Advection Project

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Introduction

The ideal gas equations form a set of hyperbolic differential equations of the form

$$\frac{\partial \mathbf{U}}{\partial t} + \Delta \cdot \mathbf{F} = 0$$

In this project, consider the 1D advection of the mass density ρ with a global constant velocity u:

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} = 0$$

The analytical solution is given by any function $q(x)$ with $\rho(x,t) = q(x - ut)$, which is $q(x)$ again translated by ut along the x-direction.

Solution Scheme: Finite Volume Method

Divide space into N cells of same size.

Make use of the divergence theorem: Convert volume integrals of finite volumes (cells) of divergence terms to surface integrals of fluxes though surfaces.

Advantage for conservation laws: Finite volume method is conservative, because $F_{i+\frac{1}{2}} = F_{(i+1)-\frac{1}{2}}$

Solution Scheme: Problems

Direction of information

In this problem, all characteristics propagate downstream: Information strictly travels in the flow direction.
$\Rightarrow \rho_i$ mustn't depend on downstream value. Otherwise, you'll be using "information that isn't there yet".
$\Rightarrow$ The discretisation will depend on the sign of the velocity u.

CFL Condition

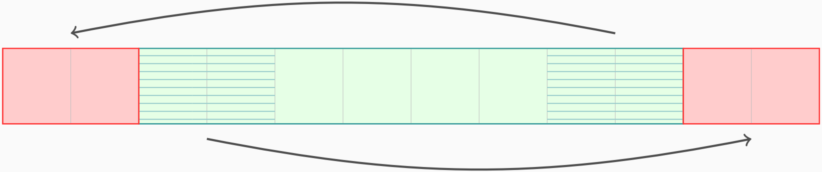
For explicit integration methods, we need to limit the time step such that information can travel at most by 1 cell during each time step to avoid instabilities.
In 1D: $\Delta t \leq \Delta x / u$; In ND: $\Delta t \leq [\sum_i^N u_i / \Delta x_i]^{-1}$

Numerical Diffusion

The solution is not advected perfectly (except for $\Delta t = \frac{\Delta x}{u}$) but smoothed out. The numerical algorithm has a diffusion term as a by-product For a first order algorithm, we're effectively solving
$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} = \frac{\Delta x u}{2} \frac{\partial^2 \rho}{\partial x^2}$

Periodic Boundary Conditions

Simulate a periodic boundary by introducing ghost cells:



Piecewise Constant Method: Donor-Cell Advection

Assume cell state within cell is constant.
For $u = 1$:

$$\rho_i^{n+1} = \rho_i^n + \frac{\Delta t}{\Delta x}(f_{i-1/2}^{n+1/2} - f_{i+1/2}^{n+1/2})$$

$$f_{i\pm 1/2}^{n+1} = u_{i\pm 1/2} \quad \rho_{i-1/2\pm 1/2}$$

First order accurate in time and space.

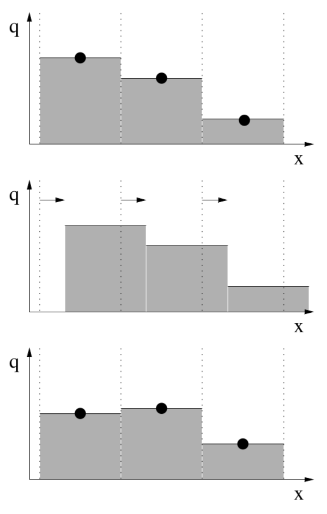
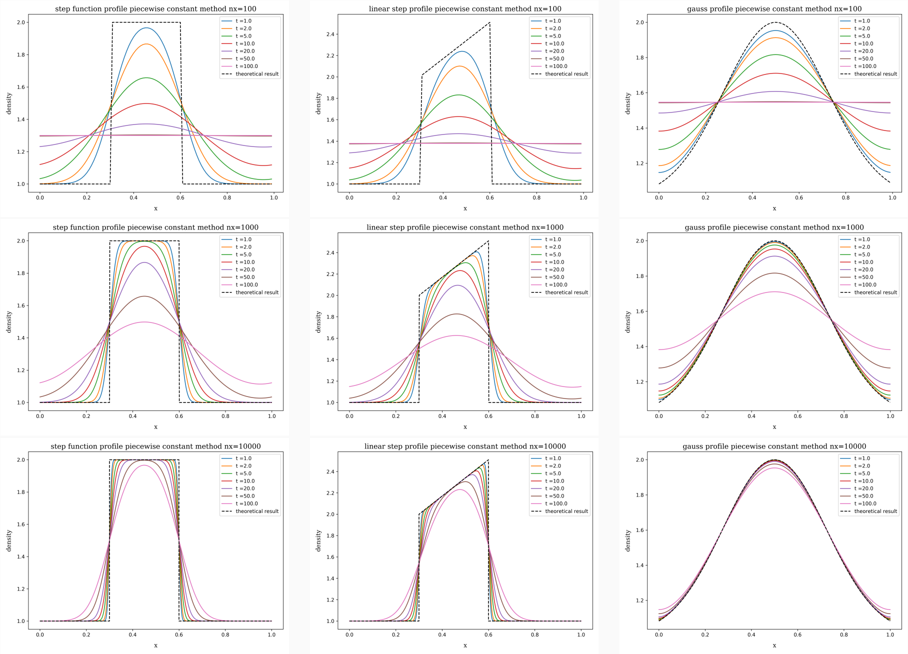


