$$

$$B \perp J \mid G$$

$$\neg E \perp F \mid K$$

D-Separation – Some Examples



$A \perp B \qquad I \perp J$

$B \perp J \mid G$

$\neg\, E \perp F \mid K$

$E \perp F \mid K, I$

The difference is

$E \perp F \mid I$

More generally: Soundness of d-separation

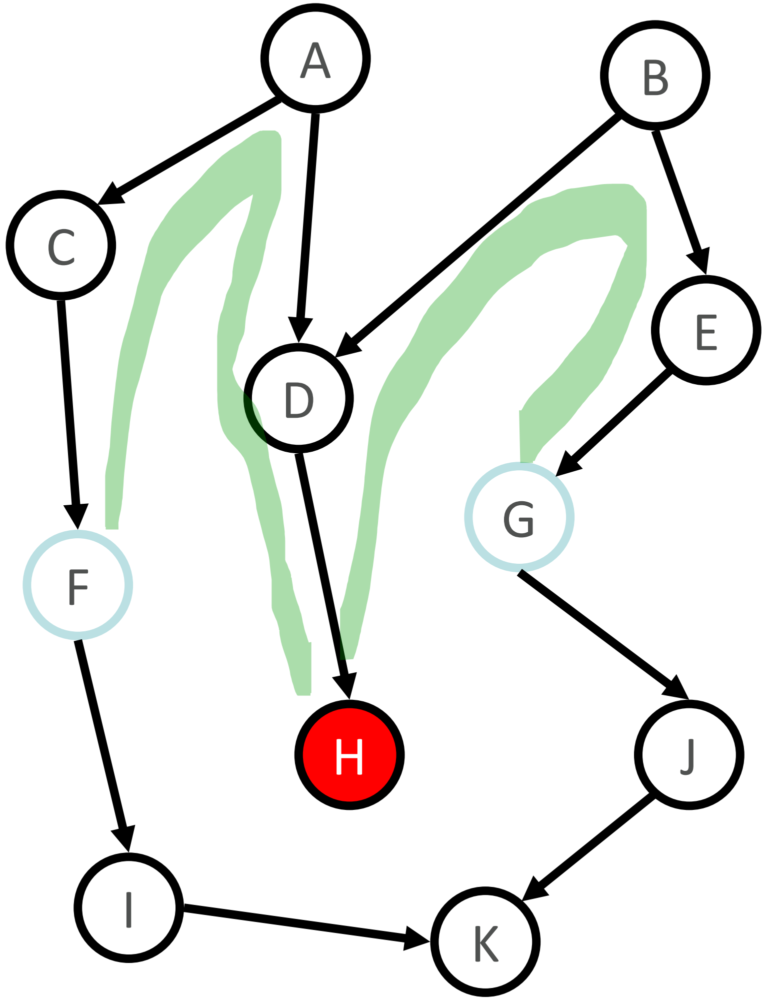
- Given BN structure G
- Set of independence assertions obtained by d-separation: $\mathrm{I}(G) = \{(\mathbf{X} \perp \mathbf{Y} \mid \mathbf{Z}) : \text{d-sep}_G(\mathbf{X}; \mathbf{Y} \mid \mathbf{Z})\}$

- **Theorem: Soundness of d-separation**
  - If P factorizes over G then $\mathbf{I}(G) \subseteq \mathbf{I}(P)$ and not only $\mathbf{I}_l(G) \subseteq \mathbf{I}(P)$

- That means **d-separation only captures true independencies**

Existence of dependency when not d-separated

- **Theorem:** If X and Y are **not** d-separated given **Z**, then X and Y are dependent given **Z** under **some *P*** that factorizes over *G*
- **Proof sketch**:
  - Choose an active trail between X and Y given **Z**
  - Make this trail dependent
  - Make all else uniform (independent) to avoid "canceling" out influence



d-sep says

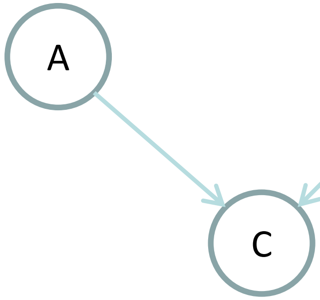
$$\neg F \perp G \mid H$$

Excursion: Faithful Distribution

- P is said to be **faithful** if it does not declare extra independence assumption that cannot be read from G

False	True
0.8	0.2

False	True
0.8	0.2



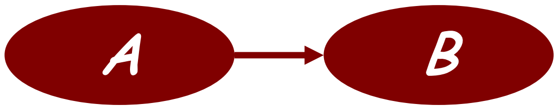
AB\C	False	True
FF	0.9	0.1
FT	0.9	0.1
TF	0.9	0.1
TT	0.9	0.1

P entails $C \perp A,B$!!

P is **not** faithful

More generally: Completeness of d-separation

- **Theorem: Completeness of d-separation**
  - For "almost all" distributions where *P* factorizes over *G*, we have that I(*G*) = I(*P*)
    - *"almost all" distributions*: except for a set of measure zero parameterizations of the CPTs (assuming no finite set of parameterizations has positive measure)
    - Means that if **X** and **Y** are **not** d-separated given **Z**, then ¬(**X**⊥**Y**|**Z**)
- **Proof sketch for very simple case:**



Polynomials are either identical zero or they are non-zero almost everywhere. Thus, the equality happens with probability 0

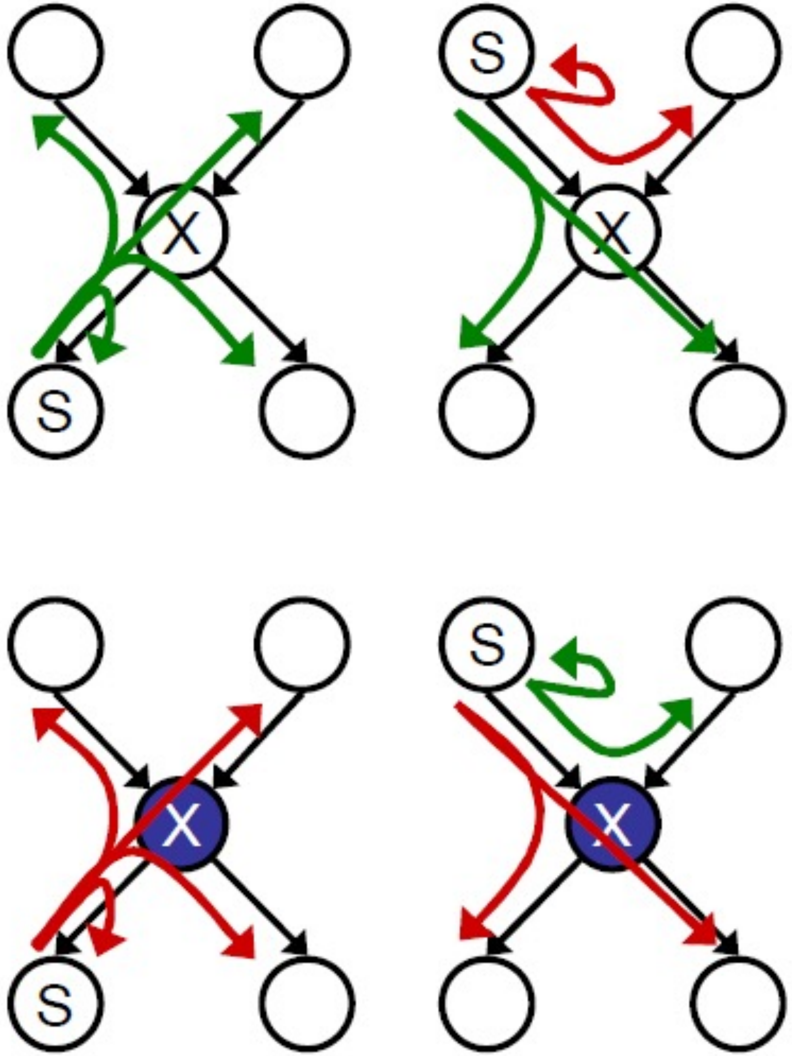
d-sep says $\neg\, \mathrm{A} \perp B$ but it holds $\mathrm{A} \perp B$

i.e., $P(B \mid A) = P(B)$, which is a polynomial equality over the space of CPDs

$$\theta_B = \theta_{B|A=t} = \theta_{A=t} \cdot \theta_{B|A=t} + (1-\theta_{A=t}) \cdot \theta_{B|A=f}$$

