

Mesh Based Numerical Hydrodynamics of Ideal Gases

The Used and Implemented Equations

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1 Ideal Gases

1.1 Governing Equations

We are mostly going to concern ourselves with ideal gases, which are described by the Euler equations:

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \rho \mathbf{v} \\ E \end{pmatrix} + \nabla \cdot \begin{pmatrix} \rho \mathbf{v} \\ \rho \mathbf{v} \otimes \mathbf{v} + p \\ (E + p) \mathbf{v} \end{pmatrix} = \begin{pmatrix} 0 \\ \rho \mathbf{a} \\ \rho \mathbf{a} \cdot \mathbf{v} \end{pmatrix} \quad (1)$$

Where

- ρ : fluid density
- \mathbf{v} : fluid (mean/bulk) velocity at a given point. I use the notation $\mathbf{v} = (u, v, w)$, or when indices are useful, $\mathbf{v} = (v_1, v_2, v_3)$
- p : pressure
- E : specific energy. $E = \frac{1}{2} \rho \mathbf{v}^2 + \rho \varepsilon$, with ε = specific internal thermal energy
- \mathbf{a} : acceleration due to some external force.

The outer product $\cdot \otimes \cdot$ gives the following tensor:

$$(\mathbf{v} \otimes \mathbf{v})_{ij} = v_i v_j \quad (2)$$

Furthermore, we have the following relations for ideal gasses:

$$p = nkT \quad (3)$$

$$p = C \rho^\gamma \quad \text{entropy relation for smooth flow, i.e. no shocks} \quad (4)$$

$$s = c_V \ln \left(\frac{p}{\rho^\gamma} \right) + s_0 \quad \text{entropy} \quad (5)$$

$$a = \sqrt{\left. \frac{\partial p}{\partial \rho} \right|_s} = \sqrt{\frac{\gamma p}{\rho}} \quad \text{sound speed} \quad (6)$$

with

- n : number density
- k : Boltzmann constant
- T : temperature
- s : entropy
- γ : adiabatic index
- c_V : specific heat

and the Equation of State

$$\varepsilon = \frac{1}{\gamma - 1} \frac{p}{\rho} \quad (7)$$

The Euler equations can be written as a conservation law as

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{U}) = 0 \quad (8)$$

where we neglect any outer forces, i.e. $\mathbf{a} = 0$, and

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho \mathbf{v} \\ E \end{pmatrix}, \quad \mathbf{F}(\mathbf{U}) = \begin{pmatrix} \rho \mathbf{v} \\ \rho \mathbf{v} \otimes \mathbf{v} + p \\ (E + p) \mathbf{v} \end{pmatrix} \quad (9)$$

1.1.1 Euler equations in 1D

In 1D, we can write the Euler equations without source terms ($\mathbf{a} = 0$) as

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} = 0 \quad (10)$$

or explicitly (remember $\mathbf{v} = (u, v, w)$)

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix} + \frac{\partial}{\partial x} \begin{pmatrix} \rho u \\ \rho u^2 + p \\ (E + p)u \end{pmatrix} = 0 \quad (11)$$

1.1.2 Euler equations in 2D

In 2D, we have without source terms ($\mathbf{a} = 0$)

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} + \frac{\partial \mathbf{G}(\mathbf{U})}{\partial y} = 0 \quad (12)$$

or explicitly (remember $\mathbf{v} = (u, v, w)$)

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix} + \frac{\partial}{\partial x} \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ (E + p)u \end{pmatrix} + \frac{\partial}{\partial y} \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ (E + p)v \end{pmatrix} = 0 \quad (13)$$

1.2 Conserved and primitive variables

For now, we have described the Euler equation as a hyperbolic conservation law using the (conserved) state vector \mathbf{U} :

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho \mathbf{v} \\ E \end{pmatrix} \quad (14)$$

for this reason, the variables ρ , $\rho\mathbf{v}$, and E are referred to as “conserved variables”, as they obey conservation laws.

However, this is not the only set of variables that allows us to describe the fluid dynamics. In particular, the solution of the Riemann problem (section 3) will give us a set of with so called “primitive variables” (or “physical variables”) with the “primitive” state vector \mathbf{W}

$$\mathbf{W} = \begin{pmatrix} \rho \\ \mathbf{v} \\ p \end{pmatrix} \quad (15)$$

Using the ideal gas equations, these are the equations to translate between primitive and conservative variables:

Primitive to conservative:

$$(\rho) = (\rho) \quad (16)$$

$$(\rho\mathbf{v}) = (\rho) \cdot (\mathbf{v}) \quad (17)$$

$$(E) = \frac{1}{2}(\rho)(\mathbf{v})^2 + \frac{1}{\gamma - 1} \frac{(p)}{(\rho)} \quad (18)$$

Conservative to primitive:

$$(\rho) = (\rho) \quad (19)$$

$$(\mathbf{v}) = \frac{(\rho\mathbf{v})}{(\rho)} \quad (20)$$

$$(p) = \frac{\gamma - 1}{(\rho)} \left((E) - \frac{1}{2} \frac{(\rho\mathbf{v})^2}{(\rho)} \right) \quad (21)$$

1.3 Implementation Details

All the functions for computing gas related quantities are written in `gas.h` and `gas.c`. Every cell is represented by a struct `struct cell` written in `cell.h`. It stores both the

primitive variables/states and the conservative states in the **pstate** and **cstate** structs, respectively.

The adiabatic index γ is hardcoded as a macro in **defines.h**. If you change it, all the derived quantities stored in macros (e.g. $\gamma-1$, $\frac{1}{\gamma-1}$) should be computed automatically.

2 Notation

We are working on numerical methods. Both space and time will be discretized.

Space will be discretized in cells which will have integer indices to describe their position. Time will be discretized in fixed time steps, which may have variable lengths. Nevertheless the time progresses step by step.

The lower left corner has indices $(0, 0)$ in 2D. In 1D, index 0 also represents the leftmost cell.

We have:

- integer subscript: Value of a quantity at the cell, i.e. the center of the cell. Example: \mathbf{U}_i , \mathbf{U}_{i-2} or $\mathbf{U}_{i,j+1}$ for 2D.
- non-integer subscript: Value at the cell faces, e.g. $\mathbf{F}_{i-1/2}$ is the flux at the interface between cell i and $i - 1$, i.e. the left cell as seen from cell i .
- integer superscript: Indication of the time step. E.g. \mathbf{U}^n : State at timestep n
- non-integer superscript: (Estimated) value of a quantity in between timesteps. E.g. $\mathbf{F}^{n+1/2}$: The flux at the middle of the time between steps n and $n + 1$.

