Exercise 8

Deadline: 01.08.2019

This exercise deals with Hidden Markov Models and Causal Analysis.

Regulations

Please create a Jupyter notebook `robot.ipynb` (along with export `robot.html`) for task 1 and a PDF file `causality.pdf` for tasks 2 and 3. Zip all files into a single archive with naming convention (sorted alphabetically by last names)

`lastname1-firstname1_lastname2-firstname2_exercise08.zip`

or (if you work in a team of three)

`lastname1-firstname1_lastname2-firstname2_lastname3-firstname3_exercise08.zip`

and upload it to Moodle before the given deadline. We will give zero points if your zip-file does not conform to the naming convention.

1 Robot on a Circle (18 Points)

Consider the following tracking problem. A Robot is constrained to a circular corridor with S discrete positions.

Transition

At each timestep, a command is sent to the robot to either move clockwise or counter-clockwise. The transition probabilities for the commands are supposed to be independent

$$A = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}.$$

However, the probability that the robot receives the signal correctly is only $\epsilon < 1$. When the command is incorrectly received, the robot will move in the opposite direction. Adjust your model accordingly.

Measurement

You do not know the robot's command and can only get a faulty measurement of its position. Your sensor fails completely with a probability of ω (model this as an additional state). It gives you a completely random position (uniformly distributed) with probability τ. In all other cases, your measurement will yield the correct position (probability $1 - \omega - \tau$).

Tasks:

- Describe this process with a hidden Markov Model (draw the nodes and include their possible states).
- Compute all transition matrices for $S = 50$, $\epsilon = 0.4$, $\omega = 0.3$ and $\tau = 0.1$.
- Create your own implementation of the Baum-Welch algorithm.
- Use the sequence of measurements given in file `train_walk.npy` on Moodle to learn the values of the transition matrices using the Baum-Welch algorithm.
- Use the Viterbi algorithm from the previous homework to compute the globally most likely path, given the measurements in file `test_walk.npy` in Moodle. Compare this to the locally most likely position in each timestep.

2 Conditional Independence (6 Points)

1. Let A, B, C three binary random variables with different distributions. Prove by counterexample that

$$A \perp B \,|\, C$$

does not imply

$$A \perp B.$$

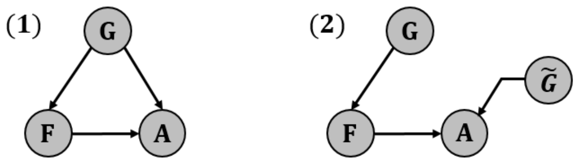
Hint: To construct the counterexample, create two tables for the conditional probabilities (one for *C=true* and one for *C=false*) that conform to the conditional independence assumptions. Then marginalize the tables over C and show that unconditional independence does not hold.

3 Berkeley Admission (16 Points)

We consider again the Berkeley admission example from the lecture. Recall that the University of Berkeley was sued because it apparently preferred men over woman. This exercise constructs a counter-argument against this claim of gender discrimination. Applications vs. admissions data for six fields of study are given in the following table:

field	men		woman	
	applied	admitted	applied	admitted
A	825	512	108	89
B	560	353	25	17
C	325	120	593	202
D	417	138	375	131
E	191	53	393	94
F	272	16	341	24

The training data can be used to compute the joint probability $p(G, F, A)$ (with G = gender $\in$ {male, female}, F = field of study $\in \{A, ..., F\}$, and A = admission $\in$ {true, false}). We can express the two competing claims in terms of causal models that treat F as a mediating variable between G and A, and either contain or do not contain a direct causal influence from G to A.



Total Causal Effect

The total causal effect from gender to admission in model (1) is defined by the interventional distributions

$$p_1(A = \text{true}|\, do(G = \text{male})) \quad \text{and} \quad p_1(A = \text{true}|\, do(G = \text{female}))$$

Since gender is a root variable in the present application, these distributions coincide with the conditional distributions here:

$$p_1(A = \text{true}|\, do(G)) = p_1(A = \text{true}|\, G)$$

Tasks:

1. Derive the formula to compute these conditional probabilities from the factorization according to model (1).
2. Use the given data to calculate the conditional probability of $A =$ true for male and female applicants and prove that men indeed had a higher chance of being accepted, indicating a total causal effect in favor of men.

