Solution to the Diophantine Equation $x^2 + 13 = y^3$

1. Summary

a. Verdict: I have successfully solved the problem. The integer solutions (x, y) to the equation $x^2+13 = y^3$ are $(70, 17)$ and $(-70, 17)$.

b. Method Sketch:

1. **Ring of Integers and Class Group:** We identify the ring of integers of $K = \mathbb{Q}(\sqrt{-13})$ as $\mathcal{O}_K = \mathbb{Z}[\sqrt{-13}]$. To find a representative for the non-trivial class in the class group (given $h_K = 2$), we analyze the factorization of the ideal (2). We show that $(2) = \mathfrak{p}_2^2$, where $\mathfrak{p}_2 = (2, 1 + \sqrt{-13})$. By proving that there are no elements of norm 2 in $\mathcal{O}_K$, we establish that $\mathfrak{p}_2$ is not principal, thus representing the non-unit element in the class group.

2. **Coprimality:** We factor the Diophantine equation in $\mathcal{O}_K$ as $(x + \sqrt{-13})(x - \sqrt{-13}) = y^3$. We prove that the ideals $J = (x+\sqrt{-13})$ and $\bar{J} = (x-\sqrt{-13})$ are coprime. This is done by assuming a common prime ideal factor $\mathfrak{p}$ exists, which implies the underlying rational prime p must be 2 or 13 and must divide y. We then derive contradictions for both $p = 2$ and $p = 13$ using modular arithmetic on the original equation.

3. **Solving the Equation:** Since $J\bar{J}$ is a perfect cube and $J, \bar{J}$ are coprime, J must be the cube of an ideal $\mathfrak{a}$. Using the class number $h_K = 2$, we deduce that $\mathfrak{a}$ must be a principal ideal. This leads to the element equation $x + \sqrt{-13} = (a + b\sqrt{-13})^3$. Equating real and imaginary parts yields a system of equations for integers a and b, which we solve to find the final values for x and y.

2. Detailed Solution

Part 1: The Ring of Integers and a Non-Principal Ideal

Let $K = \mathbb{Q}(\sqrt{-13})$. Since -13 is square-free and $-13 \equiv 3 \pmod 4$, the ring of integers of K is:

$$\mathcal{O}_K = \mathbb{Z}[\sqrt{-13}] = \{a + b\sqrt{-13} \mid a, b \in \mathbb{Z}\}.$$

The discriminant of K is $D_K = 4(-13) = -52$.

We are given that the class number $h_K = 2$. The class group $\mathrm{Cl}(K)$ is therefore isomorphic to the cyclic group of order 2, consisting of the trivial class $[(1)]$ and a non-trivial class. We seek an ideal I belonging to this non-trivial class (i.e., a non-principal ideal).

Consider the ideal generated by the rational prime 2 in $\mathcal{O}_K$. The minimal polynomial of $\sqrt{-13}$ is $f(t) = t^2 + 13$. Reducing modulo 2:

$$t^2 + 13 \equiv t^2 + 1 \equiv (t + 1)^2 \pmod 2.$$

By the Dedekind-Kummer theorem, the ideal (2) ramifies as the square of a prime ideal $\mathfrak{p}_2$:

$$(2) = \mathfrak{p}_2^2, \quad \text{where } \mathfrak{p}_2 = (2, 1 + \sqrt{-13}).$$

The norm of this ideal is $N(\mathfrak{p}_2) = 2$.

We check if $\mathfrak{p}_2$ is principal. Suppose $\mathfrak{p}_2 = (\alpha)$ for some $\alpha = a + b\sqrt{-13} \in \mathcal{O}_K$. Then the norm of the element must equal the norm of the ideal:

$$N(\alpha) = a^2 + 13b^2 = 2.$$

Since a, b are integers:

- If $b \neq 0$, then $a^2 + 13b^2 \geq 13 > 2$.
- If $b = 0$, then $a^2 = 2$, which has no integer solution.

Thus, no such element α exists, and $\mathfrak{p}_2$ is not a principal ideal. Since the class group has only two elements, the class $[\mathfrak{p}_2]$ must be the non-trivial class. Therefore, the ideal $I = (2, 1 + \sqrt{-13})$ represents the non-unit element in the class group.

Part 2: Coprimality of $(x + \sqrt{-13})$ and $(x - \sqrt{-13})$

We wish to solve $x^2 + 13 = y^3$ for integers x, y. In $\mathcal{O}_K$, this factors as:

$$(x + \sqrt{-13})(x - \sqrt{-13}) = (y)^3.$$

Let $J = (x + \sqrt{-13})$ and $\bar{J} = (x - \sqrt{-13})$. Suppose there exists a prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ such that $\mathfrak{p} \mid J$ and $\mathfrak{p} \mid \bar{J}$. Then $\mathfrak{p}$ divides the sum and difference of the generators:

$$\mathfrak{p} \mid (J + \bar{J}) \implies \mathfrak{p} \mid 2x, \quad \mathfrak{p} \mid (J - \bar{J}) \implies \mathfrak{p} \mid 2\sqrt{-13}.$$

Consequently, $\mathfrak{p}$ divides the ideal generated by the greatest common divisor of the norms of these quantities. Specifically, $N(\mathfrak{p})$ must divide $N(2\sqrt{-13}) = 4 \cdot 13 = 52$. This implies that the rational prime p lying below $\mathfrak{p}$ (i.e., $\mathfrak{p} \cap \mathbb{Z} = (p)$) must be either 2 or 13. Furthermore, since $\mathfrak{p} \mid J$ and $J \mid (y)^3$, we have $\mathfrak{p} \mid (y)$, which implies $p \mid y$.

