infinite series

- what is infinity? Until now, we've only thought about it in terms of limits. But: lets say n is $100^{100^{100^{100}}}$. Even though n is large, it's no closer to the end of the number system than is the number 1.
- What is the the result of this infinite sum? $[1 + -1] + [1 + -1] + [1 + -1]$.... Now what is this sum? $1 + [-1 + 1] + [-1 + 1] + [-1 + 1]$...You cannot say 0 or 1 just based on visualizing infinity as a sum to a very large number, because infinity is not simply a very large number. So your intuition is wrong because infinity is not what your intuition thinks it is.
- Let's take a geometric series (the difference between a series and a sequence is that a series is a sum but a sequence is just a listing of the terms) where the first term is $\frac{1}{2}$ and the common ratio is also $\frac{1}{2}$.
- We can easily show that the sum of a geometric series is $\frac{a(r^n-1)}{r-1}$. google the proof, its really easy.
- So what is the rule of adding infinitely many terms? We'll define it as the $\sum_{n=1}^{\infty} f(n) = \lim_{n\to\infty}(f(n_1) + f(n_2) + ...)$. Seems like an obvious, useful way to define it. Notice this is the series definition. We'll see the sequence definition next.
- We can define infinite sequences using limits similar to how we define regular limits. More formally, the infinite sequence is said to converge to the limit L (written $\lim_{n\to\infty} b_n = L$) if and only if for all $\epsilon > 0$, we can find a $N(\epsilon)$ such that $n > N$ → $|a_n - L| < \epsilon$. Note this is the definition for limit of sequence, not limit of series.
- If the series converges, then we know the sequence also must converge. But the reverse is obviously not true, in general.
- For a sequence S, a number M is called an upper bound for S if and only if $M \geq x$, for all $x \in S$.
- M is a least upper bound for S if M is an upper bound for S and no number smaller than M can be an upper bound for S.
- Note that by these definitions, the upper bound and least upper bound need not belong to S.
- We make ditto definitions for lower bound and greatest lower bound, and that too need to belong to S.
- A sequence S is called bounded if it has both an upper bound and a lower bound.
- This crazy theorem that Herb Gross throws out but doesn't prove: Every bounded set has a glb (greatest lower bound) and a lub (least upper bound). It's crazy just because it makes quite an assertion without proving it, but at least he explicitly acknowledges that we are not proving it.
- A sequence might be monotonic increasing, decreasing, non-increasing or non-decreasing.
- For a non-decreasing sequence $a_n = f(n)$ has an upper bound, then $lim_{n\to\infty}$ has to be the least upper bound. This is because anything less than the limit can be exceeded (by definition of limit) and hence is not an upper bound. Anything greater than the limit is obviously not a least upper bound (cuz the limit is an upper bound and is lesser than it!)
- A positive series is one where each term $a_n > 0$. Obviously for a positive series, it either diverges to ∞ or converges to a limit L where L is the least upper bound for the sequence of partial sums.
- Tests for whether a positive series diverges or converges to a limit:
  - Comparison test:

    If each term in a positive series is less than or equal to the corresponding term of anther series, and the other series converges, then this series also converges. Conversely, if each term is greater than the corresponding term of another series, and the other series diverges, then this series also diverges.
  - Ratio test:

Given $\sum_{n=1}^{\infty} a_n$, form the sequence $\frac{a_{n+1}}{a_n} = u_n$. If $lim_{n\to\infty} u_n$, and $lim_{n\to\infty} u_n < 1$, then $\sum_{n=1}^{\infty} a_n$ converges. if $lim_{n\to\infty} u_n > 1$, then it diverges. If $lim_{n\to\infty} u_n = 1$, then the test fails.