Algorithm for d-separation: The Bayes' Ball

- Correct algorithm:
  - Shade in evidence
  - Start at source node
  - Try to reach target by search
  - States: pair of (node X, previous state S)
  - Successor function:
    - X unobserved:
      - To any child
      - To any parent if coming from a child
    - X observed:
      - From parent to parent
  - If you can't reach a node, it's conditionally independent of the start node given evidence

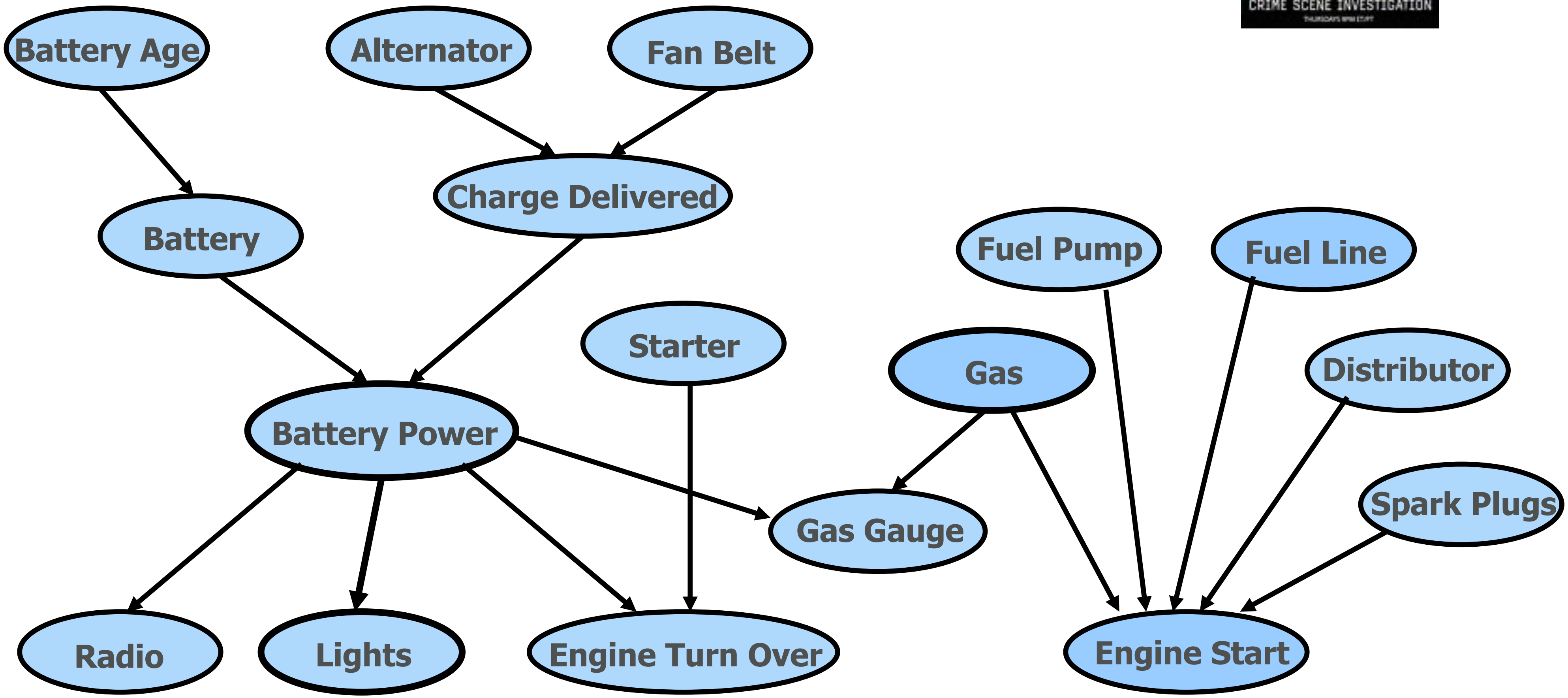


Interpretation of completeness

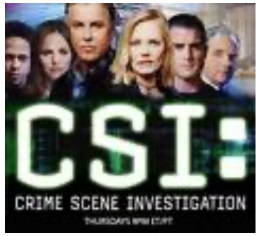
- **Theorem: Completeness of d-separation**
  - For "almost all" distributions that P factorize over G, we have that $\mathrm{I}(G) = \mathrm{I}(P)$
- **BN graph is usually sufficient to capture all independence properties of the distribution!!!!**

- But only for **complete** independence:
  - P says $(\mathbf{X}=\mathbf{x} \perp \mathbf{Y}=\mathbf{y} \mid \mathbf{Z}=\mathbf{z})$, $\forall$ $\mathbf{x} \in \mathrm{Val}(\mathbf{X})$, $\mathbf{y} \in \mathrm{Val}(\mathbf{Y})$, $\mathbf{z} \in \mathrm{Val}(\mathbf{Z})$
- Often we have **context-specific independence (CSI)**
  - $\exists \mathbf{x} \in \mathrm{Val}(\mathbf{X})$, $\mathbf{y} \in \mathrm{Val}(\mathbf{Y})$, $\mathbf{z} \in \mathrm{Val}(\mathbf{Z})$: P says $(\mathbf{X}=\mathbf{x} \perp \mathbf{Y}=\mathbf{y} \mid \mathbf{Z}=\mathbf{z})$
  - Many factors may affect your grade

