N-body Project

Mladen Ivkovic

January 2018

Introduction

Goal: Calculation of gravitational force on a system of collisionless particles

Possible methods:

- Direct calculation
- Iterative methods on a mesh
- Fourier methods on a mesh
- Tree methods (hierarchical multipole methods)

In this project, I implemented a direct calculation and a tree method.

Dataset

The dataset for which to compute the forces is a set of $\sim 50'000$ particles aranged according to the spherically symmetric “Hernquist model” :

$$\rho(r) = \frac{M}{2\pi}\frac{a}{r}\frac{1}{(r+a)^3}$$

$$M(r) = M\frac{r^2}{(r+a)^2} \qquad \Rightarrow M(a) = \frac{M}{4}$$

$$\phi(r) = -\frac{GM}{r+a}$$

Choice of Units

Use dimensionless units, with $G \equiv 1$

$\Rightarrow$ Reduce number of multiplications necessary

$\Rightarrow$ Reduce effect of finite floating point precision by moving problem to a better suited order of magnitude

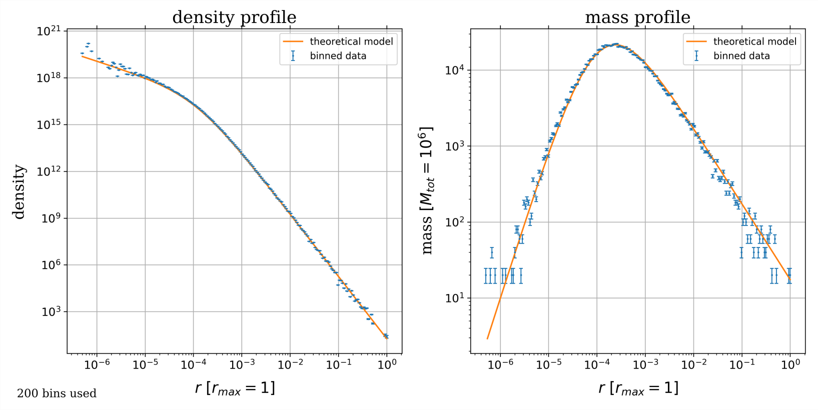
Define a scale for every physical quantity:

$$a_{phys} \equiv A_0 a_{code}$$

Setting $G = 1$ restricts the scale for either time, mass or distance. I chose

$$M_0 = M_{tot} \qquad \text{such that} \qquad M_{tot,code} = 1$$
$$R_0 = R_{max} \qquad \text{such that} \qquad R_{max,code} = 1$$
$$\Rightarrow \quad T_0 = \sqrt{\frac{R_0^3}{GM_0}}$$

Dataset



Only in this plot: $M_{tot} \equiv 10^6$ so that the errorbars don't dominate the plot.

Direct forces calculation

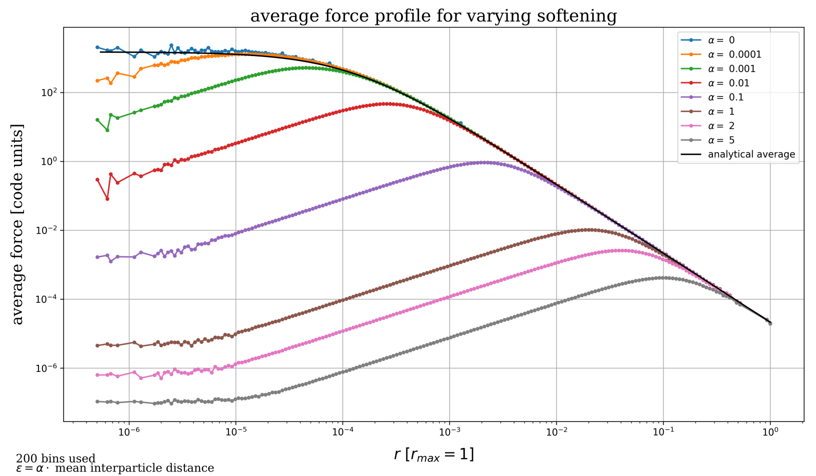
Acceleration of particle i:

$$\ddot{\mathbf{r}}_i = -G \sum_{j=1}^{N} \frac{m_j}{\left[(\mathbf{r}_i - \mathbf{r}_j)^2 + \epsilon^2\right]^{\frac{3}{2}}} (\mathbf{r}_i - \mathbf{r}_j)$$

ϵ is the *softening length*. It's purposes are:

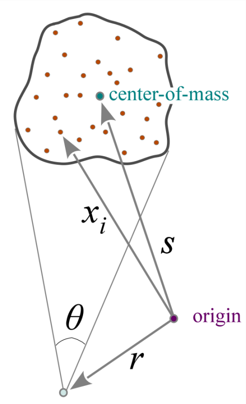
- computational efficiency
- avoid large angle scatterings
- avoid expense to calculate orbits in a singular potential
- prevent the possibility of formation of bound particle pairs

Direct force calculation results for varying softening



Hierarchical Multipole Method

Central idea: use the multipole expansion of a distant group of particles to describe its gravity, instead of summing up the forces from all individual particles. (For close groups, I use direct force calculation.)