Image adapted from "Lecture Numerical Fluid Dynamics", Lecture given by C.P. Dullemond and H.H. Wang at Heidelberg University, 2009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t =2.0
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t =20.0
t =50.0
t =100.0
theoretical result
density
x

Piecewise Linear Method

Assume that the state in the cell is piecewise linear with some slope s. This gives a second-order accurate method.

$$\text{For} \quad x_{i-1/2} < x_i < x_{i+1/2}: \quad \rho(x, t = t_n) = \rho_i^n + s_i^n (x - x_i)$$

$$\text{For} \quad t_n < t < t_{n+1}: \quad \rho(x, t) = \rho_i^n + s_i^n (x - [x_i + u(t - t_n)])$$

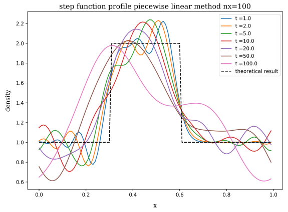
The flux over the interface is then

$$\begin{aligned} f_{i-1/2}(t) &= u\rho(x = x_{i-1/2}, t) \\ &= u\rho_{i-1} + u s_i^n ({}^{\Delta x}/_2 - u(t - t_n)) \end{aligned}$$

Finally averaging the fluxes over a time step gives:

$$\rho_i^{n+1} = \rho_i^n - \frac{u\Delta t}{\Delta x}(\rho_i^n - \rho_{i-1}^n) - \frac{u\Delta t}{\Delta x}\frac{1}{2}(s_i^n - s_{i-1}^n)(\Delta x - u\Delta t)$$

Choice of slope: $s_i^n = \frac{\rho_{i+1}^n - \rho_i^n}{\Delta x}$ (Lax-Wendroff method)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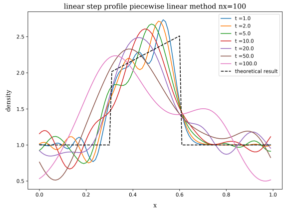
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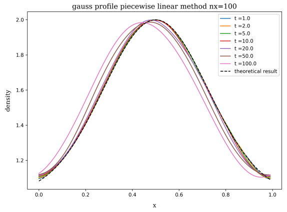
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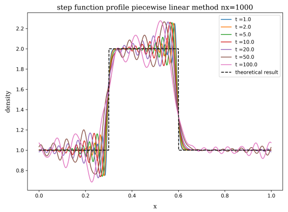
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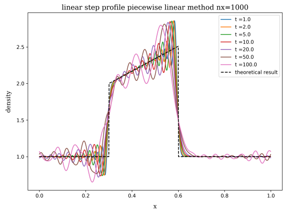
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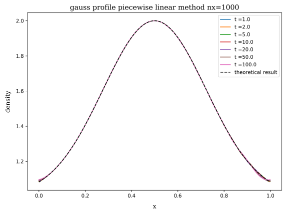
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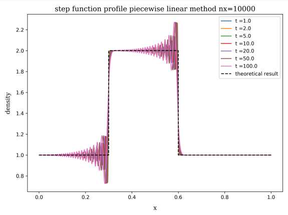
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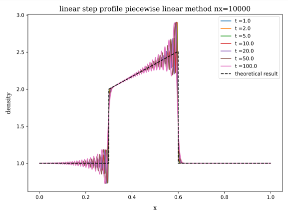
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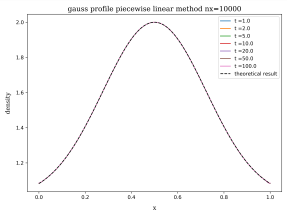
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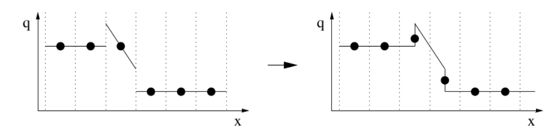
New Problem: Oscillations

The piecewise linear elements can have overshoots, leading to the oscillations seen in the previous plots.