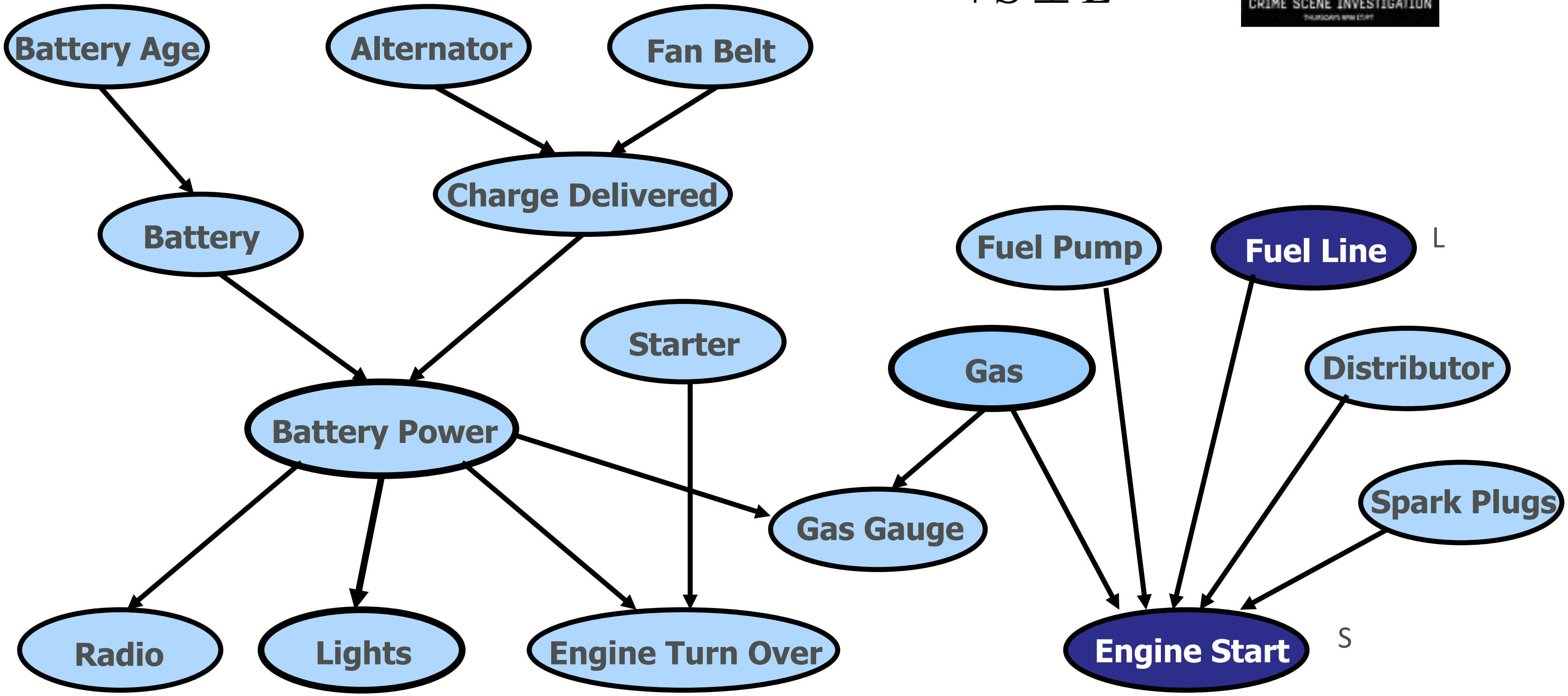
Excursion: Context specific indepecence



Excursion: Context specific indepencence



$$\neg\, S \perp L$$



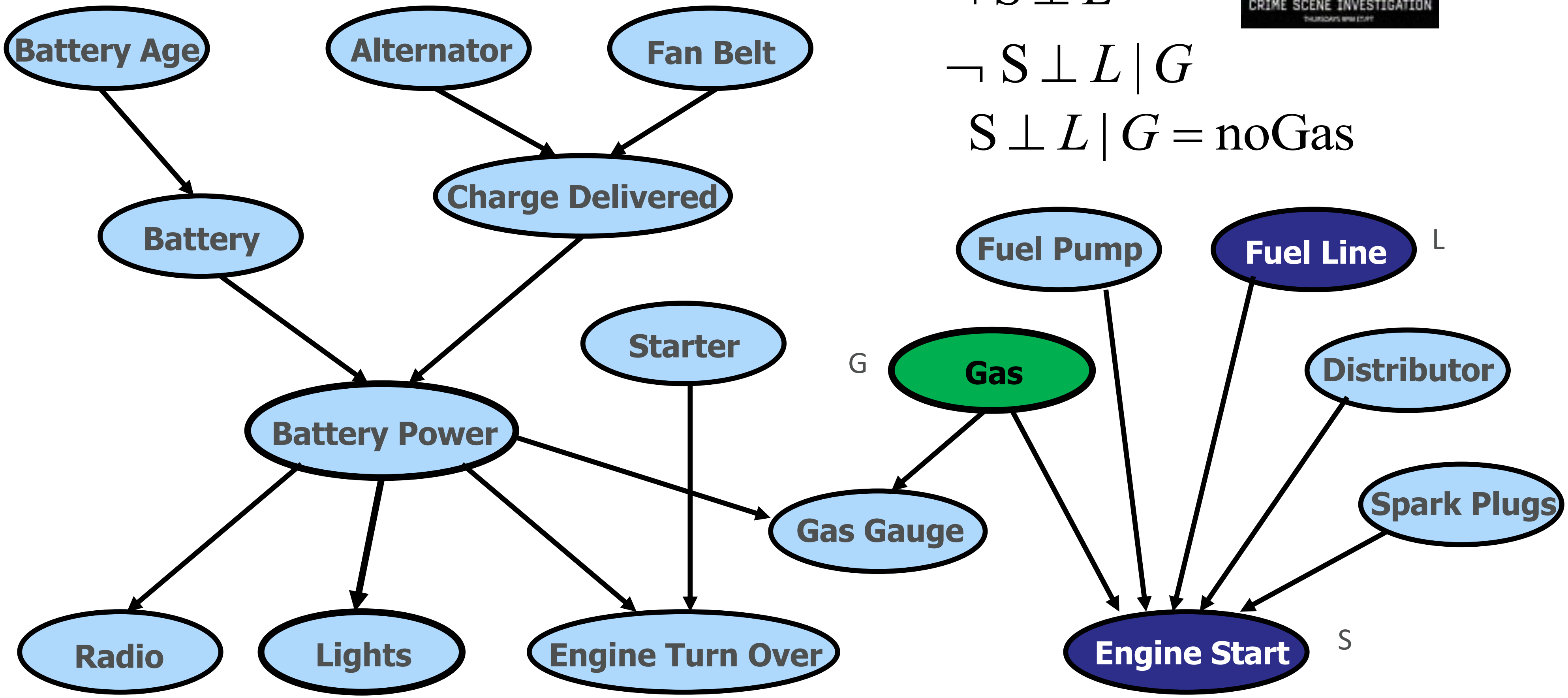
Excursion: Context specific indepencence

After observing a variable, some vars become independent

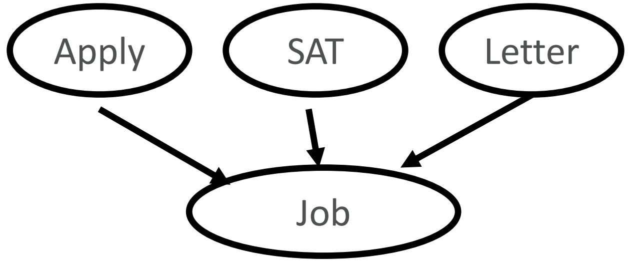
$$\neg S \perp L$$

$$\neg S \perp L \mid G$$

$$S \perp L \mid G = \text{noGas}$$



CSI Representation: Tree CPD



- Represent $P(X_i|\mathbf{Pa}_{X_i})$ using a decision tree
  - Path to leaf is an assignment to (a subset of) $\mathbf{Pa}_{X_i}$
  - Leaves are distributions over X_i given assignment of $\mathbf{Pa}_{X_i}$ on path to leaf
- **Interpretation of leaf**:
  - For specific assignment of $\mathbf{Pa}_{X_i}$ on path to this leaf – X_i is independent of other parents

- Representation can be exponentially smaller than equivalent table

Assuming binary RVs
Table: $O(2^k)$
In the strutcure: $O(k)$

A: f / t → S: low / high → L: good / bad

$P(J=t|A=f,S,L)$
$=0$

S,L irrelevant

$P(J=t|A=t, S=low, L)$
$=\varepsilon=0.1$

L irrelevant

0.9 0.2

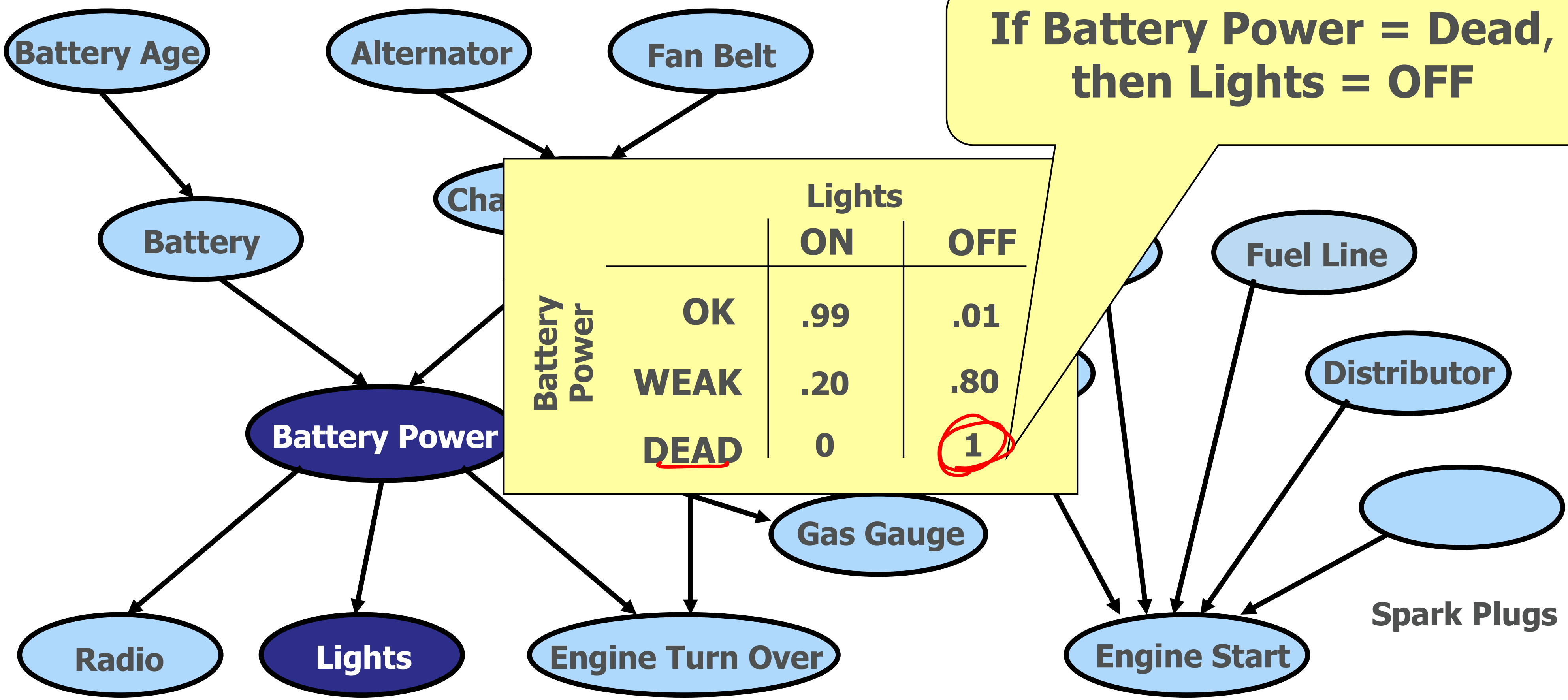
$$J \perp S, L \mid A = \text{no}$$

4 params

Other CSIs: Determinism

Determinism

If Battery Power = Dead, then Lights = OFF



Battery Power	Lights ON	Lights OFF
OK	.99	.01
WEAK	.20	.80
DEAD	0	1

Determinism and inference

- Determinism gives a little sparsity in table, but much bigger impact on inference
- Multiplying deterministic factor with other factor often introduces many new zeros
  - Operations related to theorem proving, e.g., unit resolution

