Non-existence of Compact Manifolds with Torsion Fundamental Groups and $\mathbb{Q}$-Acyclic Covers

Problem Statement

Problem 1. Does there exist a compact manifold M without boundary such that $\pi_1(M) \cong \Gamma$ (where Γ has 2-torsion) and its universal cover $\tilde{M}$ is acyclic over $\mathbb{Q}$?

Detailed Solution

1. Summary

Verdict: No, it is not possible.

Method Sketch: The proof relies on the theory of cohomological finiteness conditions for groups.

1. We assume such a manifold M exists. Since M is compact, it is homotopy equivalent to a finite CW-complex K.
2. The universal cover $\tilde{K}$ is $\mathbb{Q}$-acyclic, meaning $H_0(\tilde{K};\mathbb{Q}) \cong \mathbb{Q}$ and higher homology vanishes.
3. The cellular chain complex $C_*(\tilde{K};\mathbb{Q})$ provides a resolution of the trivial module $\mathbb{Q}$ by finitely generated **free** $\mathbb{Q}\Gamma$-modules of finite length. This implies that Γ is of type $FL(\mathbb{Q})$ (finite length resolution by free modules).
4. We compute the Hattori-Stallings trace (or character) of the Euler class of this resolution. For the trivial module $\mathbb{Q}$, the trace on a non-identity torsion element γ is 1. However, for any free module, the trace on a non-identity element is 0.
5. The existence of a torsion element $\gamma \in \Gamma$ of order 2 leads to the contradiction $1 = 0$. Thus, such a manifold cannot exist.

2. Proof

Assume for the sake of contradiction that there exists a compact manifold M such that $\pi_1(M) \cong \Gamma$ contains an element of order 2, and the universal cover $\tilde{M}$ satisfies $\tilde{H}_*(\tilde{M};\mathbb{Q}) = 0$.

Step 1: Construction of the Free Resolution

Since M is a compact manifold, it is homotopy equivalent to a finite CW-complex K. The fundamental group is $\pi_1(K) \cong \Gamma$. Let $\tilde{K}$ be the universal cover of K. Since $\tilde{M}$ is $\mathbb{Q}$-acyclic and $\tilde{K}$ is Γ-equivariantly homotopy equivalent to $\tilde{M}$, $\tilde{K}$ is also $\mathbb{Q}$-acyclic. The cellular chain complex of $\tilde{K}$ with rational coefficients, denoted $C_*(\tilde{K};\mathbb{Q})$, is an exact sequence of $\mathbb{Q}\Gamma$-modules (except at degree 0):

$$0 \to C_n(\tilde{K};\mathbb{Q}) \xrightarrow{\partial_n} C_{n-1}(\tilde{K};\mathbb{Q}) \to \cdots \to C_0(\tilde{K};\mathbb{Q}) \xrightarrow{\epsilon} \mathbb{Q} \to 0.$$

Here, ϵ is the augmentation map to the trivial module $\mathbb{Q}$. The action of Γ on $\tilde{K}$ is free (deck transformations) and cellular, and $K = \tilde{K}/\Gamma$ is a finite complex. Therefore, each chain group $C_i(\tilde{K}; \mathbb{Q})$ is a **finitely generated free** $\mathbb{Q}\Gamma$-module. Let r_i be the number of i-cells in K; then $C_i(\tilde{K}; \mathbb{Q}) \cong (\mathbb{Q}\Gamma)^{r_i}$.

This sequence constitutes a finite resolution of $\mathbb{Q}$ by finitely generated free $\mathbb{Q}\Gamma$-modules. In homological algebra terms, this implies that Γ is of type $FL(\mathbb{Q})$.

Step 2: Euler Characteristic and Trace Argument

The existence of such a resolution implies an identity in the Grothendieck group of finitely generated projective $\mathbb{Q}\Gamma$-modules, $K_0(\mathbb{Q}\Gamma)$:

$$[\mathbb{Q}] = \sum_{i=0}^{n} (-1)^i [C_i(\tilde{K}; \mathbb{Q})].$$

Substituting the structure of the free modules:

$$[\mathbb{Q}] = \sum_{i=0}^{n} (-1)^i r_i [\mathbb{Q}\Gamma] = \left(\sum_{i=0}^{n} (-1)^i r_i \right) [\mathbb{Q}\Gamma] = \chi(K)[\mathbb{Q}\Gamma].$$

Let $\chi = \chi(K) \in \mathbb{Z}$. We have the equality $[\mathbb{Q}] = \chi[\mathbb{Q}\Gamma]$.

