When is $(X \times Y)/(\sim \times \mathrm{id}_Y)$ Homeomorphic to $(X/\sim) \times Y$?

Summary

- **a. Verdict:** No, the map f is not necessarily a homeomorphism.
- **b. Method Sketch:** The map $f : (X \times Y)/(\sim \times \mathrm{id}_Y) \to (X/\sim) \times Y$ is a continuous bijection. It is a homeomorphism iff it is open, which is equivalent to $g = \pi \times \mathrm{id}_Y : X \times Y \to (X/\sim) \times Y$ being a quotient map (with $\pi : X \to X/\sim$ the projection). We construct an explicit counterexample where g is not a quotient map:
  1. Let $Y = \mathbb{Q}$ (rationals with subspace topology from $\mathbb{R}$).
  2. Let X be the disjoint union of countably many copies of $\mathbb{Q}$, i.e., $X = \mathbb{Q} \times \mathbb{N}$ with the disjoint union topology.
  3. Define $\sim$ on X by identifying the point 0 in each copy to a single point z_0.
  4. Construct an open set $V \subseteq X \times Y$ that is saturated with respect to $\sim \times \mathrm{id}_Y$.
  5. Show that $g(V)$ is not open in $(X/\sim) \times Y$ because it contains no "rectangular" neighborhood of $(z_0, 0)$. This exploits the fact that $\mathbb{Q}$ is not locally compact: any neighborhood of 0 in $\mathbb{Q}$ contains rationals arbitrarily close to a sequence of irrationals converging to 0.

Detailed Solution

Let X and Y be topological spaces, $\sim$ an equivalence relation on X, $Z = X/\sim$ the quotient space, and $\pi : X \to Z$ the canonical quotient map. The map

$$f : (X \times Y)/(\sim \times \mathrm{id}_Y) \to Z \times Y$$

is defined by $f([x, y]) = (\pi(x), y)$. Let $Q = (X \times Y)/(\sim \times \mathrm{id}_Y)$ and $q : X \times Y \to Q$ the quotient map. Then $\pi \times \mathrm{id}_Y = f \circ q$.

Since f is a continuous bijection, f is a homeomorphism iff it is open. As q is quotient, $U \subseteq Q$ is open iff $q^{-1}(U)$ is open in $X \times Y$. Thus f is open iff for every open $W \subseteq Q$, $f(W)$ is open in $Z \times Y$. For $V = q^{-1}(W)$, V is open and saturated w.r.t. $\sim \times \mathrm{id}_Y$, and $f(W) = (\pi \times \mathrm{id}_Y)(V)$. Hence f is a homeomorphism iff $\pi \times \mathrm{id}_Y$ is a quotient map.

We now construct a counterexample where $\pi \times \mathrm{id}_Y$ is *not* a quotient map.

Construction of the Counterexample

1. Let $Y = \mathbb{Q}$ with the subspace topology from $\mathbb{R}$.
2. Let $X = \bigsqcup_{n=1}^{\infty} Y_n$, where each Y_n is a copy of Y. Formally, $X = \mathbb{Q} \times \mathbb{Z}^+$ with the topology: $U \subseteq X$ is open iff $U_n = \{x \in \mathbb{Q} \mid (x, n) \in U\}$ is open in $\mathbb{Q}$ for each $n \in \mathbb{Z}^+$.
3. Define $\sim$ on X by: $(x, n) \sim (x', m)$ iff $(x, n) = (x', m)$ or $(x = 0$ and $x' = 0)$. This identifies the element 0 from each copy Y_n to a single point. Let $z_0 = [(0, 1)] \in Z = X/\sim$.

The Set V

Let $\alpha_n = \frac{\sqrt{2}}{n}$ (irrational for all n, so $\alpha_n \notin Y$). Define

$$V = \bigcup_{n=1}^{\infty} \{((x, n), y) \in X \times Y \mid |x| < |y - \alpha_n|\}.$$

Properties of V

1. **V is open in $X \times Y$:** The space $X \times Y$ is the disjoint union of $Y_n \times Y \cong \mathbb{Q} \times \mathbb{Q}$. On the n-th component,

$$V_n = \{(x, y) \in \mathbb{Q} \times \mathbb{Q} \mid |x| < |y - \alpha_n|\}.$$

   The function $h_n(x, y) = |y - \alpha_n| - |x|$ is continuous on $\mathbb{Q} \times \mathbb{Q}$, and $V_n = h_n^{-1}((0, \infty))$. Hence V_n is open, so V is open.

