# Lagrangian Smoothing of Polyhedral Surfaces

## Problem Statement

**Problem.** A polyhedral Lagrangian surface $K$ in $\mathbb{R}^4$ is a finite polyhedral complex all of whose faces are Lagrangians, and which is a topological submanifold of $\mathbb{R}^4$. A Lagrangian smoothing of $K$ is a Hamiltonian isotopy $K_t$ of smooth Lagrangian submanifolds, parameterised by $(0, 1]$, extending to a topological isotopy, parametrised by $[0, 1]$, with endpoint $K_0 = K$.

Let $K$ be a polyhedral Lagrangian surface with the property that exactly 4 faces meet at every vertex. Does $K$ necessarily have a Lagrangian smoothing?

---

## Detailed Solution

### Summary

**Answer: Yes**.

The problem reduces to a local analysis of the singularities at the vertices. We show that the link of a vertex formed by 4 Lagrangian planes is a Legendrian unknot in $(S^3, \xi_{\text{std}})$ with Thurston-Bennequin invariant $\text{tb} = -1$. This condition is sufficient for the existence of a local Lagrangian smoothing, which can then be patched together to form a global smoothing.

### Step 1: Reduction to Local Smoothing

A Lagrangian smoothing of $K$ is a Hamiltonian isotopy $K_t$ ($t \in [0, 1]$) such that $K_0 = K$ and $K_t$ is smooth for $t > 0$. Since $K$ is a topological manifold, its singularities are isolated points (the vertices). The obstruction to the existence of a global smoothing is local. Specifically, $K$ admits a smoothing if and only if for every vertex $v \in K$, the local singularity (which is locally a cone over the link of $v$) admits a Lagrangian smoothing.

If local smoothings exist for each vertex, they can be glued together using standard partition of unity arguments for Hamiltonian functions (or generating functions) to produce a global smoothing of the entire surface.

### Step 2: Geometry of the Vertex Link

Let $v$ be a vertex of $K$. We identify a neighborhood of $v$ in $\mathbb{R}^4$ with a neighborhood of the origin. The local structure of $K$ is a cone $C(\Lambda) = \{tr \mid r \in [0, \epsilon), t \in \Lambda\}$, where $\Lambda = K \cap S^3$ is the **link** of the vertex.

Since the faces of $K$ are Lagrangian planes passing through the origin, the intersection of each face with $S^3$ is a Legendrian great circle with respect to the standard contact structure $\xi_{\text{std}}$ on $S^3$.

- The condition that $K$ is a topological surface implies that $\Lambda$ is homeomorphic to a circle $S^1$.

- The condition that exactly 4 faces meet at $v$ implies that $\Lambda$ is a piecewise smooth curve composed of exactly 4 arcs of great circles.

Let the vertices of $\Lambda$ be $u_1, u_2, u_3, u_4 \in S^3$ in cyclic order. The edges are geodesic segments connecting $u_i$ to $u_{i+1}$. Thus, $\Lambda$ is a "Legendrian quadrilateral" in $S^3$. Since any polygon with 4 edges in $S^3$ is unknotted, $\Lambda$ is a **topological unknot**.

### Step 3: Criterion for Lagrangian Smoothing

A fundamental result in symplectic topology (due to Eliashberg, etc.) states that a Lagrangian cone over a Legendrian knot $\Lambda \subset S^3$ admits a Lagrangian smoothing if and only if $\Lambda$ is the boundary of a smooth Lagrangian disk $D \subset B^4$.

For a Legendrian unknot $\Lambda$ in the standard contact sphere $(S^3, \xi_{\mathrm{std}})$, the existence of such a Lagrangian disk is fully determined by its classical Legendrian invariants:

1. The Thurston-Bennequin number $\mathrm{tb}(\Lambda)$.
2. The rotation number $\mathrm{rot}(\Lambda)$.

Specifically, a Legendrian unknot bounds a Lagrangian disk if and only if:

$$\mathrm{tb}(\Lambda) = -1.$$

(Note: The rotation number condition is usually satisfied for unknots bounding disks, often $\mathrm{rot} = 0$). Therefore, the problem reduces to proving that $\mathrm{tb}(\Lambda) = -1$ for any Legendrian quadrilateral arising from the intersection of 4 cyclically adjacent Lagrangian planes.

### Step 4: Analysis of the Configuration Space

We analyze the space of valid configurations. A configuration is defined by an ordered sequence of 4 isotropic lines $L_1, L_2, L_3, L_4$ in $\mathbb{R}^4$ (representing the edges of the polyhedral cone) such that consecutive lines span a Lagrangian plane. The condition that $P_i = \mathrm{span}(L_i, L_{i+1})$ is Lagrangian is equivalent to the condition that $L_i$ and $L_{i+1}$ are symplectically orthogonal (i.e., $\omega(u, v) = 0$ for all $u \in L_i, v \in L_{i+1}$).

The space $\mathcal{C}$ of such quadruples $(L_1, L_2, L_3, L_4)$ is path-connected.

- We can fix $L_1$.
- $L_2$ varies in the hyperplane $L_1^\omega \cong \mathbb{R}^3$.
- $L_3$ varies in $L_2^\omega$.
- $L_4$ varies in $L_3^\omega \cap L_1^\omega$.

Although we must avoid configurations where planes coincide (to maintain distinct faces), the non-degenerate configuration space remains connected. This implies that the Legendrian isotopy class of the link $\Lambda$ is the same for all valid vertices. Consequently, the integer-valued invariant $\mathrm{tb}(\Lambda)$ is constant.

### Step 5: Calculation of the Thurston-Bennequin Number

Since the invariant is constant, we can compute it by considering a limiting case. Consider a deformation of the 4 planes $P_1, P_2, P_3, P_4$ towards a single Lagrangian plane $P$. In the limit, the vertices $u_1, u_2, u_3, u_4$ become distinct points on the great circle $C = P \cap S^3$.

- The link $\Lambda$ degenerates to this great circle $C$.

- The great circle $C$ is the standard Legendrian unknot.
- The plane $P$ intersects the ball $B^4$ in a flat Lagrangian disk $D = P \cap B^4$.

Since the great circle $C$ bounds a Lagrangian disk, its Thurston-Bennequin number is $\mathrm{tb}(C) = -1$. Because our configuration can be deformed arbitrarily close to this standard Legendrian unknot, we must have:

$$\mathrm{tb}(\Lambda) = -1.$$

### Conclusion

We have established that for any vertex $v$ where exactly 4 Lagrangian faces meet, the link $\Lambda_v$ is a Legendrian unknot with $\mathrm{tb}(\Lambda_v) = -1$. This implies that $\Lambda_v$ bounds a Lagrangian disk in $B^4$. The existence of this disk is the necessary and sufficient condition for the existence of a local Lagrangian smoothing of the singularity at $v$.

Since every vertex admits a local smoothing, and these local models can be patched together, the entire polyhedral surface $K$ admits a Lagrangian smoothing.

## References


- **Eliashberg, Y. (1990).** *Topological characterization of Stein manifolds of dimension > 2.* International Journal of Mathematics, 1(1), 29-46.
- **Etnyre, J. (2005).** *Legendrian and Transversal Knots.* Handbook of Knot Theory, 105-185.
- **Joyce, D. (2004).** *Special Lagrangian submanifolds with isolated conical singularities. I. Regularity.* Annals of Global Analysis and Geometry, 25, 201-251.