MOMENTA
Multi-Observations and Multi-Energy Neutrino Transient Analysis

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1 Introduction

This framework is using a Bayesian approach to convert observations from neutrino telescopes to constraints on the neutrino emission from transient astrophysical sources. It is aimed to combine observations from several neutrino samples into a single set of constraints.

2 Inputs

The user should provide various inputs.

Neutrino information

- number of observed events in each neutrino sample $\rightarrow N_s$
- expected background in each neutrino sample (see subsection 2.1) $\rightarrow B_s$
- detector effective area as a function of neutrino energy and direction $\rightarrow A_{\mathrm{eff,s}}(E, \Omega)$
- *OPTIONAL:* list of observed events with their time, direction, and energy $\rightarrow \{\mathrm{ev}_{i,s}\}$
- *OPTIONAL:* instrumental response functions such as angular/energy p.d.f. for signal/background hypotheses

Source information

- localisation of the source (see subsection 2.3) $\rightarrow \Omega_{\mathrm{src}}$
- *OPTIONAL:* time of the event
- *OPTIONAL:* other information that may be relevant for astrophysical interpretations (luminosity distance, redshift...)

Other information

- assumed neutrino energy spectrum, which may include several components $\rightarrow F(E) = \sum_{i=1}^{N_c} \phi_i \times f_i(E)$
- priors on the flux normalisation $\rightarrow \pi(\phi_i)$ (can be uniform in linear/log scale, etc...)
- *OPTIONAL:* assumed jet structure

2.1 Expected background

The expected background will be incorporated as a prior in the analysis. Three scenarios are implemented:

- fixed background: $\pi(B_s|B_{s,0}) = \delta(B_s - B_{s,0})$
  $\rightarrow$ user should provide $B_{s,0}$
- Gaussian background: $\pi(B_s|B_{s,0}, \sigma_{B_s}) = \text{Gaus}(B_s; \mu = B_{s,0}, \sigma = \sigma_{B_s})$
  $\rightarrow$ user should provide mean $B_{s,0}$ and error σ_{B_s}
- background from ON/OFF measurement: $\pi(B_s|N_s^{\text{off}}, \alpha_s^{\text{OFF/ON}}) = \text{Poisson}(N_{\text{OFF}}; B_s \times \alpha_s^{\text{OFF/ON}})$
  $\rightarrow$ user should provide number of events in OFF region N_s^{off} and ratio between OFF and ON $\alpha_s^{\text{OFF/ON}}$

The backgrounds in the different samples are assumed to be uncorrelated.

2.2 Additional uncertainties

Additional uncertainties on the effective area may be incorporated as prior. For instance, a 10% uncertainty on the effective area corresponds to: $\pi(a) = \text{Gaus}(a; 1, 0.10)$ and this factor a will be added in front of the effective area in all formula.
For simplification, this term will be neglected in the following.

2.3 Source localisation

Different source types may be considered, but the two already implemented are:

- using GW posterior samples describing the localisation of the source
- fixed equatorial coordinates

Generally, these are represented by a prior $\pi(\Omega_{\text{src}})$ (that is trivial for the second case).

3 Likelihood and posterior

3.1 Likelihood

For one given sample, the likelihood may be defined for two different cases:

- if we want to perform a simple cut & count search (cc), the likelihood will simply be a Poisson term
- if we want to perform a point-source-like analysis (ps), the likelihood will incorporate both a Poisson term and the angular/energy/time p.d.f.s

In both case, we need to define the expected number of signal events $N_{\text{sig,s}}$ that depends on all the relevant inputs:

$$N_{\text{sig,s}}(\{\phi_i\}, \Omega_{\text{src}}) = \sum_i \phi_i \int A_{\text{eff,s}}(E, \Omega_{\text{src}}) \times f_i(E) \mathrm{d}E \tag{1}$$

We can then write the two likelihoods:

$$\mathcal{L}_{cc}(N_s|\{\phi_i\}, B_s, \Omega_{\text{src}}) = \text{Poisson}\left(N_s; B_s + N_{\text{sig,s}}(\{\phi_i\}, \Omega_{\text{src}})\right) \tag{2}$$

$$\begin{aligned} \mathcal{L}_{ps}(N_s, \{\text{ev}_{j,s}\}|\{\phi_i\}, B_s, \Omega_{\text{src}}) = & \\ & \mathcal{L}_{cc}(N_s, |\{\phi_i\}, B_s, \Omega_{\text{src}}) \\ & \times \prod_{j \in s} \frac{B_s \times p_{\text{bkg}}(\text{ev}_j) + N_{\text{sig,s}}(\{\phi_i\}, \Omega_{\text{src}}) \times p_{\text{sig}}(\text{ev}_j|\{\phi_i\}, \Omega_{\text{src}})}{B_s + N_{\text{sig,s}}(\{\phi_i\}, \Omega_{\text{src}})} \end{aligned} \tag{3}$$

In the point-source case, p_{bkg} and p_{sig} are the probabilities for the event j to be background or signal. These are built from the instrumental response functions. For instance, if we just incorporate the point-spread function, we have:

- $p_{\text{bkg}}(\text{ev}_j) = g(\Omega_j)$ depends solely on event direction Ω_j and described how likely this direction is in the background hypothesis.
- $p_{\text{sig}}(\text{ev}_j) = g(\Omega_j, \sigma_j, \Omega_{\text{src}})$ depends on event direction Ω_j, uncertainty on this direction, and source direction.

These functions are normalized such that $\int g(\Omega, \ldots) d\Omega = 1$.

