HomeWork 4. Graph theory ~ Week 9.

Exercises (1)

1. If G is a graph of order n, what is the maximum number of edges in G?

Undirected Graph: $max_{edges} = \binom{n}{2} = \frac{n(n-1)}{2}$

Directed Graph: $max_{edges} = n(n-1)$

2. Prove that for any graph G of order at least 2, the degree sequence has at least one pair of repeated entries.

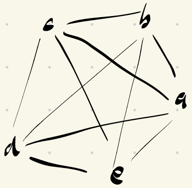
Proof: the degree of any vertex in G is bounded by $0 \le \deg(v) \le n-1$.
contains n entries, one for each vertex.

if v has degree: $n-1$ (conected to all others), no vertex can have degree 0.
Thus n vertices are assigned degrees from $\{1, \dots, n-1\}$ at most $n-1$ distinct values.

By the Pigeonhole principle, at least one degree must repeat.

Therefore: For any G of order: $n \ge 2$, the degree sequence has at least one repeated val.

3. Complete graph K_5



a) How many different paths have c as an end vertex?

4 paths: a-c, b-c, d-c, e-c

b) How many different simple paths avoid vertex c altogether?

6 paths: a-b, a-d, a-e, b-d, b-e, d-e

c) What is the maximum length of a circuit in this graph? Give an example of such a circuit

max length of circuit is 5: a-b-c-d-e-a
visits all vertices and return to starting vertex.

4. Let G be a graph where $\delta(G) \geq k$

a) Prove that G has a simple path of length at least k.

b) If $k \geq 2$, prove that G has a cycle of length at least $k+1$

a) Since $\delta(G) \geq k \Rightarrow \forall v \in V, \exists \deg(v) \geq k$.

Proof: Since the degree of this vertex is at least k, we can follow k distinct edges from this vertex and each of these k edges will lead to a different vertex (no repeated vertices in the path)

Therefore, simple path of length k in G

b) From a) we know when $k \geq 2$, G has a simple path of length at least k.

Start with the simple path of length at least k.

Since $\deg(\text{all } v) \geq k$, we follow one more edge from the last vertex in the path to create a cycle.

The cycle has a len. $\geq k+1$

Therefore when $k \geq 2$, G is guaranteed to have a cycle of len $\geq k+1$.

5. Prove that every closed odd walk in a graph contains an odd cycle.

Let W be a closed odd walk in G

Let $|W|$ denote the number of edges in W

Since W is closed and odd, $|W| = 2k+1$ for some $k \geq 0$

By Handshaking Lemma, $\exists C$ cycle in W.

Let $|C|$ number of edges in C.

Since C is a subwalk of W, $|C| \leq |W|$

if $|C|$ odd, then C is an odd cycle in walk W odd.

if $|C|$ even, then $|W| - |C| = 2m+1$ for some $m \geq 0$, $|W|$ odd.

This means the remaining subwalk of W has an odd number of edges.

the remaining subwalk must contain an cycle odd.

Therefore every odd walk W in G contains an odd cycle.

6. Let P_1 and P_2 be two paths of maximum length in a connected graph G.

Prove that P_1 and P_2 have a common vertex.

Let G be a connected graph. Let P_1 am P_2 paths of max. len. in G.

Proof: Assume P_1, P_2 have no common vertex.

$\exists Q$ path connects a vertex in P_1 to a vertex in P_2.

$W_{walk} = Cat(P_1, Q, P_2)$ Then: $|W| = |P_1| + |Q| + |P_2| > \max(|P_1|, |P_2|)$ Proved by Contradiction.

7. Prove that every 2-connected graph contains at least one cycle.

Let G a 2-connected graph
Assume G has no cycles.
Since G is 2-connected, $\forall u, v \in V(G)$, $\exists$ at least 2 vertex-disjoint paths connecting u and v.

However, if G has no cycles, then all paths in G are <u>trees</u>, then $\forall u, v \in V(G)$, $\exists$ at most 1 path between (u, v)
by contradiction proved.

<u>More Exercises:</u>

1. For $n \geq 2$ prove that K_n has $n(n-1)/2$ edges.

Each vertex v is connected to $(n-1)$ other vertices.
therefore: $n(n-1)$

To avoid double-counting for edges:
$$\frac{n(n-1)}{2} \quad \text{same as: } \binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$$

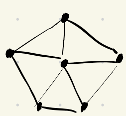
2. Determine whether K_4 is a subgraph of $K_{4,4}$

We need to check if we can find a set of 4 vertices in $K_{4,4}$, where all possible edges between them are present.
$K_{4,4} \rightarrow$ two sets of 4 vertices.

ex: let's choose $\{a, b\}$ from one partition, $\{c, d\}$ other, then subgraph by $\{a, b, c, d\}$ is a K_4 subgraph $\subset K_{4,4}$.

3. The Line graph $L(G)$ of a graph G is defined in the following way:
- the vertices of $L(G)$ are the edges of G, $V(L(G)) = E(G)$
- two vertices in $L(G)$ are adjacent iff. corresponding edges in G share a vertex.

a) Find $L(G)$ for the graph.



K_5 has: $\binom{5}{2} = 10$ edges
K_5 complete $\Rightarrow L(K_5)$ complete on 10 vertices (K_{10})

b) Complement of $L(K_5)$

$\bar{G}$ (same vertices as G), two vertices adjacent in $\bar{G}$ iff not adjacent in G

For $L(K_5) = K_{10}$, $\overline{L(K_5)}$ will have no edge (as K_{10} is complete)

Trees Quiz

which ones are Trees.