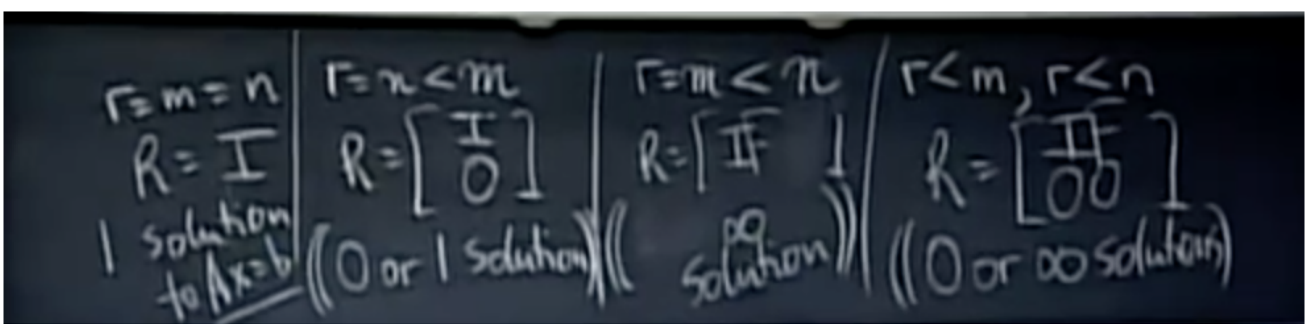
vector spaces & solutions to Ax = b

- A vector space is a set of vectors that are closed under linear combination (a linear combination of any of the vectors in the space must be in the space)
- The above means that the 0 vector must exist in every vector space. Some vector subspaces in R^2 are:
  - all of R^2
  - lines that pass thru the 0 vector.
  - the subspace containing only the 0 vector
- A subspace is a vector space (with all the properties) inside another vector space. (eg. a plane thru the origin is a subspace of R^3).
- the union of two subspaces, in general, is not a subspace. eg: a plane and a line in R^3. (since all linear combinations won't lie within it)
- the intersection of two subspaces is a subspace. proof: given two spaces S and T, the intersection is the set of vectors that are in both spaces. given any two vectors in this intersection space, all linear combinations of them are both in S and T, and hence all linear combinations are in the intersection.
- The column space of a matrix A is the set of all linear combinations of the vectors that form the columns of the matrix. It is denoted by $C(A)$.
- For $A\vec{x} = b$, is a system with A having 4 rows and 3 columns (the columns of A being in R^4), for which b's can we find a solution? Clearly, not all b's have a solution. But we can tell some b's do. Which ones do? Exactly those b's that are in $C(A)$.
- An interesting question is can we throw away some columns and still have the same column space? That is are some columns just linear combinations of others? How many columns are there such each column in there is not express-able as a linear combination of other columns?
- The nullspace of a matrix A denoted by $N(A)$ is all $\vec{x}$'s such that $A\vec{x} = \vec{0}$. (notice that this set of all $\vec{x}$'s is a vector space → this is easily provable).
- The way we have defined column space and null space tells us two important ways to describe a vector space → by giving a set of vectors and by giving an equation and saying all solutions of this form a space (of course, assuming they actually do).
- Notice that for $A\vec{x} = b$ and non-zero b, the set of solutions $\vec{x}$'s don't form a subspace → at least because $\vec{0}$ is not in that subspace.
- If we have a matrix A with 3 rows and 4 columns such that the $\vec{x}$ in $A\vec{x} = \vec{0}$ is of size 4, and we try to row reduce A to U, and we end up getting only 2 pivots, we will have 2 free variables → so called because we can give any values to these free variables and find the value of the other two variables. The set of $\vec{x}$ vectors thus obtained is the nullspace.
- The rank of a matrix is the number of pivots in the row reduced form (also called row echelon form). If we have a matrix of m rows, n columns, rank r then the number of free variables is $n - r$. By giving each free variable the value 1 while giving all other free variables the value 0, we can find the vectors whose linear combinations fully describe the nullspace (the basis of the nullspace).
- The U we obtained above is called the row echelon form. If we do more row operations so that we make all elements in the pivot columns other than than the pivots 0's and if we make all the pivots 1's (we can do this by dividing since b is $\vec{0}$ → we are dealing with nullspace), this is called the reduced row echelon form R. Notice that the solutions to A, U, R are all the same (nullspace hasn't changed).

