Laurent Series Expansion of $f(z) = \frac{1}{(z^2+1)(z-2)}$ in the Annulus $1 < |z| < 2$

Summary

- **a. Verdict:** The Laurent series expansion in the annulus $1 < |z| < 2$ has been successfully computed.
- **b. Method Sketch:**
  1. Perform partial fraction decomposition over the real numbers:
  $$f(z) = \frac{Az+B}{z^2+1} + \frac{C}{z-2}$$
  Solving yields $A = -1/5$, $B = -2/5$, and $C = 1/5$.
  2. For $|z| > 1$, expand the term containing $z^2 + 1$ in powers of $1/z$ using the geometric series with $w = -1/z^2$.
  3. For $|z| < 2$, expand the term containing $z-2$ in powers of z using the geometric series with ratio $z/2$.
  4. Combine the two series to obtain the complete Laurent series valid in the annulus $1 < |z| < 2$.

Detailed Solution

Step 1: Partial Fraction Decomposition

We perform partial fraction decomposition of $f(z)$ in the form:

$$f(z) = \frac{1}{(z^2+1)(z-2)} = \frac{Az+B}{z^2+1} + \frac{C}{z-2}.$$

Multiplying both sides by the common denominator $(z^2 + 1)(z - 2)$ gives:

$$1 = (Az + B)(z - 2) + C(z^2 + 1).$$

To determine C, evaluate at $z = 2$:

$$1 = (2A + B)(0) + C(2^2 + 1) \implies 1 = 5C \implies C = \frac{1}{5}.$$

To find A and B, expand the right-hand side and collect powers of z:

$$1 = (Az^2 - 2Az + Bz - 2B) + C(z^2 + 1) = (A + C)z^2 + (-2A + B)z + (-2B + C).$$

Comparing coefficients:

$$
\begin{aligned}
A + C = 0 &\implies A = -C = -\frac{1}{5}, \\
-2A + B = 0 &\implies B = 2A = -\frac{2}{5}.
\end{aligned}
$$

Thus, the decomposition is:

$$
f(z) = -\frac{1}{5} \cdot \frac{z+2}{z^2+1} + \frac{1}{5} \cdot \frac{1}{z-2}.
$$

Step 2: Expansion in the Annulus $1 < |z| < 2$

We expand each term into a series convergent in the given annulus.

Part A: Expansion of $-\frac{1}{5}\frac{z+2}{z^2+1}$

This term has singularities at $z = \pm i$. Since $|z| > 1$ in the annulus, we have $|z^2| > 1$, i.e., $|-1/z^2| < 1$. We rewrite:

$$
\frac{z+2}{z^2+1} = \frac{z+2}{z^2(1+z^{-2})} = (z^{-1} + 2z^{-2}) \cdot \frac{1}{1-(-z^{-2})}.
$$

Using the geometric series $\frac{1}{1-w} = \sum_{k=0}^{\infty} w^k$ with $w = -z^{-2}$:

$$
\frac{z+2}{z^2+1} = (z^{-1} + 2z^{-2}) \sum_{k=0}^{\infty} (-1)^k z^{-2k}.
$$

Distributing the sums:

$$
\frac{z+2}{z^2+1} = \sum_{k=0}^{\infty} (-1)^k z^{-2k-1} + \sum_{k=0}^{\infty} 2(-1)^k z^{-2k-2}.
$$

Multiplying by $-\frac{1}{5}$:

$$
-\frac{1}{5}\frac{z+2}{z^2+1} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{5} z^{-(2k+1)} + \sum_{k=0}^{\infty} \frac{2(-1)^{k+1}}{5} z^{-(2k+2)}.
$$

This gives the principal part of the Laurent series.

Part B: Expansion of $\frac{1}{5}\frac{1}{z-2}$

This term has a singularity at $z = 2$. Since $|z| < 2$ in the annulus, we have $|z/2| < 1$. We rewrite:

$$
\frac{1}{z-2} = \frac{1}{-2(1-z/2)} = -\frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}.
$$

Multiplying by $\frac{1}{5}$:

$$
\frac{1}{5}\frac{1}{z-2} = -\sum_{n=0}^{\infty} \frac{1}{5 \cdot 2^{n+1}} z^n.
$$

This gives the analytic part of the Laurent series.

Step 3: Final Laurent Series

Combining both parts, the Laurent series for $f(z)$ in the annulus $1 < |z| < 2$ is:

$$f(z) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{5} z^{-(2k+1)} + \sum_{k=0}^{\infty} \frac{2(-1)^{k+1}}{5} z^{-(2k+2)} - \sum_{n=0}^{\infty} \frac{1}{5 \cdot 2^{n+1}} z^n.$$

Writing the first few terms explicitly:

$$f(z) = \cdots + \frac{2}{5}z^{-4} + \frac{1}{5}z^{-3} - \frac{2}{5}z^{-2} - \frac{1}{5}z^{-1} - \frac{1}{10} - \frac{1}{20}z - \frac{1}{40}z^2 - \cdots$$