3 Riemann Problem and Solvers

3.1 The Riemann Problem and Solution Strategy

At the heart of Eulerian (not-moving) numerical fluid dynamics is the solution to a specific initial value problem (IVP) called the “Riemann problem”. For a hyperbolic system of conservation laws of the form

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} = 0 \quad (22)$$

the Riemann problem is defined as

$$\mathbf{U}(x, t = 0) = \begin{cases} \mathbf{U}_L & \text{if } x < 0 \\ \mathbf{U}_R & \text{if } x > 0 \end{cases} \quad (23)$$

see also fig. 1.

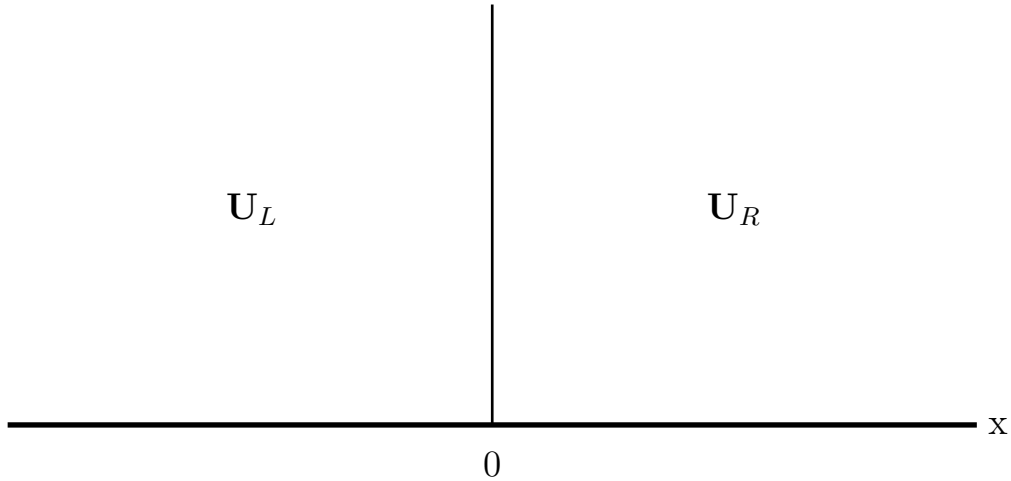


Figure 1: The Riemann Initial Value Problem in 1D.

Unfortunately, there is no exact closed-form solution to the Riemann problem for the

Euler equations. However, it is possible to devise iterative schemes whereby the solution can be computed numerically. To solve full fluid dynamics problems, this calculation needs to be repeated many many times, making the solution quite expensive. For that reason, people have developed approximate Riemann solvers, which we also will have a look at.

For a full derivation of how to solve the Riemann problem for the Euler equations, see e.g. Toro 1999. For our purposes, it suffices to accept that (assuming we have no vacuum) as time progresses, three waves will form which will separate the two initial states \mathbf{U}_L and \mathbf{U}_R . This results in two new states, \mathbf{U}_L^* and \mathbf{U}_R^* between the initial states, because \mathbf{U}_L^* and \mathbf{U}_R^* themselves will be separated by a wave. This is shown in figure 2

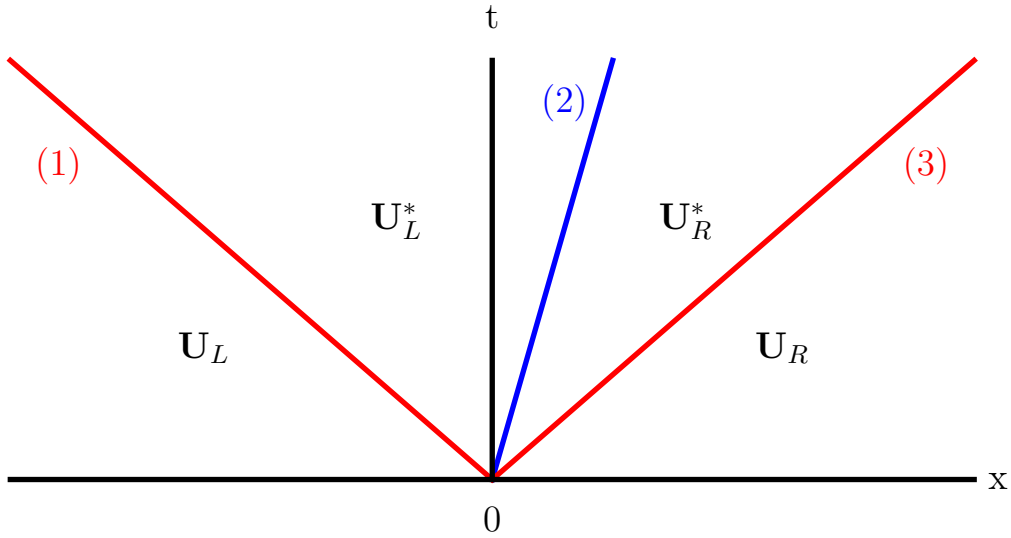


Figure 2: The solution to the Riemann problem for Euler equations: Three waves, (1), (2), and (3), arise from the origin as time evolves. (2) is always a contact wave, (1) and (3) can be either rarefaction or shock waves in each case, depending on the initial conditions. The initial states \mathbf{U}_L and \mathbf{U}_R are separated through the waves (1) and (3) from the two new arising “star states” \mathbf{U}_L^* and \mathbf{U}_R^* , which themselves are separated by the contact wave (2).

3.2 Wave types and relations

It turns out that we get three types of waves: A contact wave, a shock wave, and a rarefaction wave. The middle wave is always a contact wave. The left and right waves can be any combination of shock and/or rarefaction wave, depending on the initial conditions. A model problem containing all three waves is shown in figure ???. These are the wave properties:

3.2.1 Contact wave

The contact wave is a jump discontinuity in the density ρ only. Pressure and velocity remain constant across the contact wave. This gives us the relations

$$\begin{aligned} p_L^* &= p_R^* = p^* \\ u_L^* &= u_R^* = u^* \end{aligned}$$

for this reason, the star state pressure and velocity will have no index indicating whether they are the left or right star state, and will be referred to as p^* and u^* , respectively.