3.2 Posterior

Generally, we define the posterior probability distribution function as the product of the contribution of the different neutrino samples and all the priors:

$$\begin{aligned} P(\{\phi_i\}, \{B_s\}, \Omega_{\text{src}}|\ldots) = &\prod_s \mathcal{L}(N_s, \{\text{ev}_{j,s}\}|\{\phi_i\}, B_s, \Omega_{\text{src}}) \\ &\times \prod_s \pi(B_s) \times \pi(\Omega_{\text{src}}) \times \pi(\{\phi_i\}) \end{aligned} \tag{4}$$

3.3 Marginalised posterior

We can eventually integrate over all nuisance parameters to get the marginalised posterior:

$$P_{\text{marg}}(\{\phi_i\}) = \int \cdots \int P(\{\phi_i\}, \{B_s\}, \Omega_{\text{src}}|\ldots) \mathrm{d}\Omega_{\text{src}} \prod_s \mathrm{d}B_s \tag{5}$$

From that, one may trivially extract flux constraints:

- $X\%$ upper limits on the different parameters in the absence of any excess
- $X\%$ Highest Posterior Density (HPD) intervals on the different parameters in the presence of an excess
- best-fit value for each signal parameter
- N-D contours on the different parameters (if more than 1)

4 Bayes factors

Another feature of Bayesian analyses is the possibility to extract the Bayes factor that plays the role of significance/p-values in frequentist approaches. In this case, we do not care about the extraction of upper limits, but we simply want to compare different models and see which one the data may favour.
Let's consider two hypotheses:

- H_0: there is only background ($N_{\text{sig,s}} = 0$ for all s)
- H_1: there is both background and signal contributions with $F(E) = \phi(E/\text{GeV})^{-\Gamma}$

We may then compute the related Bayes evidence starting from the posterior probability:

- For H_0, we just need to integrate over nuisance parameters as no source is involved

$$E_0 = \int \cdots \int P(\{B_s\}|\ldots) \times \prod_s \mathrm{d}B_s \tag{6}$$

- For H_1, we need to integrate over all possible source parameters

$$E_1 = \int \cdots \int P(\phi, \{B_s\}, \Omega_{\text{src}}|\ldots) \times \mathrm{d}\Omega_{\text{src}} \times \prod_s \mathrm{d}B_s \times \mathrm{d}\phi \tag{7}$$

The Bayes factor is then naively defined as:

$$B_{10}^{\text{naive}} = E_1/E_0 \tag{8}$$

Correction When using non-informative priors on source parameters (such as flat ones $\pi(\phi) = 1/C$ for $0 \leq \phi < C$), the Bayes factor is defined up to a constant.
For illustration, let's consider the simple cut-and-count approach with one sample, $N = 0$, $A = \int A_{\text{eff,s}}(E, \Omega_{\text{src}})E^{-\Gamma}\text{d}E$, fixed background B, and fixed source position:

$$B_{10}^{\text{naive}} = \frac{\int \text{Poisson}(0, B + \phi A(\Omega_{\text{src}})) \times \pi(\phi)\text{d}\phi}{\text{Poisson}(0, B)} = (1/C) \times \int_0^C e^{-\phi A(\Omega_{\text{src}})}\text{d}\phi = \frac{1 - e^{-CA(\Omega_{\text{src}})}}{CA(\Omega_{\text{src}})} \tag{9}$$

Several approaches are available to correct for this. One of those is the usage of Arithmetic Intrinsic Bayes Factor (AIBF, [BP96]) where we use minimal training samples that cannot discriminate between the two models to compute a correction:

$$B_{10}^{\text{AI}}(\text{data}) = B_{10}^{\text{naive}}(\text{data}) \times (B_{10}^{\text{naive}}(\text{minimal set}))^{-1} \tag{10}$$

5 Implementation

The basic need for the implementation of the Bayesian analysis is to be able to sample the posterior distribution. With this, if it is then possible to:

- marginalise over nuisance parameters to get the marginalised posterior and compute constraints on flux parameters
- derive also constrained on other source parameters such as the total energy emitted in neutrinos assuming isotropic emission
- marginalise over nuisance+source parameters to get the evidence of a model and compute Bayes factors when comparing two models

The sampling of the posterior distribution is performed using nested sampling with `ultranest` [Buc21]. The implementation of the likelihoods and priors are implemented in "src/momenta/stats/model.py". All these are defined as a function of the flux parameters, but we may define as well the corresponding total isotropic energy emitted in neutrinos, total energy with a different jet model, ...
Usage is described in the following section.

6 Use cases

6.1 Transient sources

The initial use case for the package is to search for neutrino emission from transient sources in short time windows (seconds to minutes). The effective area input is the instantaneous effective area at the time of the transient, and everything can be done either in local or equatorial coordinates.
For longer durations (when the source transit is not negligible any more), if the effective area is defined using local coordinates, one has to be careful to properly average over the search time window. Moreover, for the point-source likelihood, one may define signal and background p.d.fs to account for the time of the event in the estimation of the signal and background probabilities $p_{\text{sig}}(\text{ev}_j)$ and $p_{\text{bkg}}(\text{ev}_j)$.

6.2 Steady point sources

The case of steady point sources is similar to the one of long-duration transients.

References

Berger, James O. and Luis R. Pericchi. "The Intrinsic Bayes Factor for Model Selection and Prediction". In: *Journal of the American Statistical Association* 91.433 (1996), pp. 109–122. ISSN: 01621459. DOI: 10.2307/2291387.

Buchner, Johannes. "UltraNest – a robust, general purpose Bayesian inference engine". In: (Jan. 2021). DOI: 10.21105/joss.03001.