Solution to the 8 Students 3 Subjects Problem

1 Summary

- **a. Verdict:** I have successfully solved the problem. The final answer is 5.
- **b. Method Sketch:**
  1. **Definitions:** Let S_X be the set of the top 4 students in subject X. A student can only win an ordering starting with X if they are in S_X.
  2. **Intersection Lower Bound:** Using inclusion-exclusion on the 8 students and 3 subjects, we prove that the sum of the sizes of the pairwise intersections of survival sets, $\sum |S_X \cap S_Y|$, is at least 4.
  3. **Intersection Size 2 Lemma:** We prove that if any intersection $|S_X \cap S_Y| = 2$, then the winners w_{XYZ} and w_{YXZ} are identical. This forces a reduction in the number of potential champions.
  4. **Large Intersection Analysis:** To avoid intersections of size 2, the configuration of intersection sizes must include sets of size 3 or 4 (since the sum is ≥ 4).
     - We prove that if $|S_A \cap S_B| = 3$, at least two of the winners $\{w_{ABC}, w_{BAC}, w_{BCA}\}$ must be identical.
     - We prove that if $|S_A \cap S_B| = 4$ (i.e., $S_A = S_B$), the set of winners $\{w_{ABC}, w_{BAC}, w_{ACB}, w_{BCA}\}$ contains at most 3 distinct students.
  5. **Conclusion:** Since any valid configuration of ranks implies either an intersection of size 2 or an intersection of size ≥ 3, a duplication is unavoidable. Thus, the maximum number of potential champions is $6 - 1 = 5$.
  6. **Construction:** We provide a specific set of rankings for 8 students that yields exactly 5 distinct potential champions.

2 Detailed Solution

Let the set of students be $U = \{1, \ldots, 8\}$ and the subjects be $\{A, B, C\}$. For any subject X, let $r_X(s)$ denote the rank of student s in subject X, where 1 is the best and 8 is the worst. Let $S_X = \{s \in U \mid r_X(s) \leq 4\}$ be the set of students who survive the first round of elimination if the first subject is X. Note that $|S_X| = 4$.

For a permutation of subjects (X, Y, Z), the winner w_{XYZ} is determined as follows:

1. The set of survivors after subject X is S_X.

2. From S_X, we keep the 2 students with the best ranks in subject Y. Let this set be $U_{XY} \subset S_X$.

3. From U_{XY}, we keep the 1 student with the best rank in subject Z. This student is w_{XYZ}.

We seek to maximize the size of the set of potential champions $W = \{w_\sigma \mid \sigma \in S_3\}$.

Lemma 1. $\sum_{X<Y} |S_X \cap S_Y| \geq 4$.

Proof. Let n_k be the number of students belonging to exactly k of the sets S_A, S_B, S_C. We have the following constraints:

$$\sum_{k=0}^{3} n_k = |U| = 8,$$
$$\sum_{k=0}^{3} k n_k = |S_A| + |S_B| + |S_C| = 4 + 4 + 4 = 12.$$

The sum of the sizes of the pairwise intersections is:

$$L = |S_A \cap S_B| + |S_B \cap S_C| + |S_C \cap S_A| = \sum_{k=2}^{3} \binom{k}{2} n_k = n_2 + 3n_3.$$

Subtracting the first constraint from the second:

$$(n_1 + 2n_2 + 3n_3) - (n_0 + n_1 + n_2 + n_3) = 12 - 8,$$

that is,

$$n_2 + 2n_3 - n_0 = 4.$$

Substituting $n_2 = 4 + n_0 - 2n_3$ into the expression for L:

$$L = (4 + n_0 - 2n_3) + 3n_3 = 4 + n_0 + n_3.$$

Since $n_0 \geq 0$ and $n_3 \geq 0$, we have $L \geq 4$. $\square$

Lemma 2. *If* $|S_X \cap S_Y| = 2$*, then* $w_{XYZ} = w_{YXZ}$.

