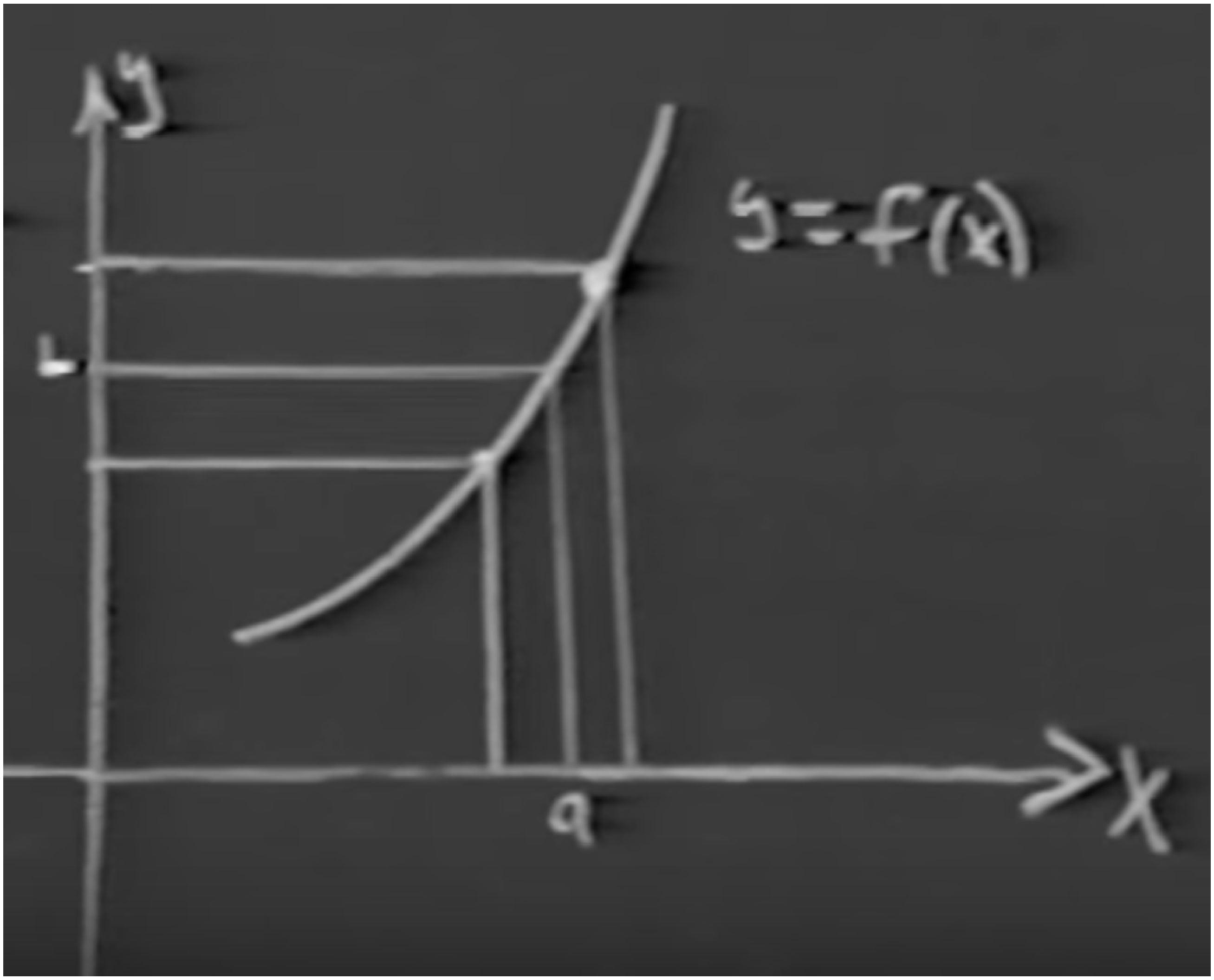
limits, derivatives and rates of change

- Math is really 3 things: definitions (that turn out to be useful as we often see after learning them), why they are useful, and things that follow from definitions (theorems and proofs)
- to understand math well, you must understand:
  - the precise definition. what are the details and boundaries?
  - what follows from that and other definitions and why it follows (proofs)?
  - why is defining something the way it is defined useful?
- to find $f'(x)$ (we'll call this the derivative) at $x = x_0$:
  - first, lets define the derivative as the slope of the tangent at $x = x_0$
  - that begs the question what is the tangent at a point? informally, think of a line that passes thru two points x_1 and x_2 that both lie on the curve. Let's call the change in x, $x_2 - x_1$ asΔx. The slope of that line is $\frac{\Delta y}{\Delta x}$. The tangent is $\lim_{x \to 0} \frac{\Delta y}{\Delta x}$.
- notice that $\frac{\Delta y}{\Delta x}$ is average rate of change of the function $y = f(x)$ and the limit is, as we'll see, the instantaneous rate of change of the function.
- now what does limit mean? we'll define below **limits and continuity.**
- $\lim_{x \to 0^+}$ is the right hand limit, that is as x approaches x_0 from the right side. Similarly, $\lim_{x \to 0^-}$ is the left hand limit.
- **definition of limit:**
  - $lim_{x \to a} f(x) = L$ if for each $\epsilon > 0$, you can find $\delta > 0$ such that:
    $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$.
  - note that the function need not actually be defined at x_0 for the limit to exist at x_0, so we do not allow $x = a$, it has to be strictly greater than.
  - When thinking of limits, the best diagram I've found useful while reasoning is this projection diagram:



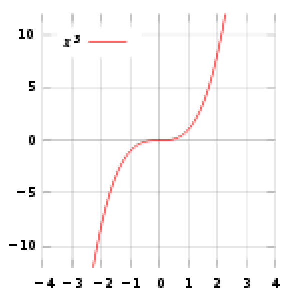
- Rules for limits:
  - limit of sum is sum of limits
  - limit of $c * f(x)$ is $c * L$ where limit of $f(x)$ is L.
- Definition for the limit of $f(x)$ as $x \to \infty$:
  $lim_{x \to \infty} f(x) = L$ means that for any $\epsilon > 0$, we can find some number N such that $x > N$ implies $|f(x) - L| < \epsilon$.
  Similar definition for $-\infty$.
- Mathematical induction is a technique used to prove some kinds of things we already suspect. The process is:
  - show conjecture true for $n = 1$.
  - show that conjecture being true for $n = k$ implies it is true for $n = k + 1$.
- mathematical induction is only useful only when:
  - you already suspect the answer (have a conjecture)
  - the truth for the $(n + 1)th$ case follows inescapably from the truth for the nth case.
- **definition of continuity.** a function is continuous at $x = x_0$ if**:**
  - $\lim_{x \to x_0}$ exists
  - $\lim_{x \to x_0} f(x)$ = $f(x_0)$
- continuity can also be defined on an interval.
- **continuity is a pre-req for differentiability** → this isn't a definition, it can be shown.

- a jump discontinuity is when the left and right limits exist but are not equal.
- a removable discontinuity is when the the two limits exist and are equal but are not equal to $f(x_0)$.
- the derivative of a function need not look anything like the function, and often doesn't!
- We can pretty easily show that $\frac{d}{dx}x^n = nx^{n-1}$. To show this, just write the fundamental definition of the derivative in terms of limits and expand out the exponent n as multiplying n times.
- Using the fact that limit of sum of functions is sum of limits, we can derive the similar rule for derivatives: *derivative of sum is sum of derivatives.*
- We can easily show that $\frac{d}{dx}cf(x) = c\frac{d}{dx}f(x)$.
- Using limits, we can derive the product rule and quotient rule of derivatives:
  $\frac{d}{dx}(f(x) * g(x)) = \frac{d(f(x))}{dx} * g(x) + \frac{d(g(x))}{dx} * f(x)$.
  To derive this, simply write the fundamental definition of derivative in terms of limits, and do some algebraic manipulation. The only other thing to use is:
  $\lim_{\delta \to 0} f(x+\delta) = f(x)$ when f is differentiable at x. You'll see why this is true if you just move the RHS to the LHS (thus making the right hand side 0) and multiply and divide the right hand side by δ, and then use the product rule for limits.
- Through similar first principle application of the definition of limits, we can derive the quotient rule:
  $\frac{d}{dx}(\frac{f(x)}{g(x)}) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$
- For small enough Δx, $f(x+\Delta x) \approx f(x) + f'(x)\Delta x$. To see why this is true, we can show that:
  $f(x+\Delta x) - (f(x) + f'(x)\Delta x)$ tends to 0 as Δx tends to 0:
  $\frac{\Delta y}{\Delta x} = \frac{dy}{dx} + k$ and so $lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\frac{dy}{dx} + \lim_{\Delta x \to 0} k$ and so $\lim_{\Delta x \to 0} k = 0$
- Also, $\Delta y = \frac{dy}{dx}\Delta x + k\Delta x$ → (1)
  where k is some function of Δx, that makes up for the error. By diving by Δt throughout and taking the limit, we get $\lim_{\Delta x \to 0} k = 0$. This means this error k tends to 0 as Δx tends to 0.
- Using the equation (1) above, we can also derive the chain rule and parametric differentiation rule:
  $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} + (lim_{\Delta x \to 0}k)\frac{dx}{dt}$ (note that $\lim_{\Delta x \to 0} k = 0$), and so

