



UNIVERSITY OF
CAMBRIDGE

Machine Learning and the Physical World

Lecture 3 : Gaussian Processes

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Model+Data->Predict

Data + Model $\xrightarrow{\text{Compute}}$ Prediction

Model+Data->Predict

$$\text{Data + Model} \xrightarrow{\text{Compute}} \text{Prediction}$$

- We do not have enough knowledge

$$\text{Model} \not\xrightarrow{\text{Compute}} \text{Prediction}$$

Model+Data->Predict



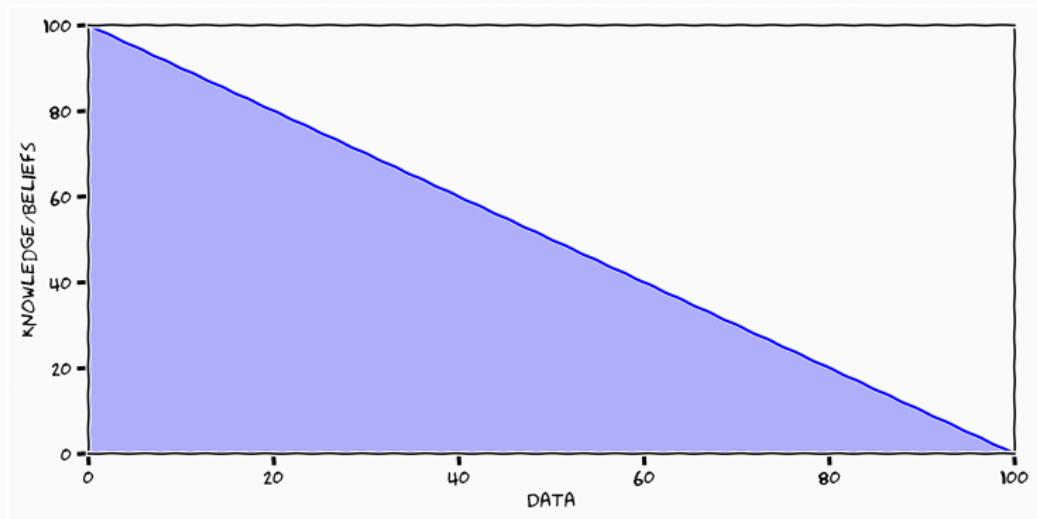
- We do not have enough knowledge



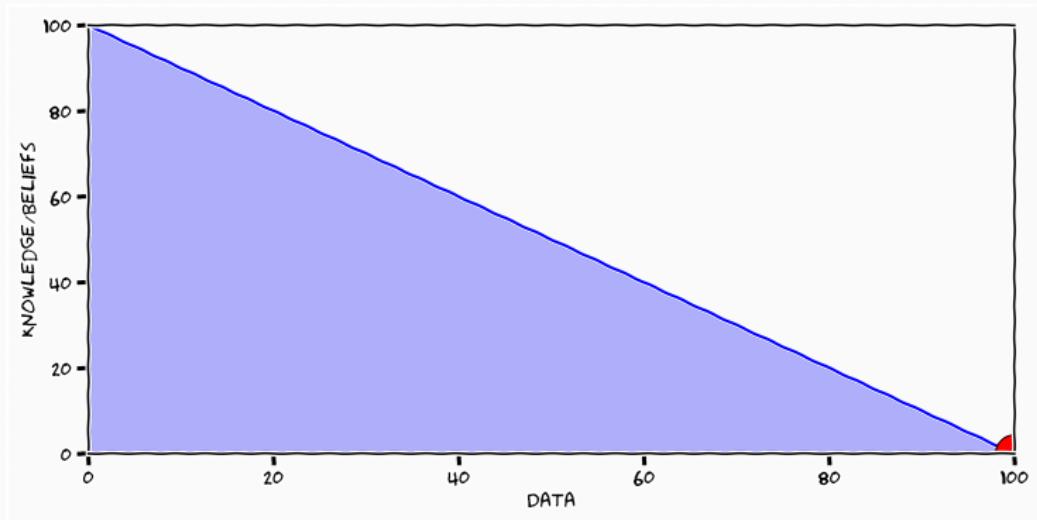
- We do not have enough data



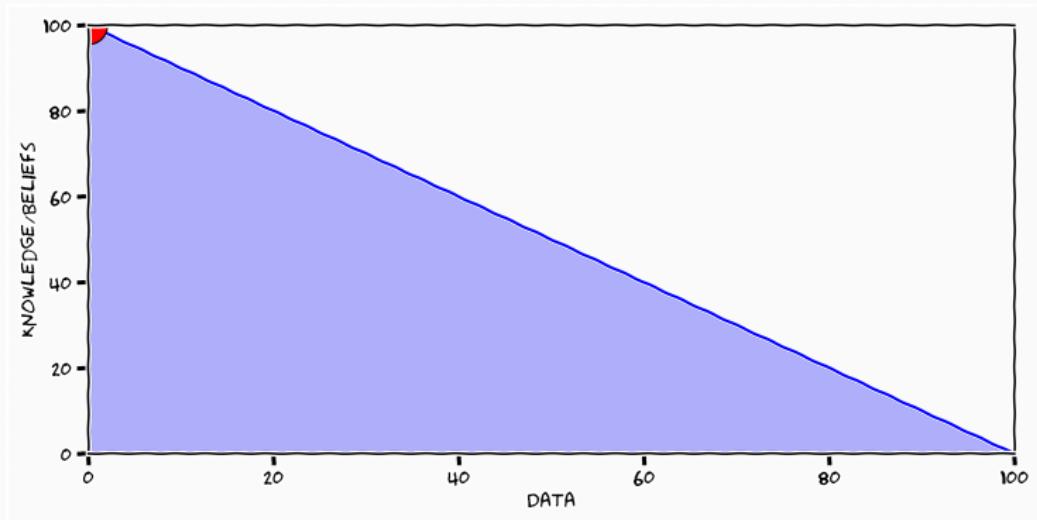
Data and Knowledge



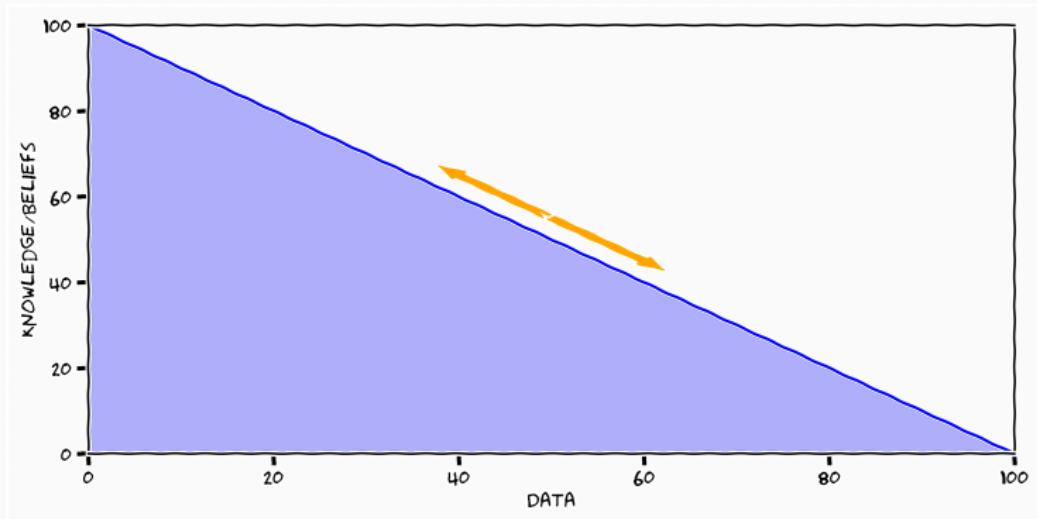
Data and Knowledge



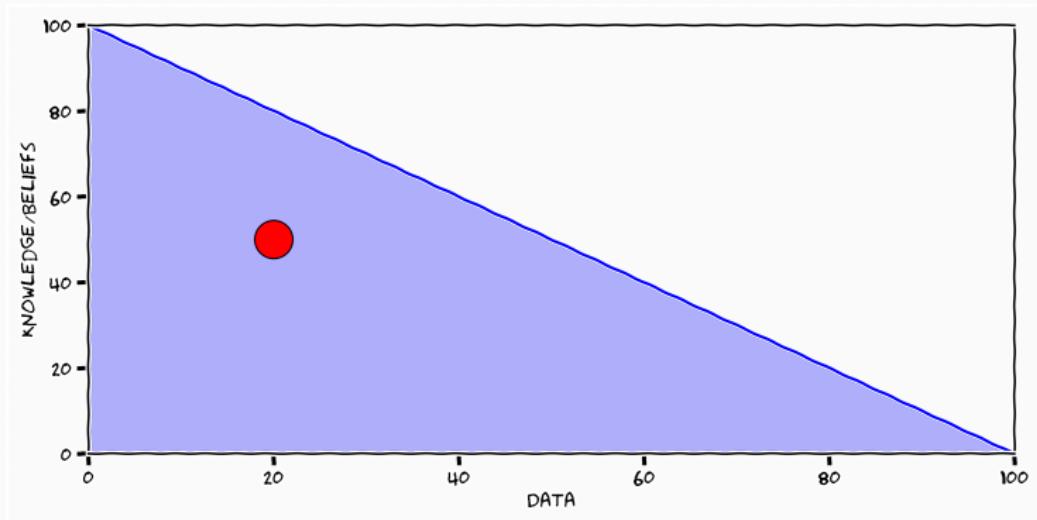
Data and Knowledge



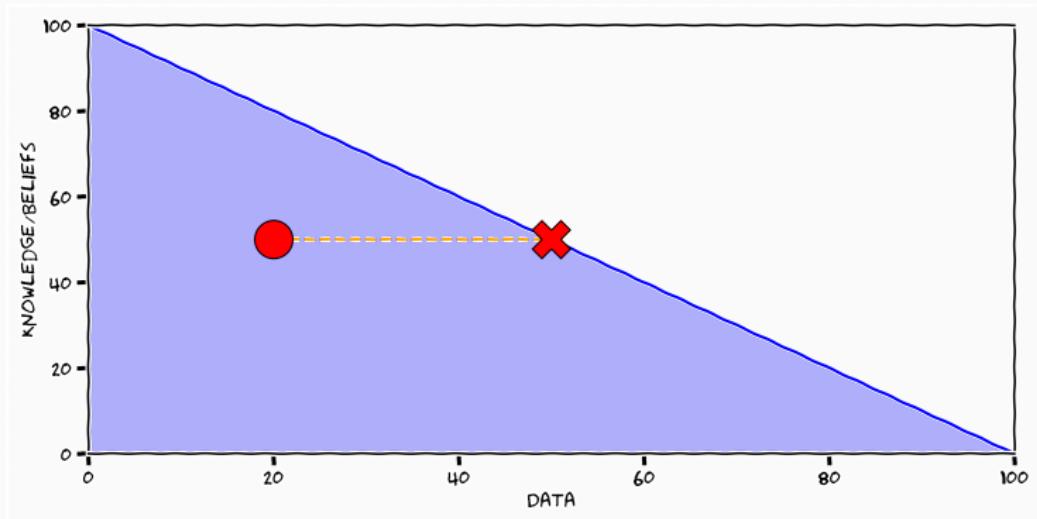
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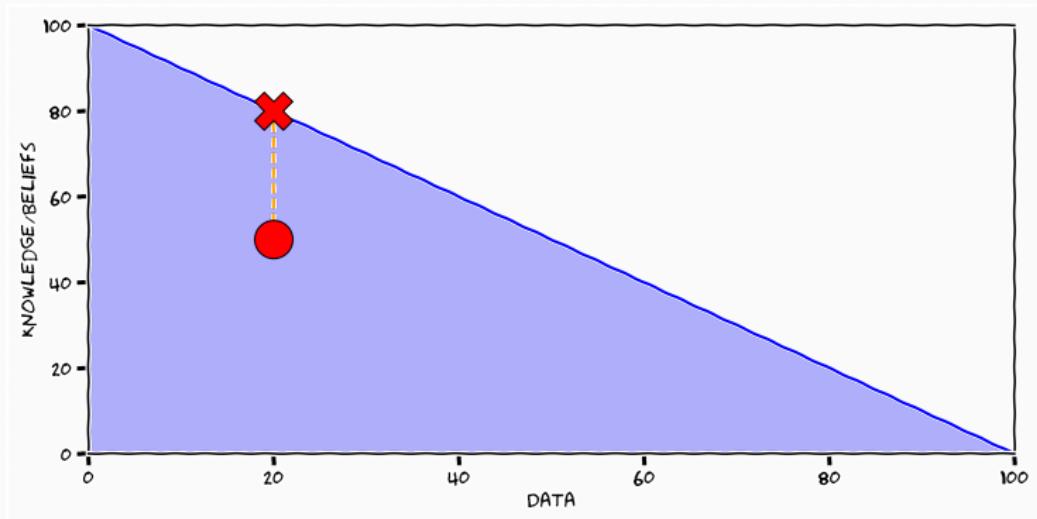
Data and Knowledge