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Godunov's theorem: *any linear algorithm for solving partial differential equations, with the property of not producing new extrema, can be at most first order.*

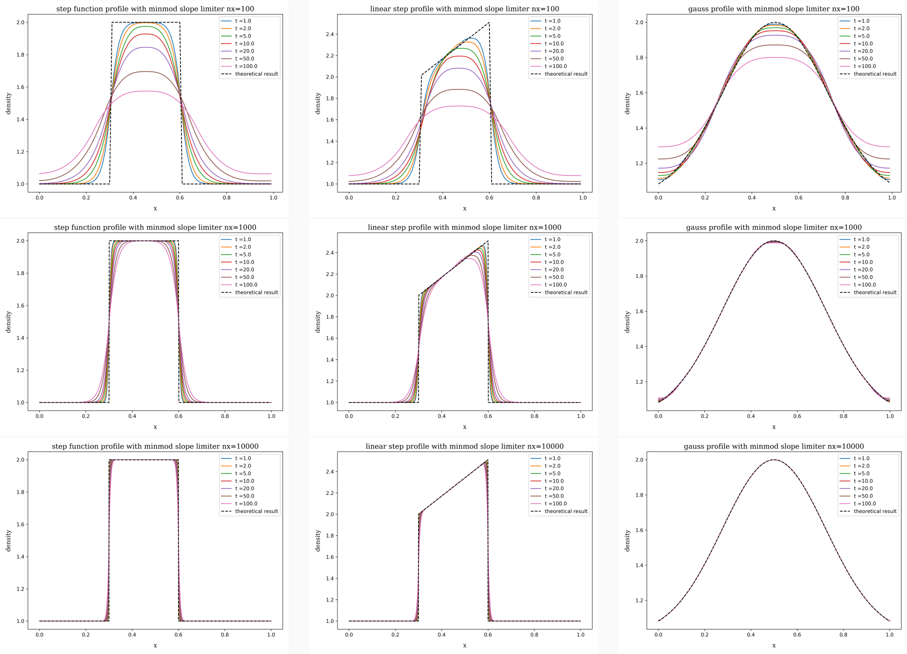
$\Rightarrow$ use non-linear conditions (slope limiters) to modify the slope s_i^n to prevent overshoots. Requirement: total variation diminishing: $TV(\rho^{n+1}) \leq TV(\rho^n) \equiv \sum |\rho_i - \rho_{i-1}|$. Such a scheme will not develop oscillations near a jump, because a jump is a monotonically in/decreasing function and a TVD scheme will not increase the TV.

Minmod Slope Limiter

$$s_i^n = \text{minmod}\left(\frac{\rho_i^n - \rho_{i-1}^n}{\Delta x}, \frac{\rho_{i+1}^n - \rho_i^n}{\Delta x}\right)$$

with

$$\text{minmod}(a, b) = \begin{cases} a & \text{if } |a| < |b| \text{ and } ab > 0 \\ b & \text{if } |a| > |b| \text{ and } ab > 0 \\ 0 & \text{if } ab < 0 \end{cases}$$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linear step profile with minmod slope limiter nx=100
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t =10.0
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t =50.0
t =100.0
theoretical result

