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Series Solutions of Linear Equations

An equation containing the derivatives of one or more unknown functions (or dependent variables), with respect to one or more independent variables, is said to be a differential equation. (page 2, of [1])

For example, $y' - y = 0$ is a differential equation, $y = f(x)$. The solution of $y' - y = 0$ is $y = e^x$, since $y' = e^x$, $y' - y = e^x - e^x = 0$. What's more, the solution of a differential equation can also be a power series.

Show that the power series $y = c_0 \sum_{n=0}^{\infty} \frac{4^n}{n!} x^n$ is the solution of linear equation

$y' - 4y = 0$

Solution:

$$y = c_0 \sum_{n=0}^{\infty} \frac{4^n}{n!} x^n = c_0 + 4c_0 x + \frac{4^2}{2!} c_0 x^2 + \frac{4^3}{3!} c_0 x^3 \ldots,$$

$$y' = \frac{d}{dx}\left(c_0 \sum_{n=0}^{\infty} \frac{4^n}{n!} x^n\right) = 0 + 4c_0 + \frac{4^2}{1!} c_0 x + \frac{4^3}{2!} c_0 x^2 \ldots$$

$$y = c_0 + 4c_0 x + \frac{4^2}{2!} c_0 x^2 + \frac{4^3}{3!} c_0 x^3 \ldots,$$

$$4y = 4c_0 + 4^2 c_0 x + \frac{4^3}{2!} c_0 x^2 + \frac{4^4}{3!} c_0 x^3 \ldots,$$

$$y' = 0 + 4c_0 + \frac{4^2}{1!} c_0 x + \frac{4^3}{2!} c_0 x^2 \ldots, \text{ so } y' - 4y = 0.$$

From the example, we know that series could be solutions of linear equations. This paper is going to introduce series solutions of linear equations. First, some background knowledge about power series and the way to obtain a power series solution of a linear first-order differential equation will be included. Then, the paper will show how to find ordinary points of linear second-order equations in the form of power series. After that, we will show how to find singular points of linear second-order equations in the form of power series.

Power Series

A power series in $x - a$ is an infinite series of the form

$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$. Such a series also said to be a power series centered at a. For example, the power series $\sum_{n=0}^{\infty} c_n (x-1)^n$ is centered at $a = 1$. The power series $\sum_{n=0}^{\infty} c_n (x)^n$ is centered at $a = 0$, like $\sum_{n=0}^{\infty} 3^n x^n = 1 + 3x + 9x^2 + 27x^3 + \cdots$, which is called a power series in x. (page 231, of [1])

There are four important facts about power series $\sum_{n=0}^{\infty} c_n (x-a)^n$:

First, a power series is convergent at a specified value of x if its sequence of partial sums $\{S_N(x)\}$ converges, which means, $\lim_{N\to\infty} S_N(x) = \lim_{N\to\infty} \sum_{n=0}^{N} c_n (x-a)^n$ exists. If the limit does not exist at x, the the series is said to be divergent.

Second, every power series has an interval of convergence. The interval of convergence is the set of all real numbers x for which the series converges. The center of the interval of convergence is the center a of the series.

Third, the radius R of the interval of convergence of a power series is called its radius of convergence. If $R > 0$, then a power series converges for $|x-a| < R$ and diverges for $|x-a| > R$. If the series converges only at its center a, then $R = 0$. If the series converges for all x, then we write $R = \infty$. Recall, the absolute-value inequality $|x - a| < R$ is equivalent to

the simultaneous inequality $a - R < x < a + R$. A power series may or may not converge at the endpoints $a - R$ and $a + R$ of this interval.

Fourth, within its interval of convergence a power series converges absolutely. In other words, if x is in the interval of convergence and is not an endpoint of the interval, the series of absolute values $\sum_{n=0}^{\infty} |c_n(x-a)^n|$ converges (page 232, [1]).

Fifth, convergence of power series can often be determined by the ratio test. Suppose $c_n \neq 0$ for all n in $\sum_{n=0}^{\infty} c_n(x-a)^n$, and that

$$\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = |x-a| \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = L.$$

If $L < 1$, the series converges absolutely; if $L > 1$ the series diverges; and if $L = 1$ the test is inconclusive. The ratio test is always inconclusive at an endpoint $a \in R$ (page 233, of [1]).

Here is an example of interval of convergence:

Find the interval and radius of convergence for $\sum_{n=1}^{\infty} \frac{(x-3)^n}{4^n n}$.

Solution: The ratio test shows that

$$\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = |x-a| \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = L$$

$$\lim_{n\to\infty}\left|\frac{\frac{(x-3)^{n+1}}{4^{n+1}(n+1)}}{\frac{(x-3)^n}{4^n n}}\right| = |x-3|\lim_{n\to\infty}\left|\frac{n+1}{4n}\right| = \frac{1}{4}|x-3|$$

When $\frac{1}{4}|x-3| < 1, -1 < x < 7,$ the series converges absolutely.