3.2.2 Shock wave

All three primitive variables ρ , p , and u change across a shock wave. A shock wave is a jump discontinuity too. If the **leftmost** wave (wave (1) in fig. 2) is a shock wave, we have

$$\begin{aligned} \rho_L^* &= \frac{\frac{p^*}{p_L} + \frac{\gamma-1}{\gamma+1}}{\frac{\gamma-1}{\gamma+1} \frac{p^*}{p_L} + 1} \rho_L \\ u^* &= u_L - \frac{p^* - p_L}{\sqrt{\frac{p^* + B_L}{A_L}}} \\ A_L &= \frac{2}{(\gamma+1)\rho_L} \\ B_L &= \frac{\gamma-1}{\gamma+1} p_L \end{aligned}$$

and the shock speed is

$$S_L = u_L - a_L \left[\frac{\gamma+1}{2\gamma} \frac{p^*}{p_L} + \frac{\gamma-1}{2\gamma} \right]^{1/2}$$

where a_L is the sound speed in the left state U_L .

For a **right shock wave**, i.e. when wave (3) is a shock wave, we have the relations

$$\begin{aligned}\rho_R^* &= \frac{\frac{p^*}{p_R} + \frac{\gamma-1}{\gamma+1}}{\frac{\gamma-1}{\gamma+1} \frac{p^*}{p_R} + 1} \rho_R \\ u^* &= u_R - \frac{p^* - p_R}{\sqrt{\frac{p^* + B_R}{A_R}}} \\ A_R &= \frac{2}{(\gamma+1)\rho_R} \\ B_R &= \frac{\gamma-1}{\gamma+1} p_R\end{aligned}$$

and the shock speed is

$$S_R = u_R + a_R \left[\frac{\gamma+1}{2\gamma} \frac{p^*}{p_R} + \frac{\gamma-1}{2\gamma} \right]^{1/2}$$

where a_R is the sound speed in the left state U_L .

TODO

check u^* : is $+ B_R$ correct?
actually, check all equations

3.2.3 Rarefaction wave

Rarefaction waves are smooth transitions, not infinitesimally thin jump discontinuities. This makes them really easy to spot in the solutions of Riemann problems, see fig. ?? . Across rarefactions, entropy is conserved.

The rarefaction waves are enclosed by the head and the tail of the wave, between which we have a smooth transition which is called the “fan”. The head is the “front” of the wave, i.e. the part of the wave that gets furthest away from the origin as time progresses. The tail is the “back” of the wave, i.e. the part of the wave that stays closest to the origin as time progresses. The wave speeds of the head, S_H , and of the tail, S_T , are both given below.

If we have a **left-facing** rarefaction, i.e. if wave (1) is a rarefaction wave, we have

$$\rho_L^* = \rho_L \left(\frac{p^*}{p_L} \right)^{\frac{1}{\gamma}}$$

$$u^* = u_L + \frac{2a_L}{\gamma - 1} \left[1 - \left(\frac{p^*}{p_L} \right)^{\frac{\gamma-1}{2\gamma}} \right]$$

where a_L is the sound speed in the left state U_L .

The wave speeds of the head, S_H , and the tail, S_T , for the left facing wave are

$$S_{HL} = u_L - a_L$$

$$S_{TL} = u^* - a_L^*$$

$$a_L^* = a_L \left(\frac{p^*}{p_L} \right)^{\frac{\gamma-1}{2\gamma}}$$

Finally, the solution inside the rarefaction fan, i.e. in regions where $S_{HL} \leq \frac{x}{t} \leq S_{TL}$, is

$$\rho_{\text{fan},L} = \rho_L \left[\frac{2}{\gamma + 1} + \frac{\gamma - 1}{\gamma + 1} \frac{1}{a_L} \left(u_L - \frac{x}{t} \right) \right]^{\frac{2}{\gamma-1}}$$

$$u_{\text{fan},L} = \frac{2}{\gamma + 1} \left[\frac{\gamma - 1}{2} u_L + a_L + \frac{x}{t} \right]$$

$$p_{\text{fan},L} = p_L \left[\frac{2}{\gamma + 1} + \frac{\gamma - 1}{\gamma + 1} \frac{1}{a_L} \left(u_L - \frac{x}{t} \right) \right]^{\frac{2\gamma}{\gamma-1}}$$

If we have a **right-facing rarefaction**, i.e. if wave (1) is a rarefaction wave, we have

$$\rho_R^* = \rho_R \left(\frac{p^*}{p_R} \right)^{\frac{1}{\gamma}}$$

$$u^* = u_R - \frac{2a_R}{\gamma - 1} \left[1 - \left(\frac{p^*}{p_R} \right)^{\frac{\gamma-1}{2\gamma}} \right]$$

where a_R is the sound speed in the left state U_R .

The wave speeds of the head, S_H , and the tail, S_T , for the left facing wave are

$$\begin{aligned} S_{HR} &= u_R + a_R \\ S_{TR} &= u^* - a_R^* \\ a_R^* &= a_R \left(\frac{p^*}{p_R} \right)^{\frac{\gamma-1}{2\gamma}} \end{aligned}$$

Finally, the solution inside the rarefaction fan, i.e. in regions where $S_{HL} \leq \frac{x}{t} \leq S_{TL}$, is

$$\begin{aligned} \rho_{\text{fan},R} &= \rho_R \left[\frac{2}{\gamma+1} - \frac{\gamma-1}{\gamma+1} \frac{1}{a_R} \left(u_R - \frac{x}{t} \right) \right]^{\frac{2}{\gamma-1}} \\ u_{\text{fan},R} &= \frac{2}{\gamma+1} \left[\frac{\gamma-1}{2} u_R - a_R + \frac{x}{t} \right] \\ p_{\text{fan},R} &= p_R \left[\frac{2}{\gamma+1} - \frac{\gamma-1}{\gamma+1} \frac{1}{a_R} \left(u_R - \frac{x}{t} \right) \right]^{\frac{2\gamma}{\gamma-1}} \end{aligned}$$

3.2.4 Which wave type do we have?

As written before, the middle wave (wave (2) in fig. 2) is always a contact wave, while the other two waves are any combination of rarefaction and/or shock wave. It turns out that the condition for a rarefaction or shock wave is remarkably simple.

For the left wave (wave (1)):

$$p^* > p_L : \quad (1) \text{ is a shock wave} \quad (24)$$

$$p^* \leq p_L : \quad (1) \text{ is a rarefaction wave} \quad (25)$$

and for the right wave (wave (3)):

$$p^* > p_R : \quad (3) \text{ is a shock wave} \quad (26)$$

$$p^* \leq p_R : \quad (3) \text{ is a rarefaction wave} \quad (27)$$

See Toro 1999 for more details.

