Number of Subgroups of Order p^3 in an Abelian p-Group

1. Summary

a. Verdict:

I have successfully solved the problem. The final answer is

$$p^{10} + 3p^9 + 4p^8 + 6p^7 + 6p^6 + 5p^5 + 4p^4 + 3p^3 + 2p^2 + p + 1.$$

b. Method Sketch:

1. **Identify Group Structure:** The group
$$G = \mathbb{Z}/p^3\mathbb{Z} \times (\mathbb{Z}/p^2\mathbb{Z})^2 \times (\mathbb{Z}/p\mathbb{Z})^3$$
is a finite abelian p-group corresponding to the partition $\lambda = (3, 2, 2, 1, 1, 1)$.

2. **Classify Subgroups:** The subgroups of order p^3 correspond to partitions μ of the integer 3. The possible partitions are
$$\mu \in \{(3), (2, 1), (1, 1, 1)\},$$
representing subgroups isomorphic to $\mathbb{Z}/p^3\mathbb{Z}$, $\mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, and $(\mathbb{Z}/p\mathbb{Z})^3$, respectively.

3. **Apply Counting Formula:** I used the standard formula for the number of subgroups of type μ in a finite abelian p-group of type λ. This formula utilizes the conjugate partitions λ' and μ' and Gaussian binomial coefficients:
$$N_\mu(\lambda) = \prod_{k=1}^{\infty} p^{\mu'_{k+1}(\lambda'_k - \mu'_k)} \begin{bmatrix} \lambda'_k - \mu'_{k+1} \\ \mu'_k - \mu'_{k+1} \end{bmatrix}_p .$$

4. **Calculate and Sum:** I computed the number of subgroups for each of the three types:
   - For $\mu = (3)$, $N_{(3)} = p^7$.
   - For $\mu = (2, 1)$,
   $$N_{(2,1)} = p^4 \begin{bmatrix} 5 \\ 1 \end{bmatrix}_p \begin{bmatrix} 3 \\ 1 \end{bmatrix}_p .$$
   - For $\mu = (1, 1, 1)$,
   $$N_{(1,1,1)} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}_p .$$

   Finally, I expanded these expressions as polynomials in p and summed them to obtain the final result.

2. Detailed Solution

Let
$$G = \mathbb{Z}/p^3\mathbb{Z} \times (\mathbb{Z}/p^2\mathbb{Z})^2 \times (\mathbb{Z}/p\mathbb{Z})^3.$$
This is a finite abelian p-group of type λ, where the parts of λ are the exponents of the cyclic factors. Thus,
$$\lambda = (3, 2, 2, 1, 1, 1).$$

We seek the number of subgroups of G of order p^3. The isomorphism type of such a subgroup is determined by a partition μ of 3. The possible partitions are:

1. $\mu = (3)$, corresponding to the cyclic group $\mathbb{Z}/p^3\mathbb{Z}$.
2. $\mu = (2,1)$, corresponding to $\mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.
3. $\mu = (1,1,1)$, corresponding to $(\mathbb{Z}/p\mathbb{Z})^3$.

The number of subgroups of type μ in a finite abelian p-group of type λ is given by the formula:

$$N_\mu(\lambda) = \prod_{k=1}^{\infty} p^{\mu'_{k+1}(\lambda'_k - \mu'_k)} \begin{bmatrix} \lambda'_k - \mu'_{k+1} \\ \mu'_k - \mu'_{k+1} \end{bmatrix}_p,$$

where λ' and μ' are the conjugate partitions of λ and μ, and

$$\begin{bmatrix} n \\ k \end{bmatrix}_p$$

is the Gaussian binomial coefficient.

First, we determine the conjugate partition λ'. The parts of λ are $3, 2, 2, 1, 1, 1$.

$$\begin{aligned} \lambda'_1 &= |\{i : \lambda_i \geq 1\}| = 6, \\ \lambda'_2 &= |\{i : \lambda_i \geq 2\}| = 3, \\ \lambda'_3 &= |\{i : \lambda_i \geq 3\}| = 1, \\ \lambda'_k &= 0 \quad \text{for } k \geq 4. \end{aligned}$$

Thus, $\lambda' = (6, 3, 1)$.

We now calculate the number of subgroups for each type μ.

Case 1: $\mu = (3)$

The conjugate partition is $\mu' = (1, 1, 1)$.

- For $k = 1$: $\mu'_1 = 1$, $\mu'_2 = 1$, $\lambda'_1 = 6$. The term is
$$p^{1(6-1)} \begin{bmatrix} 6-1 \\ 1-1 \end{bmatrix}_p = p^5 \begin{bmatrix} 5 \\ 0 \end{bmatrix}_p = p^5.$$
- For $k = 2$: $\mu'_2 = 1$, $\mu'_3 = 1$, $\lambda'_2 = 3$. The term is
$$p^{1(3-1)} \begin{bmatrix} 3-1 \\ 1-1 \end{bmatrix}_p = p^2 \begin{bmatrix} 2 \\ 0 \end{bmatrix}_p = p^2.$$
- For $k = 3$: $\mu'_3 = 1$, $\mu'_4 = 0$, $\lambda'_3 = 1$. The term is
$$p^{0(1-1)} \begin{bmatrix} 1-0 \\ 1-0 \end{bmatrix}_p = 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}_p = 1.$$

Total for $\mu = (3)$:

$$N_{(3)} = p^5 \cdot p^2 \cdot 1 = p^7.$$

Case 2: $\mu = (2, 1)$

The conjugate partition is $\mu' = (2, 1)$.