- Again, by plugging in 1's and 0's the free variables in turn and doing back substitution, we get the vectors whose linear combinations give the nullspace.
- Now that's all we have to say about nullspace. we will turn our attention to solving Ax = b
- if we take another 3 rows by 4 columns A in $Ax = b$, and we row reduce and find out that it only has 2 pivots, then the last row will be all 0's which means the last row was a linear combination of the first two rows. Now we can only solve b if the elements are such that it obeys the same linear combination. So we get a solvability constraint on b. In other words, if some combination of the rows of A gives 0 rows, then the same combination for the corresponding entries of b must give 0's.
- But we also know that $Ax = b$ is only solvable when b is in $C(A)$.
- Now to find the set of solutions, do a row reduction. Notice that the set of solutions is not a vector space in the case where $b \neq 0$. Assuming the b's meet any solvability constraint that exists, we first set all free variables to 0. Then by doing back substitution, we get a particular solution x_p. Now if we find the nullspace of A (as seen previously) and call that x_n. we see that $x_p + cx_n$ is also a solution for $Ax = b$. We can take any vector in the nullspace, add that to x_p and we get a solution for $Ax = b$. Notice that this solution set does not form a subspace. Its kind of like a subspace shifted by a bit.
- If we have a matrix with m *rows* and n *columns* and rank r, note that $r \leq m$ and $r \leq n$. Two cases arise:
  - A full column rank matrix is one where all columns have pivots ($r = n$). In this case, there are no free variables and if b meets the solvability constraint, we will get one solution. So at max, we will have a unique solution. The null space here is just $x = \vec{0}$, no other vectors in the null space.
  - A full row rank matrix is one in which $r = m$. We might have some free variables. Importantly, when we do elimination we don't get any 0 rows so there is no solvability constraint. We can basically solve $Ax = b$ for every b. We have $n - r$ free variables, since we have n total variables. For any assignment of the free variables, we can find a value for the pivot variables.
  - If $r < m, r < n$, then if solvability constraints on b are met, we will have ∞ solutions. If not met, we'll have 0 solutions.
  - What if $r = m = n$? This is the nice case where we can reduce properly and we have exactly one solution for $Ax = b$ (any non - zero b). This is the invertible matrix. This is where the reduced row echelon form is I. The null space is just $\vec{0}$.
- Now we have covered what happens in both $Ax = b$ and $Ax = 0$ for all kinds of A's. Summary of things when $A\vec{x} = \vec{b}$:



- The above shows us why "the rank tells us everything about the number of solutions".
- A set of vectors $x_1, x_2 ... x_n$ is set to be dependent if there is a non-zero $\vec{c}$ such that $c_1x_1 + ..c_nx_n = 0$. If we can't write $c_1x_1 + ..c_nx_n = 0$ for any non-zero $\vec{c}$, then the vectors are independent.
- Notice if any of $x_1, x_2, ..x_n$ are $\vec{0}$, then the set of vectors is definitely dependent.