Proof. Let $I = S_X \cap S_Y$. Since $|I| = 2$, let $I = \{u, v\}$. The set U_{XY} consists of the 2 students in S_X with the best ranks in Y. The students in I belong to S_Y, so their ranks in Y are ≤ 4. Any student in $S_X \setminus S_Y$ has a rank in Y strictly greater than 4. Therefore, the students in I have strictly better Y-ranks than any student in $S_X \setminus I$. It follows that $U_{XY} = I = \{u, v\}$. By symmetry, U_{YX} consists of the 2 students in S_Y with the best ranks in X. Since students in I have X-ranks ≤ 4 and those in $S_Y \setminus S_X$ have X-ranks > 4, we have $U_{YX} = I = \{u, v\}$. The winner w_{XYZ} is the student in U_{XY} with the best Z-rank. The winner w_{YXZ} is the student in U_{YX} with the best Z-rank. Since $U_{XY} = U_{YX}$, we have $w_{XYZ} = w_{YXZ}$. $\square$

2.1 Analysis of the Maximum Number of Champions

To obtain 6 distinct champions, we must avoid the condition of Lemma 2. Thus, we require $|S_X \cap S_Y| \neq 2$ for all pairs X, Y. Given $\sum |S_X \cap S_Y| \geq 4$, and intersection sizes can only be integers from 0 to 4, any valid configuration with no 2s must contain at least one intersection of size 3 or 4. (If all were 0 or 1, the sum would be at most 3).

Case 1. *There exists an intersection of size 3.*

Proof. Without loss of generality, let $|S_A \cap S_B| = 3$. Let $I = S_A \cap S_B$. Consider the winners $w_{ABC}, w_{BAC}, w_{BCA}$.

1. U_{AB} and U_{BA} are subsets of I of size 2. (Students in I have ranks ≤ 4 in both A and B, beating those outside).

2. Let x be the student in I with the best rank in subject C.

3. If $x \in U_{AB}$, then $w_{ABC} = x$ (since x is the best in C among all of I, and $U_{AB} \subset I$).

4. If $x \in U_{BA}$, then $w_{BAC} = x$.

5. Now consider w_{BCA}. The set U_{BC} contains the top 2 students in C from S_B. Since $I \subset S_B$ and $|I| = 3$, S_B contains at most 1 student not in I. Even if that student has a better C-rank than x, x is the best among the 3 students in I. Thus x must be one of the top 2 C-ranked students in S_B. So $x \in U_{BC}$. Let $U_{BC} = \{x, z\}$.

    - If $z \notin S_A$, then $r_A(z) > 4$. Since $x \in I \subset S_A$, $r_A(x) \leq 4$. Thus x beats z in A, so $w_{BCA} = x$.
    - If $z \in S_A$, then $z \in S_A \cap S_B = I$. In this case, $U_{BC} \subset I$. This means U_{BC} consists of the top 2 C-ranked students in I. Let y be the second best C-ranked student in I. Then $U_{BC} = \{x, y\}$. w_{BCA} is the student in $\{x, y\}$ with the better A-rank. Thus $w_{BCA} \in \{x, y\}$.

We now check for duplications in the set $\{w_{ABC}, w_{BAC}, w_{BCA}\}$.

- If $w_{BCA} = x$: We need $w_{ABC} \neq x$ and $w_{BAC} \neq x$ to avoid duplication. This requires $x \notin U_{AB}$ and $x \notin U_{BA}$. Since U_{AB} and U_{BA} are size 2 subsets of I (size 3), the only subset not containing x is $I \setminus \{x\}$. Thus $U_{AB} = U_{BA} = I \setminus \{x\}$. This implies $w_{ABC} = w_{BAC}$ (both are the best C in $I \setminus \{x\}$). Duplicate found.

- If $w_{BCA} = y$ (which implies $z = y \in I$ and y beats x in A): We have winners $\{w_{ABC}, w_{BAC}, y\}$. Recall x is best C in I, y is second best. For w_{ABC} and w_{BAC} to be distinct, one must be x and the other y. (If both are x, duplicate. If neither is x, they are equal). If one is x and the other is y, then the set of winners is $\{x, y\}$. But $w_{BCA} = y$. So w_{BCA} is equal to one of the others. Duplicate found.

