Classification of 3-Manifolds from Gluing Two Solid Tori

Problem Statement

Let V_1 and V_2 be two solid tori, each homeomorphic to $S^1 \times D^2$. Let $L = V_1 \cup_f V_2$ be the closed, oriented 3-manifold obtained by gluing their boundaries via an orientation-preserving diffeomorphism $f : \partial V_1 \to \partial V_2$. Classify the possible fundamental groups $\pi_1(L)$ and the corresponding tuples of invariants $(H_1(L), H_2(L), H_3(L), \pi_2(L), \pi_3(L))$.

Detailed Solution

Part 1: Classification of $\pi_1(L)$

Let $V_1 \cong S^1 \times D^2$ and $V_2 \cong S^1 \times D^2$ be two solid tori. Denote by μ_i and λ_i the meridian and longitude classes in $\pi_1(\partial V_i)$ for $i = 1, 2$. Under the inclusion $j_i : \partial V_i \hookrightarrow V_i$, we have $(j_i)_*(\mu_i) = 1$ (since the meridian bounds a disk in V_i) and $(j_i)_*(\lambda_i)$ generates $\pi_1(V_i) \cong \mathbb{Z}$.

The manifold L is constructed as $L = V_1 \cup_f V_2$, where $f : \partial V_1 \to \partial V_2$ is an orientation-preserving diffeomorphism. The induced map $f_* : \pi_1(\partial V_1) \to \pi_1(\partial V_2)$ is given by a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ relative to the bases $\{\mu_1, \lambda_1\}$ and $\{\mu_2, \lambda_2\}$. That is,

$$f_*(\mu_1) = \mu_2^a \lambda_2^b,$$
$$f_*(\lambda_1) = \mu_2^c \lambda_2^d.$$

Apply the Seifert–van Kampen theorem. The fundamental group $\pi_1(L)$ is the amalgamated free product of $\pi_1(V_1)$ and $\pi_1(V_2)$ over $\pi_1(\partial V_1)$ with relations imposed by f. Since $\pi_1(V_i) = \langle \lambda_i \mid \emptyset \rangle$ and the meridians are trivial in $\pi_1(V_i)$, we obtain:

$$\pi_1(V_1) = \langle \lambda_1 \rangle,$$
$$\pi_1(V_2) = \langle \lambda_2 \rangle.$$

The gluing yields the relations:

$$\mu_1 = f_*(\mu_1) \quad \text{and} \quad \lambda_1 = f_*(\lambda_1).$$

But $\mu_1 = 1$ in $\pi_1(V_1)$ and $\mu_2 = 1$ in $\pi_1(V_2)$. Substituting these and the expressions for f_*, we get:

$$1 = \mu_2^a \lambda_2^b = \lambda_2^b,$$
$$\lambda_1 = \mu_2^c \lambda_2^d = \lambda_2^d.$$

Thus $\pi_1(L)$ has presentation

$$\langle \lambda_1, \lambda_2 \mid \lambda_2^b = 1,\ \lambda_1 = \lambda_2^d \rangle \cong \langle \lambda_2 \mid \lambda_2^b = 1 \rangle.$$

Therefore,

$$\pi_1(L) \cong \begin{cases} \mathbb{Z}, & \text{if } b = 0, \\ \mathbb{Z}_{|b|}, & \text{if } b \neq 0. \end{cases}$$

For any integer b, we can choose $a = 1$, $d = 1$, and $c = 0$; then the matrix $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ belongs to $SL(2, \mathbb{Z})$. Hence every integer b is realizable, and the set of possible fundamental groups up to isomorphism is

$$\{\mathbb{Z}\} \cup \{\mathbb{Z}_n \mid n \geq 1\}.$$

Part 2: Classification of Tuples $(H_1, H_2, H_3, \pi_2, \pi_3)$

Since L is a closed oriented 3-manifold, we always have $H_3(L;\mathbb{Z}) \cong \mathbb{Z}$. We treat the two cases separately.

Case 1: $\pi_1(L) \cong \mathbb{Z}$ (i.e., $b = 0$)

In this case, the gluing identifies the meridian of V_1 with a meridian of V_2 (since $b = 0$ forces $a = d = \pm 1$). The resulting manifold is homeomorphic to $S^2 \times S^1$.

Homology.

- $H_1(L) \cong \pi_1(L)_{\mathrm{ab}} \cong \mathbb{Z}$.
- By the universal coefficient theorem, $H^1(L;\mathbb{Z}) \cong \mathrm{Hom}(H_1(L), \mathbb{Z}) \cong \mathbb{Z}$.
- Poincaré duality gives $H_2(L;\mathbb{Z}) \cong H^1(L;\mathbb{Z}) \cong \mathbb{Z}$.
- $H_3(L;\mathbb{Z}) \cong \mathbb{Z}$.

Homotopy groups. The universal cover of $S^2 \times S^1$ is $S^2 \times \mathbb{R}$, which is homotopy equivalent to S^2. Hence for $k \geq 2$,

$$\pi_k(L) \cong \pi_k(S^2 \times \mathbb{R}) \cong \pi_k(S^2).$$

In particular,

- $\pi_2(L) \cong \pi_2(S^2) \cong \mathbb{Z}$,
- $\pi_3(L) \cong \pi_3(S^2) \cong \mathbb{Z}$ (generated by the Hopf fibration).

Thus the tuple is

$$(H_1, H_2, H_3, \pi_2, \pi_3) \cong (\mathbb{Z},\ \mathbb{Z},\ \mathbb{Z},\ \mathbb{Z},\ \mathbb{Z}).$$

Case 2: $\pi_1(L) \cong \mathbb{Z}_n$, $n \geq 1$ (i.e., $|b| = n$)

Here the manifold L is a lens space $L(n, a)$ (for some integer a coprime to n); in particular, $n = 1$ gives S^3.

Homology.

- $H_1(L) \cong \pi_1(L)_{\mathrm{ab}} \cong \mathbb{Z}_n$.
- The universal coefficient theorem yields $H^1(L;\mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}) = 0$.
- Poincaré duality implies $H_2(L;\mathbb{Z}) \cong H^1(L;\mathbb{Z}) = 0$.
- $H_3(L;\mathbb{Z}) \cong \mathbb{Z}$.

Homotopy groups. Since $\pi_1(L)$ is finite, the universal cover $\widetilde{L}$ is a simply connected closed 3-manifold. For lens spaces, $\widetilde{L} \cong S^3$. Therefore for $k \geq 2$,

$$\pi_k(L) \cong \pi_k(S^3).$$

Consequently,

- $\pi_2(L) \cong \pi_2(S^3) = 0$,
- $\pi_3(L) \cong \pi_3(S^3) \cong \mathbb{Z}$.

Hence the tuple is

$$(H_1, H_2, H_3, \pi_2, \pi_3) \cong (\mathbb{Z}_n,\ 0,\ \mathbb{Z},\ 0,\ \mathbb{Z}).$$

Conclusion

The possible fundamental groups are $\mathbb{Z}$ and $\mathbb{Z}_n$ for every $n \geq 1$. The corresponding invariants are:

- If $\pi_1(L) \cong \mathbb{Z}$, then $(H_1, H_2, H_3, \pi_2, \pi_3) \cong (\mathbb{Z}, \mathbb{Z}, \mathbb{Z}, \mathbb{Z}, \mathbb{Z})$.
- If $\pi_1(L) \cong \mathbb{Z}_n$ $(n \geq 1)$, then $(H_1, H_2, H_3, \pi_2, \pi_3) \cong (\mathbb{Z}_n, 0, \mathbb{Z}, 0, \mathbb{Z})$.