Battery Power	Lights ON	Lights OFF
OK	.99	.01
WEAK	.20	.80
DEAD	0	1

$g(Y) = \prod f_j$ and $f_i(x_i) = 0$ then all values of the $g(Y)$ table where $X_i = x_i$ will be zero

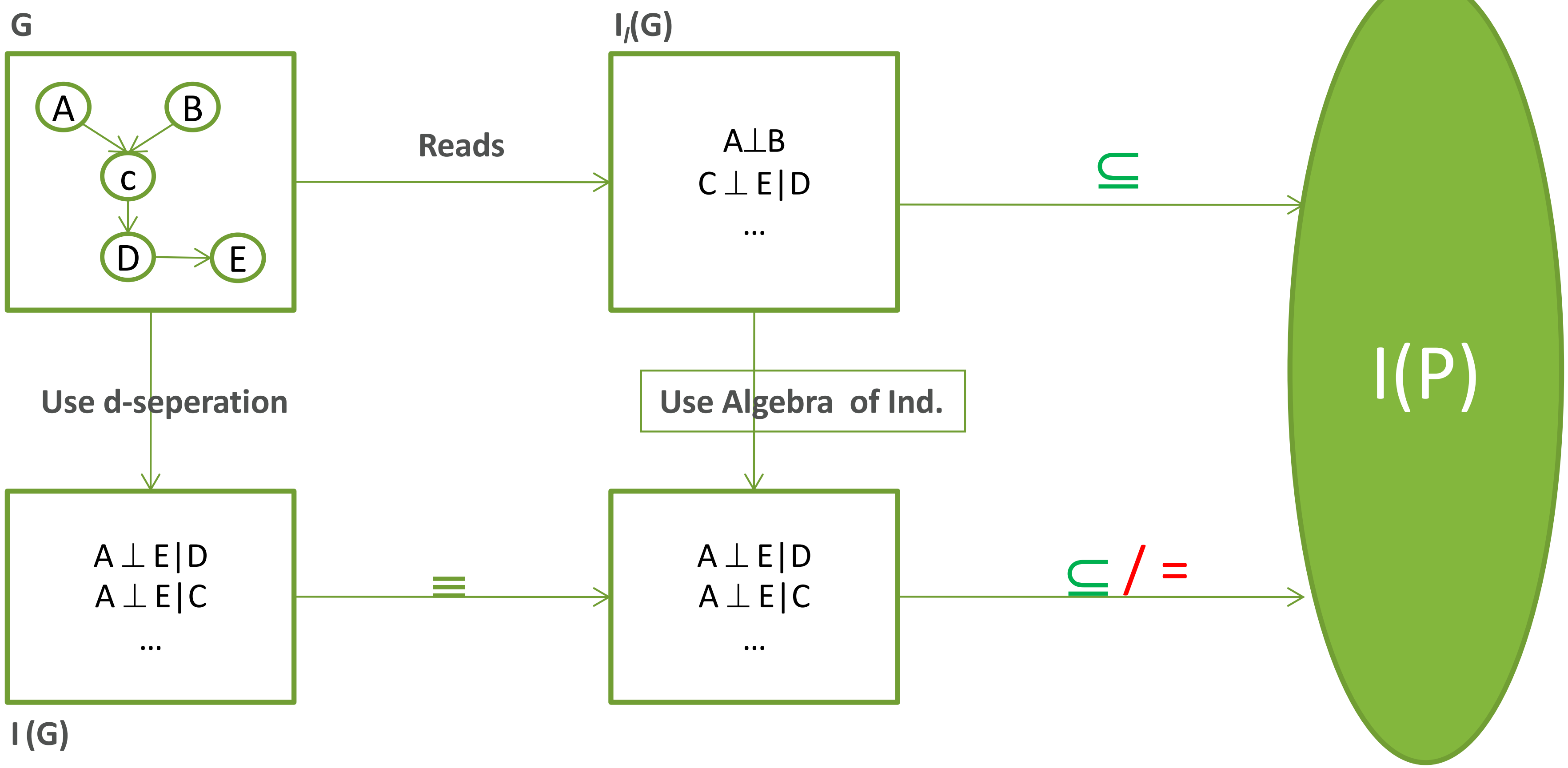
State-of-the-art Models …

- **Often characterized by:**
  - Richness in local structure (determinism, CSI)
  - Massiveness in size (10,000's variables)
  - High connectivity (treewidth, explained later in class)

- **Enabled by:**
  - High level modeling tools:
    - **relational, first order (more about this maybe later in class!!!!)**
  - Advances in machine learning
  - New application areas (synthesis):
    - Bioinformatics (e.g. linkage analysis)
    - Sensor networks

- **Exploiting local and relational structure a must!**

What you need to know



-If G is an I-map of P then $I_l(G) \subseteq I(P)$

- Also, it is always true that $I(G) \subseteq I(P)$ means d-separation is **sound**

- And for almost all Ps that factor over G, I(G) = I(P) , P is faithful to G

What you need to know

- d-separation and independence
  - sound procedure for finding independencies $I(G) \subseteq I(P)$
  - existence of distributions with these independencies
  - (almost) all independencies can be read directly from graph without looking at CPTs $I(G) = I(P)$
- Context-specific indpendence (CSI)
- Bayes' Ball

What's next

- Inference

Is Inference in BNs hopeless?

Theorem:
Inference in Bayesian networks (even approximate, without proof) is NP-hard

- In general, yes!
  - Even approximate!
- In practice
  - Exploit structure
  - Many effective approximation algorithms (some with guarantees)
- For now, we'll talk about exact inference
  - Approximate inference later this semester

General probabilistic inference

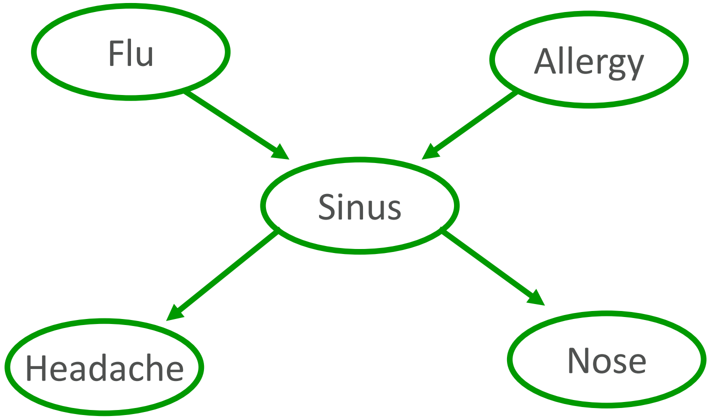
- Query: $P(X \mid e)$

- Using def. of cond. prob.:

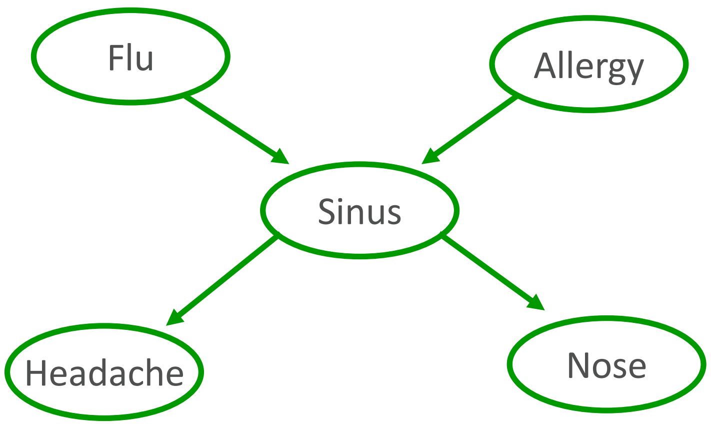
$$P(X \mid e) = \frac{P(X, e)}{P(e)}$$

- Normalization: (we stopped here)

$$P(X \mid e) \propto P(X, e)$$



Probabilistic inference example



$$P(\overset{=t}{A} \mid N=t) \propto P(\overset{=t}{A}, N=t)$$

$$= \sum_f \sum_s \sum_h P(\overset{=t}{A}, f, s, h, N=t)$$

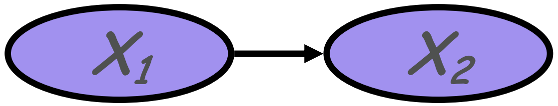
$$= \underbrace{\sum_f \sum_s \sum_h}_{8 \text{ terms}} \underbrace{P(f)\, P(\overset{=t}{A})\, P(s \mid f, \overset{=t}{A})\, P(h \mid s)\, P(N=t \mid s)}_{4 \text{ multiplies}}$$

$2^3 \leftarrow$ # eliminated vars

32 multiplies

Inference seems exponential in number of variables!

