$\boldsymbol{E}\left[\exp\left(t\sin(cW_t)\right)\right]$

Let

$$f(x) = \exp\left(t\sin\left(cx\right)\right)$$

then

$$f'(x) = \exp\left(t\sin\left(cx\right)\right)\cos\left(cx\right)c$$

$$f''(x) = c^2\left(\cos\left(cx\right) - \sin\left(cx\right)\right)\exp\left(t\sin\left(cx\right)\right).$$

Thus

$$E[f\left(W_t\right)] = \frac{1}{2}c^2\int_0^t \mathbb{E}\left[\left(\cos(cW_s) - \sin(cW_s)\right)\exp\left(t\sin\left(cW_s\right)\right)\right]ds.$$

Let

$$g(x) = \left(a\cos(x) + b\sin(x)\right)\exp(t\sin\left(cx\right))$$

then

Let Z be a standard normal distributed random variable. We seek a random variable X such that

$$\mathbb{E}[\exp(itX\sin(cZ))] = \mathbb{E}[\exp(itZ)].$$

Conditioning we obtain

$$\mathbb{E}[\mathbb{E}[\exp(itX\sin(cZ))|X]] = \mathbb{E}[\exp(itZ)].$$

Let

$$f(x,t) = \mathbb{E}[\exp(x\sin(cW_t))].$$

Then

$$f(x,t)$$