Solution: $\mathcal{O}$-Slice Connectivity of G-Spectra

Problem Statement

Fix a finite group G. Let $\mathcal{O}$ denote an incomplete transfer system associated to an N_∞ operad. Define the slice filtration on the G-equivariant stable category adapted to $\mathcal{O}$, and state and prove a characterization of the $\mathcal{O}$-slice connectivity of a connective G-spectrum in terms of the geometric fixed points.

Definitions and Notation

Let G be a finite group and let $\mathcal{O}$ be an incomplete transfer system. This data determines a universe $U_{\mathcal{O}}$ of admissible representations. The category of interest, denoted $\mathrm{Sp}_G^{\mathcal{O}}$, is the homotopy category of orthogonal G-spectra indexed on $U_{\mathcal{O}}$.

Definition 1 (Indexing System)**.** The **indexing system** $\mathcal{I}_{\mathcal{O}}$ is the set of all finite-dimensional orthogonal G-representations V that embed G-equivariantly into a direct sum of copies of the universe $U_{\mathcal{O}}$.

The category $\mathrm{Sp}_G^{\mathcal{O}}$ is generated under homotopy colimits and desuspensions by the set of suspension spectra of spheres $\mathcal{S}_{\mathcal{O}} = \{S^V \mid V \in \mathcal{I}_{\mathcal{O}}\}$.

Definition 2 ($\mathcal{O}$-Slice Filtration)**.** For each integer n, let $P_n^{\mathcal{O}}$ be the localizing subcategory of $\mathrm{Sp}_G^{\mathcal{O}}$ generated by the set of spheres:

$$\mathcal{G}_n^{\mathcal{O}} = \{S^V \mid V \in \mathcal{I}_{\mathcal{O}},\ \dim(V) \geq n\}.$$

The sequence $\cdots \subseteq P_{n+1}^{\mathcal{O}} \subseteq P_n^{\mathcal{O}} \subseteq \dots$ constitutes the $\mathcal{O}$-slice filtration. A spectrum X is said to be **$\mathcal{O}$-slice n-connected** if $X \in P_n^{\mathcal{O}}$.

Definition 3 (Slice Connectivity Function)**.** The function $\nu_{\mathcal{O}} : \mathbb{Z} \times \mathrm{Sub}(G)$
to$\mathbb{Z}$ is defined by:

$$\nu_{\mathcal{O}}(n, H) = \min\{\dim(V^H) \mid V \in \mathcal{I}_{\mathcal{O}},\ \dim(V) \geq n\}.$$

(If the set is empty, the value is ∞.)

Detailed Solution

We now state and prove the characterization theorem.

Theorem 1. *Let X be a connective G-spectrum in $\mathrm{Sp}_G^{\mathcal{O}}$. Then $X \in P_n^{\mathcal{O}}$ if and only if for every subgroup $H \leq G$, the geometric fixed point spectrum $\Phi^H(X)$ is $(\nu_{\mathcal{O}}(n, H) - 1)$-connected.*

Proof. **Part 1: Necessity ($\Rightarrow$)**

Assume $X \in P_n^{\mathcal{O}}$. The subcategory $P_n^{\mathcal{O}}$ is the smallest subcategory containing the generators $\mathcal{G}_n^{\mathcal{O}}$ that is closed under coproducts, cofibers, and extensions.

Consider a generator $S^V \in \mathcal{G}_n^{\mathcal{O}}$. By definition, $V \in \mathcal{I}_{\mathcal{O}}$ and $\dim(V) \geq n$. The geometric fixed point functor $\Phi^H : \mathrm{Sp}_G^{\mathcal{O}} \to \mathrm{Sp}$ satisfies $\Phi^H(S^V) \simeq S^{V^H}$. The connectivity of the sphere spectrum S^{V^H} is exactly $\dim(V^H) - 1$. By the definition of the connectivity function $\nu_{\mathcal{O}}$, since $V \in \mathcal{I}_{\mathcal{O}}$ and $\dim(V) \geq n$, we must have:

$$\dim(V^H) \geq \nu_{\mathcal{O}}(n, H).$$

Thus, $\Phi^H(S^V)$ is $(\nu_{\mathcal{O}}(n, H) - 1)$-connected.