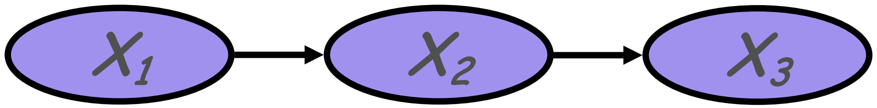
Inference in Simple Chains



How do we compute $P(x_2)$

$$P(x_2) = \sum_{x_1} P(x_1, x_2) = \sum_{x_1} \underbrace{P(x_1) P(x_2 \mid x_1)}_{\text{CPDs}}$$

Inference in Simple Chains (cont.)



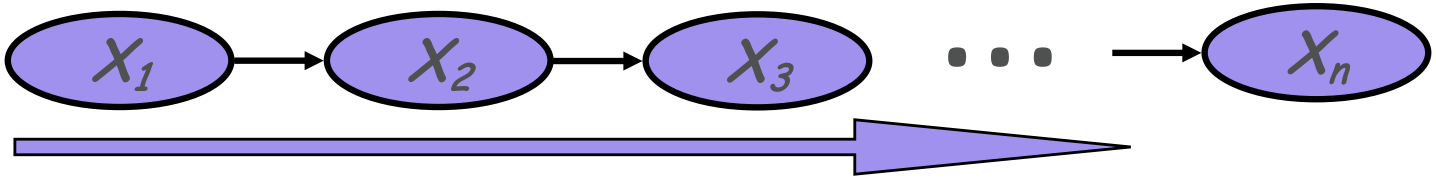
How do we compute $P(x_3)$?

$$P(x_3) = \sum_{x_2} P(x_2, x_3) = \sum_{x_2} \underbrace{P(x_2)}_{\text{computed}} \underbrace{P(x_3 \mid x_2)}_{\text{CPD}}$$

- We already know how to compute $P(x_2)$ ...

$$P(x_2) = \sum_{x_1} P(x_1, x_2) = \sum_{x_1} P(x_1) P(x_2 \mid x_1)$$

Inference in Simple Chains (cont.)

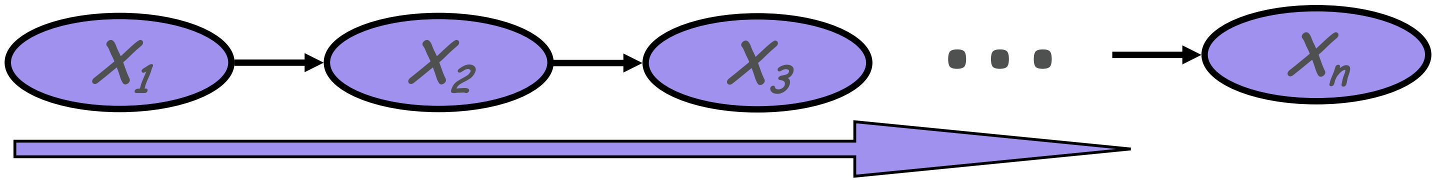


How do we compute $P(x_n)$

- Iteratively compute $P(x_1), P(x_2), P(x_3), \ldots$ using

$$P(x_{i+1}) = \sum_{x_i} \underbrace{P(x_i)}_{\text{computed}} \underbrace{P(x_{i+1} \mid x_i)}_{\text{CPD}}$$

Complexity of inference: Simple Chains



$$O\left(n \cdot k^2\right) \text{v.s. exponentially in } n$$

Variable Elimination

General idea:

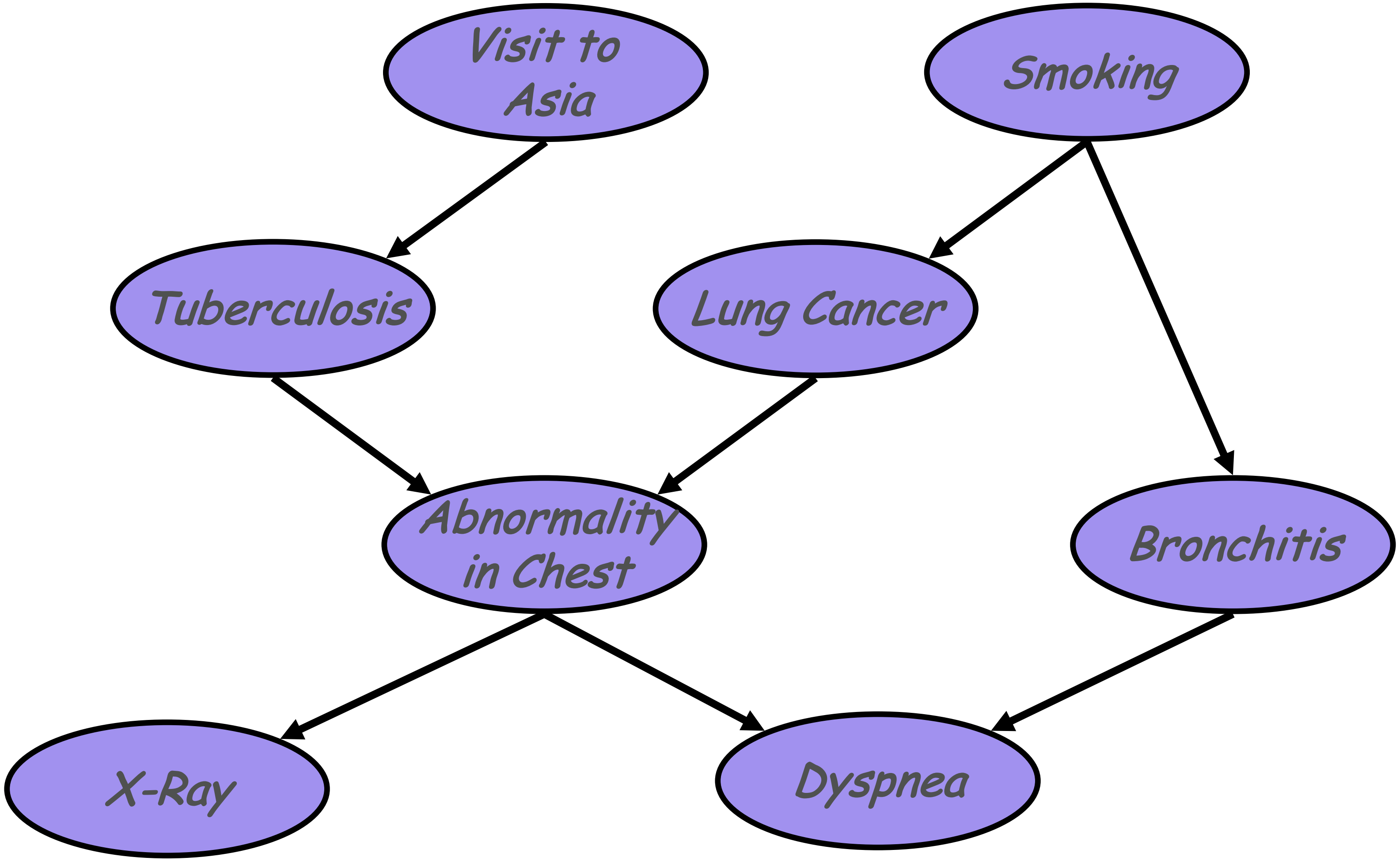
- Write query in the form

$$P(x_n, \boldsymbol{e}) = \sum_{x_k} \cdots \sum_{x_3} \sum_{x_2} \prod_i P(x_i \mid pa_i)$$

