Proof of the Inequality for Holomorphic Functions on the Unit Disk

Problem Statement

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk. Suppose $g : D \to \mathbb{C}$ is holomorphic, $g(0) = 0$, and $|\operatorname{Re} g(z)| < 1$ for all $z \in D$. Prove that

$$|g(z)| \leq \frac{2}{\pi} \log\left(\frac{1+|z|}{1-|z|}\right), \quad \forall z \in D.$$

Detailed Solution

Step 1: Conformal mapping from the strip to the disk

The condition $|\operatorname{Re} g(z)| < 1$ implies that the image of g lies in the vertical strip

$$S = \{w \in \mathbb{C} : -1 < \operatorname{Re} w < 1\}.$$

We construct a conformal map $\phi : S \to D$ from this strip onto the unit disk D such that $\phi(0) = 0$.

First, scale the strip by $\frac{\pi}{4}$: the map $w \mapsto \frac{\pi}{4}w$ sends S bijectively onto the strip

$$S' = \left\{\zeta \in \mathbb{C} : -\frac{\pi}{4} < \operatorname{Re} \zeta < \frac{\pi}{4}\right\}.$$

Next, consider the tangent function $\zeta \mapsto \tan\zeta$. We show that tan maps S' into D. Let $\zeta = x + iy$ with $|x| < \frac{\pi}{4}$. Then

$$|\tan(x+iy)|^2 = \left|\frac{\sin(x+iy)}{\cos(x+iy)}\right|^2 = \frac{\sin^2 x + \sinh^2 y}{\cos^2 x + \sinh^2 y}.$$

We want $|\tan(x+iy)| < 1$, which is equivalent to

$$\sin^2 x + \sinh^2 y < \cos^2 x + \sinh^2 y \iff \sin^2 x < \cos^2 x \iff \tan^2 x < 1.$$

Since $|x| < \frac{\pi}{4}$, we have $|\tan x| < 1$, so the inequality holds. Moreover, tan maps S' bijectively onto D. Hence,

$$\phi(w) = \tan\left(\frac{\pi}{4}w\right)$$

is a conformal map from S onto D, and $\phi(0) = 0$.

Step 2: Application of the Schwarz Lemma

Define the composite function $f : D \to D$ by

$$f(z) = \phi(g(z)) = \tan\left(\frac{\pi}{4}g(z)\right).$$

Since g is holomorphic with $g(D) \subseteq S$ and ϕ is holomorphic on S, the function f is holomorphic on D. Moreover, $g(0) = 0$ implies $f(0) = \phi(0) = 0$. Thus, f maps the unit disk into itself and fixes the origin. By the Schwarz Lemma, we have

$$|f(z)| \leq |z|, \quad \forall z \in D.$$

Step 3: Bounding $|g(z)|$

We recover $g(z)$ from $f(z)$ via the inverse of ϕ:

$$g(z) = \frac{4}{\pi} \arctan(f(z)).$$

For fixed $z \in D$, set $\zeta = f(z)$. Then $|\zeta| \leq |z|$. We need to estimate $|g(z)| = \frac{4}{\pi} |\arctan \zeta|$.

Consider the Maclaurin series expansion of $\arctan \zeta$, valid for $|\zeta| < 1$:

$$\arctan \zeta = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \zeta^{2n+1}.$$

Applying the triangle inequality,

$$|\arctan \zeta| = \left| \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \zeta^{2n+1} \right| \leq \sum_{n=0}^{\infty} \frac{1}{2n+1} |\zeta|^{2n+1}.$$

The right-hand side is the Maclaurin expansion of the inverse hyperbolic tangent $\operatorname{arctanh}(|\zeta|)$:

$$\operatorname{arctanh}(|\zeta|) = \sum_{n=0}^{\infty} \frac{|\zeta|^{2n+1}}{2n+1}.$$

Hence,

$$|\arctan \zeta| \leq \operatorname{arctanh}(|\zeta|).$$

Since the function $x \mapsto \operatorname{arctanh}(x)$ is increasing on $[0, 1)$ and $|\zeta| \leq |z|$, we have

$$\operatorname{arctanh}(|\zeta|) \leq \operatorname{arctanh}(|z|).$$

Combining these inequalities,

$$|g(z)| = \frac{4}{\pi} |\arctan \zeta| \leq \frac{4}{\pi} \operatorname{arctanh}(|z|).$$

Step 4: Final calculation

Using the logarithmic representation of the inverse hyperbolic tangent,

$$\operatorname{arctanh}(x) = \frac{1}{2} \log \left(\frac{1+x}{1-x} \right), \quad x \in (-1, 1).$$

Substituting $x = |z|$, we obtain

$$|g(z)| \leq \frac{4}{\pi} \cdot \frac{1}{2} \log \left(\frac{1+|z|}{1-|z|} \right) = \frac{2}{\pi} \log \left(\frac{1+|z|}{1-|z|} \right).$$

This completes the proof.