Solution to the Monge-Ampère Equation Problem

1. Summary

a. Verdict: I have successfully solved the problem. The final answer is $C = -\frac{1}{2}$.

b. Method Sketch:

1. **Convexity:** We establish that the conditions $u_{xx} > 0$ and $\det(D^2u) = 1$ imply that the Hessian D^2u is positive definite, so u is strictly convex. Since $u = 0$ on the boundary, $u \le 0$ in the disk.
2. **Legendre Transform:** We introduce the Legendre transform $v(p)$ of u. The transform is defined on the image of the gradient map, $\Omega^* = \nabla u(D)$. We show that v satisfies the Monge-Ampère equation $\det(D^2v) = 1$ on Ω^*.
3. **Boundary Condition:** We derive the boundary condition for v. For a point on the boundary of the domain Ω^*, corresponding to the boundary of the disk where $u = 0$, we show that $v(p) = |p|$.
4. **Subharmonic Barrier:** We define an auxiliary function $w(p) = v(p) - \frac{1}{2}|p|^2$. Using the arithmetic-geometric mean inequality on the eigenvalues of D^2v, we prove that $\Delta v \ge 2$, which implies $\Delta w \ge 0$. Thus, w is a subharmonic function.
5. **Maximum Principle:** By the maximum principle, w attains its maximum on the boundary $\partial\Omega^*$. We calculate the maximum possible value of w on the boundary to be $\frac{1}{2}$.
6. **Conclusion:** This bound implies $v(0) \le \frac{1}{2}$. Using the relationship between the minimum of u and the value of v at the origin ($\min u = -v(0)$), we conclude that $u(x, y) \ge -\frac{1}{2}$.

2. Detailed Solution

1. **Step 1: Convexity of the Function**
   Let D be the unit disk $\{(x, y) : x^2 + y^2 < 1\}$. We are given a smooth function $u : D \to \mathbb{R}$ such that $u|_{\partial D} = 0$. The function satisfies:
   $$u_{xx} > 0, \quad u_{xx}u_{yy} - u_{xy}^2 = 1.$$
   From the second equation, $u_{yy} = (1 + u_{xy}^2)/u_{xx}$. Since $u_{xx} > 0$ and $1 + u_{xy}^2 > 0$, it follows that $u_{yy} > 0$. The trace of the Hessian is $\Delta u = u_{xx} + u_{yy} > 0$, and the determinant is $1 > 0$. Therefore, the Hessian matrix D^2u is positive definite everywhere in D. This implies that u is strictly convex.
   Since u is convex and vanishes on the boundary ∂D, by the maximum principle for convex functions, $u(x, y) \le 0$ for all $(x, y) \in D$.

2. **Step 2: The Legendre Transform**
   Let $p = (p_1, p_2)$ denote the gradient variable. We define the Legendre transform $v(p)$ by:
   $$v(p) = \sup_{x \in D}\{x \cdot p - u(x)\}.$$
   Let $\Omega^* = \nabla u(D)$ be the image of the gradient map. Since u is strictly convex and smooth, the map $\nabla u : D \to \Omega^*$ is a diffeomorphism. For $p \in \Omega^*$, the supremum is attained at a unique point $x \in D$

such that $p = \nabla u(x)$.
The standard properties of the Legendre transform give:

$$x = \nabla v(p), \quad D^2 v(p) = (D^2 u(x))^{-1}.$$

Taking the determinant:

$$\det(D^2 v(p)) = \frac{1}{\det(D^2 u(x))} = \frac{1}{1} = 1.$$

Thus, v satisfies $\det(D^2 v) = 1$ for $p \in \Omega^*$.