We analyze these two cases:

Case 1: $p = 13$. If the prime ideal factor corresponds to 13, then $13 \mid y$. Substituting into the original equation:

$$x^2 + 13 = y^3 \implies x^2 \equiv -13 \equiv 0 \pmod{13}.$$

Thus $13 \mid x$. Let $x = 13k$ and $y = 13m$ for integers k, m.

$$(13k)^2 + 13 = (13m)^3 \implies 169k^2 + 13 = 2197m^3.$$

Dividing the entire equation by 13:

$$13k^2 + 1 = 169m^3.$$

Reducing modulo 13:

$$1 \equiv 0 \pmod{13}.$$

This is a contradiction. Therefore, $13 \nmid y$, and no prime ideal above 13 can be a common factor.

Case 2: $p = 2$. If the prime ideal factor corresponds to 2, then $2 \mid y$. Thus y is even, so $y^3 \equiv 0 \pmod{8}$. The equation becomes:

$$x^2 + 13 \equiv 0 \pmod{8} \implies x^2 + 5 \equiv 0 \pmod{8} \implies x^2 \equiv -5 \equiv 3 \pmod{8}.$$

However, the quadratic residues modulo 8 are $0, 1, 4$. There is no integer whose square is congruent to 3 modulo 8. This is a contradiction. Therefore, $2 \nmid y$, and no prime ideal above 2 can be a common factor.

Since no prime ideal divides both J and $\bar{J}$, the ideals $(x + \sqrt{-13})$ and $(x - \sqrt{-13})$ are coprime in $\mathcal{O}_K$.

Part 3: Finding the Solutions

We have the ideal equation:

$$(x + \sqrt{-13})(x - \sqrt{-13}) = (y)^3.$$

Since the ideals on the left are coprime, each must be the cube of an ideal. Let $\mathfrak{a}$ be an ideal such that:

$$(x + \sqrt{-13}) = \mathfrak{a}^3.$$

We consider the class of $\mathfrak{a}$ in the class group $\mathrm{Cl}(K)$.

$$[\mathfrak{a}]^3 = [(x + \sqrt{-13})] = [(1)].$$

Since the class number $h_K = 2$, the order of any element in the class group divides 2. Thus $[\mathfrak{a}]^2 = [(1)]$. Since $[\mathfrak{a}]^3 = [(1)]$ and $[\mathfrak{a}]^2 = [(1)]$, it follows that $[\mathfrak{a}] = [(1)]$. Therefore, $\mathfrak{a}$ is a principal ideal. Let $\mathfrak{a} = (\alpha)$ for some $\alpha \in \mathcal{O}_K$. The equation becomes:

$$(x + \sqrt{-13}) = (\alpha)^3 = (\alpha^3).$$

This implies that $x + \sqrt{-13}$ and α^3 differ by a unit $u \in \mathcal{O}_K^\times$.

$$x + \sqrt{-13} = u\alpha^3.$$

The units in $\mathbb{Z}[\sqrt{-13}]$ are ± 1. Since $-1 = (-1)^3$, any unit can be absorbed into the cube term. Thus, there exists $\beta \in \mathcal{O}_K$ such that:

$$x + \sqrt{-13} = \beta^3.$$

Let $\beta = a + b\sqrt{-13}$ with $a, b \in \mathbb{Z}$.

$$x + \sqrt{-13} = (a + b\sqrt{-13})^3.$$

Expanding the right side:

$$\begin{aligned}
(a + b\sqrt{-13})^3 &= a^3 + 3a^2(b\sqrt{-13}) + 3a(b\sqrt{-13})^2 + (b\sqrt{-13})^3 \\
&= a^3 + 3a^2b\sqrt{-13} - 39ab^2 - 13b^3\sqrt{-13} \\
&= (a^3 - 39ab^2) + (3a^2b - 13b^3)\sqrt{-13}.
\end{aligned}$$

Comparing the coefficients of 1 and $\sqrt{-13}$:

$$x = a(a^2 - 39b^2), \tag{1}$$
$$1 = b(3a^2 - 13b^2). \tag{2}$$

From equation (2), since b is an integer, b must divide 1. So $b = 1$ or $b = -1$.

Case $b = 1$:

$$1 = 1(3a^2 - 13(1)^2) \implies 1 = 3a^2 - 13 \implies 3a^2 = 14.$$

Since 14 is not divisible by 3, there is no integer solution for a.

Case $b = -1$:

$$1 = -1(3a^2 - 13(-1)^2) \implies 1 = -(3a^2 - 13) \implies 3a^2 - 13 = -1 \implies 3a^2 = 12 \implies a^2 = 4 \implies a = \pm 2.$$

Now we find x using equation (1) with $b = -1$ and $a = \pm 2$. If $a = 2$:

$$x = 2(2^2 - 39(-1)^2) = 2(4 - 39) = 2(-35) = -70.$$

If $a = -2$:

$$x = -2((-2)^2 - 39(-1)^2) = -2(4 - 39) = -2(-35) = 70.$$

Thus, $x = \pm 70$. We find y using the original equation $y^3 = x^2 + 13$:

$$y^3 = (\pm 70)^2 + 13 = 4900 + 13 = 4913 \implies y = \sqrt[3]{4913} = 17.$$

(Alternatively, $y = N(\beta) = a^2 + 13b^2 = (\pm 2)^2 + 13(-1)^2 = 4 + 13 = 17$.)

The integer solutions are $(x, y) = (70, 17)$ and $(-70, 17)$.