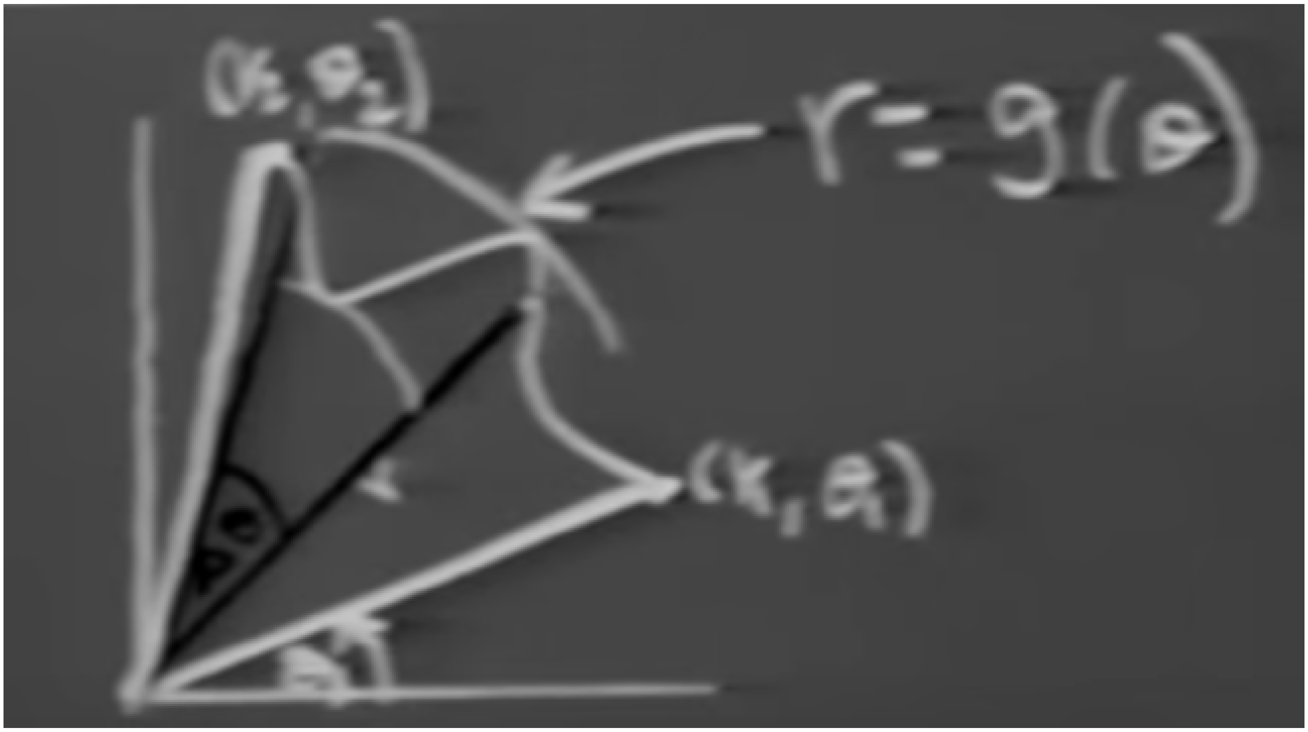
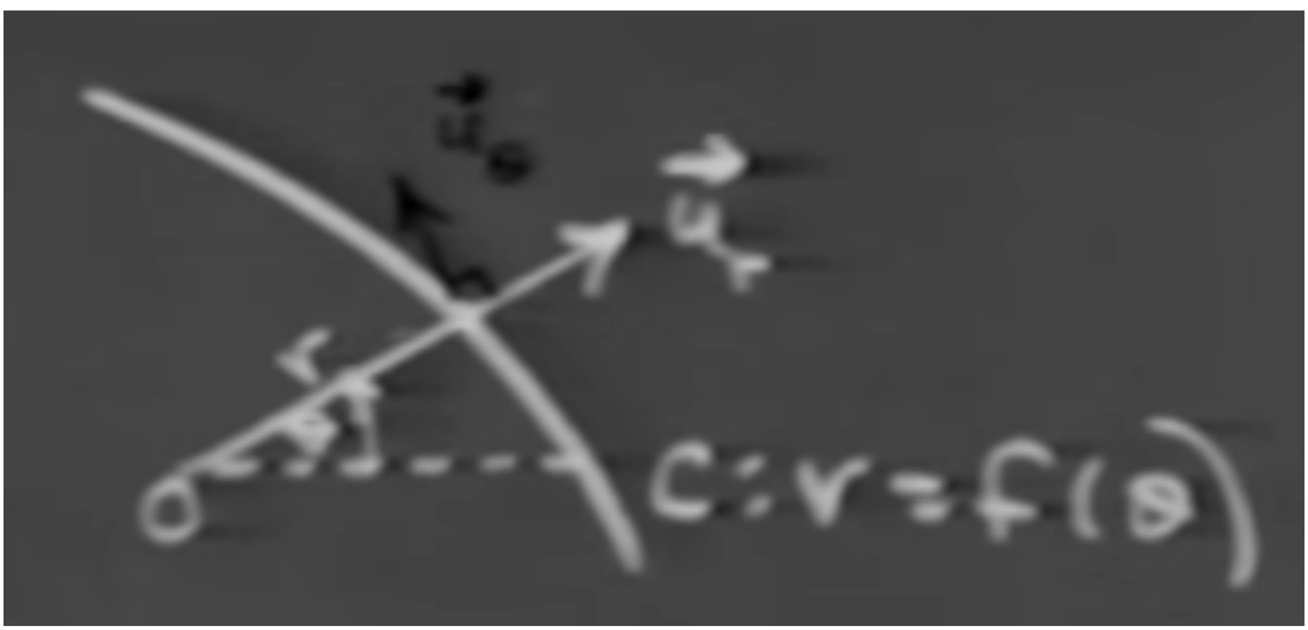
vector calculus

