Bond Percolation on an Infinite d-Regular Tree

Summary

- **a. Verdict:** The critical probability is $p_c = \dfrac{1}{d-1}$. For $p < p_c$, the cluster size decays exponentially.
- **b. Method Sketch:**
  1. Model the growth of the open cluster $C(o)$ as a Galton–Watson branching process. The root has offspring distribution $\mathrm{Bin}(d, p)$, and all other vertices have offspring distribution $\mathrm{Bin}(d-1, p)$.
  2. The process survives with positive probability iff the mean offspring of the latter exceeds 1, i.e., $(d-1)p > 1$. Hence $p_c = \frac{1}{d-1}$.
  3. For $p < p_c$, analyze the cluster size via an exploration process that maps the size to a random walk with negative drift. Apply a Chernoff bound to show exponential decay.

Detailed Solution

Let T be an infinite d-regular tree ($d \geq 2$) with a distinguished root o. Each edge of T is open independently with probability $p \in [0, 1]$. Let $C(o)$ be the set of vertices connected to o by a path of open edges. We study the probability $\theta(p) = \mathbb{P}(|C(o)| = \infty)$ and the tail of $|C(o)|$ when p is small.

Part 1: The critical probability p_c

1.1 Branching process construction

Explore T starting from o:

- Generation 0 consists of o.
- Generation 1: The root has d incident edges, each open independently with probability p. Let Z_1 be the number of open edges; then $Z_1 \sim \mathrm{Bin}(d, p)$. The vertices reached form generation 1.
- For each vertex v in generation $n \geq 1$, there are $d-1$ edges leading to new vertices (the parent is already visited). Each such edge is open with probability p, independently. If v is in generation n, let $\xi_v \sim \mathrm{Bin}(d-1, p)$ be the number of open edges to its children. Those children constitute generation $n+1$.

Thus the growth of $C(o)$ is described by a Galton–Watson process where the root has offspring distribution $\mu_0 = \mathrm{Bin}(d, p)$ and every other individual has offspring distribution $\mu = \mathrm{Bin}(d-1, p)$.

1.2 Survival analysis

Let $m = \mathbb{E}[\mu] = (d-1)p$. For a Galton–Watson process with offspring distribution μ,

$$\rho(p) := \mathbb{P}(\text{process survives}) > 0 \quad \iff \quad m > 1.$$

Now consider the whole process starting from o. The cluster $C(o)$ is infinite iff at least one of the d branches emanating from o survives. For a fixed neighbor of o, the branch exists and survives with probability $p\rho(p)$. Hence

$$\mathbb{P}(|C(o)| < \infty) = \big(1 - p\rho(p)\big)^d, \qquad \theta(p) = 1 - \big(1 - p\rho(p)\big)^d.$$

Consequently $\theta(p) > 0$ iff $\rho(p) > 0$, i.e., iff $(d-1)p > 1$.

1.3 Determination of p_c

Define

$$p_c := \frac{1}{d-1}.$$

- If $p < p_c$, then $(d-1)p < 1$, so $\rho(p) = 0$ and $\theta(p) = 0$.
- If $p > p_c$, then $(d-1)p > 1$, so $\rho(p) > 0$ and $\theta(p) > 0$.

Thus p_c is indeed the critical probability for the existence of an infinite open cluster.

Part 2: Exponential decay for $p < p_c$

Fix $p < p_c$. We will show that there exist constants $C, c > 0$ such that for all $k \geq 2$,

$$\mathbb{P}(|C(o)| \geq k) \leq Ce^{-ck}.$$

2.1 Exploration process

We explore $C(o)$ step by step, keeping a queue Q of active vertices (reached but whose neighbors have not been examined) and a set U of explored vertices.

- Initialisation: $Q_0 = \{o\}$, $U_0 = \{o\}$, $t = 0$.
- While $Q_t \neq \varnothing$, pick the first vertex $u_{t+1} \in Q_t$. Reveal the states of all edges from u_{t+1} to neighbors not in U_t. Let X_{t+1} be the number of those edges that are open. Add the endpoints of these open edges to Q and to U, and remove u_{t+1} from Q.

