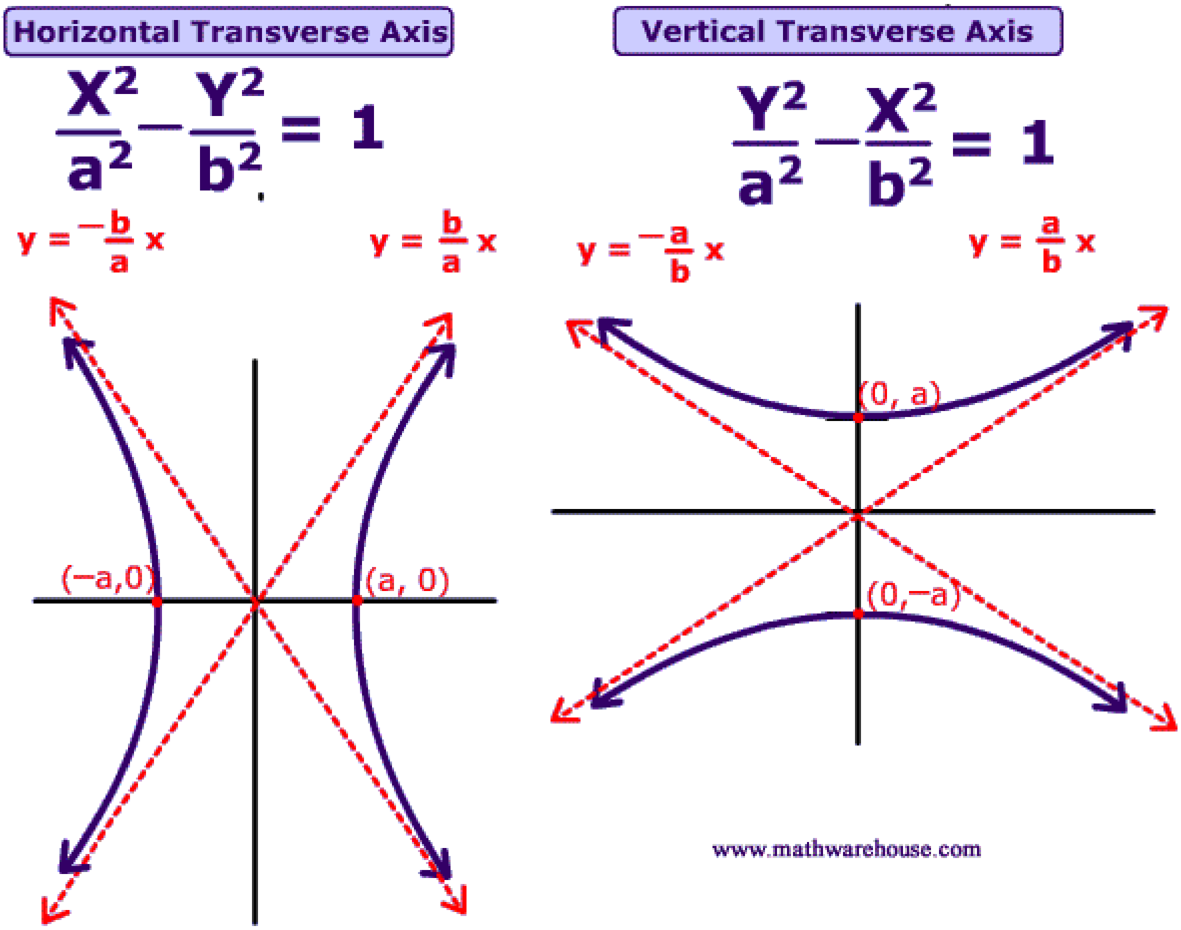
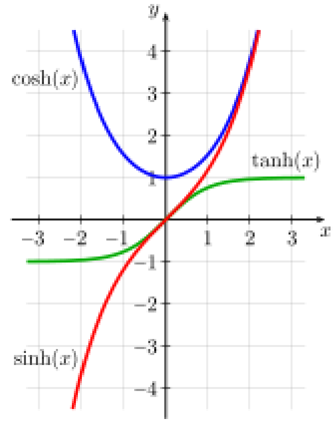
transcendental functions

