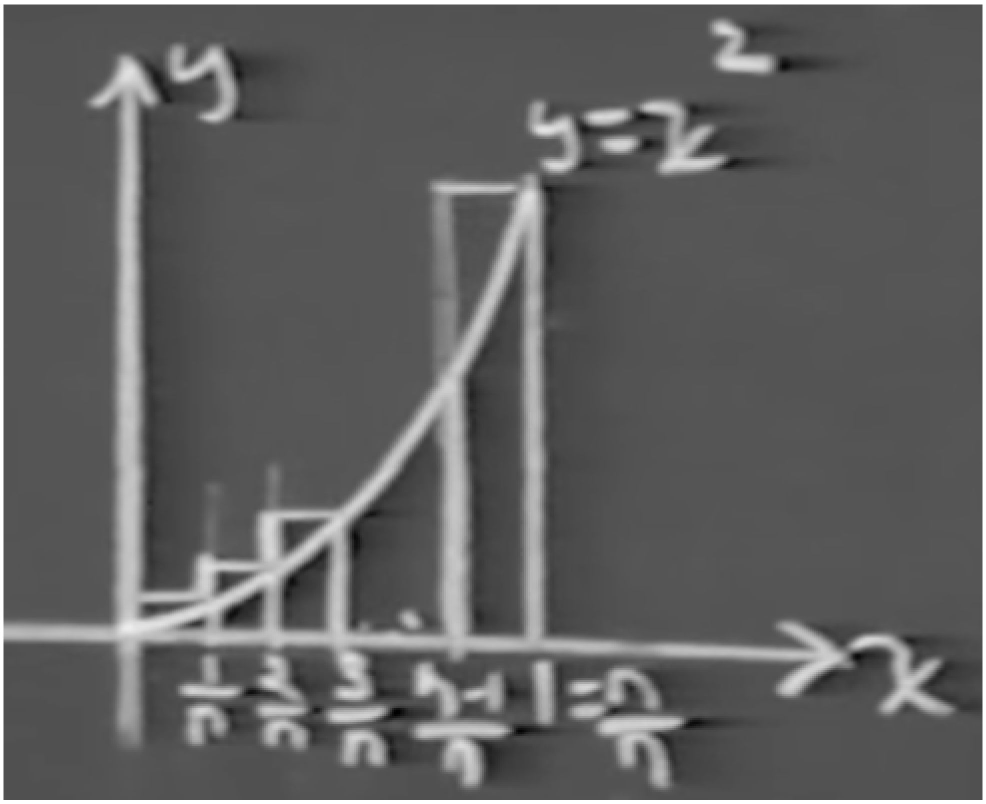
the definite integral

- There is evidence that integral calculus makes sense without differential calculus. The ancient Greeks did integral calculus 2000 years before differential calculus was first known to be made up!
- Area of a rectangle is defined as $breadth$ x $height$
- To find the area under a curve, lets divide the curve up as below:



  Summing up areas of all rectangles above will give us an upper bound on the area - look at the picture to see why.
  Similarly, moving over each rectangle one step to the right (and ignoring the right most rectangle) will give us a lower bound on the area. The actual area lies in between these two. If we can "squeeze" the actual area by taking the limit of the upper and lower bounds as $n \to \infty$, and show that $\lim_{n\to\infty} U_n - L_n = 0$ where U_n is the area computed by the first method (upper bound) and L_n is that from the second method (pushed over by 1), when each has n rectangles. If we can show this then the area is $\lim_{n\to\infty} U_n = \lim_{n\to\infty} L_n$
- $U_n = \sum_1^n f(x) * \Delta x$
- $L_n = \sum_0^{n-1} f(x) * \Delta x$
- Now, $L_n - U_n = [f(x_n) - f(x_0)] * \Delta x = [f(x_b) - f(x_a)] * \frac{b-a}{n}$, where a and b are the points between which we want to find area. Notice that if we take $\lim_{n\to\infty}$, the whole thing is a constant except the n and the $\lim_{n\to\infty}$ is 0. This means the area must equal $U_n = L_n$.
- Similar results can be obtained by using trapezoids to approximate the areas instead of rectangles.
- Note that what we need for this method to work is continuity of the function (and not differentiability).

- Observe that we have developed these results using the basic definition of limits, and totally independent of differential calculus!
- Consider the change in area of a curve between $f(x)$ and $f(x+\Delta x)$:
  - $f(x_1)\Delta x \leq \Delta A \leq f(x_1+\Delta x)\Delta x$ and so $f(x_1) \leq \frac{\Delta A}{\Delta x} \leq f(x_1+\Delta x)$
  - taking the limit as $\Delta x \to 0$, $\frac{dA}{dx} = f(x)$.
- First Fundamental Theorem of Integral Calculus:
  - Suppose we have an explicitly known function G and G' = f. Then A(x) = G(x) + c.
  - That means to find the area, we can do $\int_a^b f(x)dx$!
  - that also means, $\lim_{n\to\infty} \sum_1^n f(x) * \Delta x = \int_a^b f(x)dx = G(b) - G(a)$
  - Holy shit!!! look at how we can see the integral as an infinite sum.
- The Second Fundamental Theorem of Integral Calculus simply tells us that we can compute the definite integral without explicitly being able to integrate simply because we can compute the limit to any arbitrary precision, by making n big enough.
- The requirement on $f(x)$ for the above discussion is that it be continuous, not necessarily differentiable. Convince yourself why this is true.
- We can apply arguments very similar to what we've done to find areas to find volumes → upper bound, lower bound, squeezing, limits, integrals, infinite sums etc.
- Next we will focus on finding arc lengths
- just to start off, we'll define arc length as $\lim_{n\to\infty} \sqrt{\Delta x^2 + \Delta y^2}$
  - we'll only limit ourselves to functions for which this limit exists.
  - we'll see how to compute this limit
  - then we'll see if this definition might agree with our intuitive notion of lengths
- Notice that on dividing by Δx, the above expression (which we haven't yet shown is a good approximation for the length) becomes $\lim_{n\to\infty} \sqrt{1+(\frac{dy}{dx})^2}\Delta x$. We've shown that this is $\int \sqrt{1+(\frac{dy}{dx})^2}dx$. So that is how we compute the limit.
- Now is this a good approximation of the length?
- Well, let the length of segment k be Δw_k. Now $\Delta w_k = \int \sqrt{1+(\frac{dy}{dx})^2}dx + \alpha_k \Delta x$, where α_k is a measure of the error (this error obviously depends on our definition of length). Here, we're simply stating the obvious: for some notion of our length, lets say we can capture the error. If we can prove that this $lim_{\Delta x\to\infty} \alpha_k \Delta x = 0$, (error goes to 0 as Δx goes to 0), then this integral gives us the length. We can do some math to prove that this is the case (at least for some functions, and we can use this to compute arc length only for those functions) but we won't go into that here → we'll focus on the fundamentals: important definitions and theorems that thus follow.
- Just like its hard to conclude things from the $\frac{0}{0}$ in differential calculus, its hard to conclude things from the integral of $\infty * 0$ form in integral calculus.
-