vector arithmetic

- mathematical structures are everywhere in math and other math related subjects. a mathematical structure is like a game → it consists of definitions, rules and objectives. strategy (and thought) is used to use the definitions and rules to get to the objectives. strategy involves logic, and definitions and rules need to be memorized.
  definitions, rules → logic machine → inescapable conclusions
- definitions are made up in a way so that they are useful the way they are defined.
- Intuitively, the meaning of a vector is a quantity that has magnitude, direction and sense. direction, here, can be two sided like left and right. Sense disambiguates those two. we define these two separate terms to give us more vocabulary.
- arrow:vector::line:scalar. The vector does not depend on the arrow any more than a scalar depends on a length. the arrow is simply a geometric representation that's convenient.
- an arithmetic structure must have rules and definitions, so lets make them up:
  - vectors are equal if they are equal in magnitude, parallel and have the same sense.
  - notice that the vectors don't have to co-incide. why? just because its a definition and we can define it however we want and it turns out that this is a convenient definition.
- we make up definitions to be our servants. based on convenience and things that we know from real world physical situations and phenomena.
- addition of a vector: to add two vectors, take a one vector and place it so its tail is at the head of the previous vector. now join the tail of the first to the head of the second and you get the resultant vector.
- a vector can be conveniently placed in a 2-D cartesian coordinate system and the previous definitions apply in a straightforward manner.
- scalar multiplication: $c\vec{v}$ has magnitude $|c|$ times the magnitude of $\vec{v}$ and is in the same or opposite sense as c > 0 or c < 0. If c = 0, we get the zero vector.
- There is quite a few parallels between vector arithmetic and regular arithmetic. Likewise we will also develop vector algebra and vector calculus with definitions and results that follow from said definitions.
- Two more rules that seem right and hence we'll formalize them as axioms:
  - $\vec{A}+\vec{B}=\vec{B}+\vec{A}$. See how this is true (almost follows?) based on the definition of addition.
  - Also, similarly, $(\vec{A}+\vec{B})+\vec{C}=\vec{A}+(\vec{B}+\vec{C})$
  - Also, $\vec{A}+\vec{0}=\vec{A}$
  - and because it will be convenient, $|\vec{0}|$ = 0.
  - Let's define $-\vec{A}$ to be the vector with the same magnitude but opposite sense of $\vec{A}$. This definition is convenient because $\vec{A}+(-\vec{A})$ will now become $\vec{0}$
  - Note that $\vec{0}$ is not the same as 0. The first is a vector, the second is a scalar.
  - We define $\vec{A}-\vec{B}$ as $\vec{A}+(-\vec{B})$.
- Again, these are all just definitions and rules (axioms) that make some things convenient in the real world. eg. in physics. this is all just a mathematical structure.
- a 3-D vector is also just an arrow, but drawn thru 3-D space. the usual convention is to draw the $\vec{x}, \vec{y}$ and $\vec{z}$ axes so that if $\vec{x}$ is turned into $\vec{y}$, then the $\vec{z}$ points in the same sense as a right handed screw.
- We can break up a 3-D vector into components in the three directions like $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, where the addition is vector addition. $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors in the respective directions. We can also show that $|\vec{A}|=\sqrt{a_1^2+a_2^2+a_3^2}$.