When $\frac{1}{4}|x-3| > 1, x < -1$ or $x > 7,$ the series diverges.

When $\frac{1}{4}|x-3| = 1, x = -1$ or 7, the test is inconclusive. At the endpoint $x = -1$ of the open interval of convergence, the series $\sum_{n=1}^{\infty}((-1)^n/n) = -1 + \frac{1}{2} - \frac{1}{3} + \cdots$ is convergent by the alternating series test Because $\lim_{n\to\infty}\frac{1}{n} = 0$, and $\left\{\frac{1}{n}\right\}$ is a decreasing sequence. At the endpoint $x = 7$ of the open interval of convergence, the series of $\sum_{n=1}^{\infty}((1)^n/n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is divergent. The interval of convergence of the series is [-1,7), and the radius of convergence is $R = 3$.

Solutions About Ordinary Points

If we divide the homogeneous linear second-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{1}$$

by the lead coefficient $a_2(x)$ we obtain the standard form

$$y'' + P(x)y' + Q(x)y = 0 \tag{2}$$

We have the following definition: A point $x = x_0$ is said to be an ordinary point of the differential equation (1) if both coefficients $P(x)$ and $Q(x)$ in the standard form (2) are analytic (which means they exist) at x_0. A point that is not an ordinary point of (1) is said to be a singular point of the differential equation (page 239, of [1]).

Example: The equation $(x^2+2)y'' - 10xy' - y = 0$

$P(x) = \frac{-10x}{x^2+2}$, $Q(x) = \frac{-1}{x^2+2}$. The equation has singular point at the solution of $x^2 + 2 = 0$, which are $\pm\sqrt{2}i$. All other values of x, real or complex, are ordinary points.

Existence of Power Series Solutions

If $x = x_0$ is an ordinary point of the differential equation (1), we can always find two linearly independent solutions in the form for a power series which is centered at x_0, that is $y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$.

A power series solution converges at least on some interval defined by $|x - x_0| < R$, where R is the distance from x_0 to the closest singular point.

A solution of the form $y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$ is said to be a solution about the ordinary point x_0. The distance R in the theorem about the existence of power series solutions is the minimum value or lower bound for the radius of convergence (page 240, of [1]).

Here is an example to show power series solution:

Solve $(x^2 + 4)y'' - xy' + y = 0$.

The differential equation has singular points at $x = \pm 2i$, and a power series solution is centered at 0 and 2 is the distance in the complex plane from 0 to both $2i$ and -$2i$. Substituting $y = \sum_{n=0}^{\infty} c_n x^n$, the first derivative $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$ and the second derivative $y'' = \sum_{n=2}^{\infty} n(n+1) c_n x^{n-2}$.

$$(x^2 + 4) \sum_{n=2}^{\infty} n(n+1) c_n x^{n-2} - x \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n$$

$= \sum_{n=2}^{\infty} n(n+1) c_n x^n + 4 \sum_{n=2}^{\infty} n(n+1) c_n x^{n-2} - \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n$

$= 24c_2x^0 + 48c_3x - c_1x + c_0x^0 + c_1x$

$$+\sum_{n=2}^{\infty} n(n+1)c_nx^n + 4\sum_{n=4}^{\infty} n(n+1)c_nx^{n-2} - \sum_{n=2}^{\infty} nc_nx^n + \sum_{n=2}^{\infty} c_nx^n$$

$= 24c_2 + c_0 + 48c_3x + \sum_{k=2}^{\infty}[k(k+1)c_k + 4(k+2)(k+3)c_{k+2} - k\,c_k + c_k]\,x^k$

$= 24c_2 + c_0 + 48c_3x + \sum_{k=2}^{\infty}[(k^2+1)c_k + 4(k+2)(k+3)c_{k+2}]\,x^k = 0$

we conclude that $24c_2 + c_0 = 0$, $48c_3 = 0$, and

$$(k^2+1)c_k + 4(k+2)(k+3)c_{k+2} = 0$$

So

$$c_2 = \frac{-1}{24}c_0$$

$$c_3 = 0$$

$$c_{k+2} = -\frac{k^2+1}{4(k+2)(k+3)}c_k$$

Substituting $k = 2, 3, 4, \ . \ . \ .$ into the last formulas gives

$c_4 = -\frac{2\cdot2+1}{4\cdot4\cdot5}c_2 = \frac{-(2\cdot2+1)}{4\cdot4\cdot5}\cdot\frac{-1}{1\cdot2\cdot3\cdot4}c_0 = (-1)^2\frac{2^2+1}{5!4^2}c_0$

