# differential equations

- differential equations are those that express the relationships between variables using their rates of change.
- The mathematical symbol for writing the most general first order diff. eq. is $F(x, y, \frac{dy}{dx}) = 0$. Finding the solution to a diff. eq. means finding an equation in terms of just the dependent and independent variables, eliminating the differentials and that satisfies the given diff. eq.
- to verify if an equation is a solution to a given diff. eq, from the equation we can find $\frac{dy}{dx}$, then plug in the values of $\frac{dy}{dx}$ and $y$, both expressed in terms of $x$ into the equation $F$ and see if LHS equals RHS.
- Two questions arise:
  - Does a solution exist?
  - Is the solution unique?
- Key theorem for existence and uniqueness that Herb mentions is hard to prove but very useful:
  If $\frac{dy}{dx} = f(x, y)$, and $f, f_y$ are continuous in some region $R$, then at each $(x_0, y_0)\epsilon R$, there is a unique solution of $\frac{dy}{dx} = f(x, y)$ which passes thru $(x_0, y_0)$.
- A first degree diff. eq. is one in which the derivative terms are of the first order (raised to power 1).
- Given a first order, first degree diff. eq. we can collect the terms that do and don't contain the derivative term and write it generally as $\frac{dy}{dx}N(x, y) + M(x, y) = 0$, which in turn can be expressed as $\frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)}$. Using the above, theorem, whenever the RHS is continuous and its y-derivative is continuous in some R, then there exist unique solutions in that region.
- The above can be manipulated to $Mdx + Ndy = 0$. Now, if this happens to be an exact differential (if we can find a $w(x, y)$ such that $dw = Mdx + Ndy$), we are done. The solution is $w(x, y) = c$. This only works if its an exact diff. eq.
- What if its not an exact differential? Lets take $xdy - ydx = 0$. This isn't an exact differential. What if we divide both sides by $y^2$? We get $\frac{ydx-xdy}{y^2} = 0 \to d(\frac{x}{y}) = 0 \to y = cx$. notice that this integrating factor does not mean that we end up with the solution that doesn't work for the original equation. It is not always easy to find such integrating factors that helps us find a solution for an initially inexact differential by giving us an exact differential. If we can find an integrating factor, we can solve it!
- a linear diff. eq. is one which in which no derivative is raised to any power > 1. But the derivative itself can be second, third etc. The coefficients for the derivatives can be functions of the independent variable. It is of the general form:
  $y'' + p(x)y' + g(x)y = f(x)$. notice that $y''$ can have a coefficient → we just divide throughout by that coefficient to make sure $y''$ has coefficient 1.
- The reason the above is called *linear* is because the output of a constant times the input is that constant times the output. This is generally also the meaning of calling something "linear". The other property of linearity is that the output of the sum of two inputs is the sum of the outputs of the individual inputs. Notice that these two properties are only satisfied if the diff. eq. is of the above general form.
- From linearity, we can show that if we find two solutions $u_1$ and $u_2$ to a linear diff. eq such that $L(u_1) = 0$ and $L(u_2) = 0$, we can find infinitely many solutions by taking linear combinations of $u_1$ and $u_2$.
- Just like the theorem we saw above, a similar one exists for linear diff. eq. (of higher order, not just first order derivatives). We won't cover that here.

There are about 5 more lectures here that Herb goes thru, but we will leave out. The main purpose of these notes is to form a solid math base for machine learning and AI work → its not clear to me we'll need more of this later. I will come back and write more notes on differential equations if and when it becomes needed.

**That concludes our coverage on calculus.**