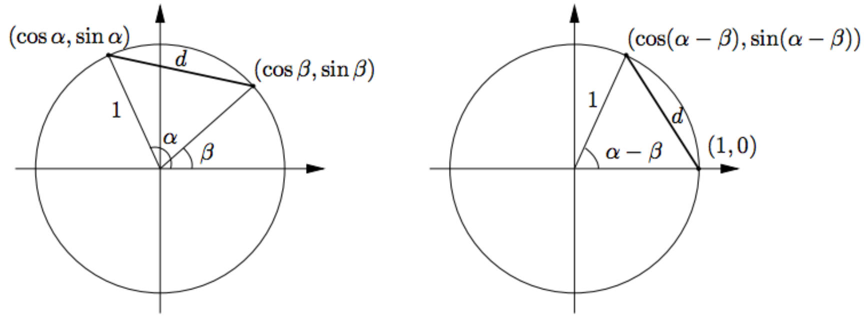
circular functions, trig refresher

- Let's define sine and cosine and other trig functions explicitly in terms of unit circle instead of triangles. Given any negative or positive real number, we can rotate anti-clockwise (for positive) and find a point on the circle. The y co-ordinate of that point is called sine and x co-ordinate is called cosine. This way of defining the functions makes it much clearer than the triangle way. Notice that this maps a real number to $[-1, 1]$.
- Once we've defined sine and cosine as above, we'll define tan and cos in terms of these:
  - $tan = \frac{sin}{cos}$
  - $cotangent(cot) = \frac{cos}{sin}$
  - $cosec = \frac{1}{sin}$
  - $sec = \frac{1}{cos}$
- This definition also gives us familiar results like sin 0 and sin pi/2 etc easily (as opposed to when defined with triangles)
- Measuring in radians, we can also think of the inputs as just lengths along the unit circle. This also happens to be the same as angles.
- Sum/difference formulas of trig and their proofs, done easy (the following image is just copy pasted from the link → the proof is so nicely done i want to preserve this even if they take down that pdf).

We will prove the following trigonometric identities.

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta \qquad \sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$
$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta \qquad \cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

Proof. Consider two angles α and β. The distance d in the following two unit circles are equal.



From the first one we obtain
$$d=\sqrt{(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2}.$$
From the second one we obtain
$$d=\sqrt{(\cos(\alpha-\beta)-1)^2+(\sin(\alpha-\beta)-0)^2}.$$
From these two expressions for d, we can deduce
$$\begin{aligned}
d^2&=d^2\\
(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2&=(\cos(\alpha-\beta)-1)^2+(\sin(\alpha-\beta)-0)^2\\
\cos^2\alpha-2\cos\alpha\cos\beta+\cos^2\beta+\sin^2\alpha-2\sin\alpha\sin\beta+\sin^2\beta&=\cos^2(\alpha-\beta)-2\cos(\alpha-\beta)+1+\sin^2(\alpha-\beta)\\
(\cos^2\alpha+\sin^2\alpha)-2\,(\cos\alpha\cos\beta+\sin\alpha\sin\beta)+(\cos^2\beta+\sin^2\beta)&=(\cos^2(\alpha-\beta)+\sin^2(\alpha-\beta))-2\cos(\alpha-\beta)+1\\
2-2\,(\cos\alpha\cos\beta+\sin\alpha\sin\beta)&=2-2\cos(\alpha-\beta).
\end{aligned}$$
Therefore,
$$\boxed{\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta.}$$
Replacing β by $-\beta$ gives us
$$\cos(\alpha-(-\beta))=\cos\alpha\cos(-\beta)+\sin\alpha\sin(-\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta.$$
Then,
$$\boxed{\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta.}$$
Now let's replace α by $\frac{\pi}{2}-\alpha$ to get
$$\cos(\tfrac{\pi}{2}-\alpha+\beta)=\cos(\tfrac{\pi}{2}-\alpha)\cos\beta-\sin(\tfrac{\pi}{2}-\alpha)\sin\beta.$$
Since we know that
$$\sin(\tfrac{\pi}{2}-\alpha)=\cos\alpha,\quad \cos(\tfrac{\pi}{2}-\alpha)=\sin\alpha,\quad\text{and}\quad \cos(\tfrac{\pi}{2}-(\alpha-\beta))=\sin(\alpha-\beta)$$
we can conclude that
$$\boxed{\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta.}$$
Finally, by replacing β by $-\beta$ we obtain
$$\sin(\alpha-(-\beta))=\sin\alpha\cos(-\beta)+\cos\alpha\sin(-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta.$$
Then,
$$\boxed{\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta.}$$

□

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- $\lim_{x\to 0}\frac{sin(x)}{x}$:

  - the inner triangle has smaller area than the outer triangle. the area of the arc t is inbetween the areas of the two triangles. Hence:
    - $\frac{sin(t)cos(t)}{2} < \frac{t}{2\pi}\pi < \frac{tan(t)}{2}$. Dividing by $sin(t)$, we get: $\cos(t) < \frac{t}{sin(t)} < \frac{1}{cos(t)}$.
    - The right and left terms tend to 1 as $t \to 0$, so the sandwiched term tends to 0 too.
    - Hence $\lim_{x\to 0} \frac{sin(x)}{x} = 1$.
    - Also, $\lim_{t\to 0} \frac{1-cos(t)}{t} = \lim_{t\to 0} \frac{1-cos^2(t)}{t(1+cos(t))} = \lim_{t\to 0} \frac{sin(t)}{t} \frac{sin(t)}{1+cos(t)} = 1 * 0 = 0$.
- Using the above, two results and using the definition of limits, we can easily show that $\frac{d}{dx} sin(x) = cos(x)$.
- Using the above and the fact that $cos(x) = sin(\frac{\pi}{2} - x)$, we can show that $\frac{d}{dx}cos(x) = -sin(x)$. Using sine and cosine, we can easily find the derivatives of all other basic trig functions using quotient rules and the recipes we've seen before!
- Obviously,
  - $\int cos(u)du = sin(u) + C$ and
  - $\int sin(u)du = -cos(u) + C$
- For two functions to be equal, they should have the same domains. We can define a new function $s_0(x)$ which is $sin(x)$, but defined over $[\frac{-\pi}{2}, \frac{\pi}{2}]$. We do this because to have an inverse function, we need to make our original function one-to-one.
- Using the fact that the derivative of the inverse of a function is the reciprocal of its derivative, we can show that:

  $y = sin^{-1}(x)$ and so $x = sin(y)$
  $\frac{dx}{dy} = cos(y)$
  $\frac{dy}{dx} = \frac{1}{cos(y)}$
  and so
  $\frac{d}{dx} sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$

and because $cos^{-1}(x) = \frac{\pi}{2} - sin^{-1}(x)$,
$\frac{d}{dx}cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$

- See how the integral of a non-trig function is an inverse trig function:
    - $\int \frac{1}{\sqrt{1-x^2}}dx = sin^{-1}(x)$