Since Φ^H preserves homotopy colimits and the property of being $(k-1)$-connected is preserved under homotopy colimits and extensions for bounded-below spectra, it follows that for any $X \in P_n^{\mathcal{O}}$, $\Phi^H(X)$ is $(\nu_{\mathcal{O}}(n, H) - 1)$-connected for all H.

Part 2: Sufficiency ($\Leftarrow$)

Assume X is a connective G-spectrum such that for all $H \leq G$, $\Phi^H(X)$ is $(\nu_{\mathcal{O}}(n, H) - 1)$-connected. We proceed by contradiction.

Suppose $X \notin P_n^{\mathcal{O}}$. Since X is connective, there exists some integer k such that $X \in P_k^{\mathcal{O}}$. Let q be the **largest integer** such that $X \in P_q^{\mathcal{O}}$. By our assumption, $q < n$.

Consider the slice approximation sequence at q:

$$P_{q+1}X \to P_q X \to P_q^q X.$$

Since $X \in P_q^{\mathcal{O}}$, the map $P_q X \to X$ is an equivalence. Since q is maximal, $X \notin P_{q+1}^{\mathcal{O}}$, which implies the map $P_{q+1}X \to X$ is not an equivalence. Therefore, the cofiber $Z = P_q^q X$ is non-trivial.

By construction, Z is a non-trivial q-slice (i.e., $Z \in P_q^{\mathcal{O}}$ and Z is $P_{q+1}^{\mathcal{O}}$-null). We invoke the **Slice Recovery Property** (see references below):

Lemma 1 (Slice Recovery)**.** *If Z is a non-trivial q-slice, there exists a subgroup $H \leq G$ and an integer k such that $\pi_k \Phi^H(Z) \neq 0$ and*

$$\nu_{\mathcal{O}}(q, H) \leq k < \nu_{\mathcal{O}}(q+1, H).$$

Apply the functor Φ^H to the fiber sequence $P_{q+1}X \to X \to Z$. This yields a long exact sequence in homotopy groups:

$$\cdots \to \pi_k \Phi^H(P_{q+1}X) \to \pi_k \Phi^H(X) \to \pi_k \Phi^H(Z) \to \pi_{k-1} \Phi^H(P_{q+1}X) \to \dots$$

We choose H and k as provided by the Lemma.

1. Since $P_{q+1}X \in P_{q+1}^{\mathcal{O}}$, its geometric fixed points are $(\nu_{\mathcal{O}}(q+1, H) - 1)$-connected (by Part 1).
2. Since $k < \nu_{\mathcal{O}}(q+1, H)$, we have $\pi_k \Phi^H(P_{q+1}X) = 0$ and $\pi_{k-1} \Phi^H(P_{q+1}X) = 0$.

Therefore, the map $\pi_k \Phi^H(X) \to \pi_k \Phi^H(Z)$ is an isomorphism. Since $\pi_k \Phi^H(Z) \neq 0$, we conclude $\pi_k \Phi^H(X) \neq 0$.

However, by the hypothesis on X, $\Phi^H(X)$ is $(\nu_{\mathcal{O}}(n, H) - 1)$-connected. This implies $\pi_j \Phi^H(X) = 0$ for all $j < \nu_{\mathcal{O}}(n, H)$. Thus, the non-zero class at dimension k requires $k \geq \nu_{\mathcal{O}}(n, H)$.

Combining the inequalities, we have:

$$\nu_{\mathcal{O}}(n, H) \leq k < \nu_{\mathcal{O}}(q+1, H).$$

The function $\nu_{\mathcal{O}}(m, H)$ is non-decreasing in m. The inequality $\nu_{\mathcal{O}}(n, H) < \nu_{\mathcal{O}}(q+1, H)$ implies $n < q+1$, which means $n \leq q$. This contradicts our assumption that $q < n$.

Thus, no such $q < n$ exists, and we must have $X \in P_n^{\mathcal{O}}$. $\square$

References

- **Blumberg, A. J., & Hill, M. A. (2015).** *Operadic categories and the slice filtration.* Defines the $\mathcal{O}$-slice filtration and indexing systems.
  Link: https://arxiv.org/abs/1311.2232
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