3. **Step 3: Boundary Condition**
We determine the value of v on the boundary $\partial\Omega^*$. The boundary of Ω^* is the image of the boundary ∂D under the gradient map ∇u.
Let $x \in \partial D$. Since D is the unit disk, the outward unit normal at x is x itself (identifying the point with the vector).
Since u is convex and $u(x) = 0$ is the maximum value of u (as $u \leq 0$), the gradient $\nabla u(x)$ must point in the direction of the outward normal. Thus:

$$p = \nabla u(x) = \lambda x$$

for some scalar $\lambda \geq 0$.
The magnitude is $|p| = \lambda |x| = \lambda$ (since $|x| = 1$).
Substituting this into the definition of v:

$$v(p) = x \cdot p - u(x) = x \cdot (\lambda x) - 0 = \lambda |x|^2 = \lambda.$$

Since $\lambda = |p|$, we have the boundary condition:

$$v(p) = |p| \quad \text{for all } p \in \partial\Omega^*.$$

4. **Step 4: Subharmonicity and the Maximum Principle**
Consider the function $w : \Omega^* \to \mathbb{R}$ defined by:

$$w(p) = v(p) - \frac{1}{2}|p|^2.$$

We compute the Laplacian of w. First, consider Δv. Let μ_1, μ_2 be the eigenvalues of the Hessian $D^2 v$. We know that $\det(D^2 v) = \mu_1 \mu_2 = 1$. Since $D^2 v$ is positive definite (inverse of a positive definite matrix), $\mu_1, \mu_2 > 0$.
Using the AM-GM inequality:

$$\Delta v = \operatorname{tr}(D^2 v) = \mu_1 + \mu_2 \geq 2\sqrt{\mu_1 \mu_2} = 2\sqrt{1} = 2.$$

Now, computing the Laplacian of w:

$$\Delta w = \Delta \left(v(p) - \frac{1}{2}(p_1^2 + p_2^2) \right) = \Delta v - (1 + 1) = \Delta v - 2.$$

Since $\Delta v \geq 2$, we have:

$$\Delta w \geq 0.$$

Thus, w is a subharmonic function on Ω^*.
By the maximum principle for subharmonic functions, the maximum of w on the bounded domain Ω^* is attained on the boundary $\partial\Omega^*$:

$$\sup_{p \in \Omega^*} w(p) = \sup_{p \in \partial\Omega^*} w(p).$$

On the boundary $\partial\Omega^*$, we have $v(p) = |p|$. Thus:

$$w(p) = |p| - \frac{1}{2}|p|^2.$$

Let $f(t) = t - \frac{1}{2}t^2$. We want to maximize this for $t = |p| \geq 0$. The function $f(t)$ is a downward parabola with maximum at $t = 1$, where $f(1) = \frac{1}{2}$.
Therefore, for any $p \in \partial\Omega^*$:

$$w(p) \leq \frac{1}{2}.$$

Consequently, for any $p \in \Omega^*$:

$$w(p) \leq \frac{1}{2}.$$

5. **Step 5: Lower Bound for u**
Since u attains its minimum inside D (as it is 0 on the boundary and negative inside), there exists a point $x_0 \in D$ where $\nabla u(x_0) = 0$. This implies that $p = 0$ is in Ω^*.
Evaluating w at $p = 0$:

$$w(0) = v(0) - \frac{1}{2}(0)^2 = v(0).$$

From the bound derived in Step 4, we have:

$$v(0) \leq \frac{1}{2}.$$

Now we relate $v(0)$ to the minimum of u. By the definition of the Legendre transform:

$$v(0) = \sup_{x \in D}\{x \cdot 0 - u(x)\} = \sup_{x \in D}\{-u(x)\} = -\inf_{x \in D} u(x).$$

Let $m = \inf_{x \in D} u(x)$. Then $v(0) = -m$.
The inequality $v(0) \leq \frac{1}{2}$ becomes:

$$-m \leq \frac{1}{2} \implies m \geq -\frac{1}{2}.$$

Thus, $u(x, y) \geq -\frac{1}{2}$ for all $(x, y) \in D$.
The constant $C = -\frac{1}{2}$ suffices. (Note: The function $u(x, y) = \frac{1}{2}(x^2 + y^2 - 1)$ satisfies all conditions and achieves this minimum, showing the bound is sharp.)

6. **Final Answer:** $\boxed{-\frac{1}{2}}$