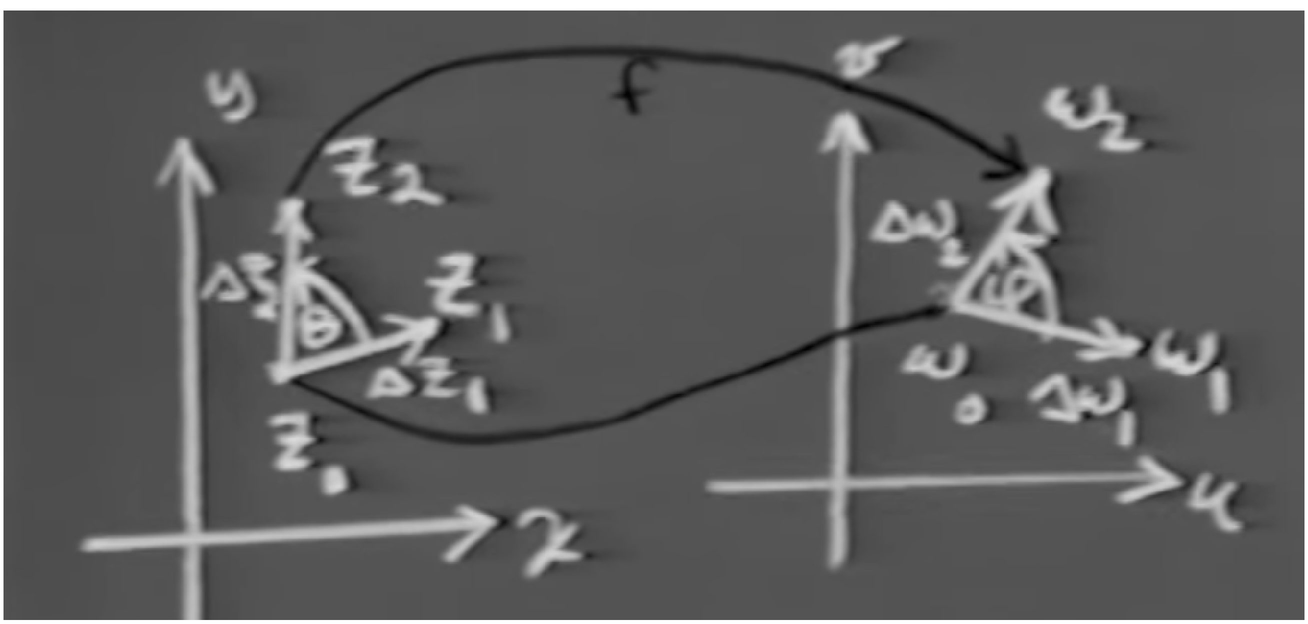
complex numbers

- “how many of you men down there?” said the floorman. “3”, came the answer. “Ok, half of you come on up”, said the floorman → realness is in the eyes of the beholder.
- If the only numbers we knew were integers and we wanted to solve $2x = 3$, it has no solutions if the only numbers we knew were integers. We can “invent”(define) something called real numbers and give this a solution. do we want this to have a solution? sometimes yes, sometimes no → depends on the situation. Going into the post office and asking for 3 cents worth of 2 cent stamps doesn’t make sense, so in that case we should limit our solutions to integers.
- Real numbers are no more “real” than the integers are real. Real numbers are numbers such {$x : x^2 \geq 0$}. Consider the equation $x^2 = -1$. Does this have a solution? It has no real solutions since the definition of a real number is that $x^2 \geq 0$. For it to have a solution, we will have to extend the system of numbers in the same way that we had to extend the system of integers so we could have a solution for $2x = 3$.
- Why would we want $x^2 + 1 = 0$ to have a solution? We sometimes run into physical situations that give us such an equation and it makes sense to define a number system that gives us a solution for the above and an interpretation for such a number system.
- The complex number system is any number that can be expressed as $x + iy$ where x, y are reals and $i = \sqrt{-1}$. To define a system, along with this system, we will need some rules that make up the structure (“game”):
  - Two complex numbers are equal if and only if their components are equal (real and imaginary parts).
  - Addition is done component wise.
  - Real multiplication with a complex: $r(x + iy) = rx + iry$ (r is real)
  - It’s magnitude is defined as $\sqrt{x^2 + y^2}$
- By the above definition, we have made complex numbers a vector space. So we can visualize the complex numbers geometrically in a 2-d plane.
- a diagram on which complex numbers are represented geometrically using Cartesian axes, the horizontal coordinate representing the real part of the number and the vertical coordinate the complex part is called an *Argand diagram.*
- We also define a polar coordinate representation for complex numbers such that $x + iy = rcos(\theta) + irsin(\theta)$. The difference between the polar coordinates we saw in vectors and this is that here, we constrain r to be positive and $\theta = arg\ z$, where z represents the complex number.
- Additional structure: we define $(a + bi)(c + di) = (ac - bd) + i(bc + ad)$.
- With the above definition, notice how real numbers fit into this definition. this means that if the i component is 0 (for numbers that lie on the x-axis in the argand diagram, real multiplication and complex multiplication yield the same results. This is similar to how integers are also real numbers.
- Also in the above rule, whenever $bc + ad = 0$, we get a real number.
- We define the complex conjugate of $x + iy$ as $x - iy$. Note that the product of the complex conjugate is a real number. this is one of those cases where $bc + ad = 0$. In polar coordinates, complex conjugates become (r, θ) and $(r, -\theta)$. this is because the y coordinate just gets negated (picture it).
- complex division is defined as $\frac{c+di}{a+bi} = \frac{c+di}{a+bi}\frac{a-bi}{a-bi} = \frac{(ac+bd)+(ad-bc)i}{a^2+b^2}$. We must have $a \neq 0$ or $b \neq 0$. If they are both 0, it is not defined (just like in the real case). Again, notice that this definition covers the case for real division (it doesn’t change the result of division if both the numerator and denominator are real)