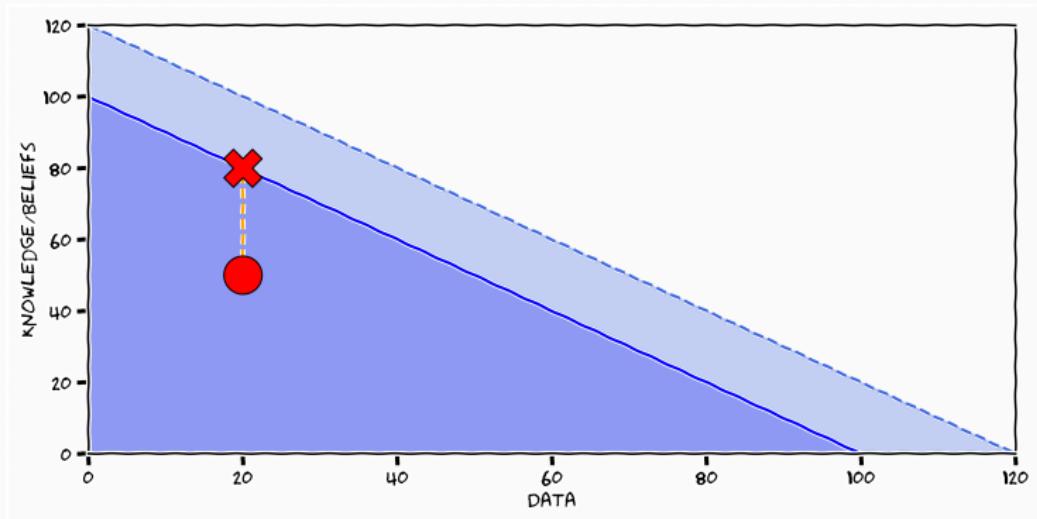
Data and Knowledge



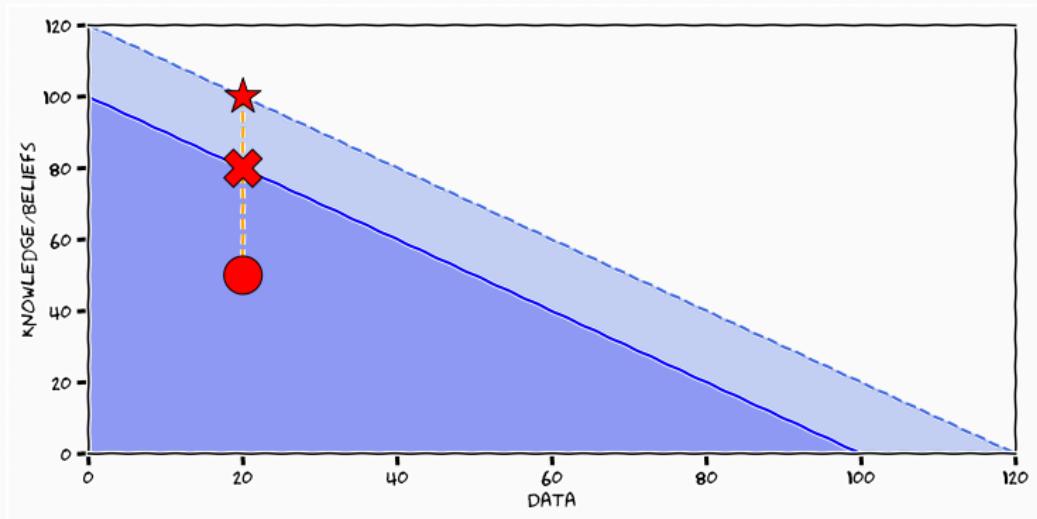
Data and Knowledge



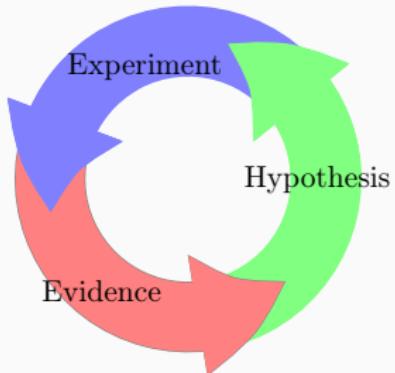
Data and Knowledge



Data and Knowledge

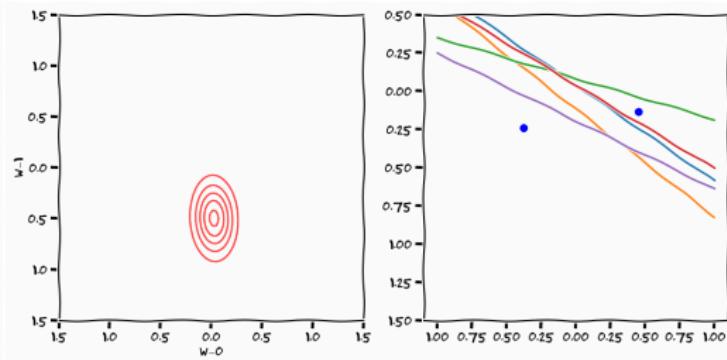
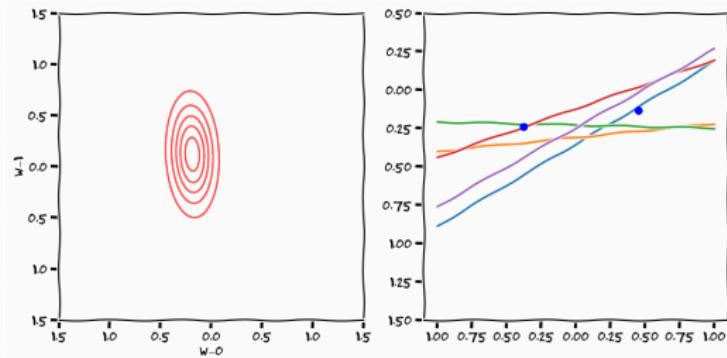


The Scientific Principle



$$p(\theta | y) = \frac{p(y | \theta)p(\theta)}{\int p(y | \theta)p(\theta)d\theta}$$

Knowledge is Relative



We must speak our minds openly, debate our disagreements honestly, but always pursue solidarity.

The leader of the free world



Machine Learning

- Machine Learning as a Framework



Machine Learning

- Machine Learning as a Framework



- Machine Learning as a Science

- how to construct "handles" to allow us to input knowledge

Today

- How to specify beliefs over larger classes of hypothesis

Today

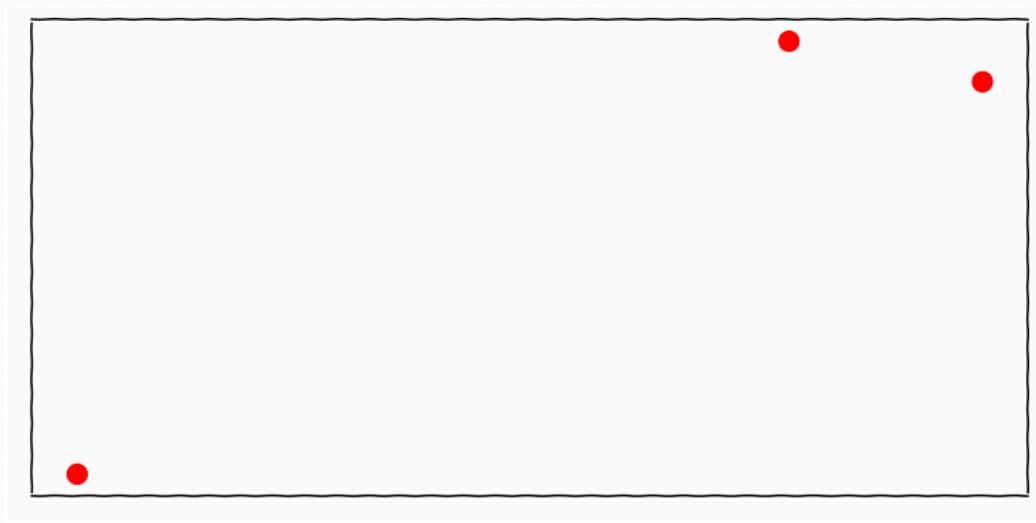
- How to specify beliefs over larger classes of hypothesis
- Non-parametrics

Today

- How to specify beliefs over larger classes of hypothesis
- Non-parametrics
- Gaussian processes