$$c_5 = -\frac{9}{4\cdot5\cdot6}c_3 = 0$$

$c_6 = -\frac{4\cdot4+1}{4\cdot6\cdot7}c_4 = \frac{-(4\cdot4+1)}{4\cdot6\cdot7}\cdot\frac{-(2\cdot2+1)}{4\cdot4\cdot5}\cdot\frac{-1}{1\cdot2\cdot3\cdot4}c_0 = (-1)^3\frac{(2^2+1)(4^2+1)}{7!4^3}c_0$

$$c_7 = -\frac{5\cdot5+1}{4\cdot6\cdot7}c_5 = 0$$

$$c_8 = -\frac{6\cdot6+1}{4\cdot8\cdot9}c_2 = \frac{-(6\cdot6+1)}{4\cdot8\cdot9}\cdot\frac{-(4\cdot4+1)}{4\cdot6\cdot7}\cdot\frac{-(2\cdot2+1)}{4\cdot4\cdot5}\cdot\frac{-1}{1\cdot2\cdot3\cdot4}c_0 = (-1)^4\frac{(2^2+1)(4^2+1)(6^2+1)}{9!4^4}c_0$$

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + c_7x^7 + c_8x^8 + \cdots$$

$$= c_0\left[1 - \frac{1}{4!}x^2 + \frac{2^2+1}{5!4^2}x^4 - \frac{(2^2+1)(4^2+1)}{7!4^3}x^6 + \frac{(2^2+1)(4^2+1)(6^2+1)}{9!4^4}x^8 - \cdots\right] + c_1x$$

$$= c_0y_1(x) + c_1y_2(x)$$

The solutions are the polynomial $y_2(x) = x$ and the power series

$$y_1(x) = 1 - \frac{1}{4!}x^2 + \sum_{n=2}^{\infty}(-1)^n\frac{(2^2+1)(4^2+1)(6^2+1)\ldots((2n-2)^2+1)}{(2n+1)!\,4^n}x^{2n},$$

$$|x| < 2$$

Solution About Singular Points

A singular point x_0 of a linear differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \qquad (1)$$

is further classified as either regular or irregular. The classification again depends on the functions P and Q in the standard form

$$y'' + P(x)y' + Q(x)y = 0$$

Regular and Irregular Singular Points

A singular point $x = x_0$ is said to be a regular singular point of the differential equation (1) if the functions $p(x) = (x - x_0)P(x)$ and $q(x) = (x - x_0)^2 Q(x)$ are both analytic at x_0. A singular point that is not regular is said to be an irregular singular point of the equation.

For example,

$$(x^2 - 9)^2 y'' + 4(x + 3)y' + 6y = 0$$

It is clear that $x = 3, -3$ are singular points of this equation.

$$P(x) = \frac{4(x+3)}{(x^2 - 9)^2} = \frac{4}{(x+3)(x-3)^2}, \qquad Q(x) = \frac{6}{(x+3)^2(x-3)^2}$$

When $x = 3$, $p(x) = (x - 3)P(x) = \frac{4}{(x+3)(x-3)}$ is not analytic at $x = 3$. $x = 3$ is an irregular singular point of the equation.

When $x = -3$, $p(x) = (x + 3)\frac{4}{(x+3)(x-3)^2} = \frac{4}{(x-3)^2}$ and

$q(x) = (x + 3)^2 \frac{6}{(x+3)^2(x-3)^2} = \frac{6}{(x-3)^2}$ are analytic at $x = -3$.

Frobenius' Theorem

If $x = x_0$ is a regular singular point of the differential equation (1), then there exists at least one solution of the form

$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$, where the number r is a constant to be determined. The series will converge at least on some interval $0 < x - x_0 < R$.

Example,

Solve

$$4xy'' - (2 + x)y' - 2y = 0$$

Solution:

Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ gives

$$
\begin{aligned}
4xy'' - (2 + x)y' - 2y \\
&= 4\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1} - 2\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} \\
&- \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} - 2\sum_{n=0}^{\infty} c_n x^{n+r} \\
&= \sum_{n=0}^{\infty}(n+r)(4n+4r-6)c_n x^{n+r-1} - \sum_{n=0}^{\infty}(n+r+2)c_n x^{n+r} \\
&= x^r\left[r(4r-6)c_0 x^{-1} + \sum_{n=1}^{\infty}(n+r)(4n+4r-2)c_n x^{n-1}\right. \\
&\left.- \sum_{n=0}^{\infty}(n+r+2)c_n x^n\right]
\end{aligned}
$$

Replace n by k,

$= x^r[r(4r-6)c_0 x^{-1} + \sum_{k=0}^{\infty}[(k+r+1)(4k+4r-2)c_{k+1} - (k+r+2)c_k]x^k]$