- Turns out, multiplication when expressed in complex numbers is simply multiplying the magnitudes and adding the angle θ_1 and θ_2. Proof:

  $(r_1, \theta_1)(r_2, \theta_2) = r_1 r_2(cos(\theta_1)cos(\theta_2) - sin(\theta_1)sin(\theta_2)) + ir_1 r_2(sin(\theta_1)cos(\theta_2) + sin(\theta_2)cos(\theta_1))$
  $= r_1 r_2 cos(\theta_1 + \theta_2) + ir_1 r_2 sin(\theta_1 + \theta_2)$.
  (the first expansion is just from following the definition of complex multiplication and the second simplification step is from the trig result we've seen in the single variable calculus course).
  So, very conveniently: $(r_1, \theta_1)(r_2, \theta_2) = (r_1 r_2, \theta_1 + \theta_2)$!!!
- Because we visualize complex numbers as vectors, this gives us a new kind of vector product (third kind after dot and cross) where we multiply the magnitudes and add up the angles. why didn't we do it for vectors? we could've if there were a physical application → we just didn't cuz there wasn't. same reason we don't define dot and cross products for complex numbers.
- As a nice aside, notice how if we multiply two negative reals expressed as complex numbers, we get a positive. This is because the angles 180 + 180 add up to 360.
- Also, $(r_1, \theta_1)...(r_n, \theta_n) = (r_1...r_n, \theta_1 + ... + \theta_n)$
  and $(r, \theta)^n = (r^n, n\theta)$
- At $r = 1$, $(r, \theta)^n = (r^n, n\theta)$ becomes $(cos\theta + isin\theta)^n = (cos(n\theta) + isin(n\theta))$ → De Moivre's theorem.
- At $n = 2$, this gives us $(cos\theta + isin\theta)^2 = cos2\theta + sin2\theta$. But if we expand the LHS, we get $cos^2\theta - sin^2\theta + 2icos(\theta)sin(\theta)$. This shows that $sin2\theta = 2cos\theta sin\theta$ and that $cos2\theta = cos^2\theta - sin^2\theta$. note that this is not a consequence from our definition of complex multiplication. in other words, this trig identity does not depend on the definition of complex numbers. our definition just happens to be useful to prove this.
- Complex numbers are closed with respect to extracting roots. That is given, something like $i^{1/6}$ (sixth root of i), we can find it using the polar coordinate representation of complex numbers, which makes it easy to raise complex numbers to roots.
  $(1, \pi/2)^{1/6} = (1^{1/6}, 6\theta)$ → $r = 1$ and $6\theta = pi/2 + 2\pi k$. Which gives us the sixth root.
- Doing this algebraically in the $x + iy$ form would have been way way harder.
- integers are not closed with respect to division, so we invented real numbers.
- real numbers are not closed with respect to roots so we invented complex numbers.
- Functions of complex numbers map complex numbers to complex numbers.
- Functions of complex numbers to complex numbers are similar to vector functions in the sense that they are 2 variable → 2 variable mappings. so a lot of those properties (which follow from parts of the definition shared by complex numbers and vectors) are shared.
- the limit definition for complex numbers is also similar to that of vectors: $lim_{z \to a} f(z) = L$ means given $\epsilon > 0$ there exists a $\delta > 0$ such that $0 < |z - a| < \delta$ → $|f(z) - L| < \epsilon$
- Because of this definition that's similar to that of vector limit definitions, "the usual theorems hold". It means vector limit results that follow from the commonality of definitions between this and vectors also apply here.
- Now, derivatives: $f'(z_0) = lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$. Notice that Δz is a complex number. Previously, in vectors we didn't define it like this and also vector division isn't defined. so this is different. Also for directional derivatives, we know the derivative can be different in different directions. But we do not allow that here, the limit of the ratio has to be the same in all directions. Note that this is a pretty strong requirement and need not in general be true, so we have to be careful before we call complex function differentiable.
- if a complex valued function of a complex variable is differentiable, it is called analytic → just a name.