3.2.5 Solution for p^*

The only thing missing to have a complete solution to the Riemann problem for the Euler equations is an expression how to obtain p^* , the pressure in the star region, depending on the initial conditions U_L and U_R . We make use of the fact that p^* and u^* are constant across the star region to relate U_L and U_R , or more precisely the primitive states \mathbf{W}_L and \mathbf{W}_R which can easily be derived from the conservative ones. For both shock and rarefaction waves on either side, we have equations for u^* depending on the outer states \mathbf{W}_L and \mathbf{W}_R and p^* . By setting $u_L^* - u_R^* = 0$, which must hold, we get the equation

$$f(p, \mathbf{W}_L, \mathbf{W}_R) \equiv f_L(p, \mathbf{W}_L) + f_R(p, \mathbf{W}_R) + (u_R - u_L) = 0 \quad (28)$$

with

$$f_{L,R} = \begin{cases} (p - p_{L,R}) \left[\frac{A_{L,R}}{p + B_{L,R}} \right]^{\frac{1}{2}} & \text{if } p > p_{L,R} \quad (\text{shock}) \\ \frac{2a_{L,R}}{\gamma - 1} \left[\left(\frac{p}{p_{L,R}} \right)^{\frac{\gamma - 1}{2\gamma}} - 1 \right] & \text{if } p \leq p_{L,R} \quad (\text{rarefaction}) \end{cases} \quad (29)$$

$$A_{L,R} = \frac{2}{(\gamma + 1)\rho_{L,R}} \quad (30)$$

$$B_{L,R} = \frac{\gamma - 1}{\gamma + 1} p_{L,R} \quad (31)$$

TODO

add image of riemann solution with all three wave types. Label it fig:riemann-solved

3.3 Exact Solver

3.4 Approximate Solvers

3.4.1 HLL Solver

3.4.2 HLLC Solver

3.4.3 Two-Rarefaction Riemann Solver (TRRS)

3.4.4 Two-Shock Riemann Solver (TSRS)

3.5 Sampling the Solution

With the solvers readily available, the final task is to sample the solution at some given point (x, t) . Assuming we have computed all the star region state variables, what is left to do is to determine in which case the point (x, t) is located. The flow chart of decision making and finally which relations to use is shown in figure 3.

Initially all we need to compute is the star states, then we sample the solution at the given point that we're interested in, and the flowchart will tell us which states we need to compute using which relations.

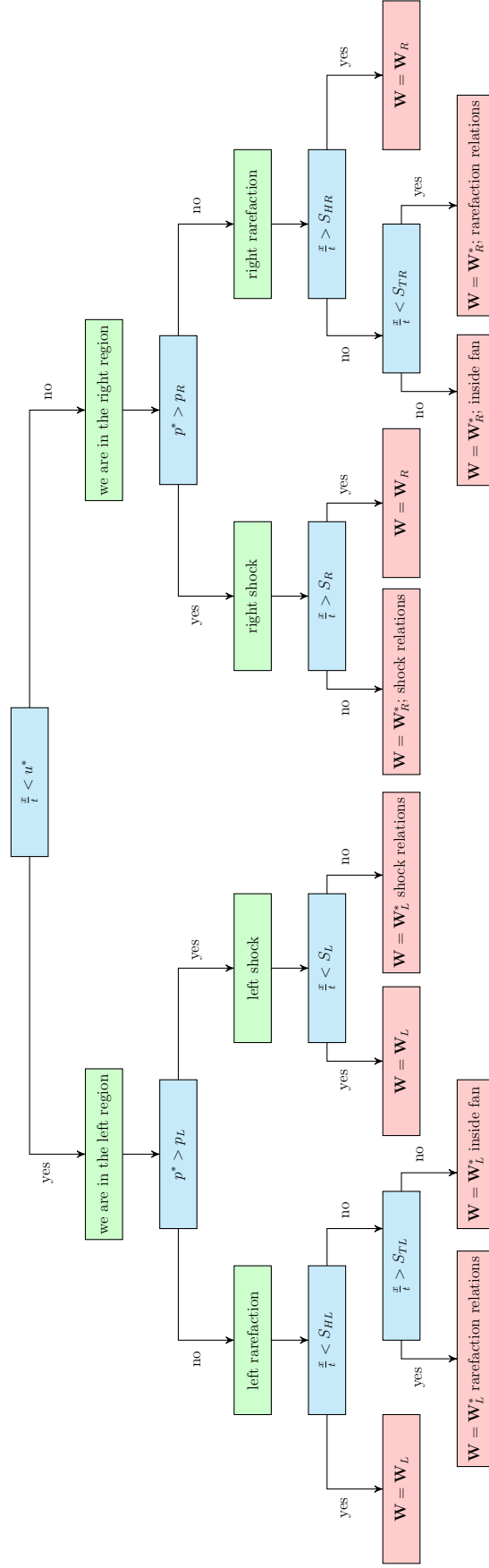


Figure 3: Flow chart to sample the solution of the Riemann problem for the Euler equations at a given point (x, t) .

4 Hydrodynamics Methods

4.1 Upwind Godunov

5 Slope and Flux Limiters

5.1 Slope Limiters

Slope limiters are employed because issues arise around numerical schemes because of their discrete nature. For example, a non-limited piecewise linear advection scheme will produce oscillations around jump discontinuities. So the idea is to compute the slope in way that is useful for us based on the current situation of the gas state that we're solving for.

The choice of the slope can be expressed via a function $\phi(r)$ (see eqns. 60, 61) with

$$r_{i-1/2}^n = \begin{cases} \frac{\mathbf{U}_{i-1}^n - \mathbf{U}_{i-2}^n}{\mathbf{U}_i^n - \mathbf{U}_{i-1}^n} & \text{for } \mathbf{v} \geq 0 \\ \frac{\mathbf{U}_{i+1}^n - \mathbf{U}_i^n}{\mathbf{U}_i^n - \mathbf{U}_{i-1}^n} & \text{for } \mathbf{v} \leq 0 \end{cases}$$

.