- like we went from mathematical structures of arithmetic of scalars to functions and calculus, we will go from arithmetic of vectors to functions and calculus of vectors.
- There are four cases involving vector functions → input scalar, output vector, input vector and output vector and input vector and output scalar, input scalar and output scalar. we discussed pretty extensively the input scalar and output scalar case the the single variable calculus course. for the rest of this course, we'll talk about the other 3.
- limit definition for functions from scalar to vector: we say that $lim_{x \to a} \vec{f}(x) = \vec{L}$ means that given any $\epsilon > 0$, we can find a $\delta > 0$ such that $0 < |x - a| < \delta \to |\vec{f}(x) - \vec{L}| < \epsilon$. Note that it is framed in terms of vector magnitude.
- Turns out, a lot of convenient properties hold:
  - $\lim_{x \to a}(\vec{f}(x) + \vec{g}(x)) = \lim_{x \to a} \vec{f}(x) + \lim_{x \to a} \vec{g}(x)$
  - $\lim_{x \to a}(\vec{f}(x) \times \vec{g}(x)) = \lim_{x \to a} \vec{f}(x) \times \lim_{x \to a} \vec{g}(x)$
  - $\lim_{x \to a}(\vec{f}(x) \cdot \vec{g}(x)) = \lim_{x \to a} \vec{f}(x) \cdot \lim_{x \to a} \vec{g}(x)$
  - $\lim_{x \to a}(f(x)\vec{g}(x)) = \lim_{x \to a} f(x) \lim_{x \to a} \vec{g}(x)$
- the derivative of vectors is defined like this:
  - $f'(x) = lim_{\Delta x \to 0} \frac{\vec{f}(x + \Delta x) - \vec{f}(x)}{\Delta x}$
  - Notice the **derivative is a vector**.
- We can show that the additive rule, product rule, chain rule, quotient rule all still hold.
- The unit tangent vector for a vector is just the derivative of the vector divided by the magnitude of the derivative. (just like you find unit vectors)
- There is a small discussion on tangents and normals to vector functions and their relation with motion in a plane. we have left that out here → nothing too complicated, watch the second lecture in part 2 if you're interested.
- An alternative representation of points on the cartesian coordinate is to use a polar coordinate so like (r, θ), where $r^2 = x^2 + y^2$ and $tan(\theta) = \frac{y}{x}$. Note that a given point can be written in many ways. Given (x, y), we can give it a certain θ and r. But there are infinite ways of describing that point. For example, $(r, \theta + 2\pi k)$, $(-r, \theta + \pi)$ etc. So given that $(r_1, \theta_1) = (r_2, \theta_2)$, it just means they both describe the same position, and not necessarily $r_1 = r_2$ and $\theta_1 = \theta_2$.
- Another big complication: Given a polar equation like $r = sin^2(\theta)$, and given a point r_1, θ_1, we cannot simply plug them in and see if it satisfies the equation. Because a point has infinitely many r, θ descriptions, it is possible that some descriptions satisfy and equation and some don't.
- We can show that area of a curve expressed in polar coordinates is derived as under:



- The lower bound is $\frac{\theta}{2\pi}\pi r^2$ and the upper bound is $\frac{\Delta\theta}{2\pi}\pi R^2$. So $\frac{\Delta\theta}{2\pi}\pi r^2 < A < \frac{\Delta\theta}{2\pi}\pi R^2$. Cancelling, $\frac{1}{2}\pi r^2 < \frac{\Delta A}{\Delta\theta} < \frac{1}{2}\pi R^2$. Now we can show that, $A = \int_{\theta_1}^{\theta_2}\frac{1}{2}r^2 d\theta$, provided r is a continuous function of θ.
- Also notice how the shape of the area this formula gives is different from the area found using cartesian coordinates.

* Unit vectors in the tangent and normal direction:



- $\vec{u}_r = cos(\theta)\hat{i} + sin(\theta)\hat{j}$ and $\vec{u}_n = cos(\theta + 90^\circ)\hat{i} + sin(\theta + 90^\circ)\hat{j}$ (because θ is the angle made by the r with the horizontal axis and they are both of magnitude 1)
- Notice that $\vec{u}_n = \frac{dr}{d\theta}$. (which is $\vec{u}_r$ rotated by 90°). We see that further differentiation gives us $-\vec{u}_r$. So it actually turns out that each differentiation rotates by 90° in the anti clockwise (positive)

direction.

- Motion in polar coordinates: $\vec{R} = r\vec{u}_r$ and so $\frac{d\vec{R}}{dt} = \frac{dr}{dt}\vec{u}_r + r\frac{d\vec{u}_r}{dt}$ (by product rule)