Block Diagonalization of Lorentz Transformations in $\mathbb{R}^4$

Problem Statement

Let $V = \mathbb{R}^4$ be equipped with the symmetric bilinear form $(\cdot, \cdot)$ of signature $(3, 1)$, given in the standard basis by $(x, y) = x_1 y_1 + x_2 y_2 + x_3 y_3 - x_4 y_4$. A Lorentz transformation is a linear operator $T : V \to V$ satisfying $(Tx, Ty) = (x, y)$ for all $x, y \in V$. Prove that there exists an orthonormal basis of V such that the matrix of T is block diagonal with blocks of the following types:

1. Type 1: $\begin{pmatrix} \pm 1 \end{pmatrix}$.
2. Type 2: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.
3. Type 3: $\begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix}$ or $\begin{pmatrix} \cosh\phi & \sinh\phi \\ -\sinh\phi & -\cosh\phi \end{pmatrix}$.
4. Type 4: $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$.

Summary

- **a. Verdict:** The statement is true; any Lorentz transformation T on $\mathbb{R}^4$ admits such a block diagonalization.
- **b. Method Sketch:**
  1. Decompose V into an orthogonal direct sum of T-invariant, indecomposable, non-degenerate subspaces.
  2. Analyze the eigenvalues of T: they are either real or on the unit circle.
  3. Classify the indecomposable subspaces according to their dimension and the nature of the eigenvalues:
     - Complex eigenvalues on the unit circle yield a 2-dimensional space-like rotation (Type 2).
     - Real eigenvalues $\lambda \notin \{\pm 1\}$ yield a 2-dimensional hyperbolic rotation (Type 3).
     - Eigenvalues ± 1 yield odd-dimensional blocks: 1-dimensional (Type 1) or 3-dimensional (Type 4).
  4. Combine the blocks while respecting the total dimension 4 and signature $(3, 1)$.

Detailed Solution

Let $V = \mathbb{R}^4$ with the bilinear form as above. Let $T : V \to V$ be a Lorentz transformation.

Step 1: Decomposition into Indecomposable Subspaces

A fundamental result in the theory of inner product spaces states that an isometry T on a finite-dimensional non-degenerate inner product space V admits an orthogonal decomposition

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$$

where each W_i is a non-degenerate, T-invariant subspace that is indecomposable (i.e., cannot be further decomposed into a direct sum of orthogonal, non-degenerate, T-invariant subspaces). The matrix of T is block diagonal with respect to a union of bases of the W_i.

Step 2: Eigenvalue Constraints

Since T is an isometry, if λ is an eigenvalue, then λ^{-1} is also an eigenvalue (with the same multiplicity). The characteristic polynomial is real. If $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then $\bar{\lambda}$ is also an eigenvalue. If $|\lambda| \neq 1$, then $\lambda, \bar{\lambda}, \lambda^{-1}, \bar{\lambda}^{-1}$ are four distinct eigenvalues, requiring a 4-dimensional subspace. Such a subspace would be spanned by null vectors and have signature $(2, 2)$, which is impossible in a space of signature $(3, 1)$. Hence all eigenvalues satisfy $|\lambda| = 1$; they are either real or lie on the unit circle.

Step 3: Classification of Indecomposable Subspaces

Let W be an indecomposable, non-degenerate, T-invariant subspace.

Case A: Complex Eigenvalues ($|\lambda| = 1$, $\lambda \notin \mathbb{R}$)

The minimal polynomial of $T|_W$ is irreducible quadratic, so $\dim W = 2$. The eigenvalues are $e^{\pm i\theta}$. Since they are not real, W contains no null eigenvectors, so the metric on W is definite. In signature $(3, 1)$, a definite subspace must be space-like (signature $(2, 0)$) because a time-like definite subspace would have two negative squares, exceeding the allowed one. On a Euclidean plane, an isometry with eigenvalues $e^{\pm i\theta}$ is a rotation. There exists an orthonormal basis of W such that

$$T|_W = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

This is a **Type 2** block.

Case B: Real Eigenvalues $\lambda \notin \{\pm 1\}$

Then the eigenvalues are a pair λ, λ^{-1}. Hence $\dim W = 2$. Let u be an eigenvector for λ. Then $(u, u) = (Tu, Tu) = \lambda^2 (u, u)$, and since $\lambda^2 \neq 1$, we have $(u, u) = 0$. Similarly, an eigenvector v for λ^{-1} satisfies $(v, v) = 0$. Because W is non-degenerate, $(u, v) \neq 0$. After scaling, we may assume $(u, v) = 1$. The subspace $\mathrm{span}\{u, v\}$ has signature $(1, 1)$. Choosing an orthonormal basis

$$f_1 = \frac{u + v}{\sqrt{2}}, \quad f_2 = \frac{u - v}{\sqrt{2}},$$

and writing $\lambda = \pm e^{\phi}$, we obtain

$$T|_W = \pm \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix}.$$

This is a **Type 3** block. The alternative form with negative bottom row also corresponds to an isometry on a $(1,1)$ subspace with eigenvalues ± 1 (a reflection combined with a boost).

Case C: Eigenvalues $\lambda \in \{1, -1\}$

Write $T|_W = \lambda I + N$ with N nilpotent. For an isometry on a non-degenerate space, indecomposable subspaces with eigenvalue ± 1 must have odd dimension. (If the Jordan block size were even, the restriction of the form to the kernel of N would be totally isotropic, contradicting non-degeneracy.)

Possible odd dimensions in $\mathbb{R}^4$ are 1 and 3.

- **Dimension 1:** W is spanned by a vector v with $(v, v) \neq 0$ and $Tv = \pm v$. This gives a **Type 1** block.
- **Dimension 3:** W corresponds to a Jordan block of size 3. The operator satisfies $(T - \lambda I)^3 = 0$ and $(T - \lambda I)^2 \neq 0$. A unipotent isometry of index 3 on a non-degenerate space requires signature $(2, 1)$ or $(1, 2)$. Given the total signature $(3, 1)$, and that W must have an orthogonal complement of dimension 1 (which is non-degenerate), the only possibility is that W has signature $(2, 1)$ and the complement is space-like (signature $(1, 0)$). In an appropriate basis, $T|_W$ takes the form

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix},$$

  which is a **Type 4** block.

Step 4: Global Synthesis

We must combine blocks while respecting $\dim V = 4$ and signature $(3, 1)$. The possible orthogonal decompositions are:

1. Four Type 1 blocks. (The signature depends on the signs of the norms of the basis vectors.)
2. One Type 2 block (signature $(2, 0)$) and two Type 1 blocks. The two Type 1 blocks must together give signature $(1, 1)$: one space-like and one time-like.
3. One Type 3 block (signature $(1, 1)$) and two Type 1 blocks. The two Type 1 blocks must be space-like to give signature $(2, 0)$.
4. One Type 2 block (signature $(2, 0)$) and one Type 3 block (signature $(1, 1)$). Total signature $(3, 1)$.
5. One Type 4 block (signature $(2, 1)$) and one Type 1 block (space-like, signature $(1, 0)$). Total signature $(3, 1)$.

No other combinations are possible because the maximum size of a unipotent block in signature $(3, 1)$ is 3.

Conclusion

We have shown that V decomposes into an orthogonal direct sum of T-invariant subspaces of the types described. For each subspace, there exists an orthonormal basis (by the definition of the signature of that subspace) in which the matrix of T takes the form of one of the blocks listed. Consequently, there exists an orthonormal basis of $\mathbb{R}^4$ such that the matrix of T is block diagonal with blocks of types 1, 2, 3, and 4.