In all subcases, $|\{w_{ABC}, w_{BAC}, w_{BCA}\}| \leq 2$. Thus, we lose at least 1 potential champion. □

Case 2. *There exists an intersection of size 4.*

Proof. Let $|S_A \cap S_B| = 4$, which implies $S_A = S_B$. Then U_{AB} is the set of the top 2 B-ranked students in S_A. U_{BA} is the set of the top 2 A-ranked students in S_A. U_{AC} and U_{BC} are the sets of the top 2 C-ranked students in S_A (since $S_A = S_B$). Let $Y = U_{AC} = U_{BC}$. The winners w_{ACB} and w_{BCA} are selected from Y. Thus $\{w_{ACB}, w_{BCA}\} \subseteq Y$. The winners w_{ABC} and w_{BAC} are the best C-ranked students in U_{AB} and U_{BA} respectively. Note that Y contains the two students in S_A with the absolute best C-ranks. If $U_{AB} \cap Y \neq \emptyset$, then the student in the intersection is better in C than any student in $U_{AB} \setminus Y$. Thus $w_{ABC} \in Y$. Similarly, if $U_{BA} \cap Y \neq \emptyset$, then $w_{BAC} \in Y$. To obtain a winner outside Y, we must have $U_{AB} \cap Y = \emptyset$, which implies $U_{AB} = S_A \setminus Y$. Similarly, we need $U_{BA} = S_A \setminus Y$. If both hold, then $U_{AB} = U_{BA}$, which implies $w_{ABC} = w_{BAC}$. Thus, we can add at most 1 distinct winner from outside Y. The set $\{w_{ABC}, w_{BAC}, w_{ACB}, w_{BCA}\}$ is a subset of $Y \cup \{w_{ABC}, w_{BAC}\}$. Its size is at most $|Y| + 1 = 2 + 1 = 3$. We lose at least 1 potential champion. □

Conclusion. *Since any valid configuration requires an intersection of size 2, 3, or 4, and each case forces at least one duplication, the number of potential champions cannot exceed 5.*

3 Construction for 5 Champions

Let the students be $1, \ldots, 8$. Let $S_A = \{1, 2, 3, 5\}$, $S_B = \{3, 4, 7, 8\}$, $S_C = \{1, 2, 4, 6\}$. Intersections: $A \cap B = \{3\}$, $B \cap C = \{4\}$, $A \cap C = \{1, 2\}$. Since $|A \cap C| = 2$, we will have $w_{ACB} = w_{CAB}$. We construct ranks to make the other 5 distinct.

Ranks (lower is better):

- **Subject A:** $2 < 3 < 5 < 1 \leq 4 < 6 < 4 < 7, 8$. Precise order: $2, 3, 5, 1, 6, 4, 7, 8$. ($S_A = \{1, 2, 3, 5\}$).
- **Subject B:** $3 < 4 < 7 < 8 \leq 4 < 5 < 6 < 1 < 2$. Precise order: $3, 4, 7, 8, 5, 6, 1, 2$. ($S_B = \{3, 4, 7, 8\}$).
- **Subject C:** $4 < 1 < 2 < 6 \leq 4 < 5 < 3 < 7, 8$. Precise order: $4, 1, 2, 6, 5, 3, 7, 8$. ($S_C = \{1, 2, 4, 6\}$).

Winners:

1. w_{ABC}: $S_A \xrightarrow{B} \{3, 5\}$ (3 is 1st, 5 is 5th; others worse). $\xrightarrow{C}$ **5** ($r_C(5) = 5 < r_C(3) = 6$).
2. w_{BAC}: $S_B \xrightarrow{A} \{3, 4\}$ (3 is 2nd, 4 is 6th; others worse). $\xrightarrow{C}$ **4** ($r_C(4) = 1$).
3. w_{BCA}: $S_B \xrightarrow{C} \{4, 3\}$ (4 is 1st, 3 is 6th; others worse). $\xrightarrow{A}$ **3** ($r_A(3) = 2 < r_A(4) = 6$).
4. w_{CBA}: $S_C \xrightarrow{B} \{4, 6\}$ (4 is 2nd, 6 is 6th; others worse). $\xrightarrow{A}$ **6** ($r_A(6) = 5 < r_A(4) = 6$).
5. w_{ACB}: $S_A \xrightarrow{C} \{1, 2\}$ (1 is 2nd, 2 is 3rd). $\xrightarrow{B}$ **1** ($r_B(1) = 7 < r_B(2) = 8$).
6. w_{CAB}: $S_C \xrightarrow{A} \{1, 2\}$ (1 is 4th, 2 is 1st). $\xrightarrow{B}$ **1**.

The set of potential champions is $\{1, 3, 4, 5, 6\}$, which has size 5.
Therefore, the maximum possible number of potential champions is 5.