We are concerned with the variance of zero-centred normal distributed random variables transformed by the sine. Therefore we consider a mean of a particular transformed Wiener process

$$f(t) = \mathbb{E}\left[\sin^2(cW_t)\right].$$

Knowing the derivatives

$$\frac{d}{dx}\sin^2(cx) = 2c\sin(cx)\cos(cx) = c\sin(2cx)$$

.

$$\frac{d^2}{dx^2}\sin(x)^2 = 2c^2\cos(2cx),$$

we may apply Itô's lemma

$$\begin{aligned}\sin^2(cW_t) &= \int_0^t c\sin(2cW_s)dW_s + \frac{1}{2}\int_0^t 2c^2\cos(2cW_s)ds = \\ &= \int_0^t c\sin(2cW_s)dW_s + \int_0^t c^2\cos(2cW_s)ds.\end{aligned}$$

Consequently

$$f(t) = c^2\int_0^t \mathbb{E}\left[\cos(2cW_s)\right]ds.$$

Now, consider

$$g(t,u) = \mathbb{E}\left[\cos(uW_t)\right].$$

Applying Itô's lemma we obtain

$$\begin{aligned}g(t,u) = \mathbb{E}\left[\cos(u(W_s))\right] &= \mathbb{E}\left[\frac{1}{2}\int_0^t -u^2\cos(uW_s)ds\right] = \\ &= -\frac{1}{2}u^2\int_0^t \mathbb{E}\left[\cos(uW_s)\right]ds = -\frac{1}{2}u^2\int_0^t g(s,u)ds.\end{aligned}$$

Thus

$$\frac{\partial g}{\partial t}(t,u) = -\frac{1}{2}u^2 g(t,u)$$

and

$$g(t,u) = C(u)e^{-\frac{1}{2}u^2 t}$$

.

Since

$$g(0,u) = 1$$

We have

$$C(u) = 1$$

and thus that

$$g(t,u) = e^{-\frac{1}{2}u^2 t}.$$

Thus

$$\begin{aligned} f(t) &= c^2 \int_0^t \mathbb{E}\left[\cos(2cW_s)\right] ds = c^2 \int_0^t g(t,2c) = \\ &= c^2 \int_0^t e^{-\frac{1}{2}4c^2 t} = c^2 \int_0^t e^{-2c^2 t} = \\ &= c^2 \frac{e^{-2tc^2} - 1}{-2c^2} = \frac{1-e^{-2c^2 t}}{2}. \end{aligned}$$

We can conclude that

$$\mathbb{E}\left[\sin(cZ)^2\right] = \frac{1-e^{-2c^2}}{2},$$

Calculation of Var $[\cos(cZ)]$

In the calculation of $\mathbb{E}[\sin(cZ)^2]$ we obtained two results that are relevant also for this calculation, that

$$\mathbb{E}[\cos(uW_t)] = e^{-\frac{1}{2}u^2 t}$$

and the conclusion of the previous calculation, that

$$\mathbb{E}[\sin(cZ)^2] = \frac{1 - e^{-2c^2}}{2}.$$

Because

$$\mathbb{E}\left[\sin(cZ)^2 + \cos(cZ)^2\right] = 1$$

it immediately follows that

$$\begin{aligned}\mathbb{E}\left[\cos(cZ)^2\right] = 1 - \mathbb{E}\left[\sin(cZ)^2\right] = \\ = 1 - \frac{1 - e^{-2c^2}}{2} = \frac{1}{2} + \frac{1}{2}e^{-2c^2}.\end{aligned}$$

Using that

$$\mathrm{Var}\left[X\right] = \mathbb{E}\left[\left(X - \mathbb{E}\left[X\right]\right)^2\right] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

we obtain

$$\begin{aligned}\mathrm{Var}\left[\cos\left(cZ\right)\right] &= \frac{1}{2} + \frac{1}{2}e^{-2c^2} - \left(e^{-\frac{1}{2}c^2}\right)^2 = \\ &= \frac{1}{2} + \frac{1}{2}e^{-2c^2} - e^{-c^2}\end{aligned}$$