Fourier transforms - history.

Joseph Fourier came up with the basic idea in 1805. He said “any” function on the interval [0,1] can be written as a sum of sines and cosines, in this form:

$f(x) = \sum a(n)\cos(2\pi n x) + \sum b(n)\sin(2\pi n x)$,

where the number a(n), b(n) are called the Fourier coefficients. He wanted to use this form to come up with solution to certain linear partial differential equations (specifically the heat equation) because sines and cosines behave nicely under differentiation. For instance, he said the derivative of the above functions should be this sum:

$f'(x) = \sum (-2\pi n)a(n)\sin(2\pi n x) + \sum (2\pi n)b(n)\cos(2\pi n x)$,

This caused a lot of confusion among mathematicians, because Fourier’s sums seemed to imply weird properties. For instance, a discontinuous function (like a step function) could be written as a sum of nice, continuous functions like sines and cosines. In fact, much of the 1800’s was spent trying to invent appropriate mathematics (e.g. real analysis) to make Fourier’s ideas rigourous. In fact, that program has been highly successful and we now have modern analysis.

Fourier transforms are properly a subdomain of harmonic analysis, which is a very general and powerful set of mathematical ideas.

Fourier transforms in general, and symmetry.

Key idea is that the Fourier transforms changes a function on one space into another function on a different space. These spaces are properly called “groups” and you can think of them as a set of numbers with an addition defined on them. A group has to include zero, and it has to include the negative of any number in it. The usual rules of addition have to hold as well (commutative, associate rules).

Groups we care about in this class include:

The integers $Z = \{0, \pm 1, \pm 2, \pm 3, \ldots \}$
The reals $R = \{$ all real numbers between -infinity and +infinity$\}$
The interval $[0,1) = \{$ all real numbers between 0 and 1$\}$ with addition modulo 1
An interval of integers $Z/n = \{0,1,2,3,\ldots,n-1\}$ with addition modulo n.

So, for instance, the set $Z/10 = \{0,1,2,3,4,5,6,7,8,9\}$ consists of the usual 10 digits from 0 to 9, and we do addition modulo 10.. eg 3 + 8 = “11” = “11-10” = 1.
The negative of 3 would be 7, since 3+7=0.

You can build bigger groups by combining the above groups. For instance,
$Z^2 = \{$pairs of integers $(m,n)\}$ with addition done component-wise.
eg (2,3) +(4,5) = (2+4,3+5) = (6,8).

Each group comes with a partner, called the dual group.
Dual of Z is [0,1)
Dual of R is R
Dual of [0,1) is Z
Dual of Z/n is Z/n

I could define the dual more abstractly, but really we only have a few examples to worry about, so just remember the examples above.

For the more complicated groups (products of the above groups), the dual of the product is the product of the duals. (Think about that carefully.)
So for instance the dual of ZxZ is the product [0,1)x[0,1), the set of pairs (s,t) of numbers in the interval [0,1). This group is useful for 2D images.

For each pair of groups G, G', (the group and its dual) there is a special function e mapping G,G' into the complex numbers of absolute value one. That is, given g in G, g' in G', we can write e(g,g') as some complex numebr of length one.

With pair Z, [0,1) : for n in Z, and w in [0,1) we write $e(n,w) = \exp(2\pi i nw)$.
With pair R, R : for t in R, w in R, we write $e(t,w) = \exp(2\pi i tw)$.
With pair [0,1), Z: for w in [0,1), n in Z we write $e(w,n) = \exp(2\pi i nw)$.
With pair Z/n, Z/n : for j in Z/n, k in Z/n, we write $e(j,k) = \exp(2\pi i jk/n)$.

The last one is interesting because we had to divide by n in the exponential to get something useful.
Notice these exponential functions can be thought of as a sum of sines as cosines, which is the link to Fourier's original work. In this exponential form, we have some nice properties.

$e(0,w) = 1$. (The zero property)
$e(t,0) = 1$.
$e(s+t,w) = e(s,t)e(t,w)$ (The additive property)
$e(t,w+v) = e(t,w)e(t,v)$
e(-t,w) = complex conjugate of e(t,-w) (the negative property)
e(t,-w) = complex conjugate of e(t,w)

These properties become important in the symmetries of the Fourier transform.

The Fourier transform is now easy to define. Given a function f(t) on some group, multiply it by the exponential e(t,w) and integrate (or sum) over all t. This gives you a new function F(w) on the dual group.

So, for instance, with groups R,R we can write the transform as

$F(w) = \int f(t)\, e(t,w)\, dt = \int f(t) \exp(2\pi i tw)\, dt$ (integrate over all of R).

For the groups [0,1), Z we take a function f(t) on the interval and get a function on the integers Z as

$F(n) = \int f(t) \exp(2\pi i nt)\, dt$ (integrate over the interval [0,1)).

Usually we like to think of this function F(n) as a sequence indexed by the integers, so we write

$c_n = \int f(t) \exp(2\pi i nt)\, dt$ (integrate over the interval [0,1)).