Which implies that $r(4r-6) = 0$ (a) and

$(k+r+1)(4k+4r-2)c_{k+1}-(k+r+2)c_k=0, k=0,1,2,3,\dots$ (b)

From equation (a), we can tell that the roots of equation (a) are $r_1=0$ and $r_2=2$

For $r_1=0$, $c_{k+1}=\frac{k+r+2}{(k+r+1)(4k+4r-2)}c_k=\frac{k+2}{(4k-2)(k+1)}c_k$, $k=0,1,2,3,\dots$

$$c_1=-c_0, \qquad c_2=\frac{3}{2\cdot 2}c_1=\frac{-3}{2\cdot 2}c_0, \quad c_3=\frac{4}{6\cdot 3}c_2=\frac{4}{6\cdot 3}\cdot\frac{-3}{2\cdot 2}c_0$$

$c_4=\frac{5}{10\cdot 4}\cdot\frac{4}{6\cdot 3}\cdot\frac{-3}{2\cdot 2}c_0$, . . . $c_n=\frac{n+1}{2^n\cdot((-1)\cdot 1\cdot 3\cdot 5\cdots(2n-3)}c_0$

For $r_2=\frac{3}{2}$, $c_{k+1}=\frac{k+r+2}{(k+r+1)(4k+4r-2)}c_k=\frac{k+\frac{7}{2}}{4(k+\frac{5}{2})(k+1)}c_k$, $k=0,1,2,3,\dots$

$$c_1=\frac{\frac{7}{2}}{4\cdot\frac{5}{2}\cdot 1}c_0=\frac{7}{20}c_0$$

$$c_2=\frac{1+\frac{7}{2}}{4\cdot(1+\frac{5}{2})\cdot 2}c_1=\frac{\frac{9}{2}}{4\cdot\frac{7}{2}\cdot 2}\cdot\frac{\frac{7}{2}}{4\cdot\frac{5}{2}\cdot 1}c_0$$

$$c_3=\frac{2+\frac{7}{2}}{4\cdot\left(2+\frac{5}{2}\right)\cdot 3}c_1=\frac{\frac{11}{2}}{4\cdot\frac{9}{2}\cdot 3}\cdot\frac{\frac{9}{2}}{4\cdot\frac{7}{2}\cdot 2}\cdot\frac{\frac{7}{2}}{4\cdot\frac{5}{2}\cdot 1}c_0$$

$$c_4=\frac{3+\frac{7}{2}}{4\cdot\left(3+\frac{5}{2}\right)\cdot 4}c_1=\frac{\frac{13}{2}}{4\cdot\frac{11}{2}\cdot 4}\cdot\frac{\frac{11}{2}}{4\cdot\frac{9}{2}\cdot 3}\cdot\frac{\frac{9}{2}}{4\cdot\frac{7}{2}\cdot 2}\cdot\frac{\frac{7}{2}}{4\cdot\frac{5}{2}\cdot 1}c_0$$

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$$c_n=\frac{5+2n}{5\cdot 4^n n!}c_0$$

Thus for root $r_1 = 0$, we can get the solution

$$y_1(x) = -1 + \sum_{n=1}^{\infty} \frac{n+1}{2^n \cdot ((-1) \cdot 1 \cdot 3 \cdot 5 \cdots (2n-3)} x^n$$

for root $r_2 = \frac{3}{2}$,

$$y_2(x) = x^{\frac{3}{2}}[\frac{7}{20} + \sum_{n=1}^{\infty} \frac{5+2n}{5 \cdot 4^n n!} x^n, \ |x| < \infty$$

On the interval $(0, \infty)$ the general solution is $y = C_1 y_1(x) + C_2 y_2(x)$.

When using the method of Frobenius, there are three cases based on the nature of r_1 and r_2:

Case I: If r_1 and r_2 are distinct and their difference $r_1 - r_2$ is not a positive integer, then there exist two linearly independent solutions of the equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

of form $y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \ a_0 \neq 0, \ \ y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r}, \ b_0 \neq 0,$

Case II: If r_1 and r_2 are distinct and their difference $r_1 - r_2$ is a positive integer, then there exist two linearly independent solutions of equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$

of form $y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r}, \ c_0 \neq 0, \ \ y_2(x) = C y_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^{n+r}, \ b_0 \neq 0,$

where C is a constant that could be zero.

Case III: If r_1 and r_2 are equal, then there always exist two linearly independent solutions of equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ of form

$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$, $c_0 \neq 0$, $y_2(x) = y_1(x) \ln x + \sum_{n=1}^{\infty} b_n x^{n+r}$ (page 254, of [1])

Reference

1 Dennis G. Zill, A First Course in Differential Equations, 10th Edition, Brooks/Cole Publishing Company, Boston, 2013.

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