Van Leer Slope Limiter

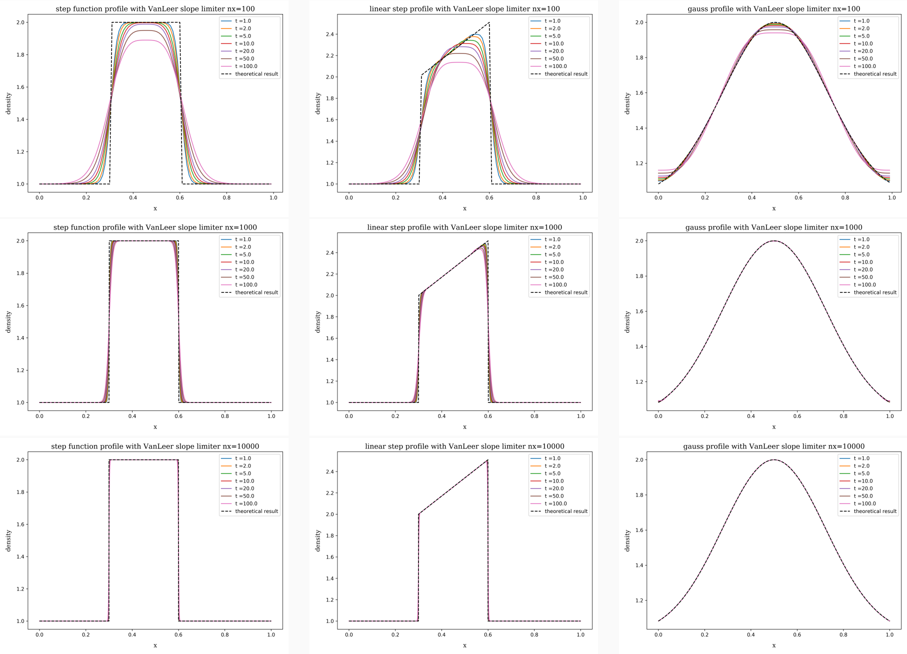
Rewrite flux (assuming $u = 1$) as

$$f_{i-1/2}^{n+1/2} = u\rho_{i-1} + \frac{1}{2}u\left(1 - \frac{u\Delta t}{\Delta x}\right)\phi(r_{i-1/2}^n)(q_i^n - q_{i-1}^n)$$
$$r_{i-1/2}^n = \frac{q_{i-1}^n - q_{i-2}^n}{q_i^n - q_{i-1}^n}$$

Here, ϕ is the slope/flux limiter.

The Van Leer flux limiter is defined as

$$\phi(r) = \frac{r - |r|}{1 - |r|}$$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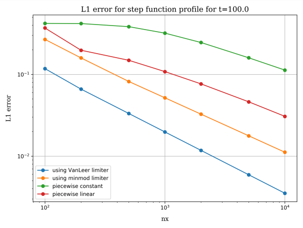
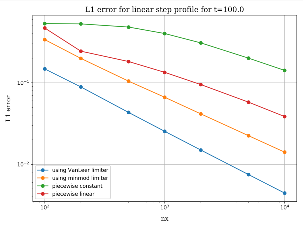
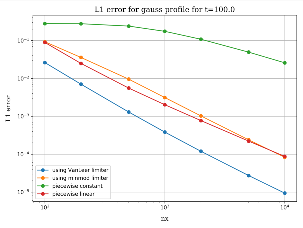
Precision of the Algorithms

Quantify error through $L1$ error norm: $L1 = \frac{1}{N} \sum_i |\rho_i - \tilde{\rho}(x_i)|$, where $\tilde{\rho}(x_i)$ is the analytical solution.

For $t = 100$:







2D Advection

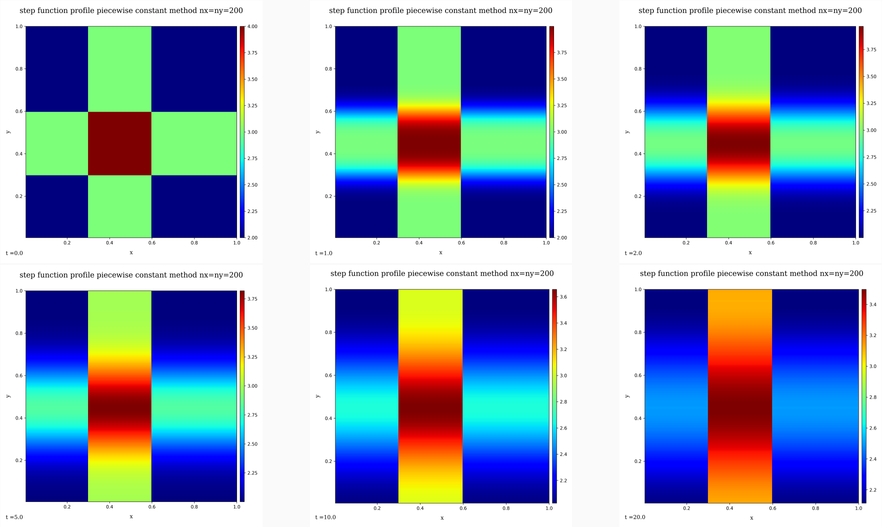
The 2D advection equation is given by

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} = 0$$

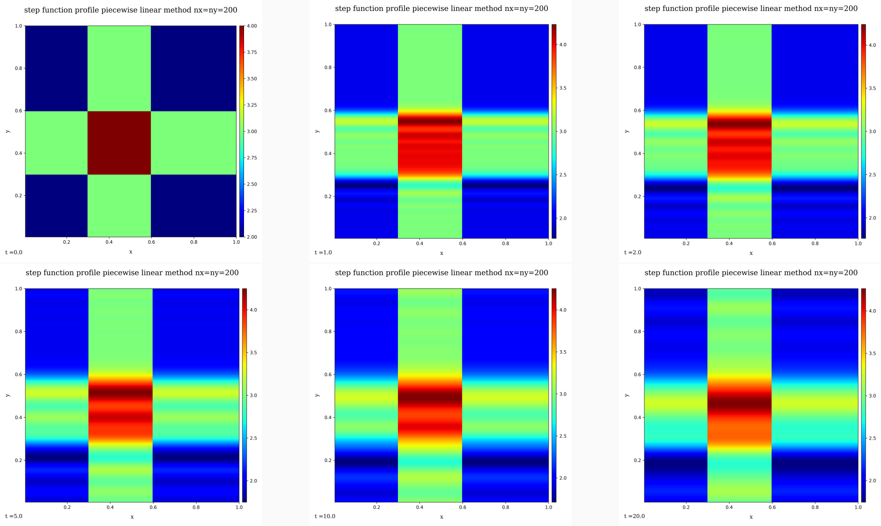
Solution using the same algorithms for each dimension, adding fluxes simultaneously.

Results are shown for a step function density field with $nx = ny = 200$.

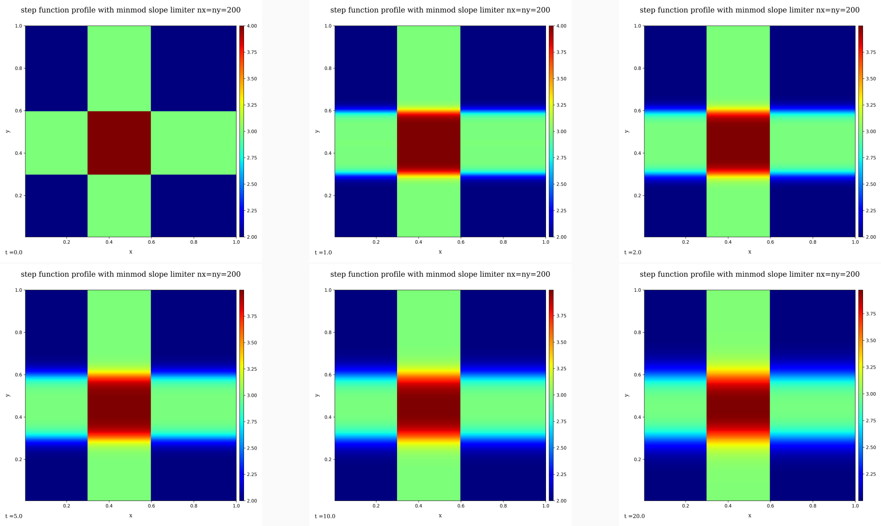
$u = 0, v = 1$ Piecewise Constant Method