Non-parametrics

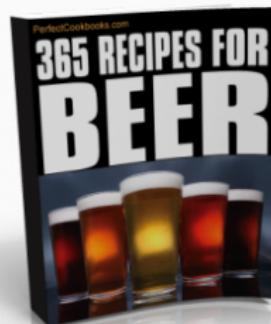
Hypothesis Classes



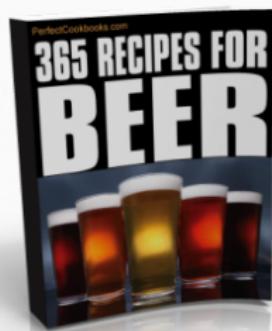
Non-parametrics



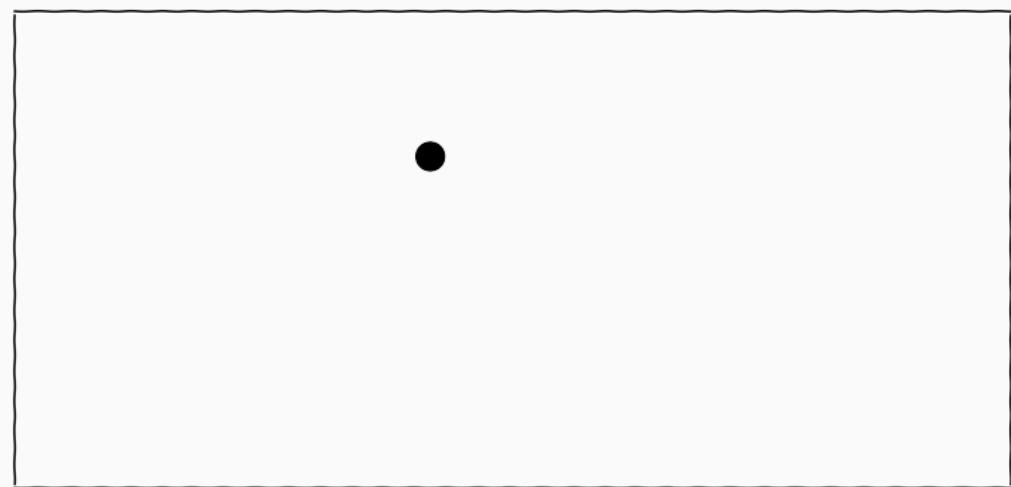
Non-parametrics



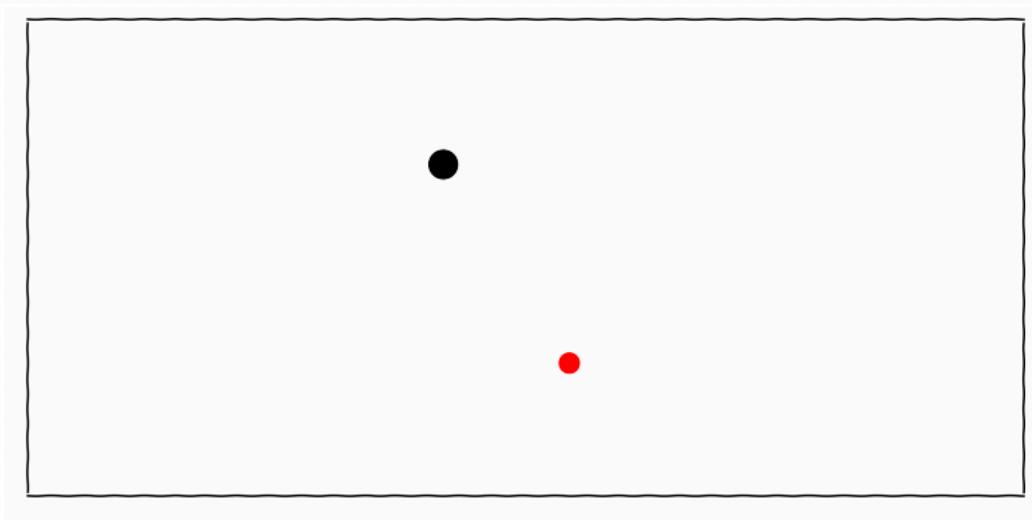
Non-parametrics



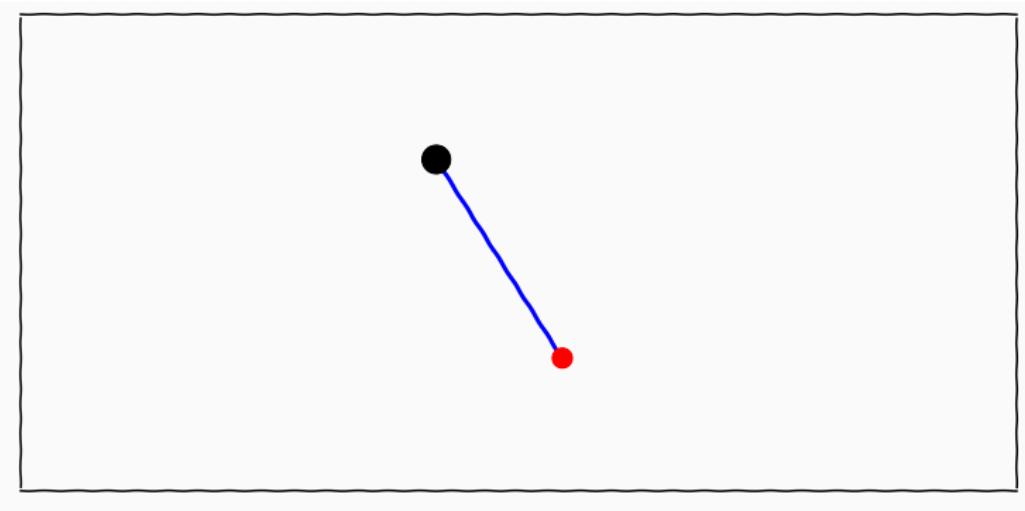
Example



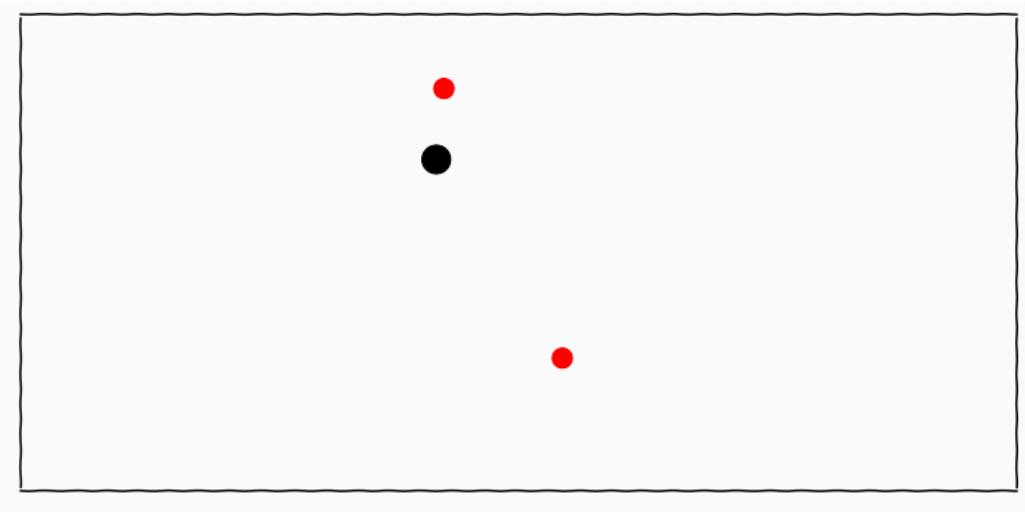
Example



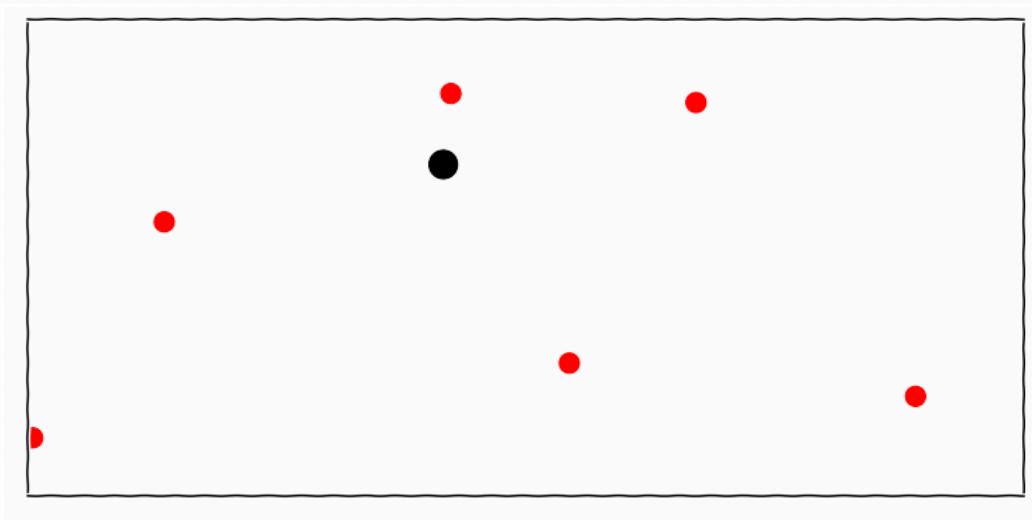
Example



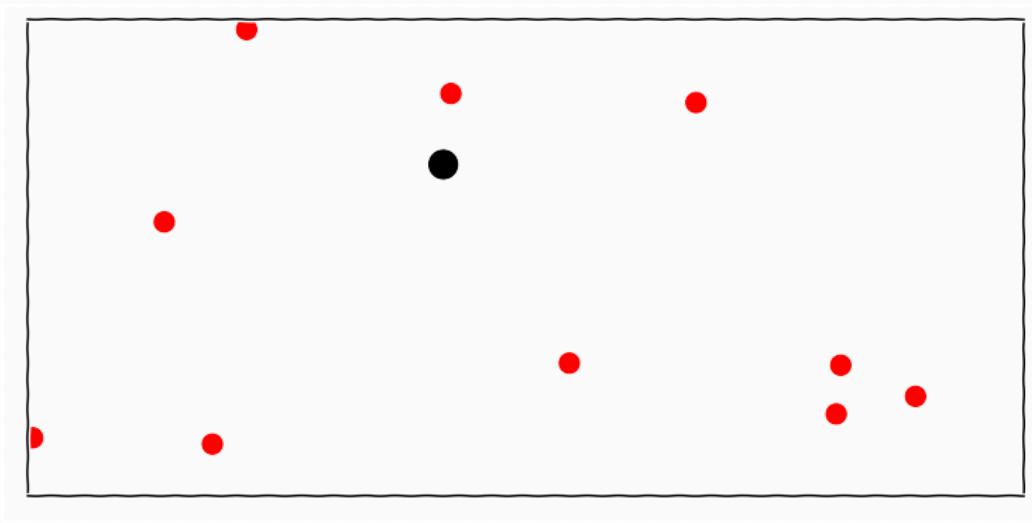
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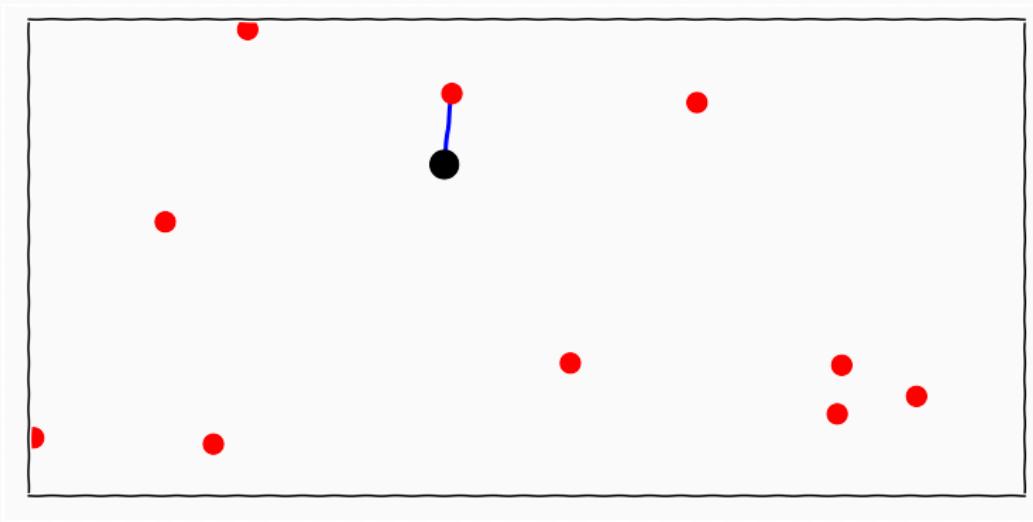
Example