To derive a contradiction, we evaluate the **Hattori-Stallings trace** $r : K_0(\mathbb{Q}\Gamma) \to T(\mathbb{Q}\Gamma)$, where $T(\mathbb{Q}\Gamma) = \mathbb{Q}\Gamma/[\mathbb{Q}\Gamma, \mathbb{Q}\Gamma]$ is the vector space of conjugacy classes. For a group ring, the trace of an element $u = \sum a_g g$ is simply the element itself in the quotient.

- For the free module $\mathbb{Q}\Gamma$, the trace is the element $1 \cdot e \in T(\mathbb{Q}\Gamma)$ (where e is the identity of Γ). The character vanishes on all non-identity elements. Thus, $r([\mathbb{Q}\Gamma])$ is concentrated at the identity.
- For the trivial module $\mathbb{Q}$, the action of any $\gamma \in \Gamma$ is the identity map on a 1-dimensional space. The trace of γ is 1. Thus, in terms of characters (or extending the trace map to modules with finite resolution), the value at a torsion element γ should be 1.

More rigorously, let $\gamma \in \Gamma$ be the element of order 2. Consider the trace map on the subalgebra $\mathbb{Q}C_\gamma \subseteq \mathbb{Q}\Gamma$ or simply evaluate the character $\chi_V(\gamma)$ for representations V. While $\mathbb{Q}\Gamma$ is infinite-dimensional, the resolution relation implies that the alternating sum of the traces of the action of γ on the chain groups must equal the trace of γ on the homology $H_*(\tilde{K}; \mathbb{Q}) \cong \mathbb{Q}$. By the Lefschetz Principle (or Euler characteristic of the group):

$$\operatorname{Tr}(\gamma \mid H_*(\tilde{K}; \mathbb{Q})) = \sum_{i=0}^{n} (-1)^i \operatorname{Tr}(\gamma \mid C_i(\tilde{K}; \mathbb{Q})).$$

- **LHS:** The homology is $\mathbb{Q}$ concentrated in degree 0 with trivial action. So $\operatorname{Tr}(\gamma \mid H_*) = 1$.
- **RHS:** The module $C_i(\tilde{K}; \mathbb{Q})$ is a free $\mathbb{Q}\Gamma$-module. As a module over the cyclic subgroup $\langle \gamma \rangle$, it is a direct sum of copies of $\mathbb{Q}\Gamma$. Since Γ is infinite (as it is the fundamental group of a manifold with acyclic cover), $\mathbb{Q}\Gamma$ is a free module over $\mathbb{Q}\langle \gamma \rangle$. The trace of a non-identity element γ on a free module over a group ring is always 0 (it permutes the basis elements freely, leaving no fixed basis elements). Thus, $\operatorname{Tr}(\gamma \mid C_i) = 0$.

Substituting these values:

$$1 = \sum_{i=0}^{n} (-1)^i \cdot 0 = 0.$$

This yields the contradiction $1 = 0$.

Conclusion

The assumption that Γ contains torsion leads to a contradiction with the existence of a finite free resolution of $\mathbb{Q}$. Therefore, no such manifold M exists.

References

- **Brown, K. S. (1982).** *Cohomology of Groups.* Springer-Verlag.
  - Chapter IX, Section 13 (Euler Characteristics) discusses the constraints on groups of type FL. Specifically, Proposition 13.2 and Corollary 13.4 establish that if Γ has a finite resolution by finitely generated projective modules (over $\mathbb{Z}$ or $\mathbb{Q}$) and possesses non-trivial torsion, the Euler characteristic must vanish in a way that often leads to contradictions (like $1 = 0$ for the trivial module).
- **Bass, H. (1976).** Euler characteristics and characters of discrete groups. *Inventiones mathematicae*, 35, 155-196.
  - Establishes the Hattori-Stallings trace methods for infinite groups and proves that for groups of type FL, the character vanishes on non-identity elements of finite order.