techniques of integration

- when we write $\int f(x)dx$, the dx conveys that our integration is with respect to x. Integration is like inverse differentiation, and with differentiation you have to ask "with respect to what?". when trying to find the inverse derivative (integral), the dx, answers the "with respect to what?". In other words, $\int f(x)dx$ means what is the function G that has to be differentiated w.r.t dx to get $f(x)$.
- a common trick is the following:
  given a function of the form $\int sin^5(x)cos(x)dx$, we attempt to express it as factors using manipuation such that it reduces to a known differentiation rule, along with the chain rule, like the above integral. This kind of pattern is called substitution.
- whenever you see something like:
  - $\int \frac{dx}{\sqrt{x^2-a^2}}$ or $\int \frac{dx}{\sqrt{a^2-x^2}}$ think of $\int \frac{1}{\sqrt{1-x^2}}dx = sin^{-1}(x)$,$\int \frac{dx}{\sqrt{1+x^2}}dx = sinh^{-1}(x)$ and $\int \frac{dx}{\sqrt{x^2-1}}dx = cosh^{-1}(x)$ and how you can reduce the first to something like the integrals we've already seen.
- completing the square:
  - to do something like $\int \frac{dx}{ax^2+bx+c}$, complete the square in the denominator and make it of the form $(x+k)^2+m$ and then use the regular rule of integrating.
- Partial fractions are polynomials of the form $\int \frac{P(x)}{Q(x)}dx$ where P and Q are polynomials and the degree of P $<$ degree of Q. If this is not true, we can easily express it as a sum of a partial fraction and a polynomial by doing polynomial long division.
- Let's take $\int \frac{dx}{(x-1)(x^2+1)}$. Now $\frac{1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+c}{x^2+1}$. We can show pretty easily that if we have a sum of the form $\sum \frac{f_i(x)}{g_i(x)}$ where f_i and g_i are polynomials. For the sum to be a partial fraction, we can show that each f_i and g_i should be partial fractions. If any of them has degree of numerator greater than degree of denominator, then degree of P will not be less than degree of Q. So we write the sum as the most general sum $\sum \frac{f_i(x)}{g_i(x)}$ where degree of any f_i is 1 less than that of corresponding g_i. Note that with the right values for constants, we can express any sum this way. Solving the equations, we might be able to find the constants A, B and C. Then, we've reduced it to something we know!
- Notice the pattern here, substitution, partial fractions → all about making it doable by reducing it to known forms.
- Integration by Parts: We know that $\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$ → this is just the product rule. Now this means that $u\frac{d}{dx}(v) = \frac{d}{dx}(uv) - v\frac{d}{dx}(u)$. Integrating both sides w.r.t x, $\int udv = uv - \int vdu + c$.
- that's all integration by parts is. Knowing one integral, we use that to compute another integral. And now we've derived the relationship between the two, we can just use it as a formula.
- A useful application of integration by parts is to integrate a function with just one factor by letting u be the function and $v = x$. eg. $\int ln(x)dx = xln(x) - \int dx$
- It is not true that $\int_{-1}^{1} \frac{dx}{x^2} = G(1) - G(-1)$. This is because this requires that the f in $\int f(x)dx$ be at least piecewise continuous, and f here isn't defined at $x = 0$.
- $\int_a^b f(x)dx$ is called *improper integral of the first kind* if and only if f is "infinite" for at least one $c\epsilon[a,b]$. At such a point c, we define the integral $\int_a^b f(x)dx = \ lim_{h\to 0^+}[\int_a^{c-h} f(x)dx + \int_{c+h}^{b} f(x)dx]$. This limit doesn't always exist, obviously. If the limit exists, then we say the integral is *convergent*. Otherwise, if the limit doesn't exist, it is called divergent.

- If $\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$ and f is continuous for $x \geq a$, then it is called an improper integral of the second kind. Notice that the requirement of continuous on $x \geq a$ means that the integrand (the function being integrated) is not infinite. In the first kind, the integrand is infinite at some point in the interval.
- Notice that a limit with the one of the bounds as infinity can exist $i.e.$, an infinite area can generate a finite volume!
- In order to determine the convergence of $\int_a^b f(x)dx$, we don't have to be able to compute it. Its sufficient to know if the integral is less than another integral that does converge. We will eventually have to compute the last integral in this "less than" chain to prove that everything before in that chain does converge.
- Before evaluating something of the form of $\int_a^\infty$, $\int_\infty^b$ or $\int_{-\infty}^\infty$, make sure the integrand doesn't blow up anywhere (go to infinity).