2. **V is saturated with respect to $\sim \times \mathrm{id}_Y$:** The equivalence classes are $\{((x, n), y)\}$ for $x \neq 0$, and $E_y = \{((0, n), y) \mid n \in \mathbb{Z}^+\}$. If $((0, n), y) \in V$ for some n, then $|0| < |y - \alpha_n|$. Since $y \in \mathbb{Q}$ and $\alpha_n \notin \mathbb{Q}$, $|y - \alpha_n| > 0$ for all n. Thus $((0, n), y) \in V$ for all n. Therefore $E_y \subseteq V$, and V is saturated.

The Image $g(V)$ is Not Open

Let $G = (\pi \times \mathrm{id}_Y)(V)$. We show G is not open in $Z \times Y$. Note $((0, 1), 0) \in V$, so $(z_0, 0) \in G$.

If G were open, there would exist a basic open set containing $(z_0, 0)$ and contained in G. A basic open neighborhood of $(z_0, 0)$ in $Z \times Y$ has the form $O_Z \times U_Y$, where U_Y is an open neighborhood of 0 in Y, and O_Z is an open neighborhood of z_0 in Z. By the quotient topology, $\pi^{-1}(O_Z)$ is open in X and contains $\pi^{-1}(z_0) = \{(0, n) \mid n \in \mathbb{Z}^+\}$. Hence $\pi^{-1}(O_Z) = \bigcup_{n=1}^{\infty}(O_n \times \{n\})$ with each O_n an open neighborhood of 0 in $\mathbb{Q}$.

Thus, if G were open, there would exist an open neighborhood $U \subseteq \mathbb{Q}$ of 0 and a sequence of open neighborhoods $O_n \subseteq \mathbb{Q}$ of 0 such that

$$\pi\left(\bigcup_{n=1}^{\infty}(O_n \times \{n\})\right) \times U \subseteq G.$$

Taking preimages under $\pi \times \mathrm{id}_Y$, this implies

$$\bigcup_{n=1}^{\infty} (O_n \times \{n\} \times U) \subseteq V.$$

Restricting to the n-th component, we obtain for each n:

$$O_n \times U \subseteq V_n = \{(x, y) \in \mathbb{Q} \times \mathbb{Q} \mid |x| < |y - \alpha_n|\}.$$

Consequently, for all $x \in O_n$ and $y \in U$, we have $|x| < |y - \alpha_n|$. In particular,

$$\sup_{x \in O_n} |x| \leq \inf_{y \in U} |y - \alpha_n|.$$

Since O_n is an open neighborhood of 0 in $\mathbb{Q}$, there exists $\epsilon > 0$ such that $(-\epsilon, \epsilon) \cap \mathbb{Q} \subseteq O_n$. Thus $\sup_{x \in O_n} |x| \geq \epsilon/2 > 0$.

Now examine the right-hand side. As $n \to \infty$, $\alpha_n = \sqrt{2}/n \to 0$. Since U is an open neighborhood of 0 in $\mathbb{Q}$, there exists $\delta > 0$ such that $(-\delta, \delta) \cap \mathbb{Q} \subseteq U$. For sufficiently large n, $|\alpha_n| < \delta$, so $\alpha_n \in (-\delta, \delta)$. Because $\mathbb{Q}$ is dense in $\mathbb{R}$, there exist rationals $y \in (-\delta, \delta) \cap \mathbb{Q} = U$ arbitrarily close to α_n. Hence $\inf_{y \in U} |y - \alpha_n| = 0$.

For large n, we obtain the contradiction:

$$0 < \sup_{x \in O_n} |x| \leq 0.$$

Thus no such neighborhoods exist, and G is not open.

Therefore $\pi \times \mathrm{id}_Y$ is not a quotient map, and f is not a homeomorphism.

Conclusion

The map $f : (X \times Y)/(\sim \times \mathrm{id}_Y) \to (X/\sim) \times Y$ is not necessarily a homeomorphism. The counterexample uses $Y = \mathbb{Q}$ and X a disjoint union of countably many copies of $\mathbb{Q}$ with identified zeros, demonstrating that the product of a quotient map with the identity on a non-locally-compact space may fail to be a quotient map.