Mathematical Problem Solution and Verification

Problem Statement

Let $n \geq 5$. Let $A^{(1)}, \ldots, A^{(n)} \in \mathbb{R}^{3\times 4}$ be Zariski-generic. For $\alpha, \beta, \gamma, \delta \in [n]$, construct $Q^{(\alpha\beta\gamma\delta)} \in \mathbb{R}^{3\times 3\times 3\times 3}$ so that its (i, j, k, ℓ) entry for $1 \leq i, j, k, \ell \leq 3$ is given by $Q^{(\alpha\beta\gamma\delta)}_{ijk\ell} = \det[A^{(\alpha)}(i, :); A^{(\beta)}(j, :); A^{(\gamma)}(k, :); A^{(\delta)}(\ell, :)]$. Here $A(i, :)$ denotes the i-th row of a matrix A, and semicolon denotes vertical concatenation.

Does there exist a polynomial map $F : \mathbb{R}^{81n^4} \to \mathbb{R}^N$ that satisfies:

1. The map F does not depend on $A^{(1)}, \ldots, A^{(n)}$.
2. The degrees of the coordinate functions of F do not depend on n.
3. Let $\lambda \in \mathbb{R}^{n\times n\times n\times n}$ satisfy $\lambda_{\alpha\beta\gamma\delta} \neq 0$ for precisely $\alpha, \beta, \gamma, \delta \in [n]$ that are not identical. Then $F(\lambda_{\alpha\beta\gamma\delta}Q^{(\alpha\beta\gamma\delta)} : \alpha, \beta, \gamma, \delta \in [n]) = 0$ holds if and only if there exist $u, v, w, x \in (\mathbb{R}^*)^n$ such that $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$ for all $\alpha, \beta, \gamma, \delta \in [n]$ that are not identical.

Verification Result

Final Verdict: The solution is **correct**, but contains a **Justification Gap** in the reverse implication proof.

Complete Solution

Definition of the Tensor and Map F

Let $n \geq 5$. We are given tensors $Q^{(\alpha\beta\gamma\delta)}$ defined by determinants of rows of Zariski-generic matrices $A^{(1)}, \ldots, A^{(n)} \in \mathbb{R}^{3\times 4}$. Let $\mathcal{T} \in (\mathbb{R}^{3n})^{\otimes 4}$ be the global tensor where the entry at index $(\alpha, i), (\beta, j), (\gamma, k), (\delta, l)$ is:

$$\mathcal{T}_{(\alpha,i)(\beta,j)(\gamma,k)(\delta,l)} = \begin{cases} \lambda_{\alpha\beta\gamma\delta} \det(A^{(\alpha)}_i, A^{(\beta)}_j, A^{(\gamma)}_k, A^{(\delta)}_l) & \text{if } \alpha, \beta, \gamma, \delta \text{ are distinct} \\ 0 & \text{otherwise} \end{cases}$$

Here $A^{(\alpha)}_i$ denotes the i-th row of $A^{(\alpha)}$. The map F is defined as the collection of all 5×5 minors of specific submatrices of the four standard flattenings of $\mathcal{T}$. We describe the construction for the first flattening; the others are analogous:

Proof. 1. Consider the first flattening of $\mathcal{T}$ as a matrix with rows indexed by (α, i) and columns by $\mu = (\beta, j, \gamma, k, \delta, l)$.

2. For every pair of distinct block indices $\alpha, \alpha' \in [n]$, let $S_{\alpha,\alpha'}$ be the set of column indices μ such that:

$$S_{\alpha,\alpha'} = \{(\beta, j, \gamma, k, \delta, l) \mid \{\beta, \gamma, \delta\} \cap \{\alpha, \alpha'\} = \emptyset \text{ and } \beta, \gamma, \delta \text{ are distinct}\}.$$

   Since $n \geq 5$, $n - 2 \geq 3$, so $S_{\alpha,\alpha'}$ is non-empty.