The queue size evolves as

$$|Q_t| = |Q_{t-1}| - 1 + X_t, \qquad t \geq 1,$$

with $|Q_0| = 1$. Hence

$$|Q_t| = 1 + \sum_{i=1}^{t} (X_i - 1).$$

Let $\tau = \min\{t \geq 1 : |Q_t| = 0\}$. Then $|C(o)| = \tau$.

2.2 Connection to a random walk

The event $\{|C(o)| \geq k\}$ is equivalent to $\{\tau \geq k\}$, which implies that the queue never empties before time k. In particular, $|Q_{k-1}| \geq 1$. Therefore

$$1 + \sum_{i=1}^{k-1} (X_i - 1) \geq 1 \quad \Longleftrightarrow \quad \sum_{i=1}^{k-1} X_i \geq k-1.$$

Thus

$$\mathbb{P}(|C(o)| \geq k) \leq \mathbb{P}\left(\sum_{i=1}^{k-1} X_i \geq k-1 \right).$$

2.3 Distribution of the X_i

- At step 1, we explore o, which has d unexplored neighbors. Hence $X_1 \sim \mathrm{Bin}(d, p)$.
- For $i \geq 2$, we explore a vertex that is a child of a previously explored vertex; it has $d-1$ unexplored neighbors (the parent is already in U). Hence $X_i \sim \mathrm{Bin}(d-1, p)$ for $i \geq 2$.

Let $Y_2, Y_3, \dots$ be i.i.d. $\mathrm{Bin}(d-1, p)$ random variables. Then

$$S_{k-1} := \sum_{i=1}^{k-1} X_i = X_1 + \sum_{i=2}^{k-1} Y_i.$$

2.4 Chernoff bound

For any $\lambda > 0$,

$$\mathbb{P}(S_{k-1} \geq k-1) \leq e^{-\lambda(k-1)} \, \mathbb{E}\left[e^{\lambda S_{k-1}}\right].$$

Using the moment generating function of the binomial distribution,

$$\mathbb{E}[e^{\lambda X_1}] = (1 - p + pe^{\lambda})^d, \qquad \mathbb{E}[e^{\lambda Y_i}] = (1 - p + pe^{\lambda})^{d-1}.$$

Hence

$$\begin{aligned}
\mathbb{P}(|C(o)| \geq k) &\leq e^{-\lambda(k-1)} (1 - p + pe^{\lambda})^d \left((1 - p + pe^{\lambda})^{d-1} \right)^{k-2} \\
&= \frac{(1 - p + pe^{\lambda})^d}{(1 - p + pe^{\lambda})^{2(d-1)}} e^{\lambda} \left(e^{-\lambda} (1 - p + pe^{\lambda})^{d-1} \right)^k \\
&=: C(\lambda) \, \gamma(\lambda)^k,
\end{aligned}$$

where

$$C(\lambda) = \frac{(1 - p + pe^{\lambda})^d}{(1 - p + pe^{\lambda})^{2(d-1)}} e^{\lambda}, \qquad \gamma(\lambda) = e^{-\lambda} (1 - p + pe^{\lambda})^{d-1}.$$

2.5 Choosing λ to make $\gamma(\lambda) < 1$

Consider the logarithm

$$g(\lambda) = \ln \gamma(\lambda) = -\lambda + (d-1)\ln(1-p+pe^{\lambda}).$$

We have $g(0) = 0$ and

$$g'(\lambda) = -1 + (d-1)\frac{pe^{\lambda}}{1-p+pe^{\lambda}}.$$

At $\lambda = 0$,

$$g'(0) = -1 + (d-1)p.$$

Because $p < p_c = \frac{1}{d-1}$, we have $(d-1)p < 1$ and hence $g'(0) < 0$. Since $g(0) = 0$ and $g'(0) < 0$, there exists $\lambda^* > 0$ such that $g(\lambda^*) < 0$, i.e., $\gamma(\lambda^*) = e^{g(\lambda^*)} < 1$. Set $c = -g(\lambda^*) > 0$ and $C = C(\lambda^*)$. Then for all $k \geq 2$,

$$\mathbb{P}(|C(o)| \geq k) \leq Ce^{-ck}.$$

This completes the proof of exponential decay.