Question 1. Let $\mathbb{T}^3$ be the three-dimensional unit size torus and let μ be the Φ^4_3 measure on the space of distributions $\mathscr{D}'(\mathbb{T}^3)$. Let $\psi : \mathbb{T}^3 \to \mathbb{R}$ be a smooth function that is not identically zero and let $T_\psi : \mathscr{D}'(\mathbb{T}^3) \to \mathscr{D}'(\mathbb{T}^3)$ be the shift map given by $T_\psi(u) = u + \psi$ (with the usual identification of smooth functions as distributions). Are the measures μ and $T^*_\psi \mu$ equivalent? Here, equivalence of measures is in the sense of having the same null sets and T^*_ψ denotes the pushforward under T_ψ.

Question 2. Let F be a non-archimedean local field with ring of integers $\mathfrak{o}$. Let N_r denote the subgroup of $\mathrm{GL}_r(F)$ consisting of upper-triangular unipotent elements. Let $\psi : F \to \mathbb{C}^\times$ be a nontrivial additive character of conductor $\mathfrak{o}$, identified in the standard way with a generic character of N_r. Let Π be a generic irreducible admissible representation of $\mathrm{GL}_{n+1}(F)$, realized in its ψ^{-1}-Whittaker model $\mathcal{W}(\Pi, \psi^{-1})$. Must there exist $W \in \mathcal{W}(\Pi, \psi^{-1})$ with the following property?

Let π be a generic irreducible admissible representation of $\mathrm{GL}_n(F)$, realized in its ψ-Whittaker model $\mathcal{W}(\pi, \psi)$. Let $\mathfrak{q}$ denote the conductor ideal of π, let $Q \in F^\times$ be a generator of $\mathfrak{q}^{-1}$, and set

$$u_Q := I_{n+1} + QE_{n,n+1} \in \mathrm{GL}_{n+1}(F),$$

where $E_{i,j}$ is the matrix with a 1 in the (i, j)-entry and 0 elsewhere. For some $V \in \mathcal{W}(\pi, \psi)$, the local Rankin–Selberg integral

$$\int_{N_n\backslash \mathrm{GL}_n(F)} W(\mathrm{diag}(g, 1)u_Q)V(g)|\det g|^{s-\frac{1}{2}}\, dg$$

is finite and nonzero for all $s \in \mathbb{C}$.

Question 3. Let $\lambda = (\lambda_1 > \cdots > \lambda_n \geq 0)$ be a partition with distinct parts. Assume moreover that λ is restricted, in the sense that it has a unique part of size 0 and no part of size 1. Does there exist a nontrivial Markov chain on $S_n(\lambda)$ whose stationary distribution is given by

$$\frac{F^*_\mu(x_1, \ldots, x_n; q = 1, t)}{P^*_\lambda(x_1, \ldots, x_n; q = 1, t)} \quad \text{for } \mu \in S_n(\lambda),$$

where $F^*_\mu(x_1, \ldots, x_n; q, t)$ and $P^*_\lambda(x_1, \ldots, x_n; q, t)$ are the interpolation ASEP polynomial and interpolation Macdonald polynomial, respectively? If so, prove that the Markov chain you construct has the desired stationary distribution. By "nontrivial" we mean that the transition probabilities of the Markov chain should not be described using the polynomials $F^*_\mu(x_1, \ldots, x_n; q, t)$.

Question 4. Let $p(x)$ and $q(x)$ be two monic polynomials of degree n:

$$p(x) = \sum_{k=0}^{n} a_k x^{n-k} \quad \text{and} \quad q(x) = \sum_{k=0}^{n} b_k x^{n-k},$$

where $a_0 = b_0 = 1$. Define $p \boxplus_n q(x)$ to be the polynomial

$$(p \boxplus_n q)(x) = \sum_{k=0}^{n} c_k x^{n-k}$$

where the coefficients c_k are given by the formula:

$$c_k = \sum_{i+j=k} \frac{(n-i)!(n-j)!}{n!(n-k)!} a_i b_j \quad \text{for } k = 0, 1, \ldots, n.$$

For a monic polynomial $p(x) = \prod_{i \le n}(x - \lambda_i)$, define

$$\Phi_n(p) := \left(\sum_{i \le n} \prod_{j \ne i} \frac{1}{\lambda_i - \lambda_j} \right)^2$$

and $\Phi_n(p) := \infty$ if p has a multiple root. Is it true that if $p(x)$ and $q(x)$ are monic real-rooted polynomials of degree n, then

$$\frac{1}{\Phi_n(p \boxplus_n q)} \ge \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)} \ ?$$

Question 5. Fix a finite group G. Let $\mathcal{O}$ denote an incomplete transfer system associated to an N_∞ operad. Define the slice filtration on the G-equivariant stable category adapted to $\mathcal{O}$ and state and prove a characterization of the $\mathcal{O}$-slice connectivity of a connective G-spectrum in terms of the geometric fixed points.

Question 6. For a graph $G = (V, E)$, let $G_S = (V, E(S, S))$ denote the graph with the same vertex set, but only the edges between vertices in S. Let L be the Laplacian matrix of G and let L_S be the Laplacian of G_S. I say that a set of vertices S is ϵ-light if the matrix $\epsilon L - L_S$ is positive semidefinite. Does there exist a constant $c > 0$ so that for every graph G and every ϵ between 0 and 1, V contains an ϵ-light subset S of size at least $c\epsilon|V|$?