$u = 0, v = 1$ Piecewise Linear Method



$u = 0, v = 1$ Minmod Slope Limiter



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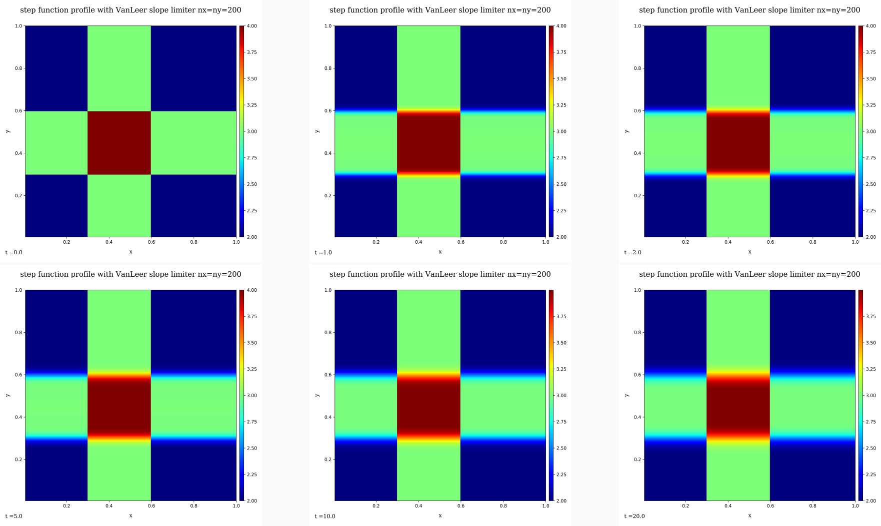
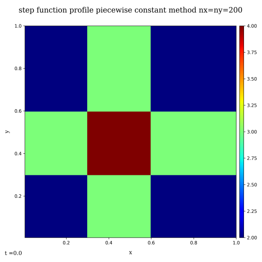
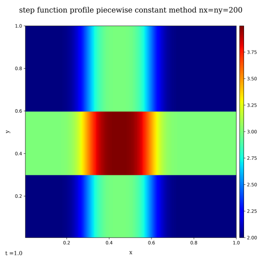
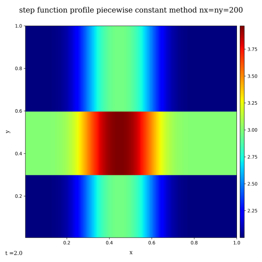
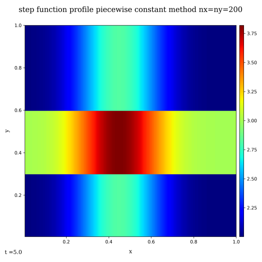


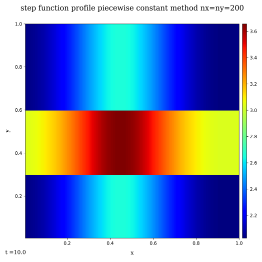
$u = 1, v = 0$ Piecewise Constant Method