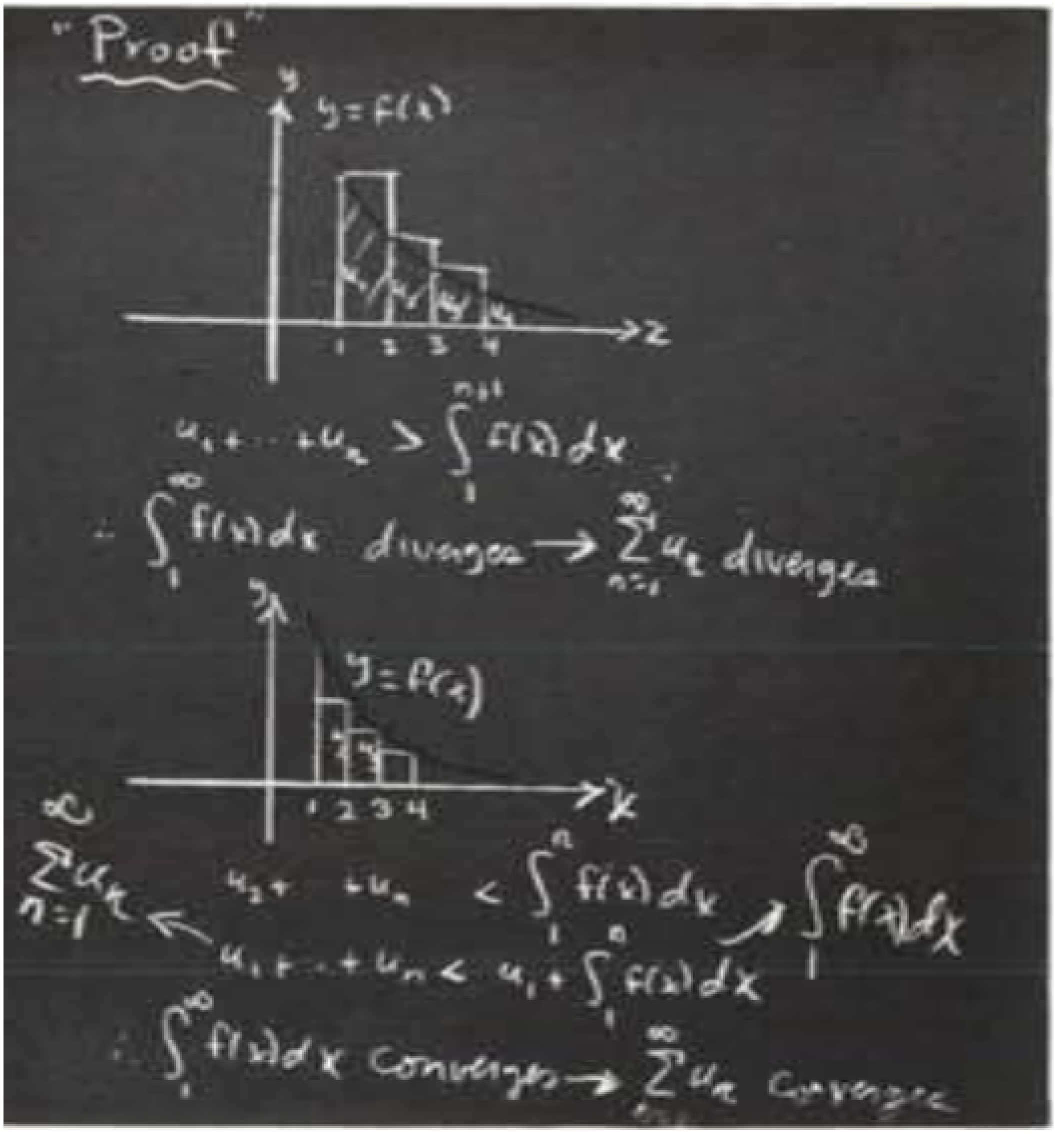
Proof:

We can show that this series (sequence sum) is less than the sum of a geometric series (with common ratio less than 1) times a constant a_N. Because that is a convergent series, and by comparison test, each term here is less than that of the convergent geometric series, this is also convergent.

- Integral test:

  Suppose there is a **decreasing continuous positive** function $f(x)$ such that $f(n) = u_n$ is the n^{th} term. Then $\sum_{n=1}^{\infty} u_n$ converges or diverges as $\int_1^{\infty} f(x)dx$ converges or diverges.