  $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$. From this, it also follows that, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$
- For inverse functions f and f^{-1}, we can easily show that $(\frac{dy}{dx})_{x=x_0}(\frac{dx}{dy})_{y=f(x_0)} = 1$. We can show this using the fundamental definition of limits, but to get a graphical feel for this, think about the fact that inverse functions are reflections in the $y = x$ line.
- The rule $\frac{d}{dx}x^n = nx^{n-1}$ can be proved for:
  - for positive integers by binomial coefficient expansion
  - for negative integers by, quotient rule
  - for fractional powers using the rule for inverse functions' derivatives
- Identity vs equality: $x^2 = 4$ is basically the set of all values $\{x : x^2 = 4\}$. An identity is something that's always true for all values of x. For example, $x^2 - 1 = (x+1)(x-1)$
- An equality or equation uses a $=$ sign whereas an identity uses a $\equiv$ sign. Differentiating both sides of an identity results in an identity.
- Differentiation both sides of an equation results in a different equation.
- Sometimes it is hard to solve for one variable in terms of another. Example: $x^8 + x^6y^4 + y^6 = 3$. To differentiate such functions, we use a technique called implicit differentiation.
- We won't go into the details of the proof for why this works but we'll state the method here (this is the only thing so far that we haven't proved → we will almost always give proofs or the idea for proofs, unless its quite simple, but Mr. Gross doesn't go into the proof here). The method is to simply

differentiate both sides treating x and y and separate functions. Note that in order to find $\frac{dy}{dx}$ (the slope) at a particular point, we first need to verify that the point exists on this curve (satisfies the equation).

- Implicit differentiation is so called because even though you have an relationship between x and y, you can't state one as the explicit function of the other.
- If a function is not continuous its possible that it doesn't have a maximum value in a closed interval. Think of $\frac{1}{(x-1)^2}$ in the interval $[0, 2]$. There is no clear max value. It takes arbitrarily larges values when x is near 1.
- Intermediate value theorem: Let f be continuous on $[a, b]$ and $f(a) < f(b)$. Let m be such that $f(a) < m < f(b)$. Then there exists a $c \, \epsilon \, (a, b)$ such that $f(c) = m$. → pretty easy to see.
- Sum and product of two continuous functions is also continuous → pretty easy to prove.
- pictorially:
  - differentiable → smooth
  - continuous → unbroken
- Curve plotting:
  - is it symmetric about either axis? symmetric about $y = x$?
  - can it take negative/positive/both values?
  - even function means $f(x) = f(-x)$. odd function is $f(-x) = -f(x)$.
  - any function can be written as the sum of an even and odd function. this is because the following is an identity:
    $f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}$. The first term is an even function and the second is odd irrespective of what f is.
- A stationary point is a point where the curve is neither rising nor falling $i.e.$ $\frac{dy}{dx} = 0$. It need not be a maxima or minima though, just that $f'(x) = 0$. This assumes that it is smooth at that x (derivative exists).
- For a smooth curve, if it is a local max or min, then derivative must be 0. On the other hand, if derivative is 0, it may not be a local max or min. It might just be, what we call a saddle point:



- If a smooth curve goes from concave to convex (or other way), its a point of inflection and $f''(x) = 0$. But, the mere fact that $f''(x) = 0$ does **not** allow us to conclude that its a point of inflection.
- If $f(c) \geq f(x)$ for all $x \, \epsilon \, N_\delta(c)$, (that means $f(c)$ is greater than any $f(x)$ for x in a given neighborhood of c), then $f'(c) = 0$, if $f'(c)$ exists. This is true even if $f(c) \leq f(x)$.
- To find maxima and minima, check for places where $f'(x) = 0$, the derivative doesn't exist and the endpoints of the domain, if they exist (if it's a closed interval). This is because maxima and minima occurring at end points need not have derivative equal to zero (or derivative even defined).
- Rolle's theorem: Let f be defined and continuous on [a, b] and differentiable in (a, b) and f(a) = f(b). Then f'(c) = 0 for at least one c ϵ (a, b). The importance of this will be seen while proving the mean value theorem.

- Mean value theorem: Let f be defined and continuous on [a, b] and differentiable in (a, b). There is a c ϵ (a, b) such that $\frac{f(b)-f(a)}{b-a} = f'(c)$. Intuition: Apply the Rolle's theorem imagining the x-axis to pass between the points (a, f(a)) and (b, f(b)). Where it falls down: the function need not be single valued (and hence not a function at all) when the axis is the new axis described above, even though it is single valued with respect to the x-axis. Proof idea for the proof that actually works: define g(x) = distance between f(x) and line connecting (a, f(a)) and (b, f(b)).
- Mean value theorem is very useful to prove a lot of things that we can suspect to be true in calculus based on looking at geometric (graph) diagrams.
- For example, we can show that if f' = 0, for all x then f is constant (not that we proved the converse before, not this). Proof idea $\frac{f(b)-f(a)}{b-a} = f'(c) = 0$. Thus f(a) = f(b), for all a and b. Thus its a constant. We can also show that f(b) - f(a) is a constant if f'(x) = 0 for all x.