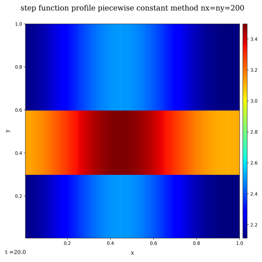




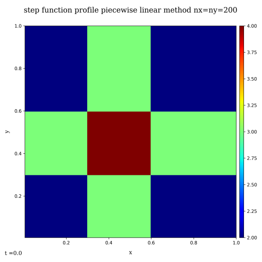


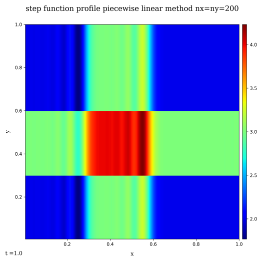


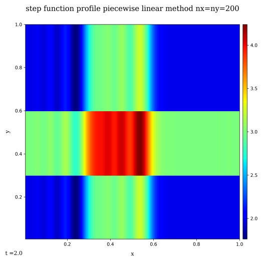


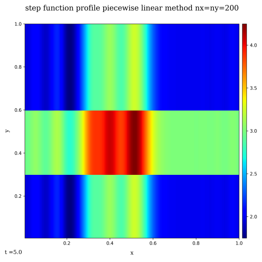


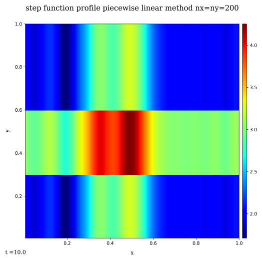
$u = 1, v = 0$ Piecewise Linear Method

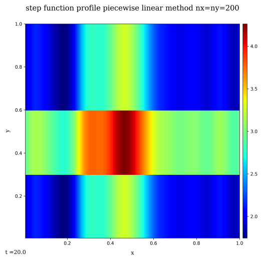




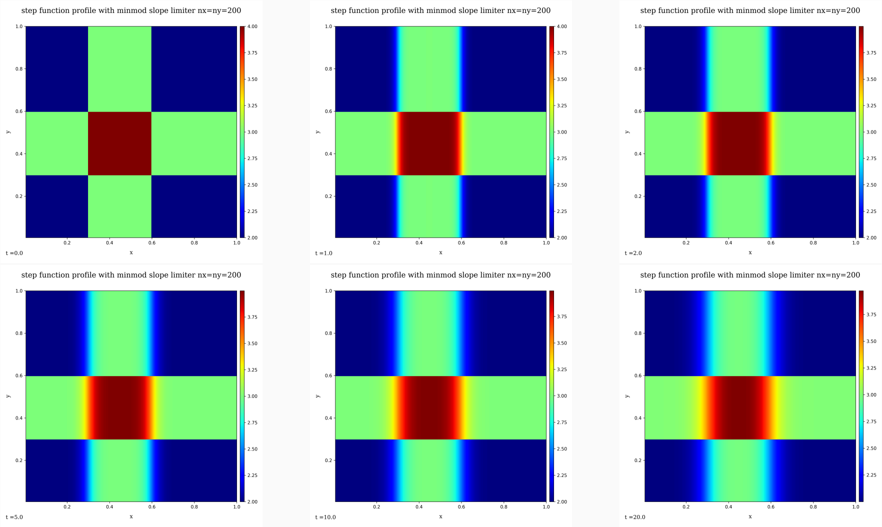




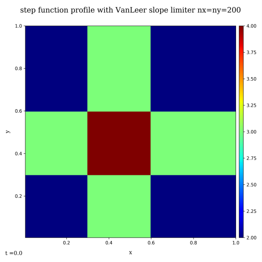


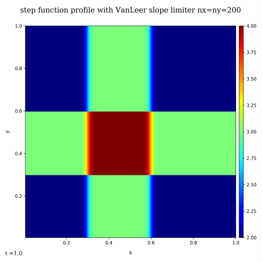


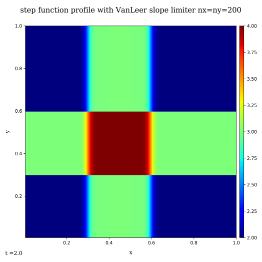
$u = 1, v = 0$ Minmod Slope Limiter

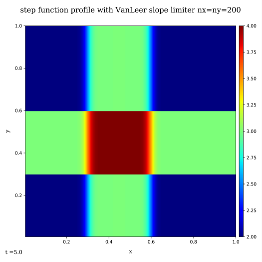


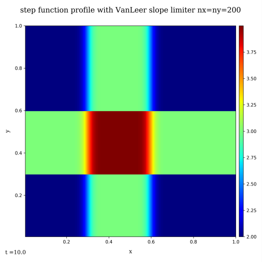
$u = 1, v = 0$ Van Leer Slope Limiter

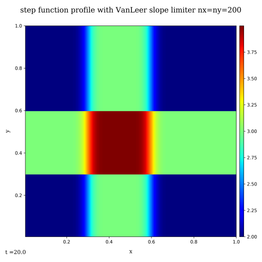




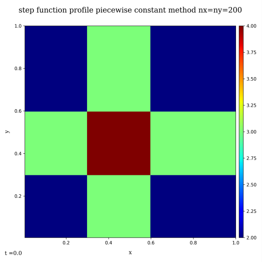


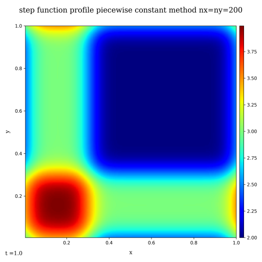


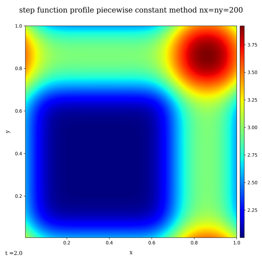


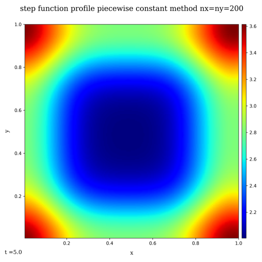


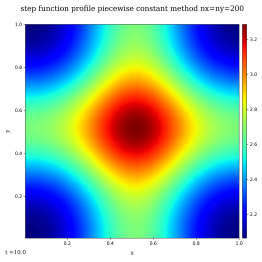
$u = v = \sqrt{2}/2$ Piecewise Constant Method