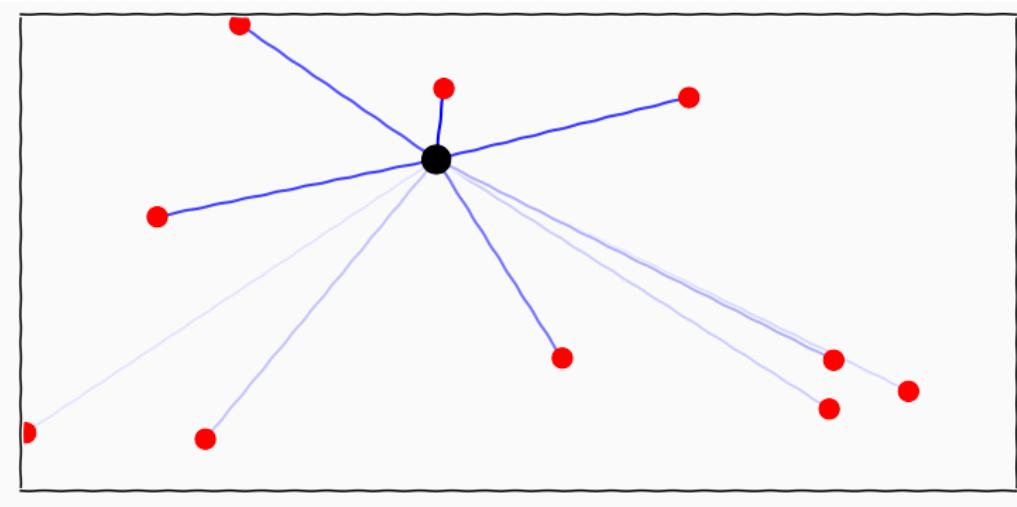
Example



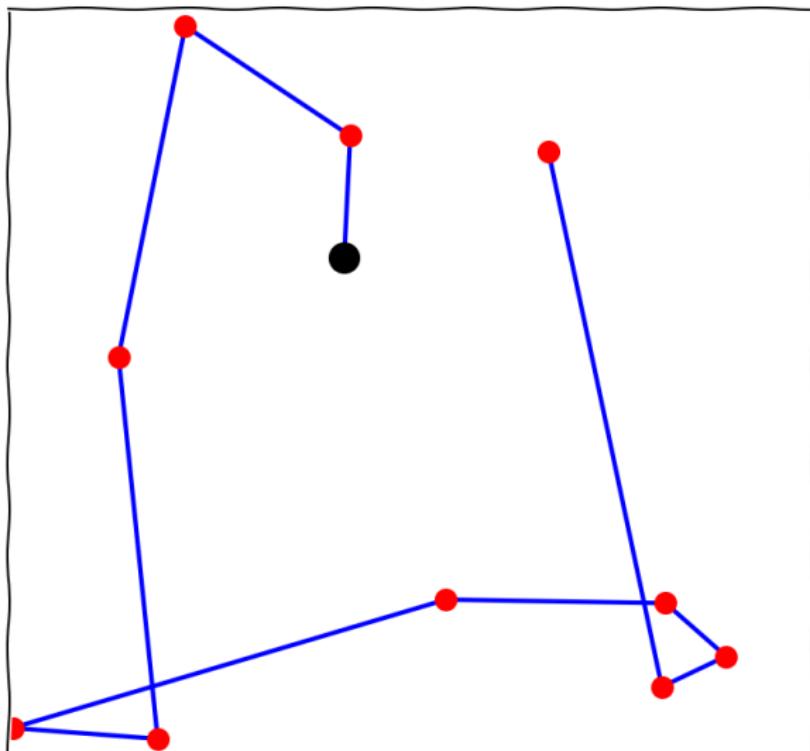
Example



Example

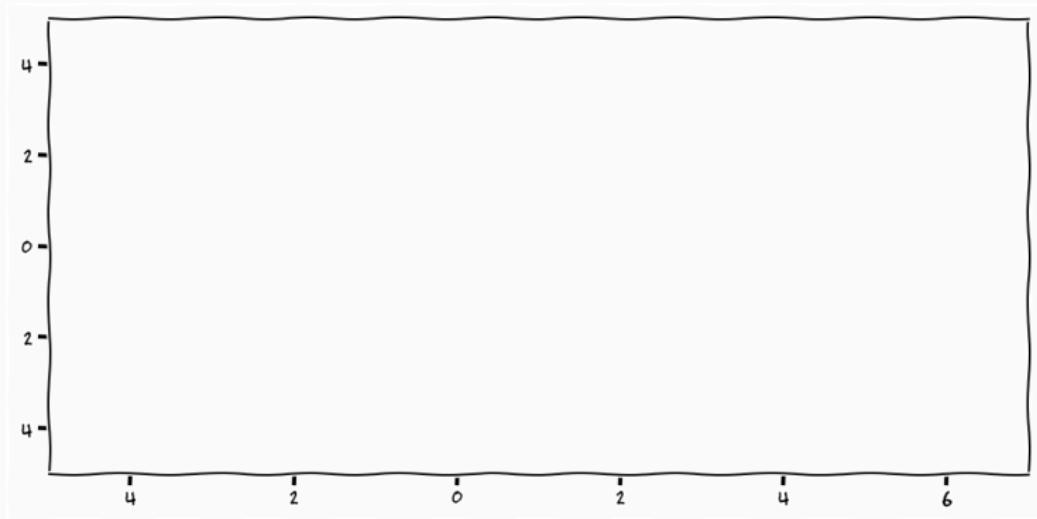


Example

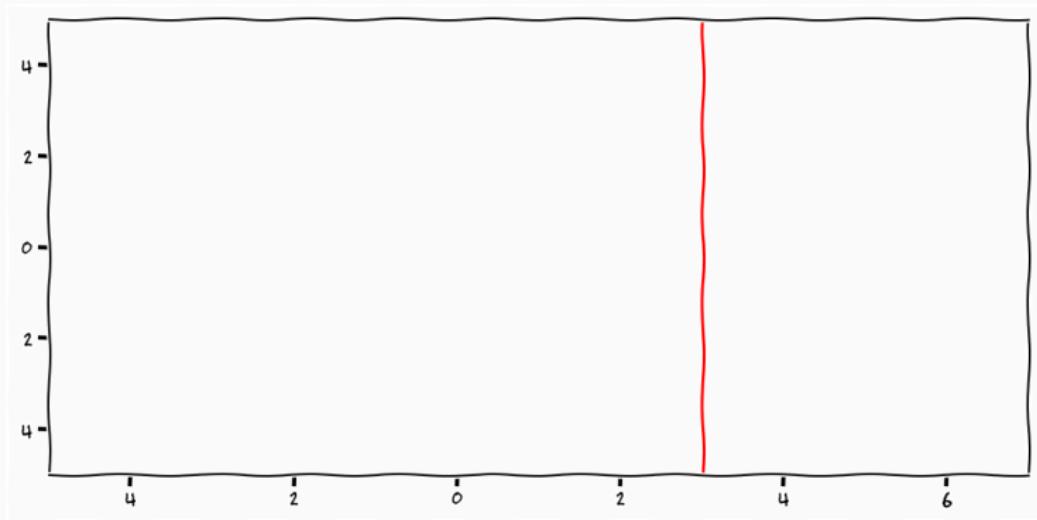


Non-parametric Functions

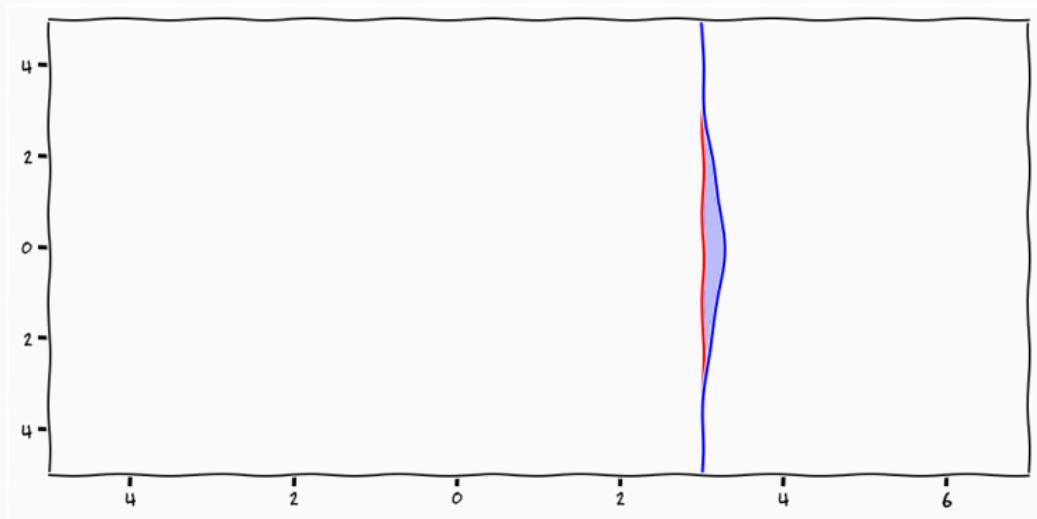
Lets talk about functions



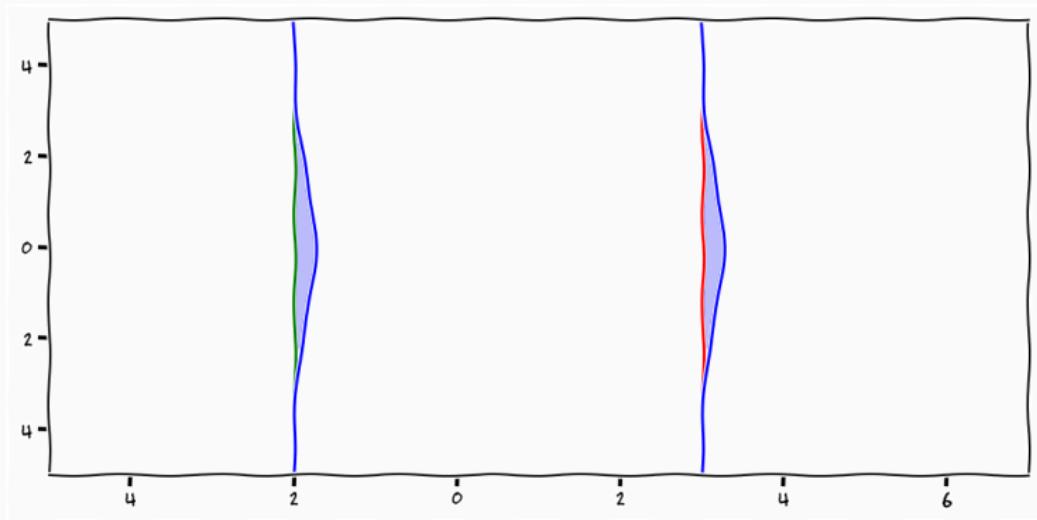
Lets talk about functions



Lets talk about functions



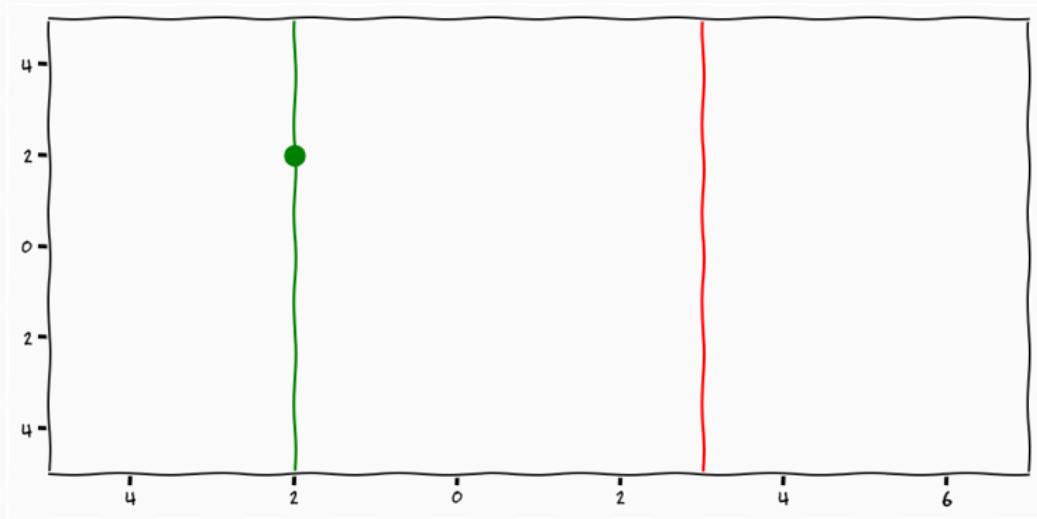
Lets talk about functions



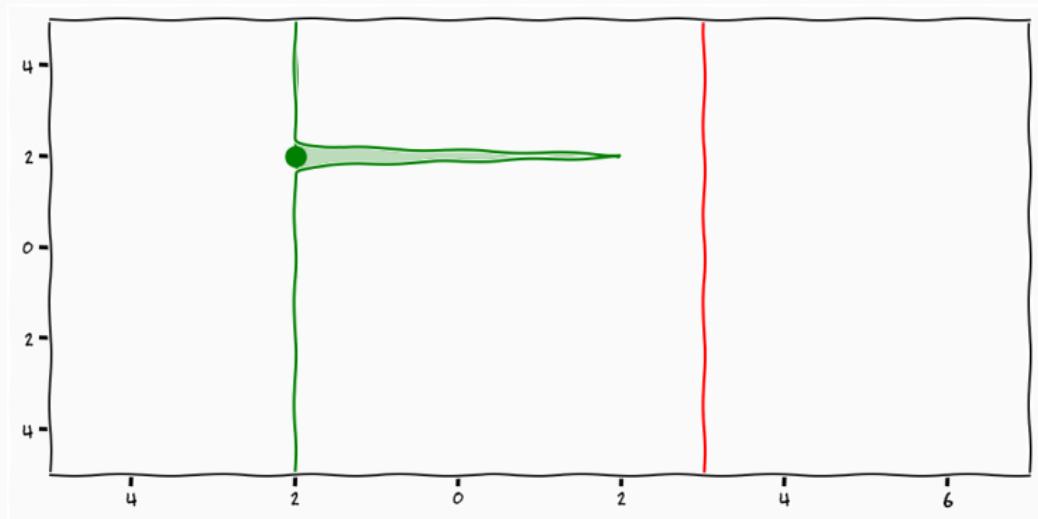
Jointly Gaussian function values

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \mathcal{N} \left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \right)$$