- For $k = 1$: $\mu'_1 = 2$, $\mu'_2 = 1$, $\lambda'_1 = 6$. The term is
$$p^{1(6-2)} \begin{bmatrix} 6-1 \\ 2-1 \end{bmatrix}_p = p^4 \begin{bmatrix} 5 \\ 1 \end{bmatrix}_p.$$
- For $k = 2$: $\mu'_2 = 1$, $\mu'_3 = 0$, $\lambda'_2 = 3$. The term is
$$p^{0(3-1)} \begin{bmatrix} 3-0 \\ 1-0 \end{bmatrix}_p = 1 \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}_p.$$

- For $k=3$: $\mu'_3 = 0$, $\mu'_4 = 0$, $\lambda'_3 = 1$. The term is

$$p^0 \begin{bmatrix} 1 \\ 0 \end{bmatrix}_p = 1.$$

Total for $\mu = (2,1)$:

$$N_{(2,1)} = p^4 \begin{bmatrix} 5 \\ 1 \end{bmatrix}_p \begin{bmatrix} 3 \\ 1 \end{bmatrix}_p .$$

Expanding the Gaussian binomials:

$$\begin{bmatrix} 5 \\ 1 \end{bmatrix}_p = 1 + p + p^2 + p^3 + p^4, \quad \begin{bmatrix} 3 \\ 1 \end{bmatrix}_p = 1 + p + p^2.$$

Multiplying the polynomials:

$$(1 + p + p^2 + p^3 + p^4)(1 + p + p^2) = 1 + 2p + 3p^2 + 3p^3 + 3p^4 + 2p^5 + p^6.$$

Multiplying by p^4:

$$N_{(2,1)} = p^4 + 2p^5 + 3p^6 + 3p^7 + 3p^8 + 2p^9 + p^{10}.$$

Case 3: $\mu = (1,1,1)$

The conjugate partition is $\mu' = (3)$.

- For $k=1$: $\mu'_1 = 3$, $\mu'_2 = 0$, $\lambda'_1 = 6$. The term is

$$p^{0(6-3)} \begin{bmatrix} 6-0 \\ 3-0 \end{bmatrix}_p = \begin{bmatrix} 6 \\ 3 \end{bmatrix}_p .$$

- For $k=2$: $\mu'_2 = 0$, $\mu'_3 = 0$, $\lambda'_2 = 3$. The term is 1.

Total for $\mu = (1,1,1)$:

$$N_{(1,1,1)} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}_p = \frac{(p^6-1)(p^5-1)(p^4-1)}{(p^3-1)(p^2-1)(p-1)}.$$

Simplifying:

$$\begin{aligned} \begin{bmatrix} 6 \\ 3 \end{bmatrix}_p &= \frac{p^6-1}{p^3-1} \cdot \frac{p^4-1}{p^2-1} \cdot \frac{p^5-1}{p-1} \\ &= (p^3+1)(p^2+1)(1+p+p^2+p^3+p^4) \\ &= (p^5+p^3+p^2+1)(p^4+p^3+p^2+p+1). \end{aligned}$$

Expanding this product:

$$N_{(1,1,1)} = p^9 + p^8 + 2p^7 + 3p^6 + 3p^5 + 3p^4 + 3p^3 + 2p^2 + p + 1.$$

Total Number of Subgroups

We sum the counts from the three cases:

$$\begin{aligned} N &= N_{(3)} + N_{(2,1)} + N_{(1,1,1)} \\ &= p^7 + (p^{10} + 2p^9 + 3p^8 + 3p^7 + 3p^6 + 2p^5 + p^4) \\ &\quad + (p^9 + p^8 + 2p^7 + 3p^6 + 3p^5 + 3p^4 + 3p^3 + 2p^2 + p + 1). \end{aligned}$$

Grouping by powers of p:

$$\begin{aligned}
&p^{10} : 1 \\
&p^9 : 2 + 1 = 3 \\
&p^8 : 3 + 1 = 4 \\
&p^7 : 1 + 3 + 2 = 6 \\
&p^6 : 3 + 3 = 6 \\
&p^5 : 2 + 3 = 5 \\
&p^4 : 1 + 3 = 4 \\
&p^3 : 3 \\
&p^2 : 2 \\
&p^1 : 1 \\
&p^0 : 1
\end{aligned}$$

Thus, the total number of subgroups of order p^3 in G is:

$$\boxed{p^{10} + 3p^9 + 4p^8 + 6p^7 + 6p^6 + 5p^5 + 4p^4 + 3p^3 + 2p^2 + p + 1}.$$