- To see if a given set of vectors is independent, put them in a matrix with the each column as one of the vectors. After this, set $A\vec{x} = \vec{0}$. Now, think about solutions to this equation. If the null space is just $\vec{0}$, then it is independent. If the null space has other things, then it is dependent.
- For a set of vectors to span R^n, their matrix A must have a solution for $Ax = b$. As we have seen for $Ax = b$ to have a unique solution, it must be a n x n invertible matrix. This means to span R^n, we need n vectors of size n each.
- A given set of vectors spans the vector space which is the linear combination of those vectors. So by definition, those vectors are the basis for that space. A set of vectors in a vector space V is called a basis, or a set of basis vectors, if the vectors are linearly independent and every vector in the vector space is a linear combination of this set. In more general terms, a basis is a linearly independent spanning set.
- Every basis for a space has the same number of vectors, that number is called the dimension of the space.
- rank of matrix with vectors in columns = # of pivot columns = dimension of $C(A)$
- dimension of $N(A)$ = # of free variables in reduced row echelon form.
- The four fundamental subspaces of a matrix A (2 we've already seen and 2 new ones) are the column space, null space, row space and the left null space (null space of the rows as columns → $N(A^T)$).
- We have seen how to find the column space and the null space given a matrix A. To find the row space, we just do a reduced row echelon form of A just as before, while noticing that the row operations do not change the row space of the original matrix since the operations are just taking linear combinations of the rows (they do change the column space). Once we have the reduced row echelon form, the rows with the pivots form the basis for the row space. Because of this, we see that the dimension of the row space is the same as the dimension of the column space and they are both equal to the rank of the matix r.
- We saw before that the dimension of the column space is the number of free variables in reduced row echelon form, which is just $n - r$ where n is the number of columns.
- To find the left null space, we just do the operations to get the reduced row echelon form as before, while keeping track of the elementary matrix E that takes A to its reduced row echelon form. Once we have E, the rows of E that generate the 0 rows in A tell us some of the vectors in the left null space. All linear combinations of these give us the full left null space. To find a basis, we just have to find a minimal basis of these rows of E that give 0 rows in the reduced row echelon form of A.
- Can we stretch the idea of a vector space to include matrices? The rules tell us we need to be able to do additions, constant multiplication and linear combinations and the result must lie in the same space. These rules are all satisfied by the following example matrix spaces: all 3x3 matrices, all upper triangular matrices, all symmetric matrices and since we can intersection of these spaces to get a valid subspace, all diagonal matrices.
- Similarly, we can define a basis for a matrix space → a set of linearly independent matrices whose linear combinations generate all matrices in a given matrix space is called the basis of that matrix space.
- This is stretching the idea of vector spaces. We call the elements of these spaces "vectors" even though they are not vectors in the traditional sense. So in that sense these matrices, which are members of the vector space, are called vectors.
- We can layout a n x n matrix as a vector with n^2 numbers, and then we can obtain things like basis, dimension and span using the regular rules for how we do this for vectors.
- For some differential equations, we can express the solution set as a vector space by specifying its dimensions. For $\frac{d^2y}{dx^2} + y = 0$, with some differential equations solution methods, it can be shown that the solution set is the set of all linear combinations $c_1 sin(x) + c_2 cos(x)$. Thus [sin x, cos x] is a basis

of the solution set. What we're trying to show here is that these ideas of vector space, basis, dimension etc. are broadly useful when the necessary operations are defined in subjects well beyond just vectors. The generalization of these ideas turns out to be useful.

- Now lets think about rank 1 matrices. Because in a rank 1 matrix, all rows can be expressed as a linear combination of the first row, we can write out any rank 1 matrix as a matrix with 1 column (the matrix that has the multiples that each row is of the first row) times the matrix with 1 row (the first row, the one with the pivot).
- there is a lecture (lecture 12) on applications of stuff we've learnt till now to graphs and circuits. no new ideas/concepts in here.
- a sparse matrix is one in which there are many 0's and only a few non-zeroes. if we can identify some inherent structure as to where the non-zeroes are, we can usually store this matrix with much lesser memory than otherwise.
- left multiplying by an invertible matrix doesn't change the nullspace since the invertible matrix can be expressed as a set of matrices that do valid row operations. (we know row operations don't change the nullspace or the solution space).