- if a function is analytic, derivatives in all directions are equal. Lets take two special derivatives in the x and y directions. Lets say we have $f(x+iy) = u + iv$. Now we're saying $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$. Applying the limit definition, we can show that if a function is analytic, then $u_x = v_y$ and $u_y = -v_x$ (there is a complex(!) name for this → its called Cauchy Riemann conditions.
- The converse can also be proved. To prove the converse, we use ideas similar to why the directional derivatives are determined by the partials with respect to x and y. If the partials exist and are continuous and are equal, knowing what happens in the x and y direction is enough for all directions and to prove that the derivative is true in all directions. Herb doesn't prove this.
- From the basic definitions of limit, just like we did for real valued functions, we can show for complex valued functions that $f'(z^n) = nz^{n-1}$ both for negative and positive n.
- One reason complex numbers are useful: just like we had a x, y plane and we could visualize curves but then we introduced vectors and that let us see things in a different way, if we have a pair of functions $u = u(x, y)$ and $v = v(x, y)$, we can introduce functions of complex numbers as mapping the x, y plane to the u, v plane. Like vectors did, this lets us view things from a different vantage point. High level speak. Lets get down to the details now.
- Lets say $f(x+iy) = u + iv$. If f is analytic, the Cauchy Riemann conditions give $u_x = v_y$ and $u_y = -v_x$. the jacobian is $u_x v_y - u_y v_x$. From the cauchy riemann conditions, the jacobian is 0. That means the pair of functions represented by the analytic function is locally invertible (because the jacobian is 0).
- However, the above does NOT mean that if it is invertible it will obey the above condition. So this complex function tool is not as strong as we'd like it to be.
- new concept: conformal mapping. a conformal mapping is one where if we have a complex function f (that is analytic and $f'(z) \neq 0$, that maps x, y to u, v, the mapping preserves the angles at which two curves intersect.



Proof:

Note that dividing a complex number involves dividing magnitudes and subtracting the angles (we can easily show this). Now, preserving the angles means, for a small enough $\Delta z_1, \Delta z_2$ the angle between them is the same as that between $\Delta w_1, \Delta w_2$. Note that the definition of the angle between the two curves is $lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$. This is the derivative!

Now since for a analytic function the derivative is the same in all directions, that would mean that $\frac{\Delta w_1}{\Delta z_1} \approx \frac{\Delta w_2}{\Delta z_2}$ for small enough magnitudes. Hence, these analytic functions are conformal.