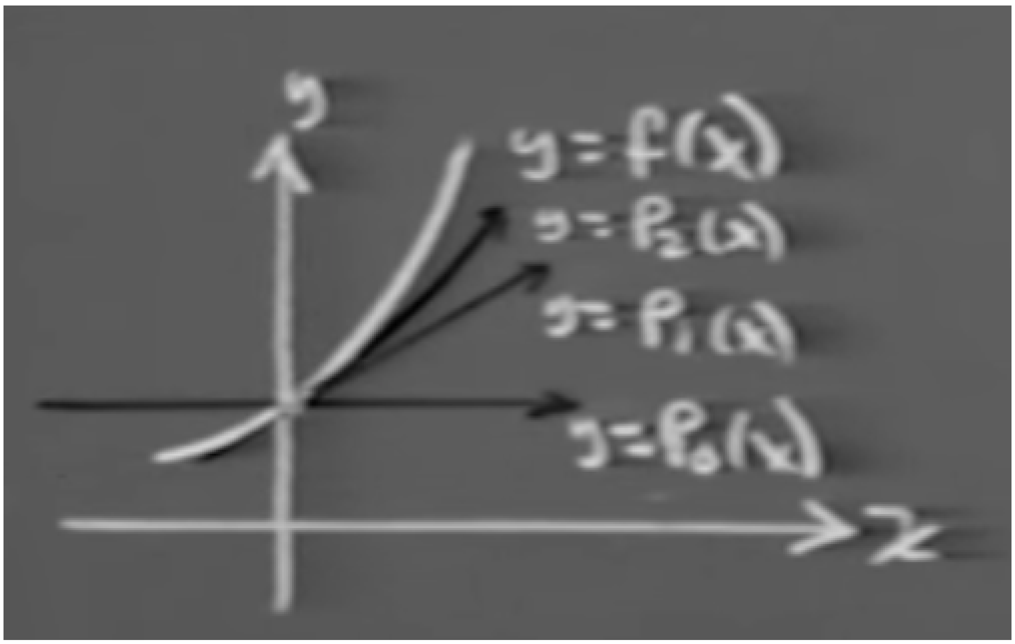
  Proof:



Note that $u_1 + u_2 + u_3..$ is the sum of areas under the rectangles. As shown in the figure, we can construct these rectangles to either be less than the integral or greater than the integral, to prove the case of convergence and divergence respectively. (note again, here also we use the comparison test to prove that the integral test works).

- $\sum_{n=1}^{\infty} a_n$ is said to converge absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges.