Possible limiters are:

$$\text{Minmod} \quad \phi(r) = \text{minmod}(1, r) \quad (32)$$

$$\text{Superbee} \quad \phi(r) = \max(0, \min(1, 2r), \min(2, r)) \quad (33)$$

$$\text{MC (monotonized cenral-difference)} \quad \phi(r) = \max(0, \min((1+r)/2, 2, 2r)) \quad (34)$$

$$\text{van Leer} \quad \phi(r) = \frac{r + |r|}{1 + |r|} \quad (35)$$

where

$$\text{minmod}(a, b) = \begin{cases} a & \text{if } |a| < |b| \text{ and } ab > 0 \\ b & \text{if } |a| > |b| \text{ and } ab > 0 \\ 0 & \text{if } ab \leq 0 \end{cases} \quad (36)$$

6 Boundary Conditions

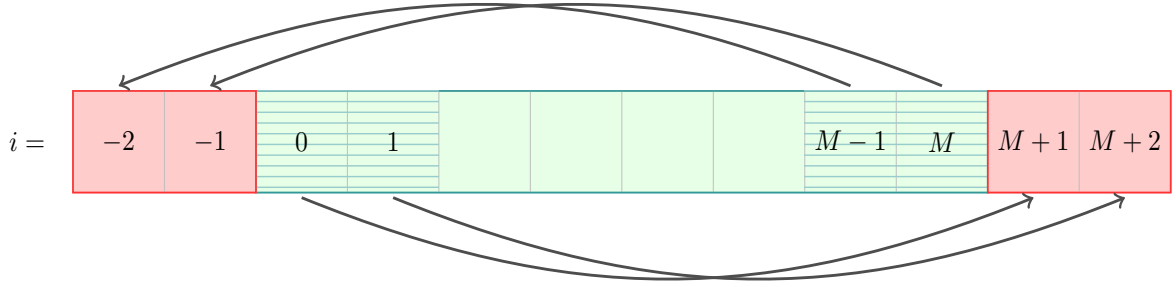


Figure 4: Method to obtain periodic boundary conditions. The ghost cells are red, the arrows show what will be copied where.

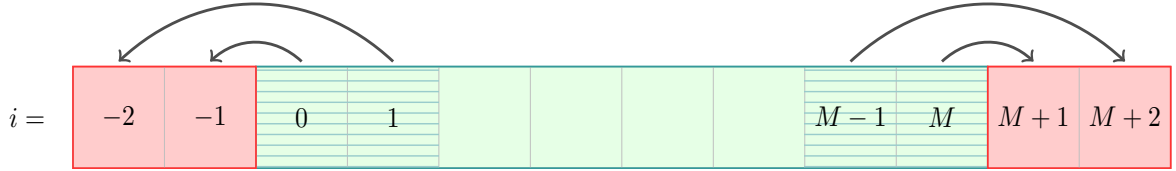


Figure 5: Method to obtain wall boundary conditions. The ghost cells are red, the arrows show what will be copied where.

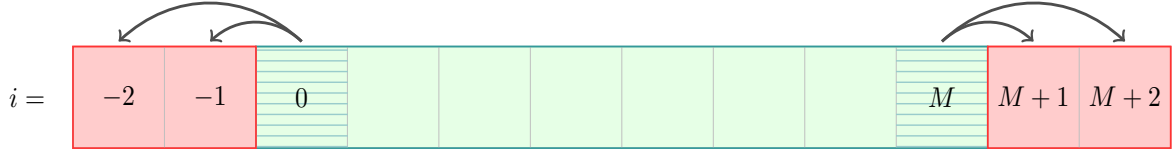


Figure 6: Method to obtain transmissive boundary conditions. The ghost cells are red, the arrows show what will be copied where.

There are tricks how to obtain different kinds of boundary conditions. In every case, we add additional cells (“ghost cells”) in every dimension so we can simulate the desired behaviour. How many cells you need to add depends on the methods (and mostly stencils) you use. If you only take into account one neighbouring cell, then one ghost cell on every boundary suffices. In figures 4, 5, and 6, two ghost cells for a 1D grid are drawn.

Suppose we have 1D grid with M cells and require 2 ghost cells each, which will have indices -2 , -1 , $M+1$, and $M+2$. Then we can get:

- **periodic boundary conditions:**

what goes over the right edge, comes back in over the left edge, and vice versa. We achieve this behaviour by enforcing (fig. 4)

$$\begin{aligned} \mathbf{U}_{-2} &= \mathbf{U}_{M-1} \\ \mathbf{U}_{-1} &= \mathbf{U}_M \\ \mathbf{U}_{M+1} &= \mathbf{U}_0 \\ \mathbf{U}_{M+2} &= \mathbf{U}_1 \end{aligned}$$

- **reflective boundary conditions:**

pretend there is a wall at the boundary. We achieve that by “mirroring” the cells next to the boundary (fig 5):

$$\begin{aligned} \mathbf{U}_{-2} &= \mathbf{U}_1 \\ \mathbf{U}_{-1} &= \mathbf{U}_0 \\ \mathbf{U}_{M+1} &= \mathbf{U}_M \\ \mathbf{U}_{M+2} &= \mathbf{U}_{M-1} \end{aligned}$$

However, every directional component (i.e. velocities/momentum) needs to have the negative value in the ghost cell compared to the real cell.

- **transmissive boundary conditions:**

Just let things flow out however they want. We achieve this by copying the last boundary cell over and over again, such that it looks that the fluid appears to have that state infinitely, and there are no net fluxes to interfere with the hydrodynamics inside the actual grid (fig. 6)

$$\begin{aligned} \mathbf{U}_{-2} &= \mathbf{U}_0 \\ \mathbf{U}_{-1} &= \mathbf{U}_0 \\ \mathbf{U}_{M+1} &= \mathbf{U}_M \\ \mathbf{U}_{M+2} &= \mathbf{U}_M \end{aligned}$$

7 Dimensional Splitting

To go from one to multiple dimensions, it is tempting to just extend the one dimensional discretisation. For example, starting with the conservation law

$$\frac{\partial}{\partial t} \mathbf{U} + \frac{\partial}{\partial x} \mathbf{F}(\mathbf{U}) + \frac{\partial}{\partial y} \mathbf{G}(\mathbf{U}) = 0 \quad (37)$$

and just apply Godunov's finite volume method:

$$\mathbf{U}_{i,j}^{n+1} = \mathbf{U}_{i,j}^n + \frac{\Delta t}{\Delta x} (\mathbf{F}_{i-1/2,j} - \mathbf{F}_{i+1/2,j}) + \frac{\Delta t}{\Delta x} (\mathbf{G}_{i,j-1/2} - \mathbf{G}_{i,j+1/2}) \quad (38)$$

However, this is a bit of a problem. The upwinding here is not complete. Consider the 2D advection equation

$$\frac{\partial}{\partial t} q + u \frac{\partial}{\partial x} q + v \frac{\partial}{\partial y} q = 0 \quad (39)$$

Now suppose we have advecting velocities $u = v = 1$, i.e. the advection velocity is along the diagonal. Then our method reads

$$q_{i,j}^{n+1} = q_{i,j}^n + u \frac{\Delta t}{\Delta x} (q_{i-1/2,j} - q_{i+1/2,j}) + v \frac{\Delta t}{\Delta x} (q_{i,j-1/2} - q_{i,j+1/2}) \quad (40)$$

This expression doesn't involve $q_{i-1,j-1}$ at all, but that's actually the value that should be advected to $q_{i,j}$ in the next timestep!

