HOMEWORK 1 – Q5

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5. Determine if $f(n) = O(g(n))$ or $g(n) = O(f(n))$ or both (i.e., $f(n) = \theta(g(n))$ or neither of the two, for the following pairs of functions

(a) $f(n) = (log_2(n))^2$; $g(n) = log_2(n^{log_2 n})^2$; (6 points)

(b) $f(n) = n^{10}$; $f(n) = 2^{\sqrt[10]{n}}$; (6 points)

(c) $f(n) = n^{1+(-1)^n}$; $g(n) = n$. (8 points)

Solution:

(a) Using $\log(a)^b = blog(a)$ we obtain

$$g(n) = log_2(n^{log_2 n})^2 = log_2(n^{2log_2 n}) = 2log_2(n)log_2(n)$$

$$= 2(log_2(n))^2 = O(log_2(n)^2) = O(f(n))$$

Therefore $g(n) = O(f(n))$.

Using $blog(a) = \log(a)^b$ we obtain

$$f(n) = (log_2(n))^2 = log_2(n)log_2(n)$$

$$g(n) = log_2(n^{log_2 n})^2 = log_2(n^{2log_2 n}) = 2log_2(n)log_2(n)$$

Therefore, there exists a positive constants c, when $c = \frac{1}{2}$, $g(n) = \Omega(f(n))$

Finally, we can obtain that $g(n) = \theta f(n)$.

(b) We want to show that $f(n) = O(g(n))$, which means we have to show that $n^{10} < c2^{\sqrt[10]{n}}$ for some positive c and all sufficiently large n. But, since the log function is monotonically increasing, this will hole just in case

$$log_2\ (n^{10}) < log_2(c) + log_2\ (2^{\sqrt[10]{n}})$$

$$10\ log_2(n) < log_2(c) + \sqrt[10]{n}$$

we now see that if we take $c = 1$ then it is enough to show that

$$10\ log_2(n) < \sqrt[10]{n}$$

for all sufficiently large n which holds because

$$(10log_2(n))^{10} < n$$

Therefore $f(n) = O(g(n))$.

(c) Just note that $1 + (-1)^n$ form a cycle, with one period equal to$\{0,\ 2\}$. Thus, for n to be an even number we have $1 + 1 = 2$ and for n to be an odd number we have $1 + (-1) = 0$. Thus for any fixed $c > 0$ for all n to be an even number eventually $n^{1+(-1)^n} = n^2 > cn$, and for n to be an odd number we have $n^{1+(-1)^n} = 1 < cn$. Thus, neither $f(n) = O(g(n))$ nor $g(n) = \Omega(f(n))$.