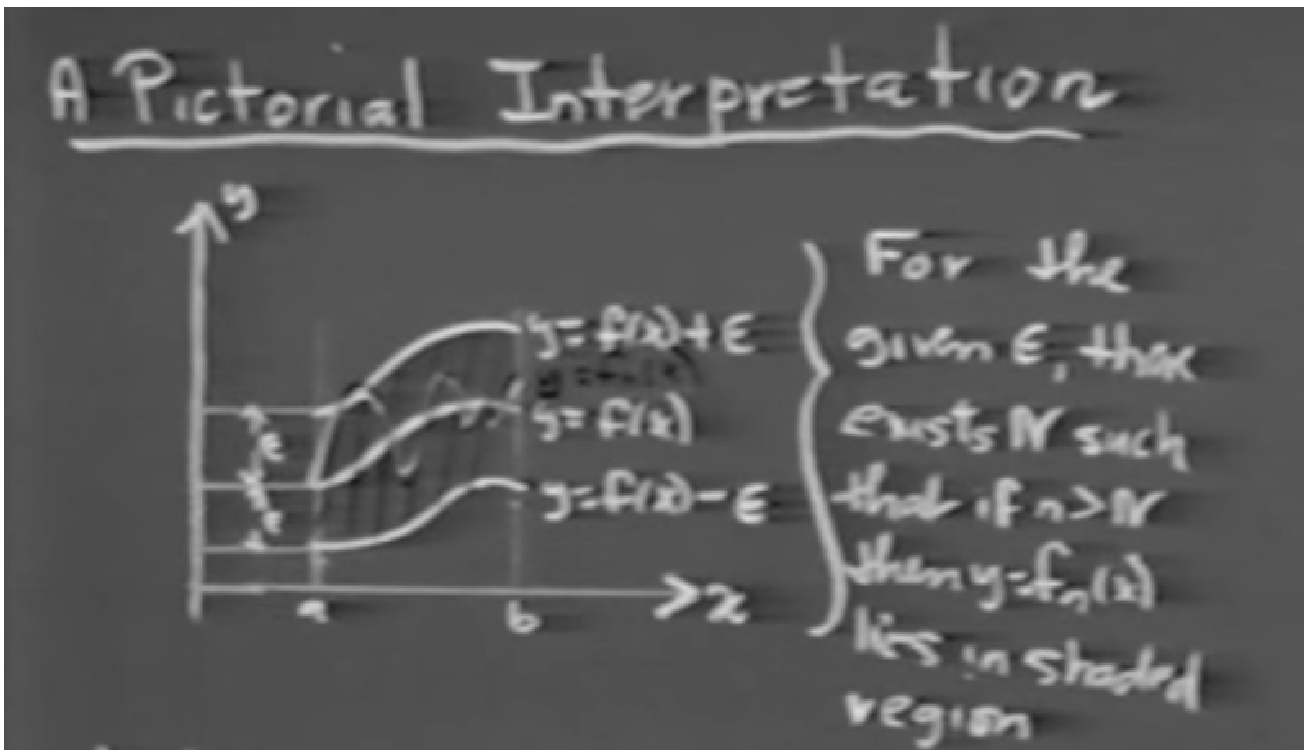
- Absolute convergence also implies normal convergence (but not necessarily the other way round). proof: Continue here.
- A series which converges, but not absolutely, is called conditionally convergent.
- It turns out, the sum of a conditionally convergent series, in general, is dependent on the order of terms in the series! what?! yes. So if $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, its limit exists, but the limit changes as the order of the terms is changed. That is, re-arranging the terms changes the series. So “don't monkey around with conditional convergence.” It's a weird thing. Herb Gross doesn't go into any more than that here 😞
- Conveniently, absolute convergence series are independent of order of terms. Huh, thank god!. But again, Herb Gross doesn't prove this.
- We now turn our interest to approximating a function with another function in the neighborhood of a given point.



- Look at the above picture. We are going to define “degree of contact” of two a curve c_2, (just a function) with another curve c_1 (another function) at a point p as:

  the number of derivatives of c_1 that are equal to the corresponding derivatives of c_2 at that point. Like if f' is equal vs f', f'' and f''' are equal etc.
- To approximate the curve at $x = c$ using a constant (0^{th} degree polynomial), we just pick the line $y = f(x)$. The y values are equal but the no derivative is equal.
- Because the rule of polynomial differentiation when $x \neq -1$ is nx^{n-1}, we can show that for the n^{th} derivative and all those before to be equal, we should have our polynomial as $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$. Now lets call $P(x) = lim_{n\to\infty} P_n(x)$. Note that $P(x)$ is the approximation as n goes to infinity, and that is different from the function we are approximating $f(x)$. This all assumes that the required number of derivatives exist, of course.
- Now by our definition of degree of contact, it is obvious why $P_n(x)$ has a good degree of contact with $f(x)$. Now lets see if this means anything about P approximating f. We can ask:
  - Does the limit $P(x) = lim_{n\to\infty} P_n(x)$ exist?
  - Does this equal $f(x)$?
  - Does P possess the nice polynomial properties that P_n possesses? eg. continuity