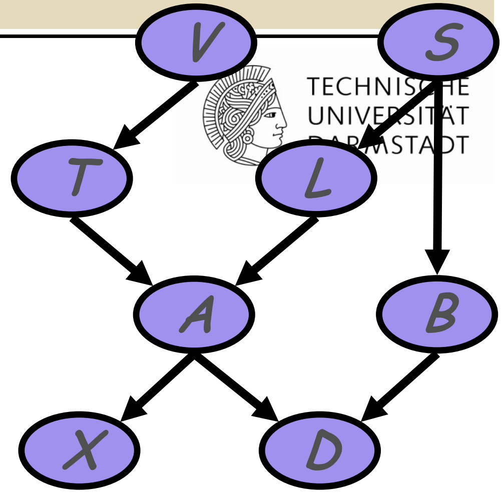
- Iteratively
  - Move all irrelevant terms outside of innermost sum
  - Perform innermost sum, getting a new term
  - Insert the new term into the product

A More Complex Example

- "Asia" network:



- We want to compute *P(d)*
- Need to eliminate: *v,s,x,t,l,a,b*



Initial factors

$$\underline{P(v)}P(s)\underline{P(t\,|\,v)}P(l\,|\,s)P(b\,|\,s)P(a\,|\,t,l)P(x\,|\,a)P(d\,|\,a,b)$$

Eliminate: *v*

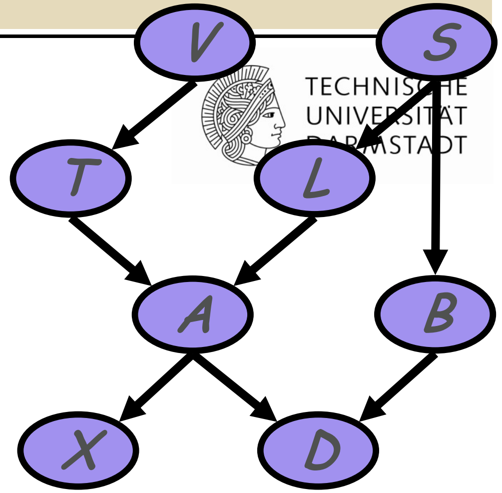
Compute: $f_v(t) = \sum_v P(v)P(t\,|\,v)$

$$\Rightarrow \underline{f_v(t)}P(s)P(l\,|\,s)P(b\,|\,s)P(a\,|\,t,l)P(x\,|\,a)P(d\,|\,a,b)$$

Note: $f_v(t) = P(t)$

In general, result of elimination is not necessarily a probability term

- We want to compute *P(d)*
- Need to eliminate: *v,s,x,t,l,a,b*



Initial factors

$$P(v)P(s)P(t|v)P(l|s)P(b|s)P(a|t,l)P(x|a)P(d|a,b)$$

$$\Rightarrow f_v(t)\underline{P(s)}\,\underline{P(l|s)}\,\underline{P(b|s)}P(a|t,l)P(x|a)P(d|a,b)$$

Eliminate: *s*

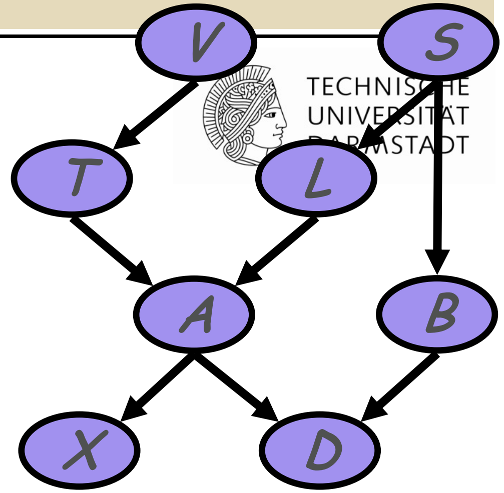
Compute: $f_s(b,l) = \sum_s P(s)P(b|s)P(l|s)$

$$\Rightarrow f_v(t)\underline{f_s(b,l)}P(a|t,l)P(x|a)P(d|a,b)$$

Summing on *s* results in a factor with two arguments $f_s(b,l)$

In general, result of elimination may be a function of several variables

- We want to compute *P(d)*
- Need to eliminate: *v,s,x,t,l,a,b*



Initial factors

$$P(v)P(s)P(t|v)P(l|s)P(b|s)P(a|t,l)P(x|a)P(d|a,b)$$

$$\Rightarrow f_v(t)P(s)P(l|s)P(b|s)P(a|t,l)P(x|a)P(d|a,b)$$

$$\Rightarrow f_v(t)f_s(b,l)P(a|t,l)\underline{P(x|a)}P(d|a,b)$$

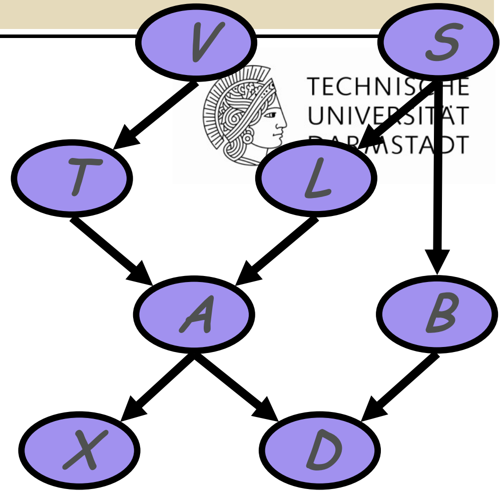
Eliminate: *x*

Compute: $f_x(a) = \sum_x P(x|a)$

$$\Rightarrow f_v(t)f_s(b,l)\underline{f_x(a)}P(a|t,l)P(d|a,b)$$

Note: $f_x(a) = 1$ for all values of *a* !!

- We want to compute *P(d)*
- Need to eliminate: *v,s,x,t,l,a,b*



Initial factors

$$P(v)P(s)P(t|v)P(l|s)P(b|s)P(a|t,l)P(x|a)P(d|a,b)$$

$$\Rightarrow f_v(t)P(s)P(l|s)P(b|s)P(a|t,l)P(x|a)P(d|a,b)$$

$$\Rightarrow f_v(t)f_s(b,l)P(a|t,l)P(x|a)P(d|a,b)$$

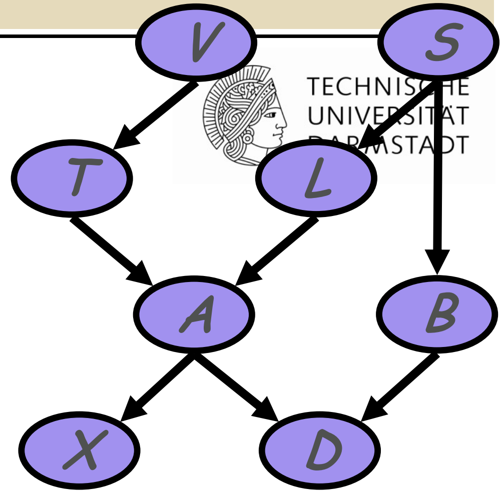
$$\Rightarrow \underline{f_v(t)}f_s(b,l)f_x(a)\underline{P(a|t,l)}P(d|a,b)$$

Eliminate: *t*

Compute: $f_t(a,l) = \sum_t f_v(t)P(a|t,l)$

$$\Rightarrow f_s(b,l)f_x(a)\underline{f_t(a,l)}P(d|a,b)$$

- We want to compute *P(d)*
- Need to eliminate: *v,s,x,t,l,a,b*



Initial factors

$$P(v)P(s)P(t\,|\,v)P(l\,|\,s)P(b\,|\,s)P(a\,|\,t,l)P(x\,|\,a)P(d\,|\,a,b)$$

$$\Rightarrow f_v(t)P(s)P(l\,|\,s)P(b\,|\,s)P(a\,|\,t,l)P(x\,|\,a)P(d\,|\,a,b)$$

$$\Rightarrow f_v(t)f_s(b,l)P(a\,|\,t,l)P(x\,|\,a)P(d\,|\,a,b)$$

$$\Rightarrow f_v(t)f_s(b,l)f_x(a)P(a\,|\,t,l)P(d\,|\,a,b)$$

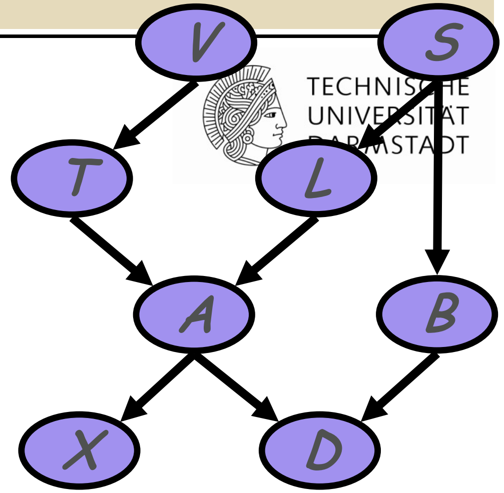
$$\Rightarrow \underline{f_s(b,l)}f_x(a)\underline{f_t(a,l)}P(d\,|\,a,b)$$