Question 7. Suppose that Γ is a uniform lattice in a real semi-simple group, and that Γ contains some 2-torsion. Is it possible for Γ to be the fundamental group of a compact manifold without boundary whose universal cover is acyclic over the rational numbers $\mathbb{Q}$?

Question 8. A polyhedral Lagrangian surface K in $\mathbb{R}^4$ is a finite polyhedral complex all of whose faces are Lagrangians, and which is a topological submanifold of $\mathbb{R}^4$. A Lagrangian smoothing of K is a Hamiltonian isotopy K_t of smooth Lagrangian submanifolds, parameterised by $(0, 1]$, extending to a topological isotopy, parametrised by $[0, 1]$, with endpoint $K_0 = K$.

Let K be a polyhedral Lagrangian surface with the property that exactly 4 faces meet at every vertex. Does K necessarily have a Lagrangian smoothing?

Question 9. Let $n \geq 5$. Let $A^{(1)}, \dots, A^{(n)} \in \mathbb{R}^{3\times 4}$ be Zariski-generic. For $\alpha, \beta, \gamma, \delta \in [n]$, construct $Q^{(\alpha\beta\gamma\delta)} \in \mathbb{R}^{3\times 3\times 3\times 3}$ so that its (i, j, k, ℓ) entry for $1 \leq i, j, k, \ell \leq 3$ is given by $Q^{(\alpha\beta\gamma\delta)}_{ijk\ell} = \det[A^{(\alpha)}(i, :); A^{(\beta)}(j, :); A^{(\gamma)}(k, :); A^{(\delta)}(\ell, :)]$. Here $A(i, :)$ denotes the i-th row of a matrix A, and semicolon denotes vertical concatenation. We are interested in algebraic relations on the set of tensors $\{Q^{(\alpha\beta\gamma\delta)} : \alpha, \beta, \gamma, \delta \in [n]\}$.

More precisely, does there exist a polynomial map $F : \mathbb{R}^{81n^4} \to \mathbb{R}^N$ that satisfies the following three properties?

- The map F does not depend on $A^{(1)}, \dots, A^{(n)}$.
- The degrees of the coordinate functions of F do not depend on n.
- Let $\lambda \in \mathbb{R}^{n\times n\times n\times n}$ satisfy $\lambda_{\alpha\beta\gamma\delta} \neq 0$ for precisely $\alpha, \beta, \gamma, \delta \in [n]$ that are not identical. Then $F(\lambda_{\alpha\beta\gamma\delta} Q^{(\alpha\beta\gamma\delta)} : \alpha, \beta, \gamma, \delta \in [n]) = 0$ holds if and only if there exist $u, v, w, x \in (\mathbb{R}^*)^n$ such that $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$ for all $\alpha, \beta, \gamma, \delta \in [n]$ that are not identical.

Question 10. Given a d-way tensor $T \in \mathbb{R}^{n_1\times n_2\times\cdots\times n_d}$ such that the data is unaligned (meaning the tensor T has missing entries), we consider the problem of computing a CP decomposition of rank r where some modes are infinite-dimensional and constrained to be in a Reproducing Kernel Hilbert Space (RKHS). We want to solve this using an alternating optimization approach, and our question is focused on the mode-k subproblem for an infinite-dimensional mode. For the subproblem, the CP factor matrices $A_1, \dots, A_{k-1}, A_{k+1}, \dots, A_d$ are fixed, and we are solving for A_k.

Our notation is as follows. Let $N = \prod_i n_i$ denote the product of all sizes. Let $n \equiv n_k$ be the size of mode k, let $M = \prod_{i\neq k} n_i$ be the product of all dimensions except k, and assume $n \ll M$. Since the data are unaligned, this means only a subset of T's entries are observed, and we let $q \ll N$ denote the number of observed entries. We let $\mathbf{T} \in \mathbb{R}^{n\times M}$ denote the mode-k unfolding of the tensor T with all missing entries set to zero. The vec operation creates a vector from a matrix by stacking its columns, and we let $S \in \mathbb{R}^{N\times q}$ denote the selection matrix (a subset of the $N \times N$ identity matrix) such that $S^T \operatorname{vec}(\mathbf{T})$ selects the q known entries of the tensor T from the vectorization of its mode-k unfolding. We let $Z = A_d \odot \cdots \odot A_{k+1} \odot A_{k-1} \odot \cdots \odot A_1 \in \mathbb{R}^{M\times r}$ be the Khatri–Rao product of the factor matrices corresponding to all modes except mode k. We let $B = \mathbf{T}Z$ denote the MTTKRP of the tensor $\mathbf{T}$ and Khatri–Rao product Z.

We assume $A_k = KW$ where $K \in \mathbb{R}^{n\times n}$ denotes the psd RKHS kernel matrix for mode k. The matrix W of size $n \times r$ is the unknown for which we must solve. The system to be solved is

$$\big((Z \otimes K)^T SS^T (Z \otimes K) + \lambda(I_r \otimes K)\big) \operatorname{vec}(W) = (I_r \otimes K) \operatorname{vec}(B).$$

Here, I_r denotes the $r \times r$ identity matrix. This is a system of size $nr \times nr$. Using a standard linear solver costs $O(n^3r^3)$, and explicitly forming the matrix is an additional expense.

Explain how an iterative preconditioned conjugate gradient linear solver can be used to solve this problem more efficiently. Explain the method and choice of preconditioner. Explain in detail how the matrix-vector products are computed and why this works. Provide complexity analysis. We assume $n, r < q \ll N$. Avoid any computation of order N.