Direct Causal Effect

The University of Berkeley argued that only a *direct* causal effect from gender to admission would constitute a case of discrimination. To this end, we compare the conditional probabilities derived from models (1) and (2). If both conditionals are (nearly) the same, there is no discrimination. If the elimination of the direct link increases the acceptance chances for woman or men in model (2), a direct causal effect and therefore a case of discrimination in the respective direction is revealed.

When the factorization for model (1) is known, we can derive the factorization of model (2) by means of the "cut" operator. The critical factor to be considered is the conditional probability of A:

$$\begin{aligned} p_2(A \,|\, F) =& p_2(A \,|\, \mathrm{PA}_2(A)) = p_1(A \,|\, \mathrm{PA}_1(A), \mathit{cut}(G \to A)) = p_1(A \,|\, G, F, \mathit{cut}(G \to A)) \\ =& \sum_{\widetilde{G}} p_1(A \,|\, \widetilde{G}, F) p_1(\widetilde{G}) \end{aligned}$$

where $\mathrm{PA}(A)$ indicates the parents of A, and $p_1(\widetilde{G})$ is the marginal distribution of G in model (1). In words, we introduce an independent copy $\widetilde{G}$ of variable G into the model and reconnect A's incoming arc from G to $\widetilde{G}$. The variable $\widetilde{G}$ is distributed according to the marginal distribution of G and can be obtained from G by a random shuffle of the gender feature, which destroys any assiciation between gender and admission. Since we are not interested in $\widetilde{G}$, we immediately marginalize it out.

Tasks:

1. Derive the complete formula for computing $p_2(A \,|\, G)$ by applying the cut-operator to model (1).
2. Use the given data to calculate the conditionals $p_2(A = \text{true} |\, G)$ according to model (2). Check if there is discrimination by comparing these results with $p_1(A = \text{true} |\, G)$.

G-Test

Even if the university as a whole doesn't discriminate woman, there is still the possibility that woman are discriminated in one field of study, provided that other fields compensate for this by discrimination in the opposite direction. We can directly check for discrimination in individual fields using the *G-test* (`https://en.wikipedia.org/wiki/G-test`). The entropy (in units of "nats") of a discrete random variable X with C_X possible states is defined as

$$H(X) = \sum_{k=1}^{C_X} -p(X = k) \ln p(X = k)$$

and the joint entropy of two variables X, Y with number of states C_X, C_Y is

$$H(X, Y) = \sum_{k=1}^{C_X} \sum_{l=1}^{C_Y} -p(X = k, Y = l) \ln p(X = k, Y = l)$$

The *mutual information*

$$\mathrm{MI}(X, Y) = H(X) + H(Y) - H(X, Y)$$

tells us how much information we gain about variable Y if we learn the value of variable X and vice versa. It is zero if and only if X and Y are independent. The G-test uses the mutual information in order to decide if a deviation of observed counts from their expected values may be due to chance, assuming as a null hypothesis that X and Y are in fact independent. If the p-value of this test (i.e. the probability of the actual observations under the null hypothesis) is small, we reject the null and conclude that X and Y must be dependent. The G-test statistic

$$G = 2N\,\widehat{\mathrm{MI}}(X, Y)$$

has a χ^2-distribution with $(C_X - 1)(C_Y - 1)$ degrees of freedom, where N is the number of observations, and $\widehat{\mathrm{MI}}$ is the empirical estimate of the mutual information (obtained by counting the different cases). Thus, the standard machinery of the χ^2-test can be used to determine the p-value. Applied to the Berkely admission problem, X is the gender of the applicants to a particular field, Y is admission into that field, and N is the total number of applicants to this field. If X and Y are found to be dependent, this would constitute a case of discrimination.

Tasks:

1. Compute the empirical mutual information $\widehat{\mathrm{MI}}$ between gender and admission for each of the six fields of study and for the university as a whole.
2. Calculate corresponding p-values and decide if the data are explainable by chance variation.