- we will define the class of logarithmic functions, without first talking about exponents.
- what is the class of functions whose derivative is $\frac{1}{x}$? We can't answer this yet from what we've learned.
- Lets put the restriction that the domain of that function is only positive reals (why? herb gross doesn't say why)
- Let's call the derivative f and let's call the function we are finding L. we can roughly plot what it might look like because we know f and f' (first and second order derivatives of L).
- Now, $L(x) = \int f(x)dx + c$ and $[\int_a^x f(x)dx]' = f(x)$. Since we know f, given an a (the constant of integration), we can estimate the value of the function L to any precision using the sum method (by making n arbitrarily small)
- Lets define a logarithmic function as any function that obeys $f(x_1x_2) = f(x_1) + f(x_2)$. We know at least that this is a non-empty set. $e.g.\ log_2(x)$.
  - From the above definition we can easily show that:
    - $f(1) = 0$
    - $f(x) = -f(\frac{1}{x})$
    - $f(x^n) = nf(x)$.
- Now lets see if L is logarithmic (surprise, surprise). All we know about L is $\frac{d}{dx}L(x) = f(x)$.
- Given our definition of a logarithmic function, we see that $\frac{d}{dx}L(bx)$ is $\frac{d}{dx}L(x) + \frac{d}{dx}L(b)$. These both turn out to be $\frac{1}{x}$. Then, it must be that $L(bx) = L(x) + L(b) + c$. At $x = 1$, $L(1) = -c$. Let's **define** this to be 0, ($L(1) = 0$) so that L is now logarithmic!
- We give a name to $L(x)$, we call it $ln(x)$. We find the "base" x of this by finding the x at which $log_x(x) = 1$. We can compute x to any arbitrary precision, using the sum approximation technique of integrals. We call the result e. It turns $e \approx 2.718$.
- We call ln the natural logarithm.
- Since ln, is logarithm, its inverse is exponential. Since the base is e, $ln^{-1}(x) = e^x$.
- What is $\frac{d}{dx}ln^{-1}(x)$? Notice $y = ln^{-1}(x)$ so $x = ln(y)$, so $\frac{dx}{dy} = \frac{1}{y}$ and $\frac{dy}{dx} = y$.
- That shows that $\frac{d}{dx}e^x = e^x$. Cool!
- $x^2 - y^2 = 1$ is a hyperbola that looks like this. It's the "unit hyperbola"



- We define hyperbolic sine and cosine as $cosh(t)$ and $sinh(t)$. Using the same idea as radians we start at $(1, 0)$ and traverse a distance of t:
  - $cosh(t) = x$.
  - $sinh(t) = y$
  - Obviously $cosh^2(t) - sinh^2(t) = 1$
- We can define these functions another way:
  - $cosh(t) = \frac{e^t + e^{-t}}{2}$
  - $sinh(t) = \frac{e^t - e^{-t}}{2}$
  - We can verify that if we define a curve such that $x = cosh(t), y = sinh(t)$, we get the curve $x^2 - y^2 = 1$.
- These two definitions are not necessarily the same, or at least we haven't shown them to be. It turns out, that the t in the first definition (which is distance from x-axis) turns out to be the same t in the second definition. Herb Gross doesn't show why this is true, though.
- Notice that at very large values of x, both $cosh(x)$ and $sinh(x)$ behave like $\frac{e^x}{2}$.
- Graphs of $sinh(x)$ and $cosh(x)$:



- By differentiating circular functions (sin and cos), we can show that circular functions are the solution to the equation $\frac{d^2x}{dt^2} = -k^2x$. This describes the case where a particle's acceleration is proportional to its
- By differentiating hyperbolic functions ($cosh$ and $sinh$), we show that hyperbolic functions are the solution to the equation $\frac{d^2x}{dt^2} = k^2x$. this describes a case where a particle's acceleration is proportional to displacement, but in the same direction as displacement.
- We can similarly define inverse hyperbolic functions. Using the same ideas for differentiation of inverse functions as we've seen before, we can find the expressions for derivatives of $sinh^{-1}(x)$ and $cosh^{-1}(x)$.
- Hence, we can show that:
  - $\int \frac{dx}{\sqrt{1+x^2}}dx = sinh^{-1}(x)$.
  - $\int \frac{dx}{\sqrt{x^2-1}}dx = cosh^{-1}(x)$
- Because we need one-to-one functions to talk about inverses, we restrict the domain of $cosh^{-1}(x)$ to $x \geq 1$, and the above integrals only apply with the right domain restrictions.