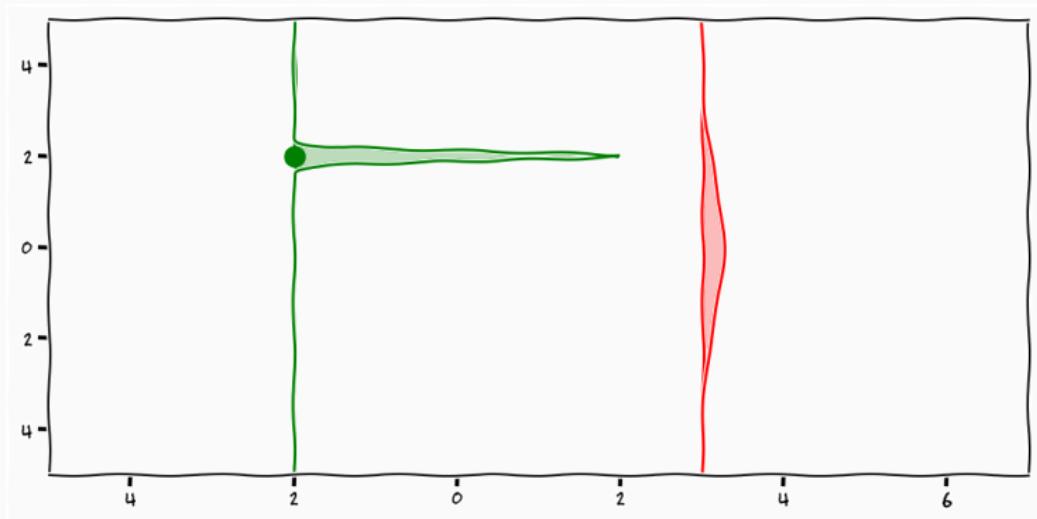
Non-parametric functions



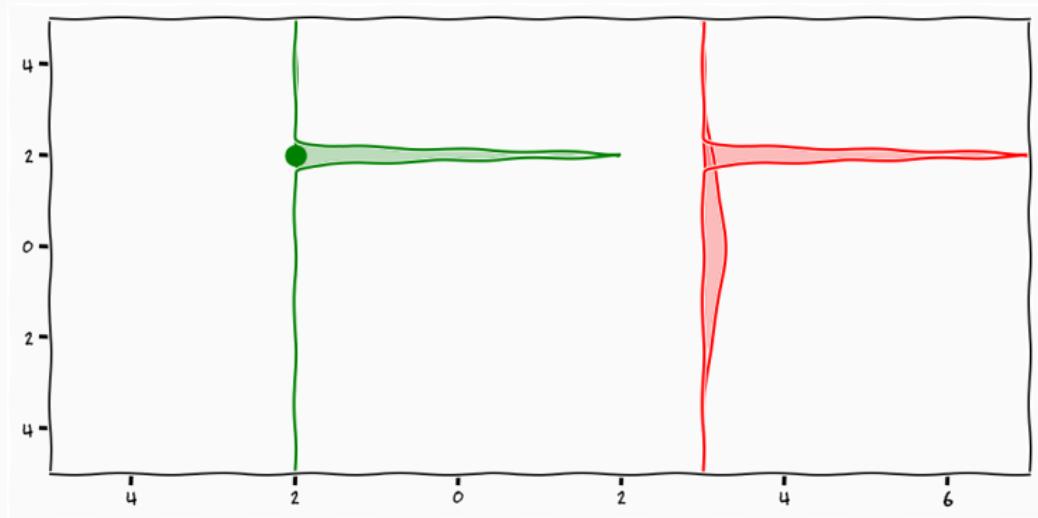
Non-parametric functions



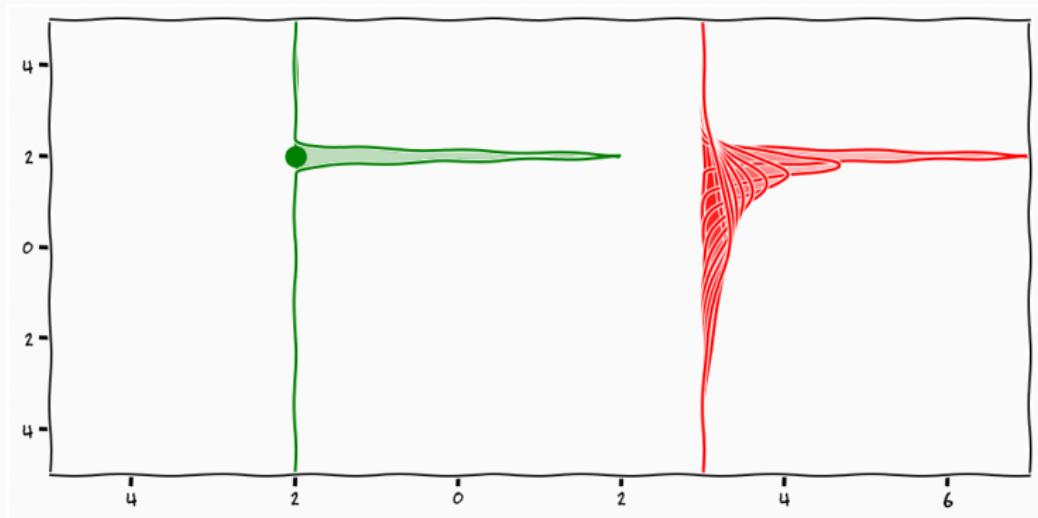
Non-parametric functions



Non-parametric functions



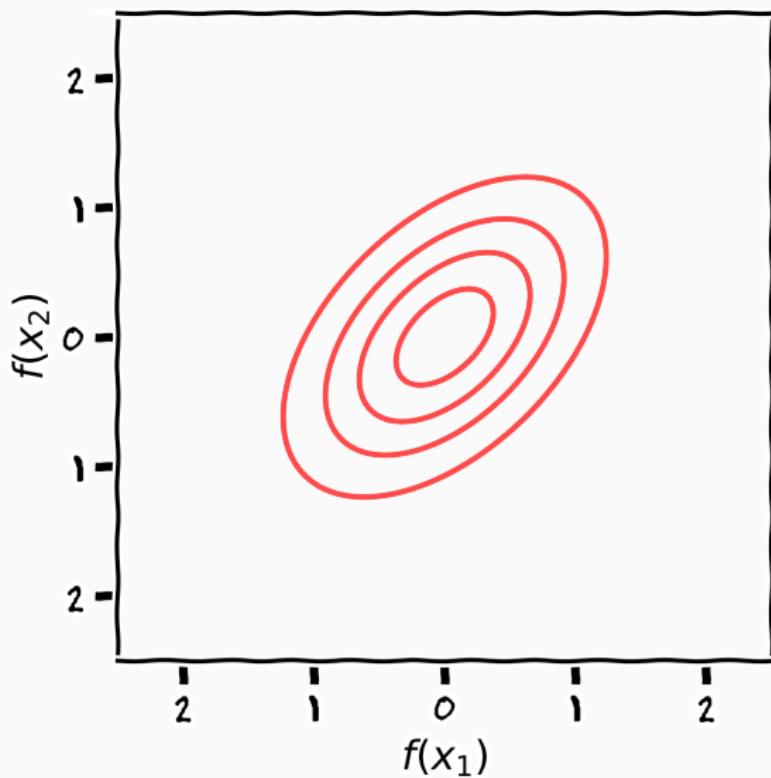
Non-parametric functions



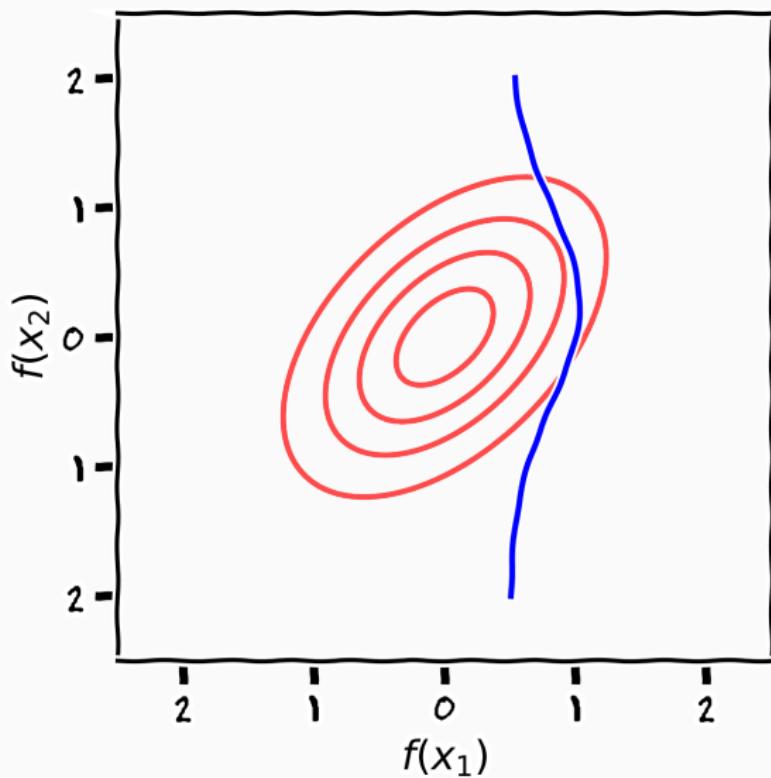
Conditional Gaussians

$$\mathcal{N} \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \right)$$

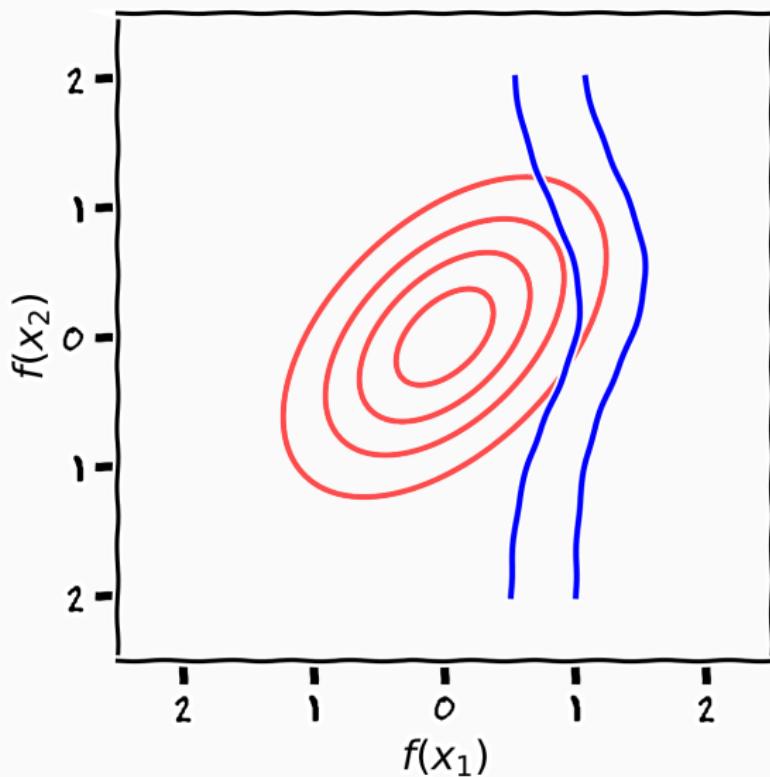
Conditional Gaussians



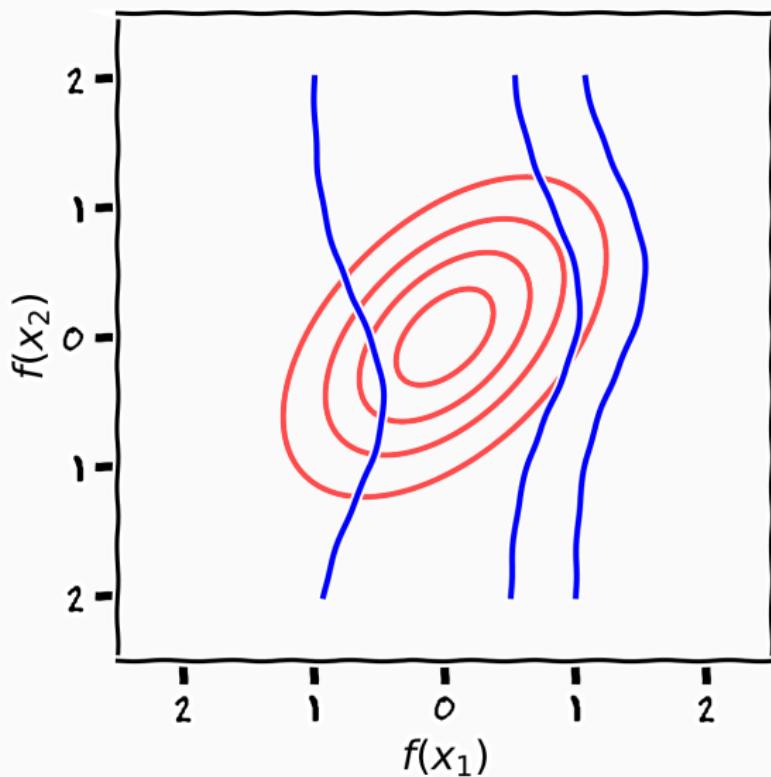
Conditional Gaussians



Conditional Gaussians



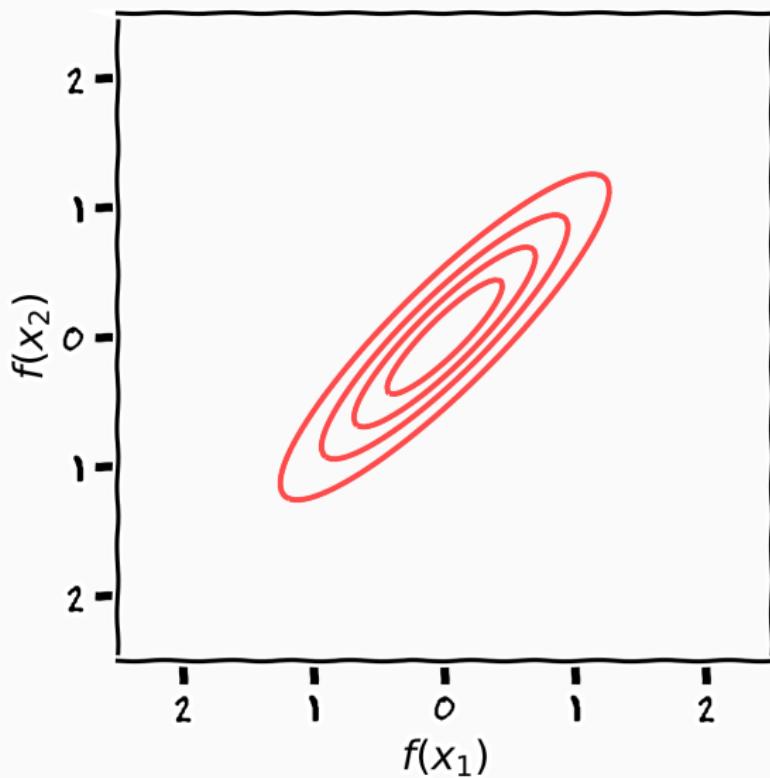
Conditional Gaussians



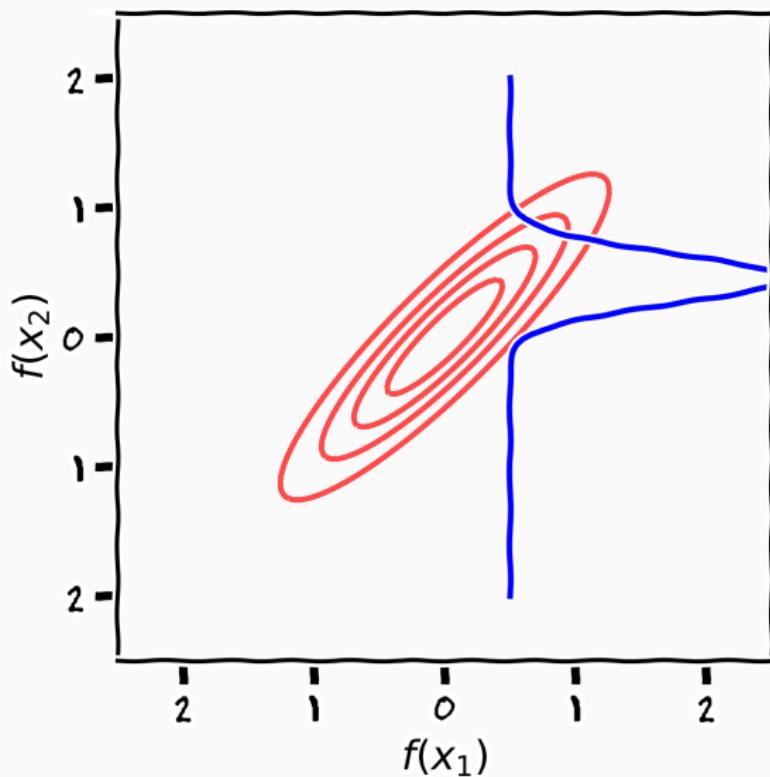
Conditional Gaussians

$$\mathcal{N} \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \textcolor{red}{0.9} \\ \textcolor{red}{0.9} & 1 \end{bmatrix} \right)$$

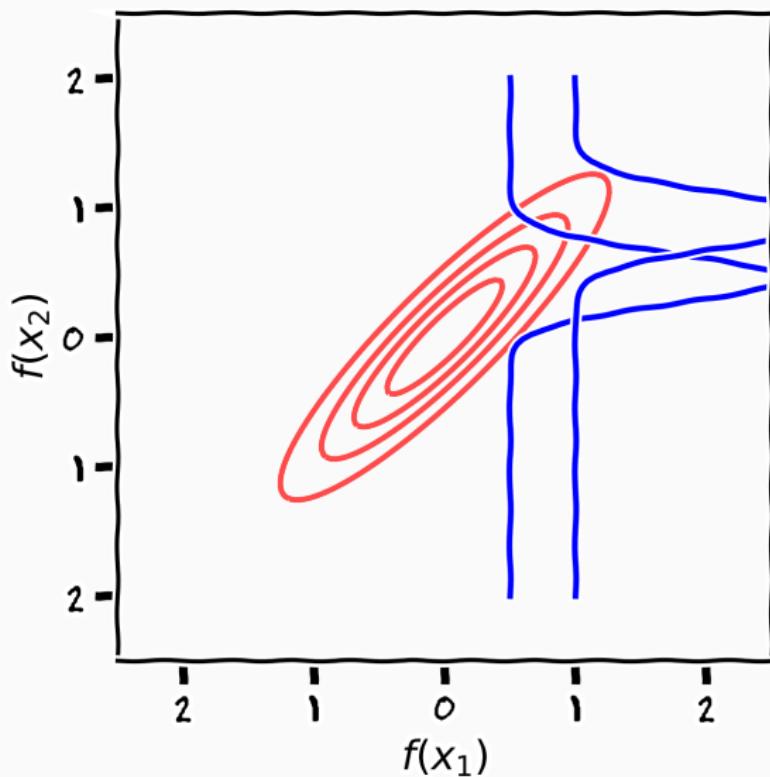
Conditional Gaussians



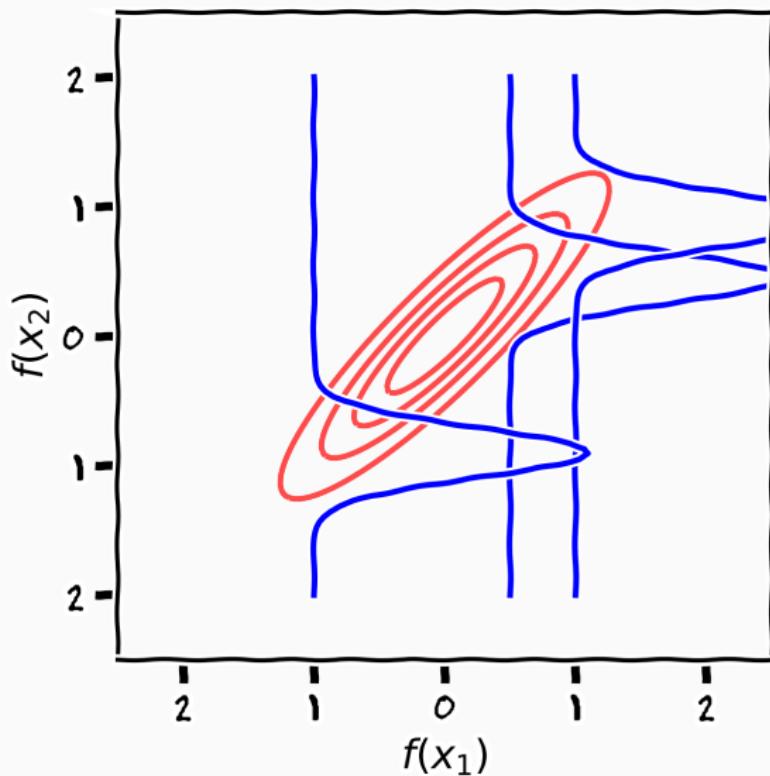
Conditional Gaussians



Conditional Gaussians



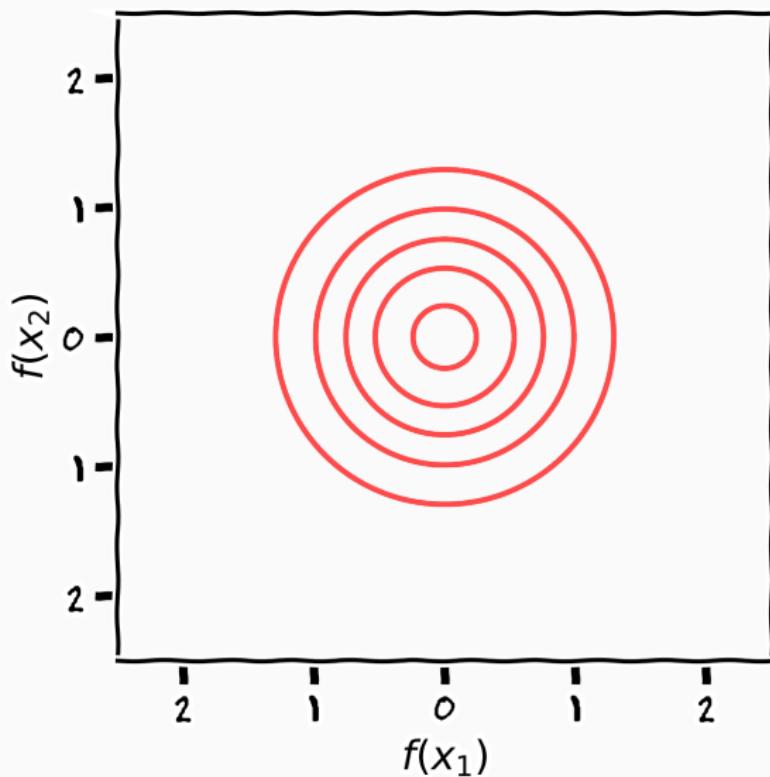
Conditional Gaussians



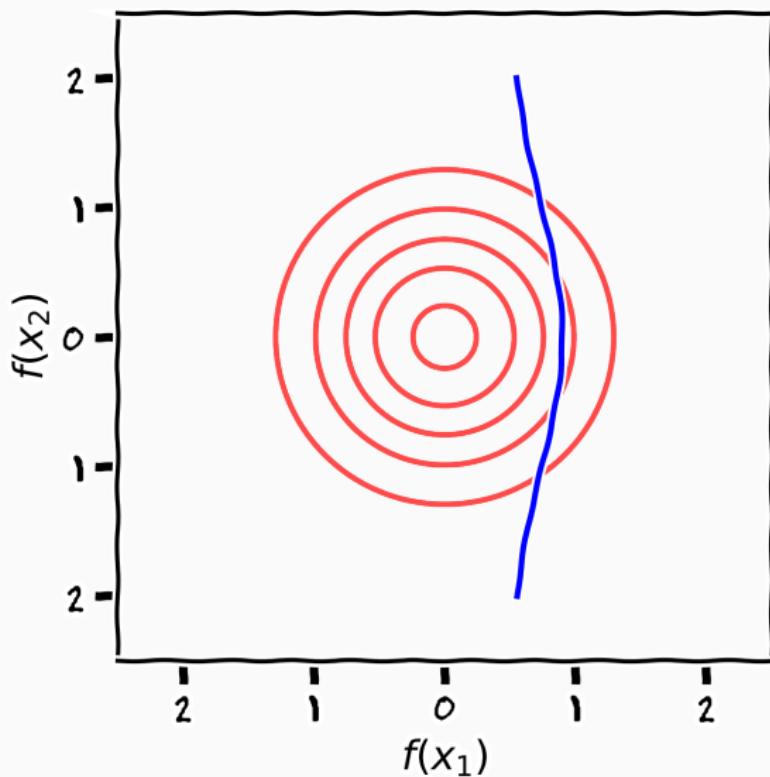
Conditional Gaussians

$$\mathcal{N} \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \textcolor{red}{0} \\ \textcolor{red}{0} & 1 \end{bmatrix} \right)$$