Eliminate: *l*

Compute: $f_l(a,b) = \sum_l f_s(b,l) f_t(a,l)$

$$\Rightarrow \underline{f_l(a,b)}f_x(a)P(d\,|\,a,b)$$

- We want to compute *P(d)*
- Need to eliminate: *v,s,x,t,l,a,b*



Initial factors

$$P(v)P(s)P(t|v)P(l|s)P(b|s)P(a|t,l)P(x|a)P(d|a,b)$$

$$\Rightarrow f_v(t)P(s)P(l|s)P(b|s)P(a|t,l)P(x|a)P(d|a,b)$$

$$\Rightarrow f_v(t)f_s(b,l)P(a|t,l)P(x|a)P(d|a,b)$$

$$\Rightarrow f_v(t)f_s(b,l)f_x(a)P(a|t,l)P(d|a,b)$$

$$\Rightarrow f_s(b,l)f_x(a)f_t(a,l)P(d|a,b)$$

$$\Rightarrow \underline{f_l(a,b)}\,\underline{f_x(a)}\,\underline{P(d|a,b)} \Rightarrow \underline{f_a(b,d)} \Rightarrow \underline{f_b(d)}$$

Eliminate: *a,b*

Compute: $f_a(b,d) = \sum_a f_l(a,b) f_x(a) p(d|a,b) \quad f_b(d) = \sum_b f_a(b,d)$

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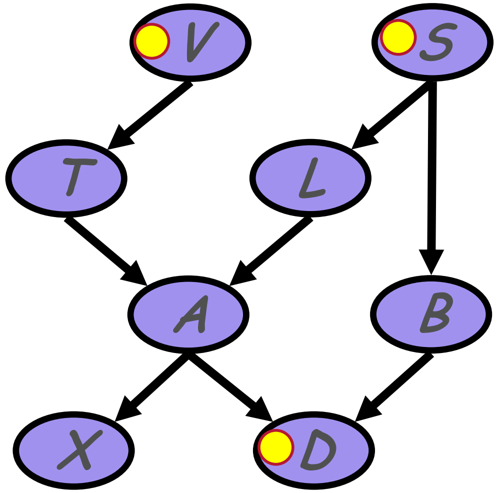
Variable Elimination

- We now understand variable elimination as a sequence of **rewriting** operations
- Computation depends on order of elimination

Dealing with Evidence



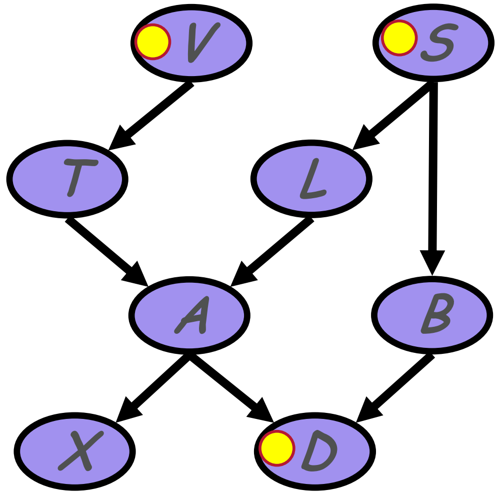
- How do we deal with evidence?
- Suppose get evidence

$$V = t, S = f, D = t$$

- We want to compute

$$P(L, V = t, S = f, D = t)$$

Dealing with Evidence



- We start by writing the factors:

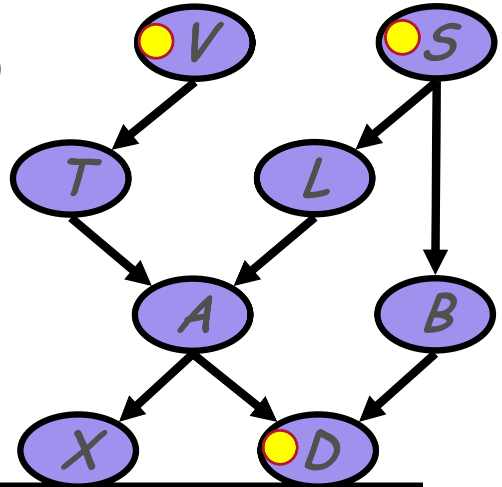
$$P(v)P(s)P(t \mid v)P(l \mid s)P(b \mid s)P(a \mid t,l)P(x \mid a)P(d \mid a,b)$$

- Since we know that $V = t$, we don't need to eliminate V
- Instead, we can replace the factors $P(V)$ and $P(T|V)$ with

$$f_{P(V)} = P(V = t) \quad f_{p(T|V)}(T) = P(T \mid V = t)$$

- This "selects" the appropriate parts of the original factors given the evidence
- Note that $f_{p(V)}$ is a constant, and thus does not appear in elimination of other variables

- Given evidence $V = t, S = f, D = t$
- Compute $P(L, V = t, S = f, D = t)$



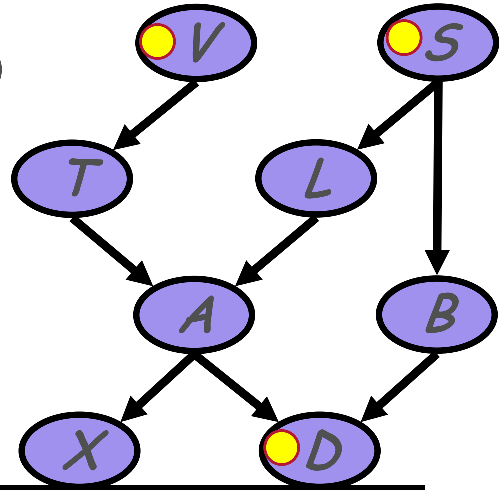
Dealing with Evidence

- Initial factors, after setting evidence:

$$f_{P(v)} f_{P(s)} f_{P(t|v)}(t) f_{P(l|s)}(l) f_{P(b|s)}(b) P(a \mid t,l) P(x \mid a) f_{P(d|a,b)}(a,b)$$

- Given evidence $V = t, S = f, D = t$
- Compute $P(L, V = t, S = f, D = t)$

Dealing with Evidence



- Initial factors, after setting evidence:

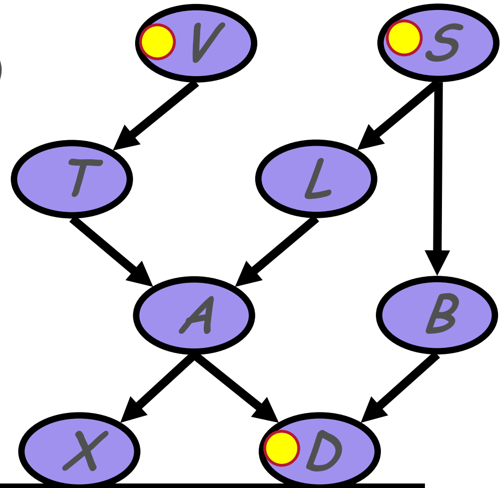
$$f_{P(v)} f_{P(s)} f_{P(t|v)}(t) f_{P(l|s)}(l) f_{P(b|s)}(b) P(a|t,l) P(x|a) f_{P(d|a,b)}(a,b)$$

- Eliminating x, we get

$$f_{P(v)} f_{P(s)} f_{P(t|v)}(t) f_{P(l|s)}(l) f_{P(b|s)}(b) P(a|t,l) f_x(a) f_{P(d|a,b)}(a,b)$$

- Given evidence $V = t, S = f, D = t$
- Compute $P(L, V = t, S = f, D = t)$

Dealing with Evidence



- Initial factors, after setting evidence:

$$f_{P(v)} f_{P(s)} f_{P(t|v)}(t) f_{P(l|s)}(l) f_{P(b|s)}(b) P(a|t,l) P(x|a) f_{P(d|a,b)}(a,b)$$