One way of doing things is to actually formulate more sophisticated unsplit methods that involve the appropriate stencils. We shan't do that here though. We make use of dimensional splitting. Instead of solving

$$\begin{cases} \text{PDE:} & \frac{\partial}{\partial t} \mathbf{U} + \frac{\partial}{\partial x} \mathbf{F}(\mathbf{U}) + \frac{\partial}{\partial y} \mathbf{G}(\mathbf{U}) = 0 \\ \text{IC:} & \mathbf{U}(x, y, t^n) = \mathbf{U}^n \end{cases} \quad (41)$$

we do it in 2 (3 for 3D) steps:

Step 1: We obtain an intermediate result $\mathbf{U}^{n+1/2}$ by solving the “x - sweep” over the full time interval Δt :

$$\begin{cases} \text{PDE:} & \frac{\partial}{\partial t} \mathbf{U} + \frac{\partial}{\partial x} \mathbf{F}(\mathbf{U}) = 0 \\ \text{IC:} & \mathbf{U}^n \end{cases} \quad (42)$$

and then we evolve the solution to the final \mathbf{U}^{n+1} by solving the “y - sweep” over the full time interval Δt :

$$\begin{cases} \text{PDE:} & \frac{\partial}{\partial t} \mathbf{U} + \frac{\partial}{\partial y} \mathbf{G}(\mathbf{U}) = 0 \\ \text{IC:} & \mathbf{U}^{n+1/2} \end{cases} \quad (43)$$

using the 1D methods that are described.

What’s even better is that it can be shown (see LeVeque 2002) that by switching the order of the sweeps every timestep (and keeping the time step interval Δt constant) leads to a second order accurate method in time. This is called “Strang splitting”.

8 Advection

8.1 Analytical Equation

Advection is a bit of an exception as a hydrodynamics method because we're not actually solving the (ideal) gas equations, but these instead:

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{v} \cdot \frac{\partial \mathbf{U}}{\partial x} = 0 \quad (44)$$

Which is still a conservation law of the form

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0 \quad (45)$$

with the flux tensor

$$\mathbf{F} = \mathbf{v} \cdot \mathbf{U} \quad (46)$$

We assume $\mathbf{v} = \text{const.}$

The analytical solution is given by any function $q(x)$ with $\mathbf{U}(x, t) = q(\mathbf{x} - \mathbf{v}t)$, which is just $q(x)$ translated by $\mathbf{v}t$.

8.2 Piecewise Constant Method

We assume that the cell state within a cell is constant (fig. 7). Furthermore, we also assume that the velocity \mathbf{v} is constant and positive.

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n + \frac{\Delta t}{\Delta x} \left(\mathbf{F}_{i-1/2}^{n+1/2} - \mathbf{F}_{i+1/2}^{n+1/2} \right) \quad (47)$$

$$\mathbf{F}_{i\pm 1/2}^{n+1/2} = \mathbf{v}_{i\pm 1/2} \cdot \mathbf{U}_{i-1/2\pm 1/2} \quad (48)$$

The method is first order accurate in time and space.

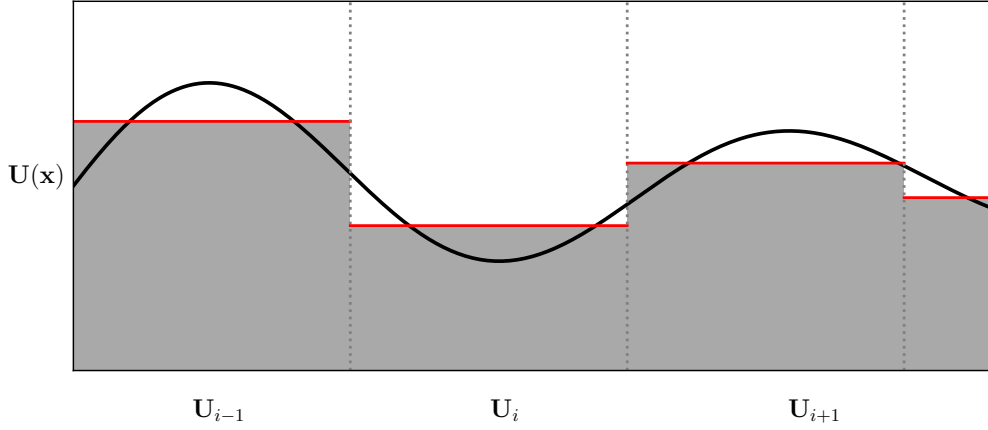


Figure 7: Piecewise constant reconstruction of the field

We assumed that the velocity is positive and constant. What if it's negative?

The important point is that we always do **downwind differencing**. To obtain a finite difference, as we do here, you must never use the value that is downstream, i.e. that is in the direction of the flux. Doing this means taking a value for your computation that won't be valid as soon as an infinitesimal time interval passes, because the ingoing flux will change the downwind state. This is unphysical and leads to violent instabilities.

So if we have negative velocity, all we need to do is change the expression 48 to

$$\mathbf{F}_{i\pm 1/2}^{n+1/2} = \mathbf{v}_{i\pm 1/2} \cdot \mathbf{U}_{i+1/2\pm 1/2} \quad (49)$$

8.3 Piecewise Linear Method

This time, we assume that the state is not constant within a cell, but follows a piecewise linear profile with some slope \mathbf{s} (fig. 8):

For $\mathbf{x}_{i-1/2} < \mathbf{x}_i < \mathbf{x}_{i+1/2}$:

Centered method:

$$\begin{aligned} \mathbf{U}(\mathbf{x}, t = t_n) &= \mathbf{U}_i^n + \mathbf{s}_i^n (\mathbf{x} - \mathbf{x}_i) \\ \mathbf{s}_i^n &= \frac{\mathbf{U}_{i+1}^n - \mathbf{U}_{i-1}^n}{2\Delta x} \end{aligned}$$

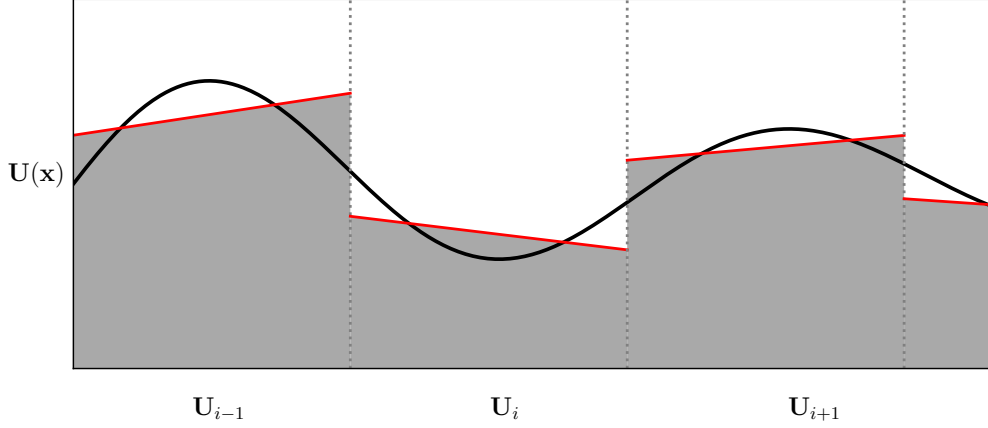


Figure 8: Piecewise linear reconstruction of the field

Other choices for the slope are possible and stable.