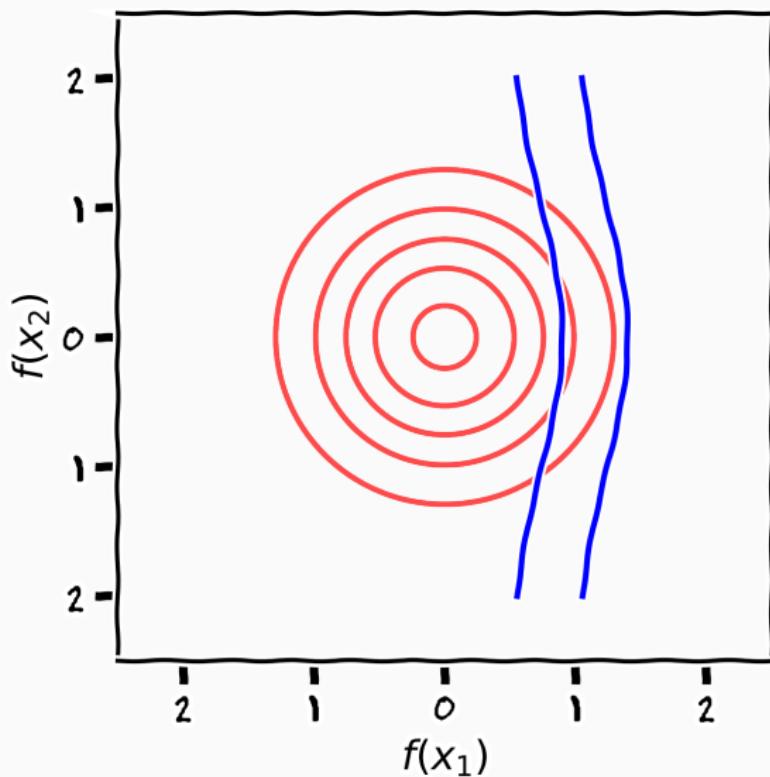
Conditional Gaussians



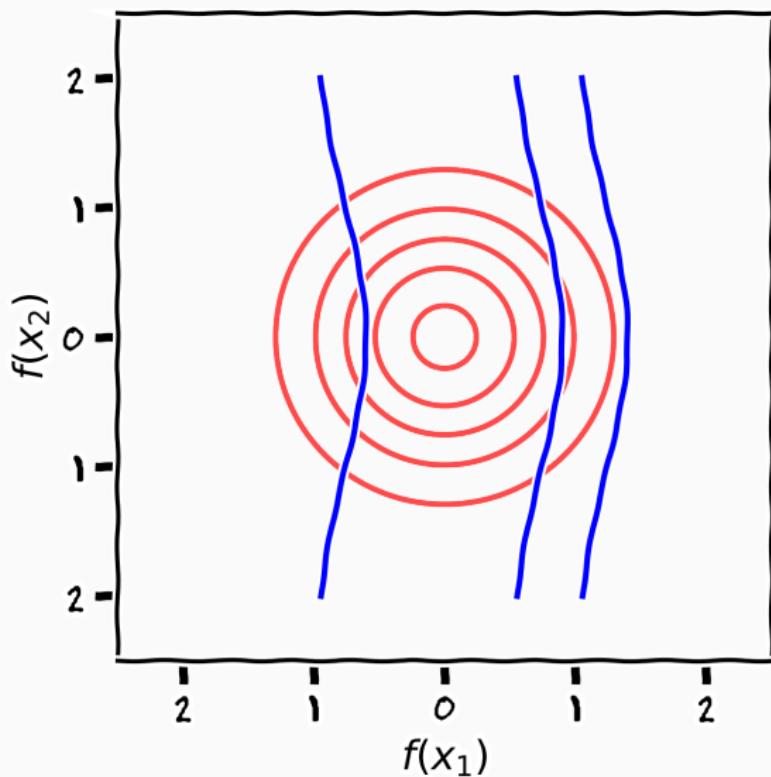
Conditional Gaussians



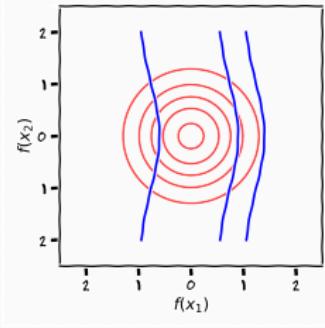
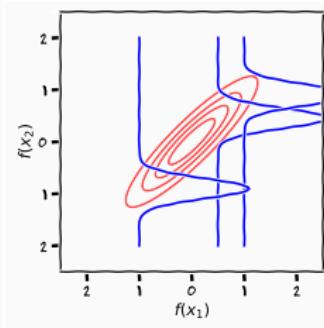
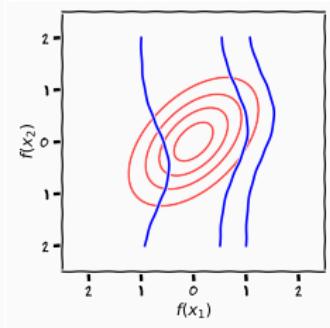
Conditional Gaussians



Conditional Gaussians



Conditional Gaussians

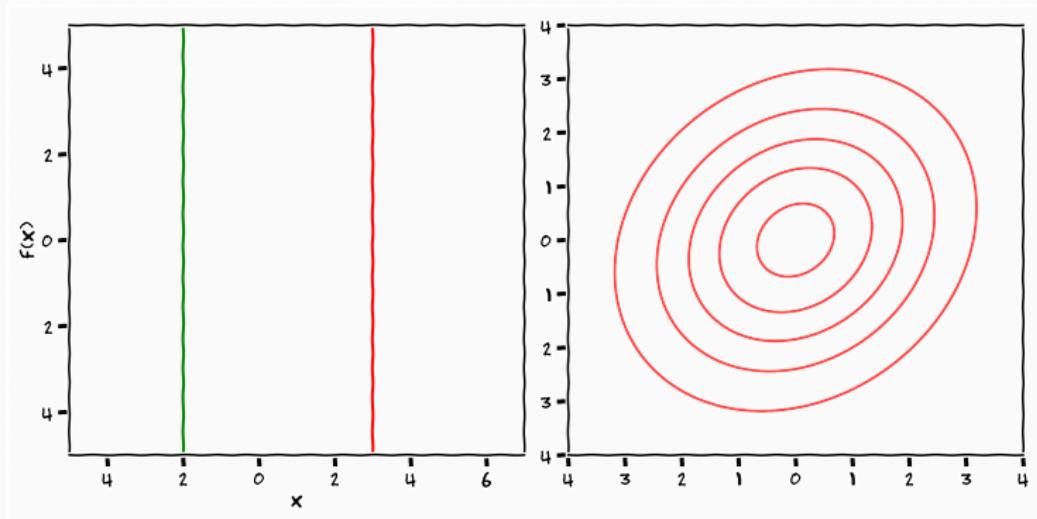


$$N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}\right)$$

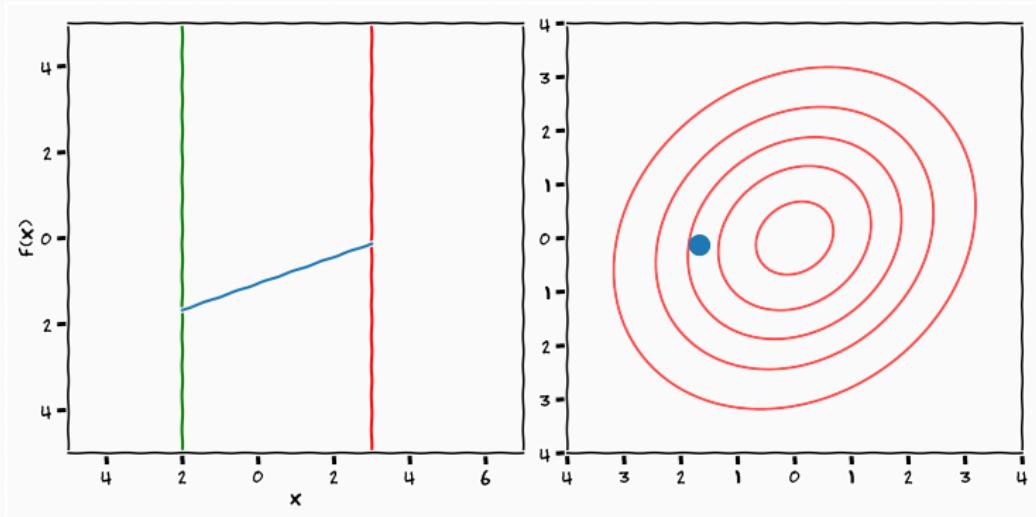
$$N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0.9 \\ 0.9 & 1 \end{bmatrix}\right)$$

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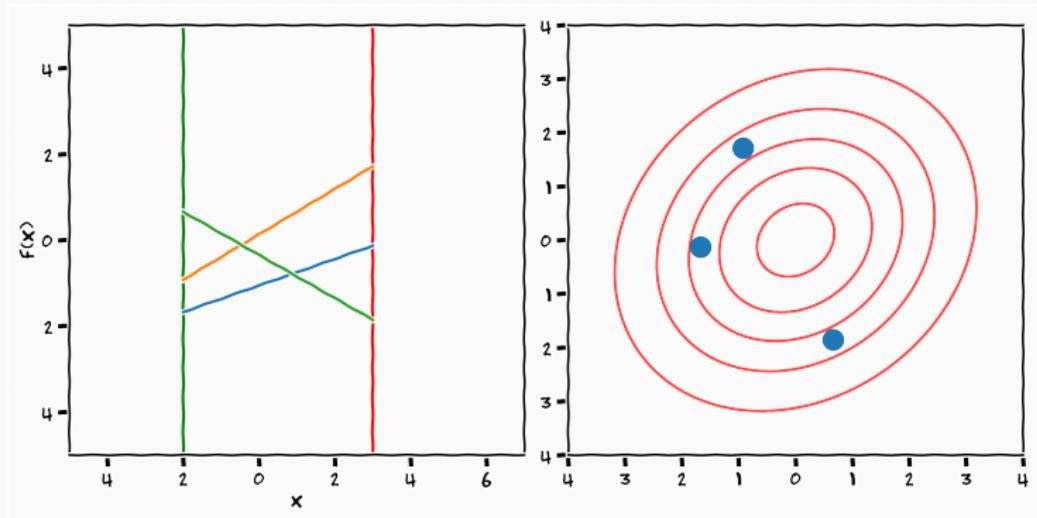
Gaussian Samples