- Eliminating x, we get

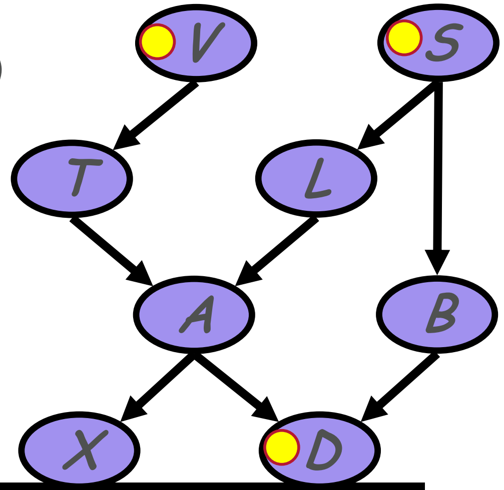
$$f_{P(v)} f_{P(s)} f_{P(t|v)}(t) f_{P(l|s)}(l) f_{P(b|s)}(b) P(a|t,l) f_x(a) f_{P(d|a,b)}(a,b)$$

- Eliminating t, we get

$$f_{P(v)} f_{P(s)} f_{P(l|s)}(l) f_{P(b|s)}(b) f_t(a,l) f_x(a) f_{P(d|a,b)}(a,b)$$

- Given evidence $V = t, S = f, D = t$
- Compute $P(L, V = t, S = f, D = t)$

Dealing with Evidence



- Initial factors, after setting evidence:

$$f_{P(v)} f_{P(s)} f_{P(t|v)}(t) f_{P(l|s)}(l) f_{P(b|s)}(b) P(a \mid t,l) \underline{P(x \mid a)} f_{P(d|a,b)}(a,b)$$

- Eliminating x, we get

$$f_{P(v)} f_{P(s)} f_{P(t|v)}(t) f_{P(l|s)}(l) f_{P(b|s)}(b) P(a \mid t,l) f_x(a) f_{P(d|a,b)}(a,b)$$

- Eliminating t, we get

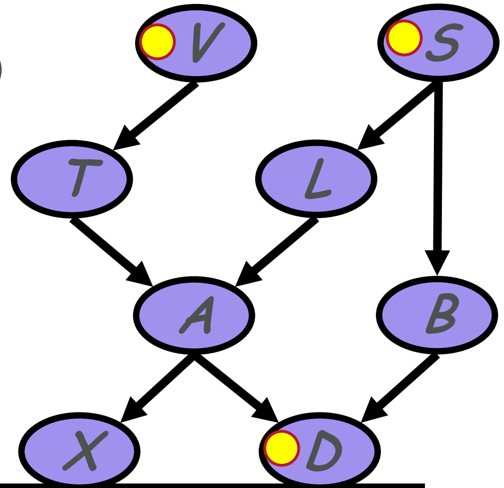
$$f_{P(v)} f_{P(s)} f_{P(l|s)}(l) f_{P(b|s)}(b) f_t(a,l) f_x(a) f_{P(d|a,b)}(a,b)$$

- Eliminating a, we get

$$f_{P(v)} f_{P(s)} f_{P(l|s)}(l) f_{P(b|s)}(b) f_a(b,l)$$

- Given evidence $V = t, S = f, D = t$
- Compute $P(L, V = t, S = f, D = t)$

Dealing with Evidence



- Initial factors, after setting evidence:

$$f_{P(v)} f_{P(s)} f_{P(t|v)}(t) f_{P(l|s)}(l) f_{P(b|s)}(b) P(a \mid t,l) P(x \mid a) f_{P(d|a,b)}(a,b)$$

- Eliminating x, we get

$$f_{P(v)} f_{P(s)} f_{P(t|v)}(t) f_{P(l|s)}(l) f_{P(b|s)}(b) P(a \mid t,l) f_x(a) f_{P(d|a,b)}(a,b)$$

- Eliminating t, we get

$$f_{P(v)} f_{P(s)} f_{P(l|s)}(l) f_{P(b|s)}(b) f_t(a,l) f_x(a) f_{P(d|a,b)}(a,b)$$

- Eliminating a, we get

$$f_{P(v)} f_{P(s)} f_{P(l|s)}(l) f_{P(b|s)}(b) f_a(b,l)$$

- Eliminating b, we get

$$f_{P(v)} f_{P(s)} f_{P(l|s)}(l) f_b(l)$$

Summary: Variable elimination algorithm

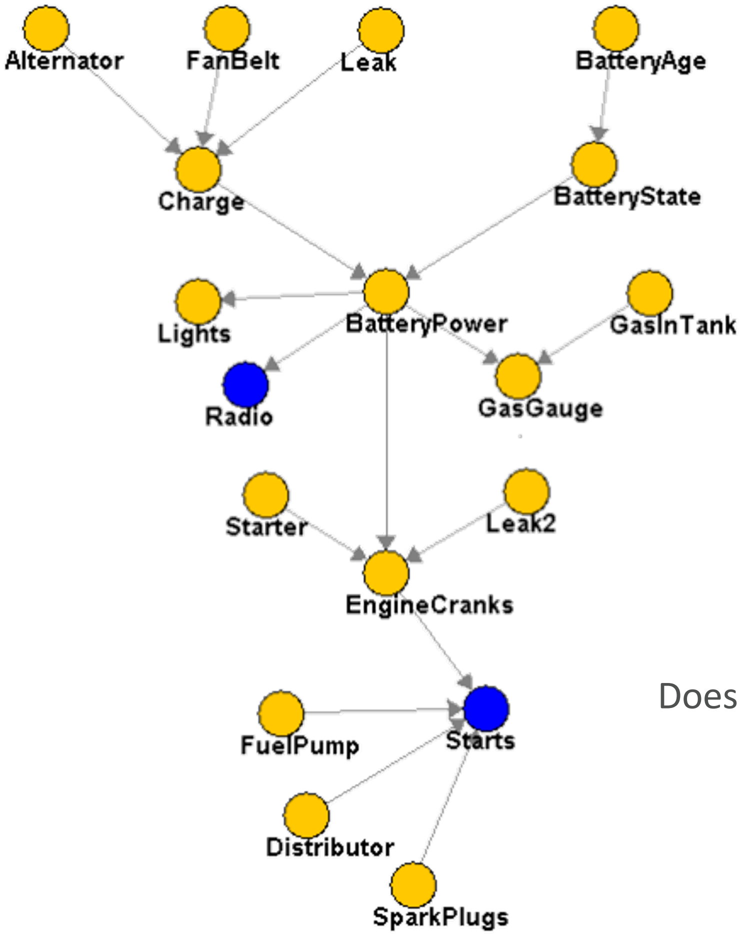
- Given a BN and a query $P(X|e)$ / $P(X,e)$
- Instantiate evidence **e**
- Prune non-active vars for $\{X,\mathbf{e}\}$
- Choose an ordering on variables, e.g., $X_1, \ldots, X_n$
- Initial *factors* $\{f_1,\ldots,f_n\}$: $f_i = P(X_i|\mathbf{Pa}_{Xi})$ (CPT for X_i)
- For i = 1 to n, If $X_i \notin \{X,\mathbf{E}\}$ ← must be eliminated
  - Collect factors $f_1,\ldots,f_k$ that include X_i
  - Generate a new factor by eliminating X_i from these factors

$$g = \sum_{X_i} \prod_{j=1}^{k} f_j$$

  - Variable X_i has been eliminated! Add g to the set of factors
- Normalize $P(X,\mathbf{e})$ to obtain $P(X|\mathbf{e})$

IMPORTANT!!!

Complexity of VE: (Poly)-tree graphs



Variable elimination order:
Start from "leaves" inwards:

- Start from skeleton!
- Choose a "root", any node
- Find topological order for root
- Eliminate variables in reverse order

Does not creat factors any bigger than original CPTs

Linear in CPT sizes!!! (versus exponential)

Complexity of VE: General Case

- During VE, we multiply and marginalize factors

$$g = \sum_{X_i} \prod_{j=1}^{k} f_j$$

- Mult.: exponential in the number of involved vars
- Marg. : linear in the size of the product factor

- **Exponential in the # of vars in intermediate factors**, i.e., dominated by the largest intermediate factor

What's next

- Thus far: Variable elimination
  - (Often) Efficient algorithm for inference in graphical models
- Next: Understanding complexity of variable elimination in more detail
  - Will lead to cool junction tree algorithm later