Assuming a positive constant velocity \mathbf{v} , we derive the flux \mathbf{F} at the time $t^n < t < t^{n+1}$ at the interface position $i - 1/2$. At time t , the cell will have been advected by a distance $\mathbf{v}(t - t^n)$, and the current state at the interface will be

$$\begin{aligned} \mathbf{U}(x = x_{i-1/2}, t) &= \mathbf{U}_{i-1}^n + \mathbf{s}_{i-1}(\mathbf{x}_{i-1/2} - \mathbf{v}(t - t^n) - \mathbf{x}_{i-1}) \\ &= \mathbf{U}_{i-1}^n + \mathbf{s}_{i-1}(1/2\Delta x - \mathbf{v}(t - t^n)) \end{aligned}$$

To understand how the $\mathbf{x}_{i-1/2} - \mathbf{v}(t - t^n)$ comes into play, imagine the state doesn't change (i.e. isn't advected), but you move the boundaries to the left instead over a distance $\mathbf{v}(t - t^n)$.

So if we have a **negative** constant velocity, the term changes to

$$\begin{aligned} \mathbf{U}(x = x_{i-1/2}, t) &= \mathbf{U}_i^n + \mathbf{s}_i(\mathbf{x}_{i-1/2} - \mathbf{v}(t - t^n) - \mathbf{x}_i) \\ &= \mathbf{U}_i^n + \mathbf{s}_i(-\mathbf{v}(t - t^n) - \Delta x) \end{aligned}$$

Note that the minus sign remains, and that the indices changed by one because we need

to always make sure to do upwind differencing, i.e. take only values where the flow comes from, not from the direction where it's going.

Finally, we can compute the average flux over the time step $\Delta t = t^{n+1} - t^n$:

$$\mathbf{F}_{i-1/2}^{n+1/2} = \langle \mathbf{F}_{i+1/2}(t) \rangle_{t^n}^{t^{n+1}} = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathbf{v} \mathbf{U}(\mathbf{x} = \mathbf{x}_{i-1/2}, t) \quad (50)$$

$$= \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathbf{v} (\mathbf{U}_{i-1}^n + \mathbf{s}_{i-1}(1/2\Delta x - \mathbf{v}(t - t^n))) \quad (51)$$

$$= \mathbf{v} \left(\mathbf{U}_{i-1}^n + \mathbf{s}_{i-1} \left(1/2\Delta x - \mathbf{v} \left(\left[\frac{1}{2\Delta t} t^2 \right]_{t^n}^{t^{n+1}} - t^n \right) \right) \right) \quad (52)$$

$$= \mathbf{v} \left(\mathbf{U}_{i-1}^n + \mathbf{s}_{i-1} \left(1/2\Delta x - \mathbf{v} \left[\frac{1}{2} (t^{n+1} + t^n) - t^n \right] \right) \right) \quad (53)$$

$$= \mathbf{v} (\mathbf{U}_{i-1}^n + 1/2\mathbf{s}_{i-1} (\Delta x - \mathbf{v}\Delta t)) \quad (54)$$

Finally averaging the fluxes over a time step gives:

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \mathbf{v} \cdot \frac{\Delta t}{\Delta x} (\mathbf{U}_i^n - \mathbf{U}_{i-1}^n) - \mathbf{v} \cdot \frac{\Delta t}{\Delta x} \frac{1}{2} (\mathbf{s}_i^n - \mathbf{s}_{i-1}^n) (\Delta x - \mathbf{v}\Delta t) \quad (55)$$

This is the same as eq. 47 where we used

$$\begin{aligned} \mathbf{F}_{i+1/2}^{n+1/2} &= \mathbf{v}_{i+1/2} \cdot \mathbf{U}_{i+1/2}^{n+1/2} \\ &= \mathbf{v} \cdot \mathbf{U}(\mathbf{x}_{i+1/2} - 1/2\mathbf{v}\Delta t) \\ &= \mathbf{v} \cdot (\mathbf{U}_i^n + \mathbf{s}_i^n[(\mathbf{x}_{i+1/2} - 1/2\mathbf{v}\Delta t) - \mathbf{x}_i]) \\ &= \mathbf{v} \cdot (\mathbf{U}_i^n + 1/2\mathbf{s}_i^n(\Delta \mathbf{x} - \mathbf{v}\Delta t)) \end{aligned}$$

and analoguely

$$\mathbf{F}_{i-1/2}^{n+1/2} = \mathbf{v} \cdot \left(\mathbf{U}_{i-1}^n + \frac{1}{2}\mathbf{s}_{i-1}^n(\Delta \mathbf{x} - \mathbf{v}\Delta t) \right)$$

To summarize the formulae:

$$\begin{aligned}
\mathbf{U}_i^{n+1} &= \mathbf{U}_i^n + \frac{\Delta t}{\Delta x} \left(\mathbf{F}_{i-1/2}^{n+1/2} - \mathbf{F}_{i+1/2}^{n+1/2} \right) \\
\mathbf{F}_{i-1/2}^{n+1/2} &= \begin{cases} \mathbf{v}_{i-1/2} \cdot \mathbf{U}_{i-1}^n + \frac{1}{2} \mathbf{v}_{i-1/2} \cdot \mathbf{s}_{i-1}^n (\Delta \mathbf{x} - \mathbf{v}_{i-1/2} \Delta t) & \text{for } \mathbf{v} \geq 0 \\ \mathbf{v}_{i-1/2} \cdot \mathbf{U}_i^n - \frac{1}{2} \mathbf{v}_{i-1/2} \cdot \mathbf{s}_i^n (\Delta \mathbf{x} + \mathbf{v}_{i-1/2} \Delta t) & \text{for } \mathbf{v} \leq 0 \end{cases} \\
\mathbf{F}_{i+1/2}^{n+1/2} &= \begin{cases} \mathbf{v}_{i+1/2} \cdot \mathbf{U}_i^n + \frac{1}{2} \mathbf{v}_{i+1/2} \cdot \mathbf{s}_i^n (\Delta \mathbf{x} - \mathbf{v}_{i+1/2} \Delta t) & \text{for } \mathbf{v} \geq 0 \\ \mathbf{v}_{i+1/2} \cdot \mathbf{U}_{i+1}^n - \frac{1}{2} \mathbf{v}_{i+1/2} \cdot \mathbf{s}_{i+1}^n (\Delta \mathbf{x} + \mathbf{v}_{i+1/2} \Delta t) & \text{for } \mathbf{v} \leq 0 \end{cases}
\end{aligned}$$