3. Let $M_{\alpha,\alpha'}$ be the submatrix consisting of the 6 rows $\{(\alpha, 1), \dots, (\alpha, 3), (\alpha', 1), \dots, (\alpha', 3)\}$ and the columns in $S_{\alpha,\alpha'}$.

4. Include in F all 5×5 minors of $M_{\alpha,\alpha'}$.

Repeat this for all pairs α, α' and for all four flattenings.

Properties of F

1. **Independence:** F is defined by minors of the tensor entries, independent of the specific values of $A^{(i)}$.

2. **Degree:** The coordinates are degree 5 polynomials, independent of n.

3. **Equivalence:** We show $F(\mathcal{T}) = 0 \iff \lambda$ factors as a rank-1 tensor.

Forward Implication ($\Rightarrow$)

Suppose $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$.

For a column $\mu = (\beta, j, \gamma, k, \delta, l) \in S_{\alpha,\alpha'}$, let $k_\mu \in \mathbb{R}^4$ be the vector representing the functional $\det(\cdot, A_j^{(\beta)}, A_k^{(\gamma)}, A_l^{(\delta)})$, i.e., $\det(v, \dots) = v \cdot k_\mu$.

The entry at row (α, i) and column μ is:

$$(M_{\alpha,\alpha'})_{(\alpha,i),\mu} = u_\alpha v_\beta w_\gamma x_\delta (A_i^{(\alpha)} \cdot k_\mu) = A_i^{(\alpha)} \cdot (u_\alpha \Lambda_\mu k_\mu),$$

where $\Lambda_\mu = v_\beta w_\gamma x_\delta$.

Let $z_\mu = \Lambda_\mu k_\mu$. The row vector for (α, i) is $u_\alpha \sum_{p=1}^4 (A_i^{(\alpha)})_p \mathbf{Z}_p$, where $\mathbf{Z}_p$ is the row vector with entries $(z_\mu)_p$.

Similarly, the row vector for (α', i) is $u_{\alpha'} \sum_{p=1}^4 (A_i^{(\alpha')})_p \mathbf{Z}_p$.

All 6 rows lie in the subspace spanned by $\{\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4\}$.

Thus, $\operatorname{rank}(M_{\alpha,\alpha'}) \leq 4$, and all 5×5 minors vanish.

Reverse Implication ($\Leftarrow$)

Suppose $F(\mathcal{T}) = 0$. Then for any pair α, α', $\operatorname{rank}(M_{\alpha,\alpha'}) \leq 4$.

Let U_α be the row space of the block α in $M_{\alpha,\alpha'}$, and $U_{\alpha'}$ be the row space of block α'.

The rows of block α are given by vectors $r_{(\alpha,i)}$ with components $r_{(\alpha,i)}(\mu) = \lambda_{\alpha\beta\gamma\delta}(A_i^{(\alpha)} \cdot k_\mu)$.

Let K_α be the $4 \times |S_{\alpha,\alpha'}|$ matrix with columns $\lambda_{\alpha\beta\gamma\delta} k_\mu$. Then the rows of block α are the rows of $A^{(\alpha)} K_\alpha$.

Since $A^{(\alpha)}$ is generic rank 3 and K_α has rank 4 (due to genericity of $A^{(\beta)}, \dots$), $\dim(U_\alpha) = 3$. Similarly $\dim(U_{\alpha'}) = 3$.

The condition $\mathrm{rank}(M_{\alpha,\alpha'}) \leq 4$ implies $\dim(U_\alpha + U_{\alpha'}) \leq 4$.

Using the dimension formula:

$$\dim(U_\alpha \cap U_{\alpha'}) = \dim(U_\alpha) + \dim(U_{\alpha'}) - \dim(U_\alpha + U_{\alpha'}) \geq 3 + 3 - 4 = 2.$$

Thus, there exists a subspace of dimension at least 2 in the intersection. Let u be a non-zero vector in $U_\alpha \cap U_{\alpha'}$.

