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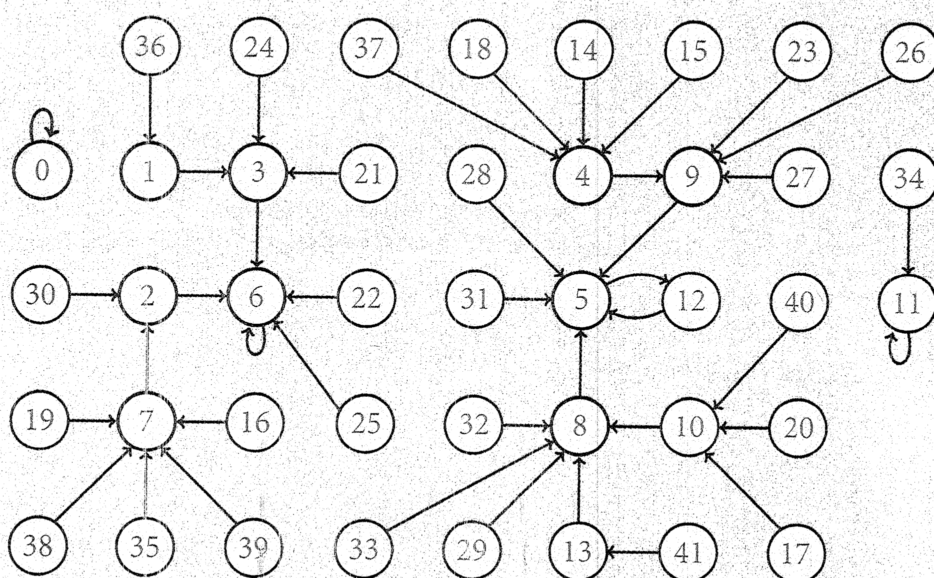

- The House of Representatives meets Lagrange's Identity
- Think cycloids are fun? Have a go at polycycloids!
- The orbits of the sum of remainders map
- The 2020 Carl B. Allendoerfer awards

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CONTENTS

ARTICLES

PROBLEMS AND SOLUTIONS

REVIEWS

Taylc
papei
uses

The AM-GM Inequality: A Proof to Remember

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My proof does not require Calculus, just ~~sort~~ and redefines ~~arrange~~ for a monotonically decreasing function

Perhaps, the most celebrated inequality of them all is the arithmetic mean-geometric mean (AM-GM) inequality. Let

$$\mathrm{AM} = \frac{a_1 + \cdots + a_n}{n}.$$

$\sum_{i=1}^{N} a_i/N$

and

$$\mathrm{GM} = \sqrt[n]{a_1 \cdots a_n}.$$

$\prod_{i=1}^{N} a_i^{1/N}$

If $a_1, \ldots, a_n$ are nonnegative real numbers, then we have that $\mathrm{AM} \geq \mathrm{GM}$. This inequality has been described as "the fundamental theorem of the theory of inequalities, the keystone on which many other important results rest" [2]. Many mathematicians find it pedagogically useful to present applications of the inequality. However, there seems to be just a small percentage of students who actually know how to prove it.

Of course, over the years many proofs of this result have been published (see Alzer [1], Cauchy [3], or Gwanyama [4], among many other examples), but many of them use sophisticated tools such as Cauchy's forward–backward induction or the method of Lagrange multipliers. The aim of this note is to supply a more flexible and easy-to-remember proof which does not seem to have appeared in the literature.

The following proof consists of only two main (and common) ingredients: induction and basic calculus.

$a_i \in R_0^+$

Theorem. For all $a_1, \ldots, a_n \geq 0$, we have

AM ≥ GM

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n} \tag{1}$$

Proof. We first consider the base cases. The theorem is trivially true for $n = 1$. If $n = 2$, then equation (1) is equivalent to

$$(\sqrt{a_1} - \sqrt{a_2})^2 \geq 0,$$

which is obviously true.

Suppose now that the theorem holds true for $n - 1$, that is

Induction

$$\frac{a_1 + \cdots + a_{n-1}}{n-1} \geq \sqrt[n-1]{a_1 \cdots a_{n-1}}, \tag{2}$$

for appropriate choices of $a_1, \ldots, a_{n-1}$. In order to show that (1) is true for n, we will use the following (crucial) trick: Set $a = a_1$. Then define the numbers $b_1, \ldots, b_{n-1}$ by

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Shows Change of Variable / Change of index
COV/COI

the formula $b_i = \frac{a_{i+1}}{a}$, with $1 \le i \le n-1$. With this notation, the AM − GM inequality is equivalent to

$$\frac{a \cdot (1 + b_1 + \cdots + b_{n-1})}{n} \ge \sqrt[n]{a^n \cdot b_1 \cdots b_{n-1}}.$$

Sort (a)

$a_{(n)} = \max \{a_i\}$

order statistic

We can cancel a from both sides and multiply by n. We now only need to show that

$$1 + b_1 + \cdots + b_{n-1} \ge n \cdot \sqrt[n]{b_1 \cdots b_{n-1}}$$

redefine

This can be done by manipulating the induction hypothesis (2) in the following way: We change a_i to b_i, $i = 1, \ldots, n-1$, multiply by $n-1$, and add 1 to both sides. We obtain that

$$1 + b_1 + \cdots + b_{n-1} \ge 1 + (n-1) \sqrt[n-1]{b_1 \cdots b_{n-1}}$$

COV $\rightarrow$ $b_i = \frac{a_{(i)}}{a_{(n)}}$

holds true, so it suffices to show that

$$1 + (n-1) \sqrt[n-1]{b_1 \cdots b_{n-1}} \ge n \sqrt[n]{b_1 \cdots b_{n-1}}$$

$0 \le b_i \le 1$

If we set

COI $\rightarrow$ $0 \le i \le n-1$

$$x = \sqrt[n(n-1)]{b_1 \cdots b_{n-1}},$$

then the last inequality can be written in the form

$0 \le x \le 1$

$$1 + (n-1)x^n \ge nx^{n-1}$$

Finally, we let

$$f(x) = 1 + (n-1)x^n - nx^{n-1}, \ x \ge 0$$

$\forall x, 0 \le f(x) \le 1$

and prove the last inequality using the first derivative test. We can see that

$$f'(x) = n(n-1)x^{n-2}(x-1).$$

It is an easy task to see that this function has a unique critical point, namely $x = 1$. Since $f'(x) > 0$ if $x > 1$, and $f'(x) < 0$ if $0 \le x < 1$, the function has a global minimum at $x = 1$, the value $f(1) = 0$. This proves that $f(x) \ge 0$ and the inequality holds true for n. This completes the proof. ∎

Not necessary given new definition of b_i and monotonic decreasing function with domain range $0 \le x, f(x) \le 1$

REFERENCES

[1] Alzer, H. (1996). A proof of the arithmetic mean-geometric mean inequality. *Amer. Math. Monthly*. 103(7): 585. doi.org/10.1080/00029890.1996.12004790

[2] Beckenbach, E. F., Bellman R. (1961). *Inequalities*. Berlin: Springer Verlag.

[3] Cauchy, A.-L. (1821). *Cours d'analyse de l'École Royale Polytechnique, Premiére Partie, Analyse Algébrique*. Paris: Chez Debure Fréres.

[4] Gwanyama, P. W. (2004). The HM-GM-AM-QM inequalities. *Coll. Math. J.* 35(1): 47–50. doi.org/10.1080/07468342.2004.11922051

Summary. We give a short and easy-to-remember proof of the arithmetic mean-geometric mean (AM-GM inequality), which does not seem to have appeared previously in the literature.

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