We can now insert a more general expression for the slopes. Let

$$\theta_{i-1/2} = \begin{cases} +1 & \text{for } \mathbf{v} \geq 0 \\ -1 & \text{for } \mathbf{v} \leq 0 \end{cases} \quad (56)$$

Then

$$\Delta x_{i-\{0,1\}} \mathbf{s}_{i-\{0,1\}} = \frac{1}{2} \Delta x \left[(1 + \theta_{i-1/2}) \mathbf{s}_{i-1}^n + (1 - \theta_{i-1/2}) \mathbf{s}_i^n \right] \quad (57)$$

$$\equiv \phi(r_{i-1/2}^n) (\mathbf{U}_i^n - \mathbf{U}_{i-1}^n) \quad (58)$$

$$r_{i-1/2}^n = \begin{cases} \frac{\mathbf{U}_{i-1}^n - \mathbf{U}_{i-2}^n}{\mathbf{U}_i^n - \mathbf{U}_{i-1}^n} & \text{for } \mathbf{v} \geq 0 \\ \frac{\mathbf{U}_{i+1}^n - \mathbf{U}_i^n}{\mathbf{U}_i^n - \mathbf{U}_{i-1}^n} & \text{for } \mathbf{v} \leq 0 \end{cases} \quad (59)$$

ϕ is discussed later. Finally:

$$\begin{aligned}
\mathbf{F}_{i-1/2}^{n+1/2} &= \frac{1}{2} \mathbf{v}_{i-1/2} \left[(1 + \theta_{i-1/2}) \mathbf{U}_{i-1}^n + (1 - \theta_{i-1/2}) \mathbf{U}_i^n \right] + \\
&\quad \frac{1}{2} |\mathbf{v}_{i-1/2}| \left(1 - \left| \frac{\mathbf{v}_{i-1/2} \Delta t}{\Delta x} \right| \right) \phi(r_{i-1/2}^n) (\mathbf{U}_i^n - \mathbf{U}_{i-1}^n) \quad (60)
\end{aligned}$$

$$\begin{aligned}
\mathbf{F}_{i+1/2}^{n+1/2} &= \frac{1}{2} \mathbf{v}_{i+1/2} \left[(1 + \theta_{i+1/2}) \mathbf{U}_i^n + (1 - \theta_{i+1/2}) \mathbf{U}_{i+1}^n \right] + \\
&\quad \frac{1}{2} |\mathbf{v}_{i+1/2}| \left(1 - \left| \frac{\mathbf{v}_{i+1/2} \Delta t}{\Delta x} \right| \right) \phi(r_{i+1/2}^n) (\mathbf{U}_{i+1}^n - \mathbf{U}_i^n) \quad (61)
\end{aligned}$$

Depending on our choice of ϕ , we can get different slopes. Here for positive velocity only, and for $r = r_{i-1/2}$:

$\phi(r) = 0 \rightarrow \mathbf{s}_i = 0$	No slopes; Piecewise constant method.
$\phi(r) = 1 \rightarrow \mathbf{s}_i = \frac{\mathbf{U}_i - \mathbf{U}_{i-1}}{\Delta x}$	Downwind slope (Lax-Wendroff)
$\phi(r) = r \rightarrow \mathbf{s}_i = \frac{\mathbf{U}_{i-1} - \mathbf{U}_{i-2}}{\Delta x}$	Upwind slope (Beam-Warming)
$\phi(r) = \frac{1}{2}(1 + r) \rightarrow \mathbf{s}_i = \frac{\mathbf{U}_i - \mathbf{U}_{i-2}}{2\Delta x}$	Centered slope (Fromm)

8.4 CFL Condition

To keep things stable and physical, we must not allow any flux in the simulation to go further than one single cell size. Otherwise, you're skipping interactions between fluxes on cells. This time restriction is known as the CFL condition.

In 1D, it's straightforward:

$$\Delta t_{max} = C_{cfl} \frac{\Delta x}{v_{max}} \quad (62)$$

$C_{cfl} \in [0, 1)$ is a user-set factor. The lower it is, the more precise the results, but the more computations you need to do.

In 2D, it is:

$$\Delta t_{max} = C_{cfl} \left(\frac{|v_{x,max}|}{\Delta x} + \frac{|v_{y,max}|}{\Delta y} \right)^{-1} \quad (63)$$

This condition is more strict than what one would expect from the restriction based on physical arguments, i.e. not allowing the flux to pass more than one cell, which would be $\Delta t_{max} = C_{cfl} \min \left\{ \frac{\Delta x}{|v_{x,max}|}, \frac{\Delta y}{|v_{y,max}|} \right\}$. It follows from a convergence condition in (von Neumann) stability analysis of the method.

For N dimensions, the condition translates to

$$\Delta t_{max} = C_{cfl} \left(\sum_{i=1}^N \frac{|v_{i,max}|}{\Delta x_i} \right)^{-1} \quad (64)$$

8.5 Implementation Details

What is implemented is the equation

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{v} \cdot \frac{\partial \mathbf{U}}{\partial \mathbf{x}} = 0 \quad (65)$$

where we assume that the velocity \mathbf{v} is constant. Therefore, the fluid velocity is never updated, but kept identical to the initial conditions.

You can change that behaviour by removing the `ADVECTION_KEEP_VELOCITY_CONSTANT` macro definition in `defines.h`

References

- LeVeque, Randall J. (2002). *Finite Volume Methods for Hyperbolic Problems*. Cambridge Texts in Applied Mathematics. Cambridge University Press. DOI: 10.1017/CB09780511791253.
- Toro, Eleuterio F (1999). *Riemann solvers and numerical methods for fluid dynamics: a practical introduction; 2nd ed.* Berlin: Springer. URL: <http://cds.cern.ch/record/404378>.