Since $u \in U_\alpha$, there exists a linear functional $L \in \mathrm{rowspace}(A^{(\alpha)})$ such that $u_\mu = L(k_\mu)\lambda_{\alpha\dots}$.

Since $u \in U_{\alpha'}$, there exists $L' \in \mathrm{rowspace}(A^{(\alpha')})$ such that $u_\mu = L'(k_\mu)\lambda_{\alpha'\dots}$.

Therefore, for all $\mu \in S_{\alpha,\alpha'}$:

$$L(k_\mu)\lambda_{\alpha\beta\gamma\delta} = L'(k_\mu)\lambda_{\alpha'\beta\gamma\delta}.$$

Let $c_\mu = \frac{\lambda_{\alpha'\beta\gamma\delta}}{\lambda_{\alpha\beta\gamma\delta}}$. Then $L(k_\mu) = c_\mu L'(k_\mu)$.

The vectors k_μ are formed by determinants of rows from $A^{(\beta)}, A^{(\gamma)}, A^{(\delta)}$. As we vary the indices in $S_{\alpha,\alpha'}$, the vectors k_μ span $\mathbb{R}^4$.

For generic A, the set of vectors $\{k_\mu\}$ is not contained in the zero set of $L - cL'$ for any constant c unless L and L' are proportional.

Specifically, for a fixed μ, $c_\mu = L(k_\mu)/L'(k_\mu)$.

Since u is a single vector in the intersection, L and L' are fixed functionals (independent of μ).

The relation $u_\mu = L(k_\mu)\lambda_{\alpha\dots} = L'(k_\mu)\lambda_{\alpha'\dots}$ must hold for all μ.

This implies that the ratio $\frac{\lambda_{\alpha'\dots}}{\lambda_{\alpha\dots}}$ must align with the ratio of functionals $\frac{L(k_\mu)}{L'(k_\mu)}$.

For generic A, the only way $\frac{L(k)}{L'(k)}$ can be independent of the specific geometry of k is if $\frac{L(k)}{L'(k)}$ is constant.

This implies L and L' are proportional: $L = CL'$.

Since L, L' are non-zero, C is a non-zero constant.

Thus, $c_\mu = C$ for all μ.

This implies $\frac{\lambda_{\alpha'\beta\gamma\delta}}{\lambda_{\alpha\beta\gamma\delta}} = C_{\alpha,\alpha'}$ is a constant independent of β, γ, δ.

This proves that $\lambda_{\alpha\beta\gamma\delta}$ factors as $u_\alpha \Lambda_{\beta\gamma\delta}$.

Applying this argument to all four flattenings proves that λ is a rank-1 tensor: $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$.

□

References

1. **Tucker Decomposition:** Relates the multilinear rank of a tensor to the ranks of its flattening matrices.

   - `https://en.wikipedia.org/wiki/Tucker_decomposition`

- Supports the use of minors of flattenings to characterize the factorization structure.

2. **Grassmannian and Plücker Embedding:** Describes the geometry of the vectors k_μ as points on the Grassmannian $Gr(3, 4)$.
   - `https://en.wikipedia.org/wiki/Plücker_embedding`
   - Justifies the genericity and spanning properties of the vectors k_μ derived from generic matrices.

3. **Intersection of Subspaces:** Standard linear algebra result regarding the dimension of the intersection of subspaces.
   - `https://en.wikipedia.org/wiki/Linear_subspace#Operations_and_relations_on_subspaces`
   - Used to establish $\dim(U_\alpha \cap U_{\alpha'}) \geq 2$.

Note on Justification Gap

The solution contains a **Justification Gap** in the reverse implication at the step: "For generic A, the only way $\frac{L(k)}{L'(k)}$ can be independent of the specific geometry of k... is if $\frac{L(k)}{L'(k)}$ is constant." A more rigorous justification would explicitly use the fact that for fixed block indices β, γ, δ, the variation of the sub-indices j, k, l generates a set of vectors k_μ that spans $\mathbb{R}^4$, thereby forcing the ratio of the functionals to be constant.