- If we take $f(x) = \frac{1}{1-x}$, we see that $P_n(x) = 1 + x + x^2 + x^3 + \ldots$ etc. At $x = 2$, $f(x)$ is defined, but $P(x)$ (the limit as n tends to infinity) definitely does not approximate it! Also, the limit function is not continuous at $x = 1$ since $lim_{n\to\infty} x^n = 0$, if $x < 1$, and it's not 0 if $x \geq 1$ So the answers to the above 3 questions is "No" for this $f(x)$.
- Are there cases where the answers to these questions is yes? First, we'll discuss the first two questions. Later, we'll discuss the third question.
- If $f(x)$ is a polynomial, then $P(x)$ exactly approximates $f(x)$ since it can be easily shown that $P(x) = f(x)$, when f is a polynomial.
- Given a polynomial series, $\sum a_n(x)$, we can show that there exists a number M, such that if $|x| < M$, then the infinite sum converges, else if $|x| > M$, it diverges - think of the ratio test. This M is called the radius of convergence. "radius" because for $-M < x < M$, the series converges.
- Derivation of Taylor polynomial. It's just the fundamental theorem of calculus($f(x) = f(0) + \int_0^x f(t)dt$), followed by integration by parts. We can easily show that the remainder $R_n = (-1)^n \int_0^x \frac{(t-x)^n}{n!} f^{n+1}(t)dt$. The approximation only holds when the remainder $\lim_{n\to\infty} R_n = 0$. Full proof: http://www.math.umd.edu/~jmr/141/remainder.pdf
- Let f_n be defined on $[a, b]$. We say the sequence f_n converges point wise to f on $[a, b]$ if and only if $\lim_{n\to\infty} f_n(x) = f(x)$ for each x in $[a, b]$. Notice that, by this definition, different x values are allowed to have different sequence limits. By limit terminology, this means that for a given ϵ, the N in $n > N$ can be different for different values of x
- However, if we can find just one N for every x in $[a, b]$, then we say that the convergence is uniform.
- For each value of n, we get a $f(x)$ curve. For uniform convergence, for all $n > N$, all the curves generated by different values of n, lie between $f(x) + \epsilon$ and $f(x) - \epsilon$.



Pictorial representation of the above point

- From our picture if f_n converges uniformly to f on $[a, b]$, and if each f_n is also continuous in that interval, then f is also continuous on that interval. See how uniform convergence is helping us answer our third (and last) question? Herb doesn't prove this theorem, but we can "see it". (although the one big thing I learnt this course is not to trust my intuition 😃)

- Now since f is continuous at $x = x_1$ when $x_1 \epsilon [a, b]$, we know that $\lim_{x \to x_1} f(x) = f(x_1)$ and since $f(x) = \lim_{n \to \infty} f_n(x)$,
  $\lim_{x \to x_1}[lim_{n \to \infty} f_n(x)] = f(x_1) = \lim_{n \to \infty}[lim_{x \to x_1} f_n(x)]$
- That means, if we have uniform convergence, we can interchange the order of the two limits when $x_1 \epsilon [a, b]$.
- **Weierstrauss m-test:** if for all $n \geq 1$, $x \epsilon A$ (a fixed domain), $|f_n(x)| \leq M_n$ and $\sum_{n=1}^{\infty} M_n$ is convergent, then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[a, b]$.
  Proof:
  For it converge, it means that $|f(x) - \sum_{k=1}^{n} f_k(x)|$ can be brought down to a sufficiently small number by picking large enough n. Now,
  $|f(x) - \sum_{k=1}^{n} f_k(x)| = |\sum_{k=1}^{\infty} f_k(x) - \sum_{k=1}^{n} f_k(x)| = |\sum_{k=n+1}^{\infty} f_k(x)|$
  Now $|\sum_{k=n+1}^{\infty} f_k(x)| \leq \sum_{k=n+1}^{\infty} |f_k(x)| \leq \sum_{k=n+1}^{\infty} M_k$, which means $f_k(x)$ converges uniformly (since it converges independent of x), since we know that $\sum_{n=1}^{\infty} M_n$ is convergent.
- There are a couple of results from the last two lectures that we have left out here mainly because Herb simply mentions them and uses them without proving them.