Counterexample to a Polynomial Inequality Involving Φ_n

Summary

a. Verdict

I have successfully solved the problem. The statement is **false**.

b. Method Sketch

1. We analyze the functional $\Phi_n(p)$. The sum inside the square, $S_n(p) = \sum_{i=1}^{n} \prod_{j \neq i} \frac{1}{\lambda_i - \lambda_j}$, is identified as the sum of residues of the rational function $1/p(z)$.

2. Using the Laurent expansion of $1/p(z)$ at infinity, we prove that $S_n(p) = 1$ for $n = 1$ and $S_n(p) = 0$ for all $n \geq 2$ (assuming distinct roots). Consequently, $\Phi_1(p) = 1$ and $\Phi_n(p) = 0$ for $n \geq 2$.

3. We test the inequality for the case $n = 1$. Compute the polynomial $p \boxplus_1 q$ explicitly for monic linear polynomials p and q, finding that $\Phi_1(p) = \Phi_1(q) = \Phi_1(p \boxplus_1 q) = 1$.

4. Substituting these values into the proposed inequality yields $1 \geq 1 + 1$, a contradiction. Thus the statement is false.

Detailed Solution

Definition 1 (The Functional Φ_n)**.** *For a monic polynomial $p(x) = \prod_{i=1}^{n}(x - \lambda_i)$ with distinct roots, define*

$$\Phi_n(p) := \left(\sum_{i=1}^{n} \prod_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \right)^2 .$$

If p has a multiple root, set $\Phi_n(p) := \infty$.

Definition 2 (The Operation $\boxplus_n$)**.** *For monic polynomials* $p(x) = \sum_{k=0}^{n} a_k x^{n-k}$ *and* $q(x) = \sum_{k=0}^{n} b_k x^{n-k}$ *with* $a_0 = b_0 = 1$*, define* $(p \boxplus_n q)(x) = \sum_{k=0}^{n} c_k x^{n-k}$ *where*

$$c_k = \sum_{i+j=k} \frac{(n-i)!\,(n-j)!}{n!\,(n-k)!} a_i b_j \quad (k = 0, 1, \ldots, n).$$

Theorem 3 (Values of Φ_n)**.** *Let* p *be a monic polynomial of degree* n *with distinct roots. Then*

$$\Phi_n(p) = \begin{cases} 1, & n = 1, \\ 0, & n \geq 2. \end{cases}$$

Proof. Write $p(x) = \prod_{i=1}^{n}(x - \lambda_i)$ with λ_i distinct. Since $p'(\lambda_i) = \prod_{j \neq i}(\lambda_i - \lambda_j)$, we have

$$S_n(p) := \sum_{i=1}^{n} \prod_{j \neq i} \frac{1}{\lambda_i - \lambda_j} = \sum_{i=1}^{n} \frac{1}{p'(\lambda_i)}.$$

Consider the partial fraction decomposition of $1/p(x)$:

$$\frac{1}{p(x)} = \sum_{i=1}^{n} \frac{1}{p'(\lambda_i)(x - \lambda_i)}.$$

Expand both sides as Laurent series near $x = \infty$.

For the left-hand side, write $p(x) = x^n + a_1 x^{n-1} + \cdots$. Then

$$\frac{1}{p(x)} = \frac{1}{x^n}\left(1 + \frac{a_1}{x} + \cdots\right)^{-1} = \frac{1}{x^n} - \frac{a_1}{x^{n+1}} + O\left(\frac{1}{x^{n+2}}\right).$$

For the right-hand side, use the geometric series expansion

$$\frac{1}{x - \lambda_i} = \frac{1}{x} \sum_{\ell=0}^{\infty} \left(\frac{\lambda_i}{x}\right)^{\ell}.$$

Thus

$$\sum_{i=1}^{n} \frac{1}{p'(\lambda_i)(x - \lambda_i)} = \frac{1}{x} \sum_{i=1}^{n} \frac{1}{p'(\lambda_i)} + \frac{1}{x^2} \sum_{i=1}^{n} \frac{\lambda_i}{p'(\lambda_i)} + \cdots.$$

Comparing the coefficients of $1/x$ on both sides:

- If $n = 1$, the left-hand side has a $1/x$ term with coefficient 1, hence $S_1(p) = 1$.
- If $n \geq 2$, the left-hand side has no $1/x$ term (the smallest power is $1/x^n$), so $S_n(p) = 0$.

Therefore $\Phi_n(p) = (S_n(p))^2$ equals 1 for $n = 1$ and 0 for $n \geq 2$. $\square$

Theorem 4 (Counterexample for $n = 1$)**.** *Let $p(x) = x + a_1$ and $q(x) = x + b_1$ be monic real-rooted polynomials of degree 1. Then*

$$\frac{1}{\Phi_1(p \boxplus_1 q)} \geq \frac{1}{\Phi_1(p)} + \frac{1}{\Phi_1(q)}$$

is false.

Proof. Compute $(p \boxplus_1 q)(x)$. Using the definition with $n = 1$:

$$\begin{aligned}
c_0 &= \frac{(1-0)!\,(1-0)!}{1!\,(1-0)!} a_0 b_0 = 1 \cdot 1 = 1, \\
c_1 &= \frac{(1-0)!\,(1-1)!}{1!\,(1-1)!} a_0 b_1 + \frac{(1-1)!\,(1-0)!}{1!\,(1-1)!} a_1 b_0 \\
&= 1 \cdot b_1 + 1 \cdot a_1 = a_1 + b_1.
\end{aligned}$$

Hence $p \boxplus_1 q = x + (a_1 + b_1)$, which is also monic of degree 1.

By the previous theorem,

$$\Phi_1(p) = 1, \quad \Phi_1(q) = 1, \quad \Phi_1(p \boxplus_1 q) = 1.$$

Substituting into the proposed inequality gives

$$\frac{1}{1} \geq \frac{1}{1} + \frac{1}{1} \quad \Longleftrightarrow \quad 1 \geq 2,$$

which is false. □

Corollary 5. *The statement*

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}$$

*does **not** hold for all monic real-rooted polynomials p, q of degree n.*

Remark 6. *For $n \geq 2$, if p and q have distinct roots, then $\Phi_n(p) = \Phi_n(q) = 0$, so both sides of the inequality become $+\infty$, and the inequality holds trivially (as $\infty \geq \infty$). However, the failure at $n = 1$ provides a definitive counterexample to the universal claim.*