Finite Generation of Homology of a Commutative Differential Graded Algebra over $\mathbb{F}_p$

Abstract

We prove that if A is a finitely generated commutative differential graded algebra over a finite field $\mathbb{F}_p$ of characteristic p, then its homology $H(A)$ is also a finitely generated algebra.

1 Summary

- **a. Verdict:** The homology $H(A)$ is necessarily a finitely generated algebra.
- **b. Method Sketch:**
  1. Define a subalgebra $S \subseteq A$ generated by the p-th powers of the algebra generators of A.
  2. Show that S is contained in the center of A and that every element in S is a cycle.
  3. Prove that A is a finitely generated module over S. Since S is a finitely generated commutative algebra over a field, it is Noetherian.
  4. Conclude that A is a Noetherian S-module, and since the differential d is S-linear, the kernel $Z(A)$ and image $B(A)$ are S-submodules. Hence, $H(A) = Z(A)/B(A)$ is a finitely generated S-module.
  5. Finally, since $H(A)$ is a finitely generated module over the finitely generated algebra S, $H(A)$ is itself a finitely generated algebra over $\mathbb{F}_p$.

2 Detailed Solution

Let A be a finitely generated commutative differential graded algebra (cdga) over the finite field $\mathbb{F}_p$. Let $d : A \to A$ denote the differential, satisfying $d^2 = 0$ and the graded Leibniz rule:

$$d(xy) = d(x)y + (-1)^{|x|}xd(y).$$

The algebra A is graded-commutative, i.e., $xy = (-1)^{|x||y|}yx$.

2.1 Construction of the Subalgebra S

Since A is finitely generated, let $\{x_1, \dots, x_n\}$ be a set of homogeneous generators for A over $\mathbb{F}_p$. Define S as the subalgebra of A generated by the p-th powers of these generators:

$$S = \mathbb{F}_p[x_1^p, \dots, x_n^p] \subseteq A.$$

2.2 Properties of S

We establish that S consists of central cycles.

Lemma 1. *Every element of S is a cycle, i.e., $S \subseteq \ker(d) = Z(A)$.*

Proof. For any generator x_i, the Leibniz rule implies:

$$d(x_i^p) = p x_i^{p-1} d(x_i).$$

Since the underlying field is $\mathbb{F}_p$, we have $p = 0$, and thus $d(x_i^p) = 0$. Because d is a derivation and S is generated by elements with vanishing differential, it follows that $d(s) = 0$ for all $s \in S$. □

Lemma 2. *The subalgebra S is central in A.*

Proof. We need to show that for any $s \in S$ and $a \in A$, $sa = as$.

- If $p = 2$, then the sign $(-1)^{|x||y|}$ is always 1 (since $1 \equiv -1 \pmod 2$). Thus A is strictly commutative, and S is central.
- If $p > 2$:
  - If $|x_i|$ is odd, then graded commutativity implies $x_i^2 = -x_i^2$, so $2x_i^2 = 0$. Since p is odd, 2 is invertible, implying $x_i^2 = 0$. Consequently, $x_i^p = 0$.
  - If $|x_i|$ is even, then x_i commutes with all elements of A.

  Thus, the non-zero generators of S are powers of even-degree elements (or zero), which are central in A.

Therefore, S lies in the center of A. □

2.3 A as a Finitely Generated S-Module

Proposition 1. *A is a finitely generated module over S.*

Proof. Any element in A can be written as a linear combination of monomials $x_1^{k_1} \dots x_n^{k_n}$. Using Euclidean division, we write each exponent as $k_i = q_i p + r_i$, where $0 \leq r_i < p$. Then

$$x_1^{k_1} \dots x_n^{k_n} = (x_1^p)^{q_1} \dots (x_n^p)^{q_n} \cdot (x_1^{r_1} \dots x_n^{r_n}).$$

The factor $(x_1^p)^{q_1} \dots (x_n^p)^{q_n}$ belongs to S. Since S is central, we can factor it out. Thus, A is generated as an S-module by the finite set of monomials:

$$\mathcal{B} = \{x_1^{r_1} \dots x_n^{r_n} \mid 0 \leq r_i < p \text{ for all } i\}.$$

Hence, A is a finitely generated S-module. □

2.4 Noetherian Argument

The algebra S is a finitely generated commutative algebra over the field $\mathbb{F}_p$. By Hilbert's Basis Theorem, S is a Noetherian ring. Since A is a finitely generated module over the Noetherian ring S, A is a Noetherian S-module. This implies that every S-submodule of A is finitely generated.

2.5 Homology as a Finitely Generated Algebra

Now consider the cycle module $Z(A) = \ker(d)$ and the boundary module $B(A) = \mathrm{im}(d)$. Since $S \subseteq Z(A)$ and S is central, the differential d is S-linear. Indeed, for $s \in S$ and $a \in A$,

$$d(sa) = d(s)a + (-1)^{|s|}sd(a) = 0 \cdot a + sd(a) = sd(a),$$

because if $p > 2$, non-zero s has even degree, so $(-1)^{|s|} = 1$; and if $p = 2$, signs are irrelevant.

Thus, d is an S-module homomorphism. Consequently:

1. $Z(A) = \ker(d)$ is an S-submodule of A.

2. $B(A) = \mathrm{im}(d)$ is an S-submodule of A.

Since A is a Noetherian S-module, the submodule $Z(A)$ is finitely generated over S. The homology $H(A) = Z(A)/B(A)$ is a quotient of $Z(A)$, so it is also a finitely generated S-module.

Let $\{h_1, \ldots, h_m\}$ be a set of generators for $H(A)$ as an S-module. Since S is a finitely generated $\mathbb{F}_p$-algebra (generated by $\{x_1^p, \ldots, x_n^p\}$), the combined set of generators $\{x_1^p, \ldots, x_n^p, h_1, \ldots, h_m\}$ generates $H(A)$ as an algebra over $\mathbb{F}_p$.

Theorem 1. *The homology $H(A)$ of a finitely generated commutative differential graded algebra A over $\mathbb{F}_p$ is a finitely generated algebra over $\mathbb{F}_p$.*

Proof. The argument above shows that $H(A)$ is a finitely generated module over the finitely generated commutative algebra S. Hence, $H(A)$ is itself a finitely generated algebra over $\mathbb{F}_p$. □