On the other hand, when we start with a a sequence c_n, you can think of it as a function f(n) on the group of integers Z, so its dual group is the interval [0,1). The Fourier transform gives you a function on the interval

$F(w) = \sum c_n \exp(2\pi i nw) = \sum f(n) \exp(2\pi i nw)$, (where you sum over all integers n).

This sum is just Joseph Fourier's original sum, using complex notation.

Again, the point is, in all cases, the Fourier transform is done the same way. Take the function, multiply by the complex exponential, integrate (or sum), and you get the new function.

The "nice" properties of the exponentials give us nice properties of the Fourier transform. With f(t) the function, and F(w) the Fourier transform, using whichever group pairs you like:

The zero property e(t,0) = 1 shows use that

$F(0) = \int f(t)\, dt$, the integral of the function.

The negative property e(t,-w) = complex conjugate of e(t,w) shows that

F(-w) = complex conjugate of F(w), when f(t) is a real-valued function.

F(-w) = F(w), and real, when f(t) is a real-valued, even function

F(-w) = -f(w), and imaginary, when f(t) is a real-valued, odd function.

Remember, a function is even if f(-t) = f(t) for all t; odd if f(-t) = -f(t) for all t.

In many examples we do in class, for instance in building filters, we often start with real, even functions. So you should expect to see a real, even function in the transform.

You might have noticed we also have the additive property for the exponential. This tells us that the Fourier transform of a convolution becomes a product. That is, if we have two functions f(t), g(t), then the Fourier transform of the convolution

$f*g(t) = \int f(t-s)g(s)\, ds$ is the product F(w)G(w).

(Yes, this does mean convolution is defined on any group. In the Z or Z/n case, the integral is replaced with a sum.)

Fourier inverse step.

If you apply the Fourier transform to function f(t), you get a new function F(w). Apply the transform again, you get f(-t), up to a constant. Notice the minus sign! Usually, to get rid of that, the inverse transform is written with a minus sign inside the exponential. So,

$f(t) = k\int F(w) \exp(-2\pi i\, tw)\, dw$, where k is some positive constant.

For the groups Z, [0,1), and R, the constant k is just 1. Which is very nice.
For the group Z/n, the constant is k=1/n. Which is not so nice, but pretty easy to remember.

Energy conservation.

The square integral of a function and its Fourier transform are the same -- this is called energy conservation. (It also says the Fourier transform is a unitary operator.) Up to a constant, the same one that showed up in the inverse step. So we have

$$k \int |F(w)|^2 \, dw = \int |f(t)|^2 \, dt.$$

You should try to verify this for a few examples.

Fun with Fourier transforms.

We work a lot with functions on an interval, and the corresponding sequence of Fourier coefficients. Remember, this is just the example of an interval [0,1) and its dual group Z of the integers.

The Fourier inverse theorem says that a function f(t) on the interval [0,1) can be written as a series

$$f(t) = \sum c_n \exp(2\pi i\, nt)$$

where the coefficients are computed as $c_n = \int f(t) \exp(-2\pi i\, nt)\, dt$. (Notice the minus sign. It has to go either here in the integral, or in the sum above. But not in both.)

One way to think of this is that the exponential functions $\exp(2\pi i nt)$, $n = 0, \pm 1, \pm 2, \ldots$, form a basis for the vector space of functions on the interval $[0,1)$.

Since these are just combinations of cosines and sines, we can also say that the functions $\cos(2\pi nt)$, $\sin(2\pi nt)$ $n = 0,1,2,3,\ldots$ form a basis for the vector space of functions on the interval.
(But note the n=0 case for sine is a stupid function, so we don't use it in the basis.)

So, we can write the function f(t) in the form

$$f(t) = \sum a_n \cos(2\pi nt) + \sum b_n \sin(2\pi nt),$$

where the first sum is for n=0,1,2,3,.. and the second one for n=1,2,3,...

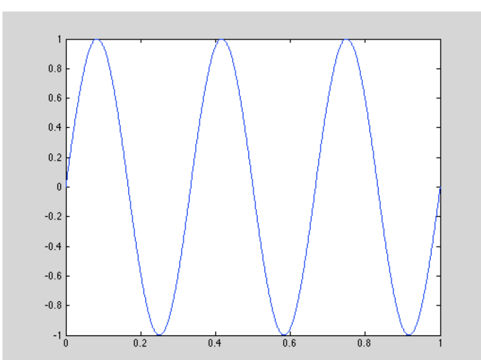
Like with the exponentials, we have integral formulas for the a_n b_n.

$$a_n = 2\int f(t) \cos(2\pi nt)\, dt, \quad b_n = 2\int f(t) \sin(2\pi nt)\, dt, \quad n=1,2,3,\ldots$$

and $a_o = \int f(t)\, dt$.

The point is, you can express f(t) as a sum of sines and cosines. Notice if f(t) is an even function, you can write it as a sum of cosines only. If f(t) is an odd function, you can write it as a sum of sines only.

Also notice all these sines and cosines have a fixed number of periods that fit exactly into the interval [0,1). So, for instance, $\sin(2\pi 3t)$ has exactly 3 cycles in the unit interval. So you don't need lots of sines and cosines, just the ones that fit nicely into the interval.