Multipole expansion of the potential gives:

$$\Phi(\mathbf{r}) = -G\left(\frac{M}{|\mathbf{y}|} + \frac{1}{2}\frac{\mathbf{y}^T\mathbf{Q}\mathbf{y}}{|\mathbf{y}|^5}\right)$$

$$\mathbf{y} = \mathbf{r} - \mathbf{s}$$

$$Q_{ij} = \sum_k m_k[3(\mathbf{s}-\mathbf{x_k})_i(\mathbf{s}-\mathbf{x_k})_j - \delta_{ij}(\mathbf{s}-\mathbf{x_k})^2]$$

The expansion is valid for $\theta \approx \frac{l}{y} \ll 1$

The particles are grouped hierarchically and the multipole moments are pre-computed for later use.

I used the Barnes-Hut oct-tree: Assume particles are in a cube. The cube is then recursively subdivided into 8 sub-cubes of half the size in each spatial dimension, until each sub-cube contains only a single particle (or some other user-set limit).

Force calculation:

For all leaf cells:

  For all root cells:

    walk tree (this leaf cell, root cell)

Walking the tree (target cell, source cell):

If source cell is leaf cell: Use direct force calculation, end walk for this source.

If target (leaf) cell is inside this source cell:

  for all children of the source cell:

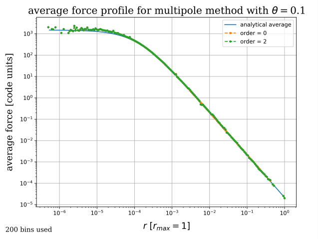
    walk the tree (target, child of source)

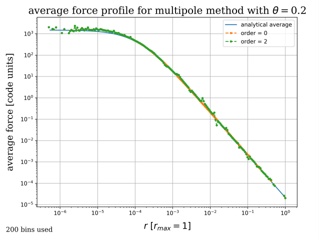
If target (leaf) cell is *not* inside this source cell:

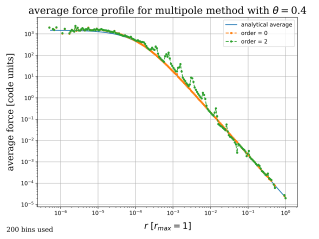
  if $\theta < \theta_{max}$: Calculate multipole force, stop walk for this source

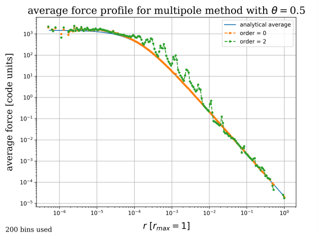
  else: for all children of the source cell:

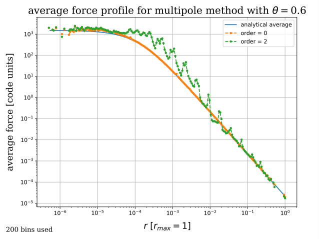
    walk the tree (target, child of source)

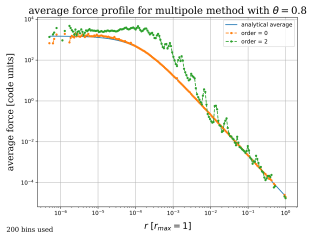

average force profile for multipole method with $\theta = 0.1$
analytical average
order = 0
order = 2
average force [code units]
r [$r_{max} = 1$]
200 bins used


average force profile for multipole method with $\theta = 0.2$
analytical average
order = 0
order = 2
average force [code units]
r [$r_{max} = 1$]
200 bins used


average force profile for multipole method with $\theta = 0.4$
analytical average
order = 0
order = 2
average force [code units]
r [$r_{max} = 1$]
200 bins used


average force profile for multipole method with $\theta = 0.5$
analytical average
order = 0
order = 2
average force [code units]
r [$r_{max} = 1$]
200 bins used

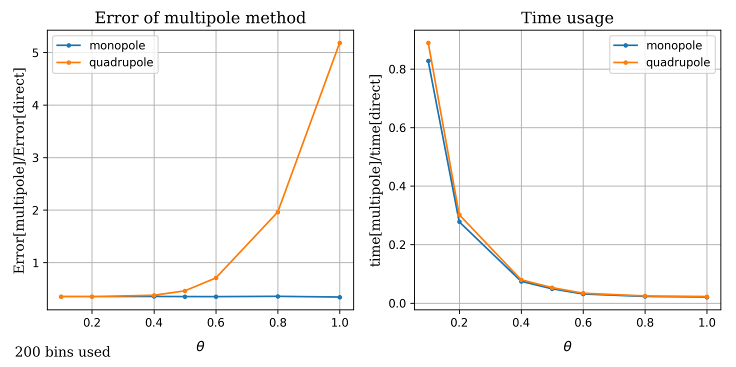

average force profile for multipole method with $\theta = 0.6$
analytical average
order = 0
order = 2
average force [code units]
r [$r_{max} = 1$]
200 bins used


average force profile for multipole method with $\theta = 0.8$
analytical average
order = 0
order = 2
average force [code units]
r [$r_{max} = 1$]
200 bins used

Comparison Direct vs multipole calculation

Estimating accuracy with the $L1$ norm:
$L1 = \frac{1}{N}\sum_i |\bar{F}_i - \bar{F}_{analytical}(r_i)|$, where $\bar{F}_{analytical}(r_i)$ is the analytical average force at r_i.

For both the direct forces and multipole method, $\epsilon = 0.01 \cdot mid$. Shown are the relative values for both monopole and quadrupole oders, relative to the values for the direct forces calculation.



Program, plotting scripts and this presentation available on
`https://bitbucket.org/mivkov/computational_astrophysics`