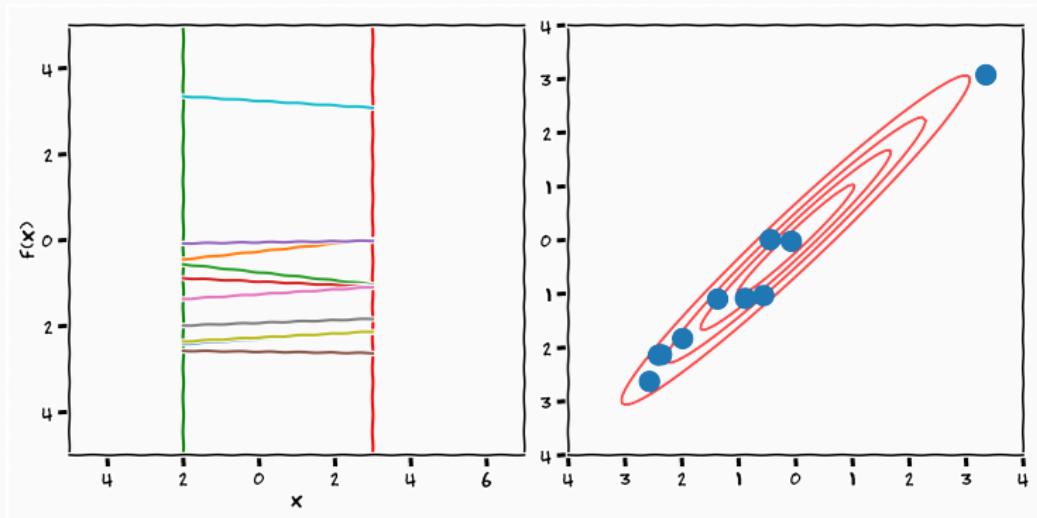
Gaussian Samples



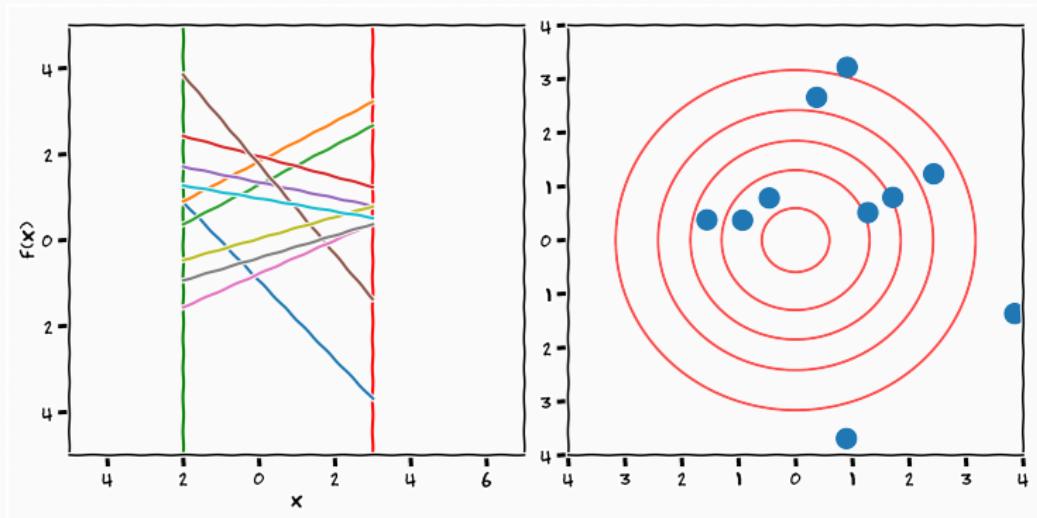
Gaussian Samples



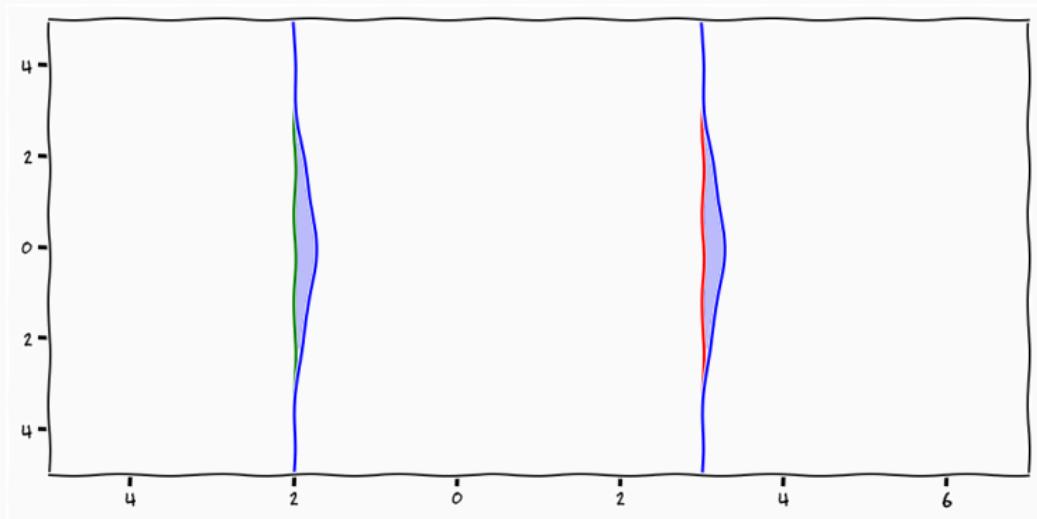
Gaussian Samples



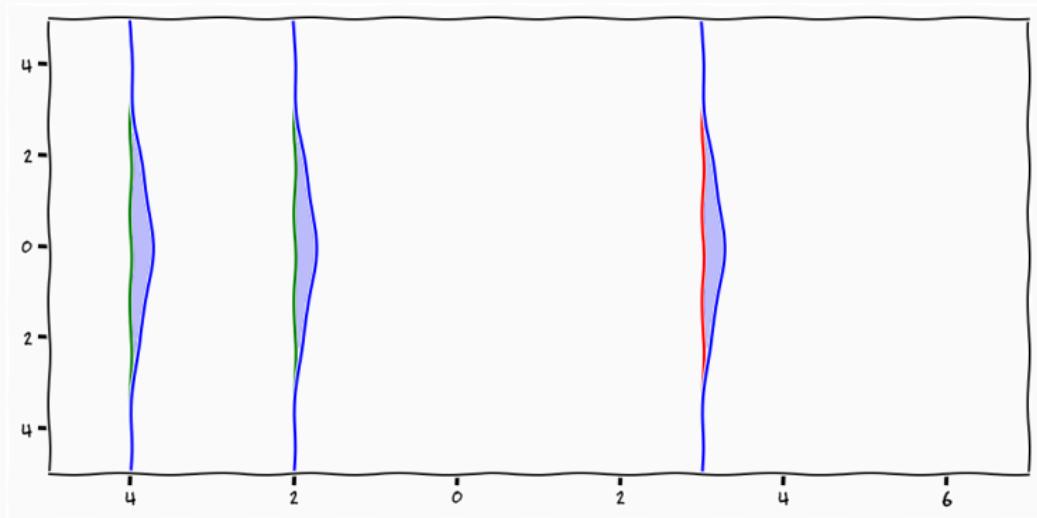
Gaussian Samples



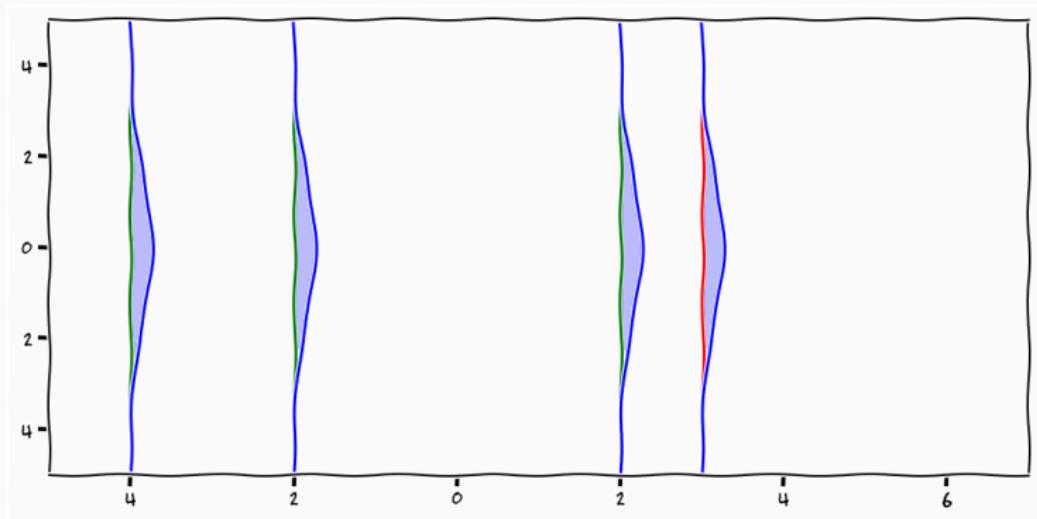
Lets talk about functions



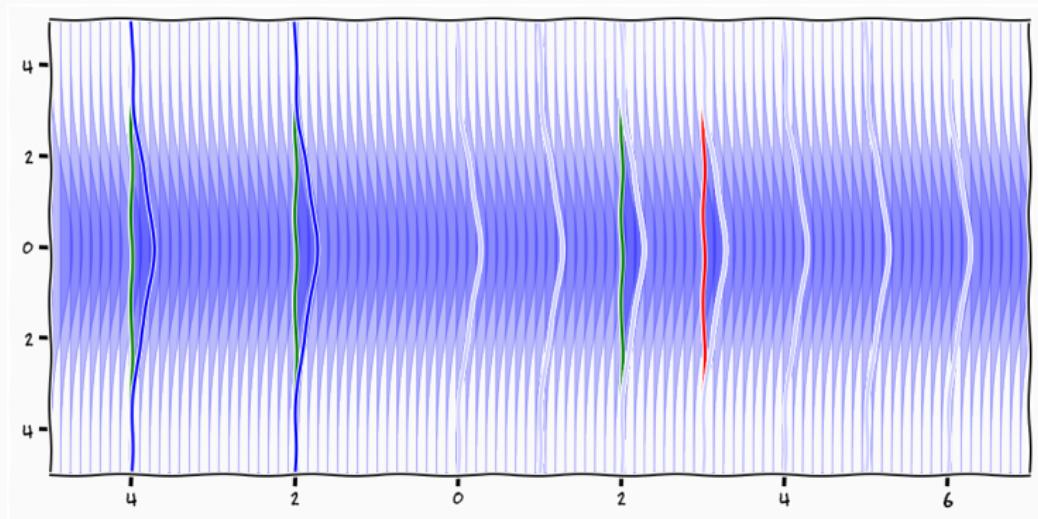
Non-parametric functions



Non-parametric functions



Non-parametric functions



Jointly Gaussian functions II

$$p(\mathbf{f}) = \mathcal{N} \left(\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{bmatrix} \middle| \begin{bmatrix} \mu(x_1) \\ \mu(x_2) \\ \vdots \\ \mu(x_N) \end{bmatrix}, \begin{bmatrix} k_{11} & k_{12} & \dots & k_{1N} \\ k_{21} & k_{22} & \dots & k_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ k_{N1} & k_{N2} & \dots & k_{NN} \end{bmatrix} \right)$$

Gaussian Distribution - Marginal

$$p(\textcolor{red}{x}_1, x_2) = \mathcal{N} \left(\begin{array}{c|cc} \textcolor{red}{x}_1 & \mu_1 & k_{11} & k_{12} \\ x_2 & \mu_2 & k_{21} & k_{22} \end{array} \right)$$

Gaussian Distribution - Marginal

$$p(\textcolor{magenta}{x}_1, x_2) = \mathcal{N} \left(\begin{array}{c|cc} \textcolor{magenta}{x}_1 & \mu_1 & k_{11} & k_{12} \\ x_2 & \mu_2 & k_{21} & k_{22} \end{array} \right)$$
$$\Rightarrow p(\textcolor{magenta}{x}_1) = \int_{x_2} p(\textcolor{magenta}{x}_1, x_2) = \underline{\mathcal{N}(\textcolor{magenta}{x}_1 | \mu_1, k_{11})}$$

Gaussian Distribution - Marginal

$$p(\textcolor{red}{x}_1, x_2) = \mathcal{N} \left(\begin{array}{c|cc} x_1 & \mu_1 & k_{11} & k_{12} \\ x_2 & \mu_2 & k_{21} & k_{22} \end{array} \right)$$

$$\Rightarrow p(\textcolor{red}{x}_1) = \int_{x_2} p(\textcolor{red}{x}_1, x_2) = \underline{\mathcal{N}(\textcolor{red}{x}_1 \mid \mu_1, k_{11})}$$

$$p(\textcolor{red}{x}_1, x_2, \dots, x_N) = \mathcal{N} \left(\begin{array}{c|cccccc} x_1 & \mu_1 & k_{11} & k_{12} & \cdots & k_{1N} \\ x_2 & \mu_2 & k_{21} & k_{22} & \cdots & k_{2N} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_N & \mu_N & k_{N1} & k_{N2} & \cdots & k_{NN} \end{array} \right)$$

Gaussian Distribution - Marginal

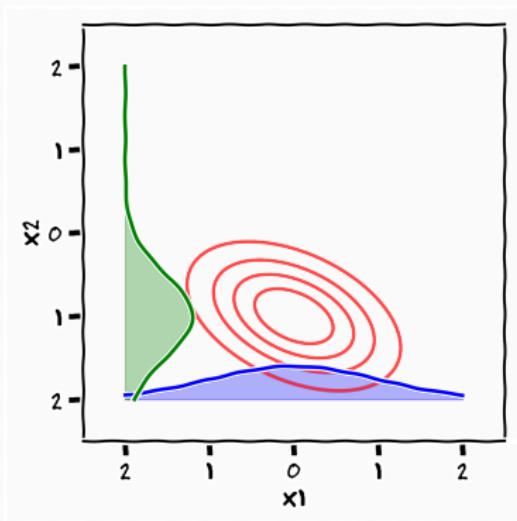
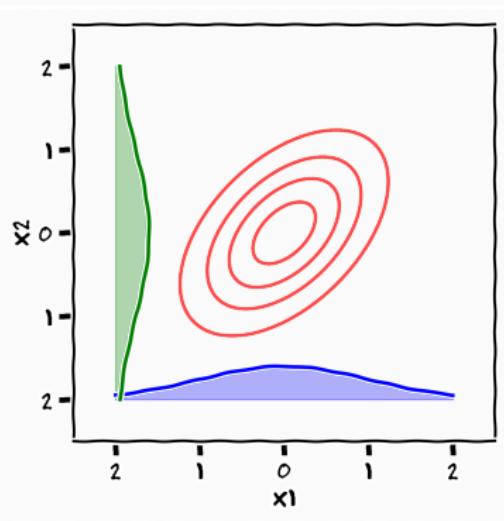
$$p(\mathbf{x}_1, x_2) = \mathcal{N} \left(\begin{array}{c|cc} \mathbf{x}_1 & \mu_1 & k_{11} & k_{12} \\ x_2 & \mu_2 & k_{21} & k_{22} \end{array} \right)$$

$$\Rightarrow p(\mathbf{x}_1) = \int_{x_2} p(\mathbf{x}_1, x_2) = \underline{\mathcal{N}(\mathbf{x}_1 | \mu_1, k_{11})}$$

$$p(\mathbf{x}_1, x_2, \dots, x_N) = \mathcal{N} \left(\begin{array}{c|cccc} \mathbf{x}_1 & \mu_1 & k_{11} & k_{12} & \cdots & k_{1N} \\ x_2 & \mu_2 & k_{21} & k_{22} & \cdots & k_{2N} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_N & \mu_N & k_{N1} & k_{N2} & \cdots & k_{NN} \end{array} \right)$$

$$\Rightarrow p(\mathbf{x}_1) = \int_{x_2, \dots, x_N} p(\mathbf{x}_1, x_2, \dots, x_N) = \underline{\mathcal{N}(\mathbf{x}_1 | \mu_1, k_{11})}$$

Gaussian Distribution - Marginal

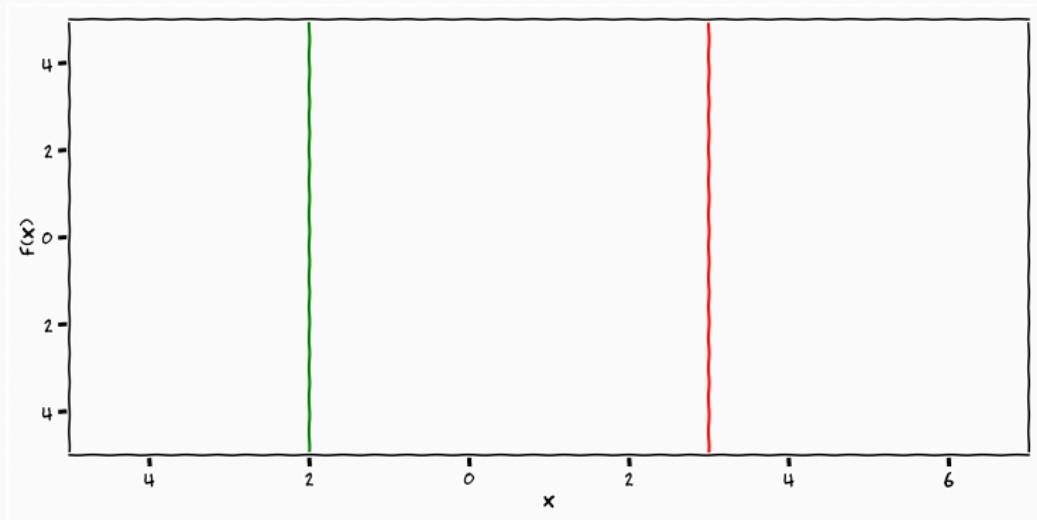


Marginal Property (Consistency)

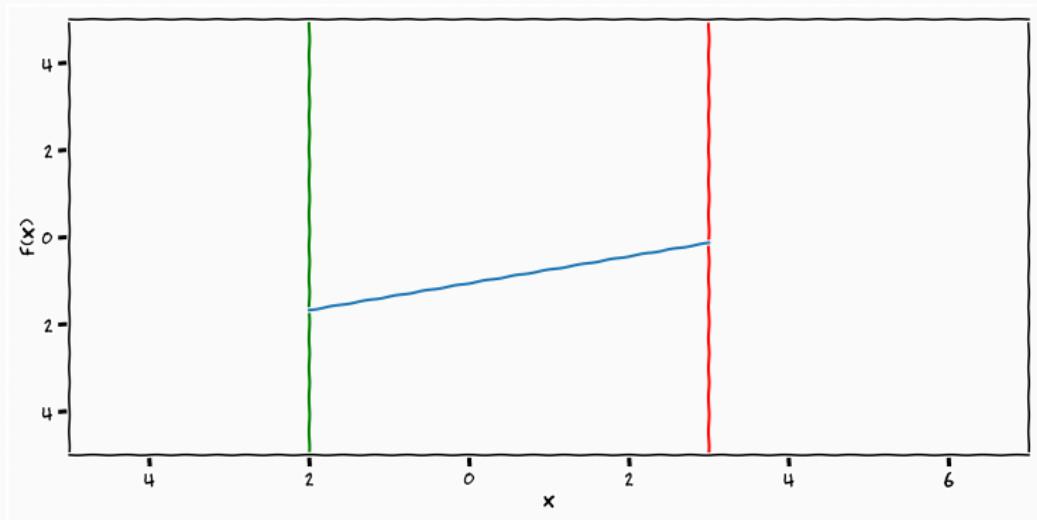
For all measurable sets $F_i \subseteq \mathbb{R}^n$ and probability measure \mathcal{N}

$$\mathcal{N}_{t_1 \cdot t_k}(F_1 \times \cdots \times F_k) = \mathcal{N}_{t_1 \dots t_k, t_{k+1} \cdot t_{k+m}}(F_1 \times \cdots \times F_k \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n)$$