Three cycles in the unit interval.

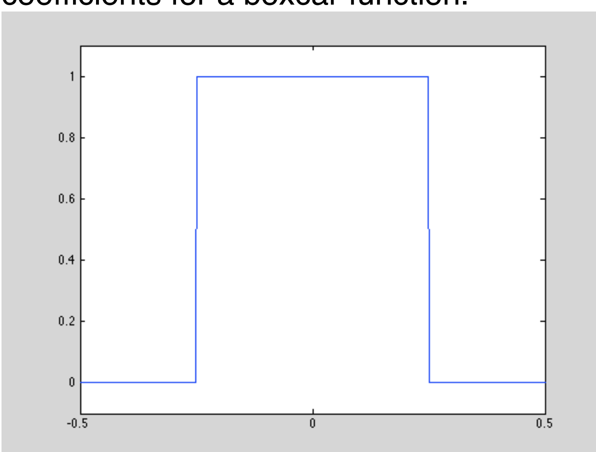
Which functions work?

You can ask, what functions on the interval can we actually write down as a sum of sines and cosines. Joseph Fourier said “all functions” but he was wrong. Certainly all continuous functions work. So do all piecewise continuous functions, that only have jump discontinuties. Although something weird happens at the jumps. For most applications, that is good enough.

However, we can get the series to converge for any square-integrable, measureable function, provided we use “convergence in norm.” We won’t worry about that in this class.

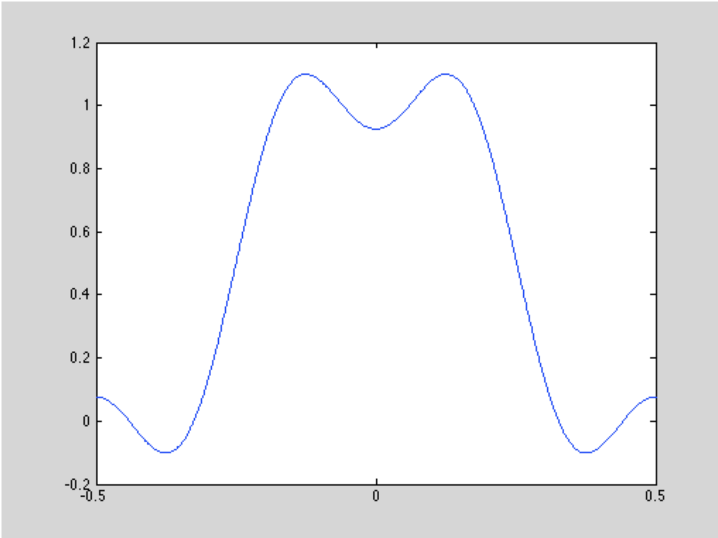
Note that when we design filters for wavelets, we only have a few non-zero coefficients, so the function f(t) is a short sum of sines and cosines, so it will be nice and smooth, with a few wiggles in it.

How about a discontinuous function. In the exam, you were to compute the Fourier coefficients for a boxcar function.

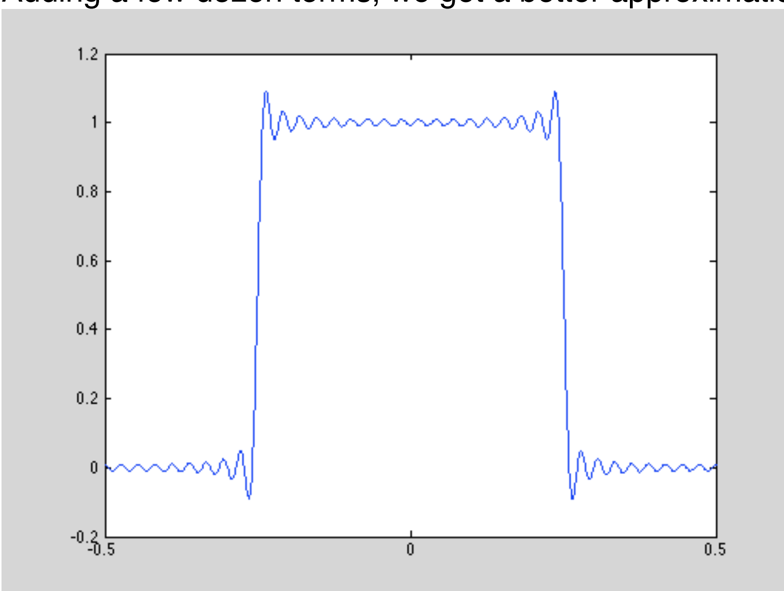


The coefficients obtained were something like plus or minus $1/k\pi$, for k odd, zero otherwise.

If we take the first few terms in the expansion $f(t) = \sum a_n \cos(2\pi nt)$, we get an approximation to the boxcar that looks like this.



Adding a few dozen terms, we get a better approximation:



The rapid oscillations near the jump from 0 to 1 is called “ringing” in electrical circuit theory, and is known as Gibbs phenomena. This ringing is an actual physical effect that can cause damage to your computer and other electronic circuitry. It is a result of removing the higher frequencies, or the laters terms in the Fourier expansion.