- The reason f' shouldn't be 0 in the above is because we can re-arrange $\frac{\Delta w_1}{\Delta z_1} \approx \frac{\Delta w_2}{\Delta z_2}$ to $\frac{\Delta z_2}{\Delta z_1} \approx \frac{\Delta w_2}{\Delta z_1}$ and the RHS will be $\frac{0}{0}$ if that is the case.
- Now Laplacian equation is defined as $T_{xx} + T_{yy} = 0$. We're, for now, going to ignore what this, why this is so defined etc and just accept the definition (we will come back to that in the notes for the linear algebra course). If we map this Laplacian from the x, y plane to the u, v plane, we can prove that if the mapping is an analytic function (conformal mapping), the Laplacian equation if true in the x, y plane is true in the u, v plane as well. Its not usual, in general, for properties that hold in one plane to hold in the mapping. So if this makes some problem easier to solve in a different plane, but requires that the equation hold, we can use this mapping to solve such problems.
- we want our definitions of $e^z, sin(z)$ etc to work when z is just a real number. so we will make it the same definitions, just so it works for real numbers.
- In the real case, we saw a taylor expansion of e^x and $sin(x)$. for the complex case we haven't really defined how to make something to the power of a complex or defined sin. We will use the taylor expansion in the real case and just plug in the complex. This way, the complex definition holds when a real number is plugged in.
- the limit of a sequence definition is also the same (for any ϵ, however close, we can find a δ such that we can bring the n^{th} term in the sequence within ϵ of the limit.
- because we have the real and imaginary components, the neighborhood ϵ is not a line segment anymore but a disc (circle) in the xy plane.
- limit of a sum (limit of sum to ∞) is also defined similar to the real case.
- because the structures are are similar a lot of proofs of the results and theorems we saw in the real case can also be shown here.
- just like we had a radius of convergence in the real case, here we have a disc of convergence, where if $|z| < R$, then the series will converge (R is the radius of that disc).
- Using the two power series expansions of e^z and $sin(z)$ (they are defined by their power series expansion), we can show with some manipulation that $e^{iz} = cos(z) + i\ sin(z)$. Since we can use any real number as z in this result, substituting π for z we get $e^{i\pi} + 1 = 0$. Just from these definitions and some algebraic manipulation (which we've left out here), we've derived this pretty famous equation.

  side note: when I looked at this equation before taking this course, it seemed like a crazy thing that I could never prove. But now, it feels like that craziness is so mostly because it is defined that way (the reason its defined that way happens to be because the definition can also work for reals and we can keep the real numbers a subset of the complex numbers). Maybe a lot of that super daunting looking math and equations are actually pretty easy to follow if one knows the structure and logically goes about using definitions and rules.
- Also, $log(re^{i\theta}) = log(re^{i\theta+2\pi k}) = log\ r + i(\theta + 2\pi k)$. Note that this means $log(re^{i\theta})$ is multivalued, but we define a principal value.
- With the same motivation as above, $cosh\ ix$ is defined as $cosh\ (ix) = \frac{e^{ix} + e^{-ix}}{2}$. Using this definition and expanding e^{ix} from above, we can show that $cosh(ix) = cos(x)$!
- Note that the above definitions are all power series based, so they are all differentiable, which means the functions are analytic, which means we invented a bunch of conformal mappings. These can be useful in many applications.
- By studying series in complex numbers, we've got more analytic functions and hence more conformal mappings which serve as tools in applications.

- To review, when we defined the definite integral (not the integral → that was just the antiderivative) as $\int_{x_1}^{x_2} f(x)dx = \sum lim_{\Delta x \to 0} f(x)\Delta x$. Note that from the above definition we proved that this is just $F(x_1) - F(x_1)$ where f is the derivative of F.
- For a complex number's definite integral, if we choose a similar definition ($\sum lim_{\Delta x \to 0} f(x)\Delta x$), we see that it is not clear along what direction Δz goes (there are multiple curves along which we can take that within the disc like region Δz).
- Now, to evaluate complex definite integrals, we will parametrize it. We write $\sum lim_{\Delta z \to 0} f(z)\Delta z$ as $\sum lim_{\Delta z \to 0} f(z)\frac{\Delta z}{\Delta t}\Delta t$, which is just $\int_{z_1}^{z_2} f(z(t))z'(t)dt$. this step just parametrizes z in terms of t ($z = x(t) + iy(t)$). Now its just like a line integral, except we follow the rules of complex number multiplication since in general, $z'(t)$ is a complex number. We just break up the integral into sum of real and imaginary parts.
- If the line integral is exact (the associated differential is exact), we can show that the value of the line integral does not depend on the path and it depends only on the endpoints of integration. Similar to that argument we can show quite easily that if a complex function f is analytic, then $\int_{z_1}^{z_2} f(z)dz$ is independent of the curve C. In particular, that over a closed curve is 0.
- Using the first fundamental theorem of integral calculus for reals, and because we have decomposed definite integrals of complex numbers into definite integrals of reals, we can show that if f is analytic, $\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1)$.
- Note that, while proving the independence we assumed that f is analytic at every point in the region. If this is not true, then the above conclusions don't hold.
- Herb makes a point about complex integrals over closed curve "donuts" that we have left out here.