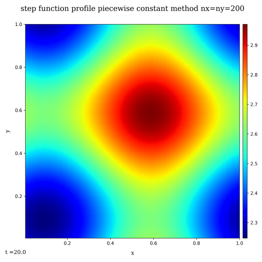




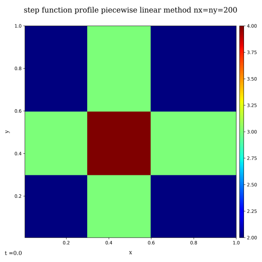


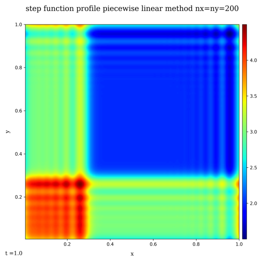


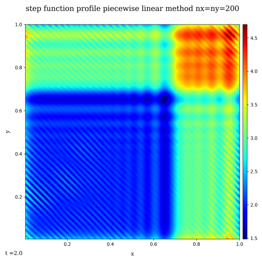


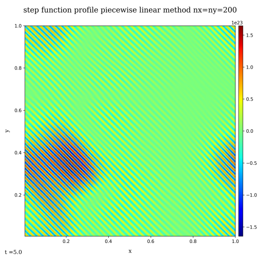


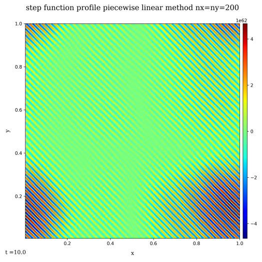
$u = v = \sqrt{2}/2$ Piecewise Linear Method

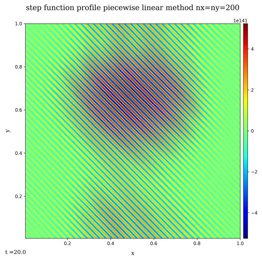




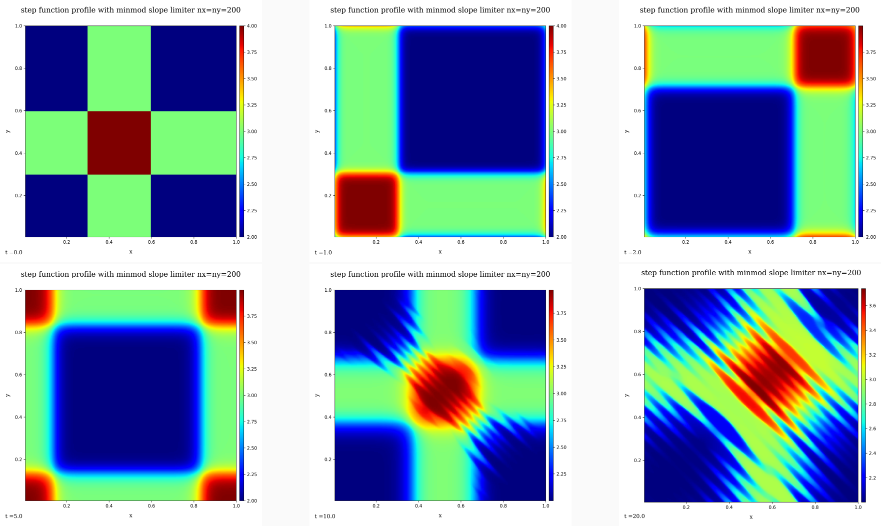




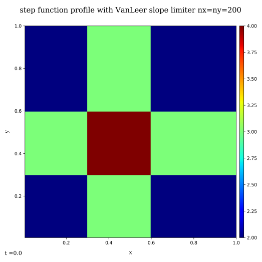


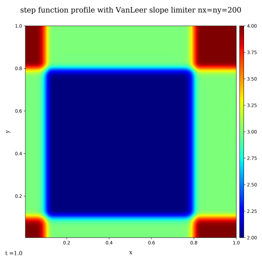


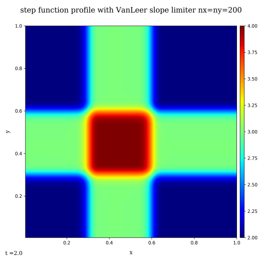
$u = v = \sqrt{2}/2$ Minmod Slope Limiter

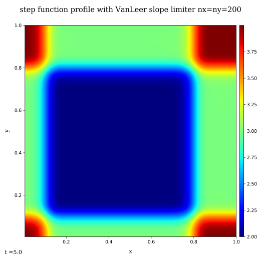


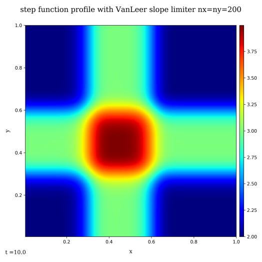
$u = v = \sqrt{2}/2$ Van Leer Slope Limiter

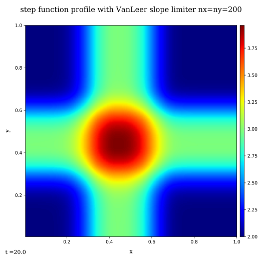












Program, plotting scripts and this presentation available on
`https://bitbucket.org/mivkov/computational_astrophysics`