- It can be shown that $\vec{A} + \vec{B} = (a_1 = b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$
- In 2-D polar coordinates, we use the (r, θ) representation of a vector.
- From law of cosines (easy to prove with a simple triangle), $|\vec{r}_2 - \vec{r}_1| = \sqrt{r_1^2 + r_2^2 - 2r_1r_2cos(\theta_2 - \theta_1)}$. Look how much more messy this is for vector addition/subtraction.
- name versus concept: the vector axioms do not change. the computational recipes **do** depend on the coordinate system we decide to choose.
- Structurally, the vector axioms are same in 2-D and 3-D. The geometric interpretation may be different/harder in 3-D space than in 2-D space. We can project the structural definitions and rules into cartesian and polar coordinates and make some things convenient. but remember, even when you can't project it onto geometry (like when we go beyond 3-D), the structural axioms remain. we just don't have the convenience of projecting it into geometry.
- Dot product: $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|cos(\theta)$, where θ is the angle when you turn the first vector into the second. Note that by this definition, the ordering matters! Notice that this is just a definition. Now how do we find θ for 3-D? its more difficult. It's even more difficult for >3-D.
- Using the law of triangles, we can easily show that $\vec{A} \cdot \vec{B} = \frac{|\vec{A}-\vec{B}|^2 - |\vec{A}|^2 - |\vec{B}|^2}{2}$. again, this stuff becomes convenient in cartesian coordinates but is true anyway.
- It can shown with some straightforward manipulation that if written in cartesian coordinates, this becomes $\vec{A} \cdot \vec{B} = a_1b_1 + a_2b_2 + a_3b_3$.
- The projection of $\vec{A}$ onto $\vec{B}$ is $\vec{A} \cdot \vec{u}_B$, where $\vec{u}_B$ is the unit vector in the direction of $\vec{B}$.
- It is interesting that 3-D geometry and something "directional cosines" existed before vector techniques, but vector techniques greatly simplified them.
- We can show the following properties pretty easily:
  - $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$ (commutative property)
  - $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$ (distributive property)
  - $c\vec{A} \cdot \vec{B} = c(\vec{A} \cdot \vec{B})$
- If $\vec{A}$ is perpendicular to $\vec{B}$, then $\vec{A} \cdot \vec{B} = 0$.
- the cross product $\vec{A} \times \vec{B}$ is a vector such that the magnitude is $|\vec{A}||\vec{B}|sin(\theta)$. the direction is perpendicular to the plane determined by $\vec{A}$ and $\vec{B}$. If they are parallel, they do not determine a plane and cross product is $\vec{0}$ in that case. the sense is determined by the right hand screw rule, when the first vector is rotated into the second, thru the *smaller* of the two angles.
- cross product turns out to be an arithmetic nuisance since a lot of the nice properties of multiplication or dot product don't hold.
  - Obviously, $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$
  - $\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$
  - $\vec{A} \times (\vec{B} \times \vec{C})$ is parallel to the plane determined by $\vec{B}$ and $\vec{C}$.
- The distributive property does turn out to hold:
  $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{A} \times B + \vec{A} \times \vec{C}$
- Turns out, $\vec{A} \times \vec{B} = (a_2b_3 - a_3b_1)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$, which is easily expressable as a determinant.
- Analytic geometry, graphs, lines in 2-D and 3-D and planes act as tools to deepen our understanding, build some sense of intuition and act as a nice area for applications.
- Planes:surface::lines:curves → the first is a tangent to the second.
- A point and normal to the plane fully describes a plane. Given a point $P_0 = (x_0, y_0, z_0)$ and a vector $\vec{N}$ that specifies the normal, a point p lies on the plane if and only if $\vec{N} \cdot \vec{P_0p} = 0$. That is, if a point lies on the plane, then the dot product would be zero AND if the dot product is zero, then it must lie on the plane. So, $(A, B, C) \cdot (x - x_0, y - y_0, z - z_0) = 0$ → $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

- Given a plane equation $Ax + By + Cz + k = 0$, we can figure out the normal vector cuz its the coefficients A, B, C. Notice that the k moves the plane around parallel to itself, but not the "direction" of it. So A, B, C really determine a family of parallel planes.
- Notice that the 2 variable equation $ax + by = k$ and $ax + by + cz = k$. Notice that these are both "linear" looking things. In general, even if we have more than 3 variables, if the variables are all of degree 1, we call the equations linear (even though we can't visualize them).
- A plane has two degrees of freedom. What does degrees of freedom mean? It means given a plane equation $ax + by + cz = k$, we can pick two out of 3 variables. Picking two fixes the third variable. So the plane has two degrees of freedom. So by picking these different assignment of values to the degree of freedom variables, we get different points on the plane/line.
- To find the equation of a 3-D line, we can use the fact that given a vector, that's parallel to the line and given a point on the line, the line is fully determined, and its equation is $\vec{P_0 p} = t\vec{V}$. That is, $(x - x_0, y - y_0, z - z_0) = (tA, tB, tC)$ and so $\frac{x-x_0}{A} = \frac{y-y_0}{B} = \frac{z-z_0}{C}$. Given the equation of a line, we see that the inverse of the coefficients of the variables x, y and z give us the vector that's parallel the line. If either A, B or C is zero, then the ramifications are pretty straightforward. And given the equation, we know a point that passes thru it → x_0, y_0 and z_0.
- For a 3-D line, there is only 1 degree of freedom. You pick one of the variables, the other two get set.