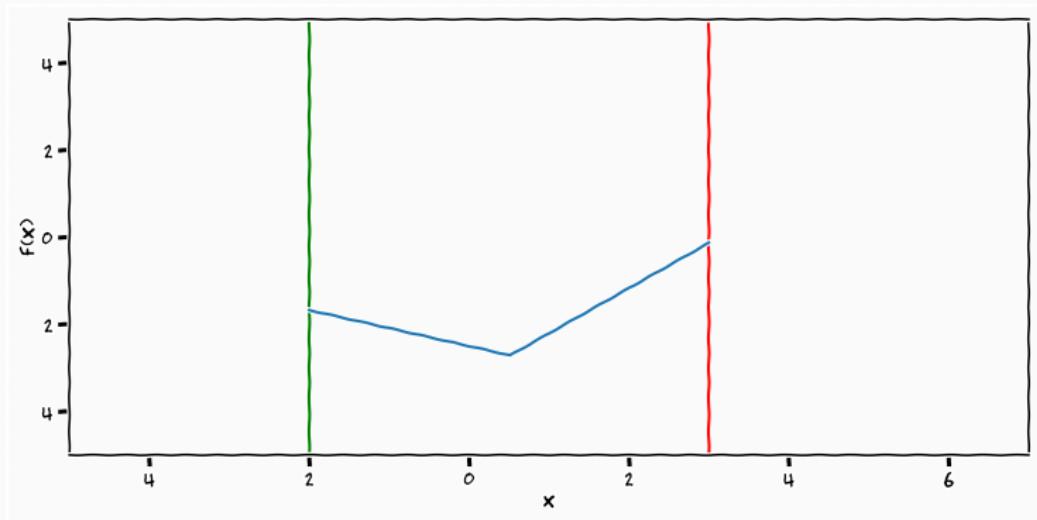
Gaussian Samples



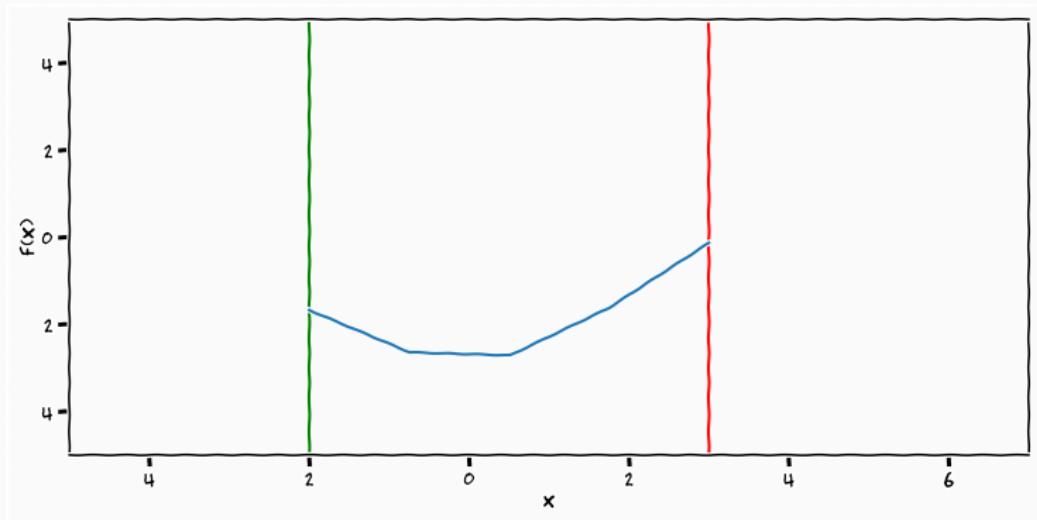
Gaussian Samples



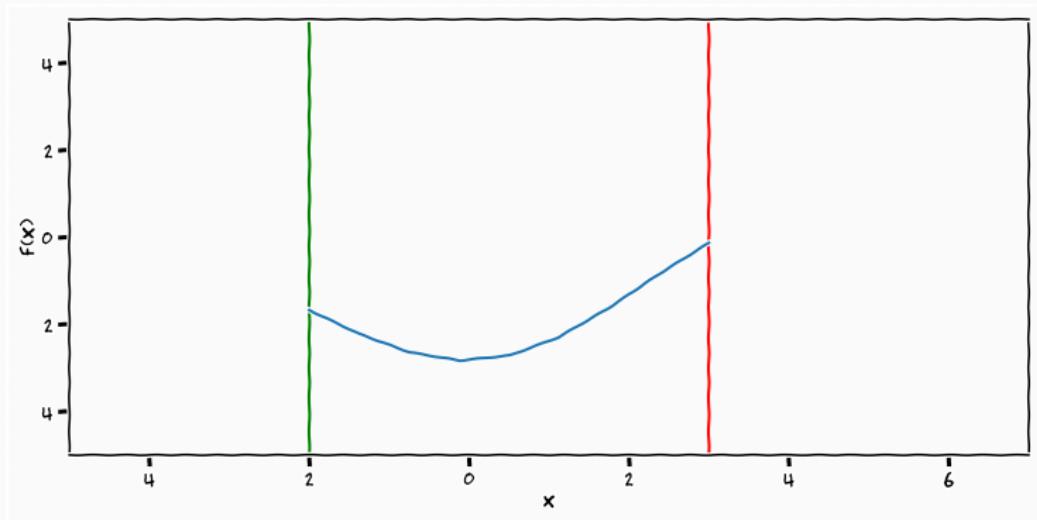
Gaussian Samples



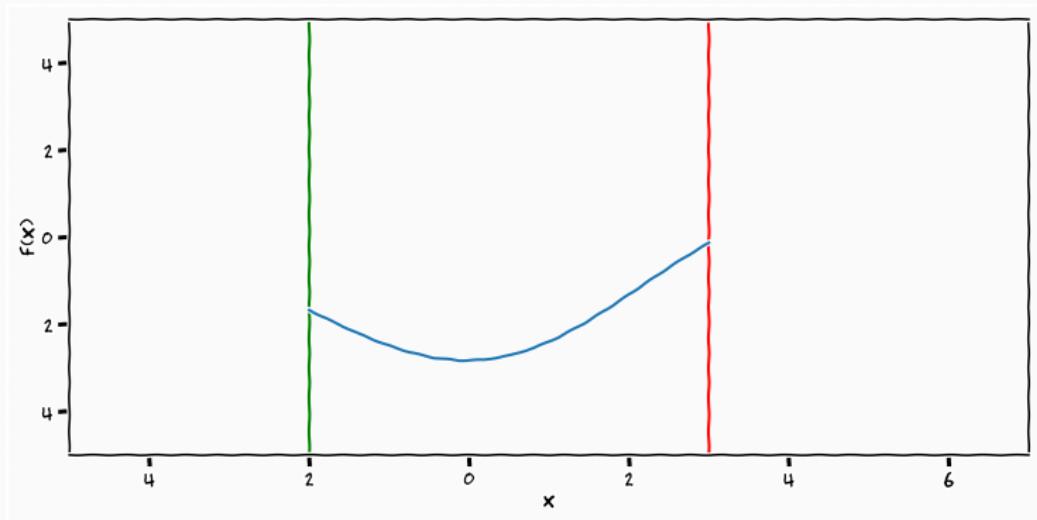
Gaussian Samples



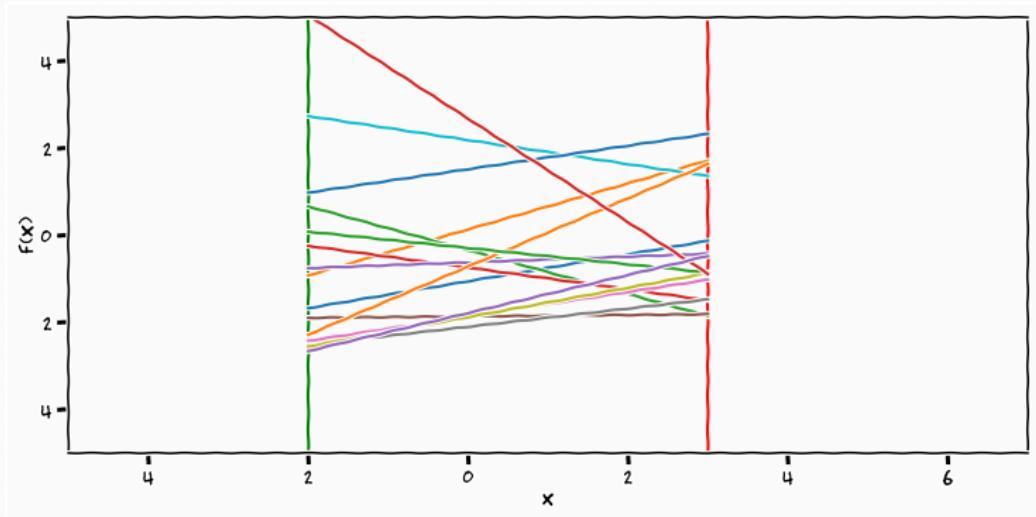
Gaussian Samples



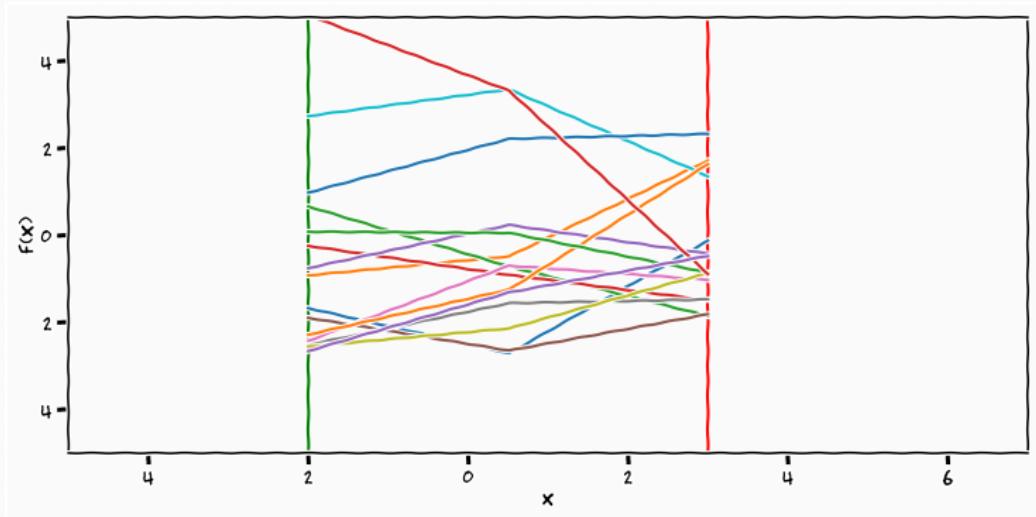
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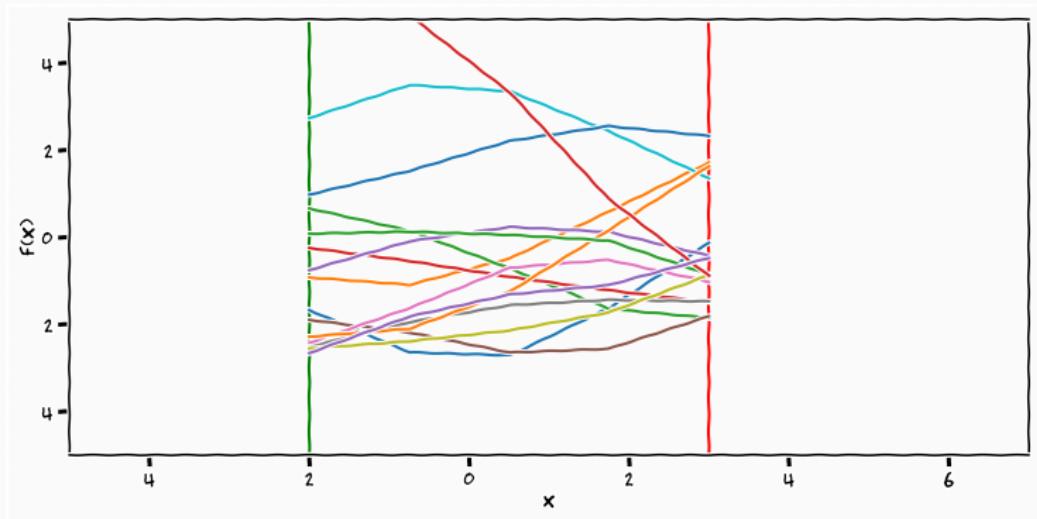
Gaussian Samples



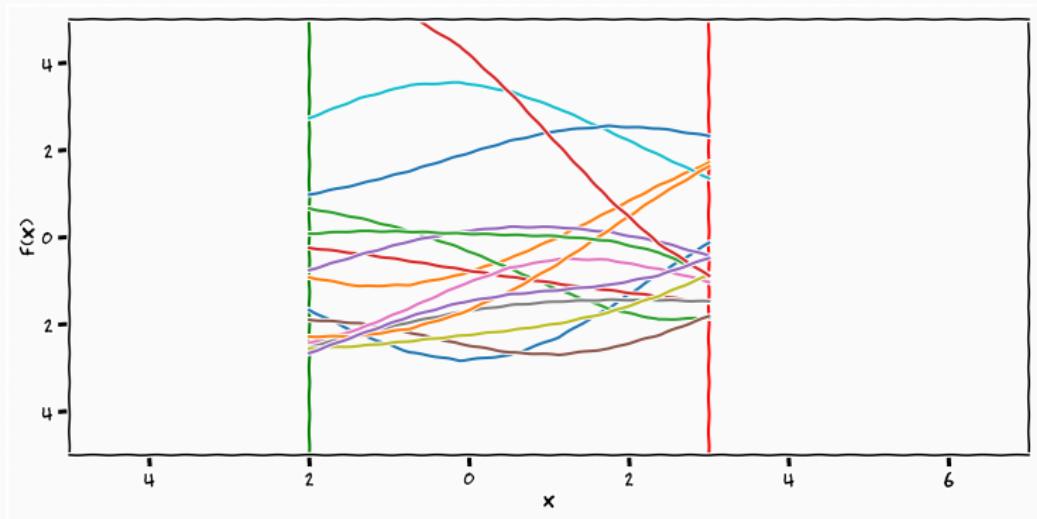
Gaussian Samples



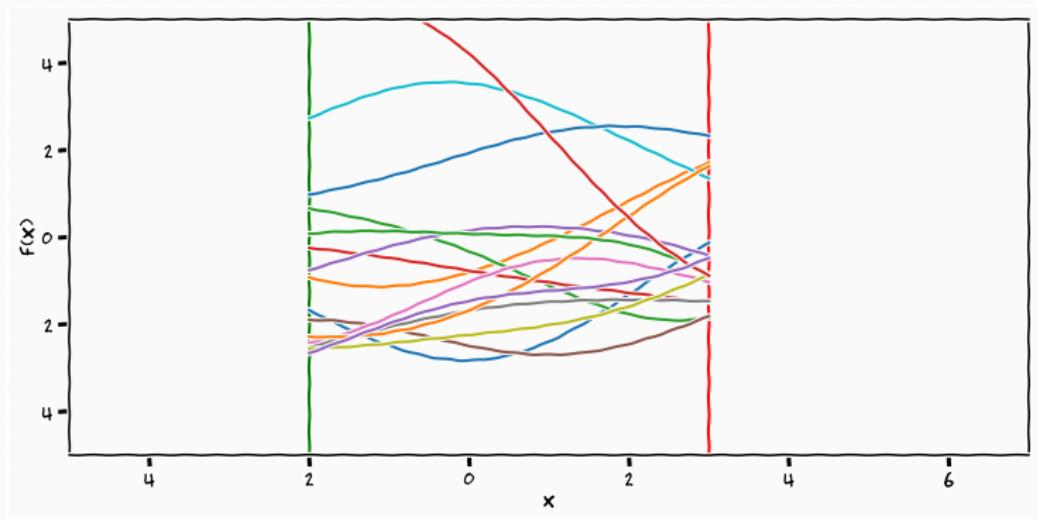
Gaussian Samples



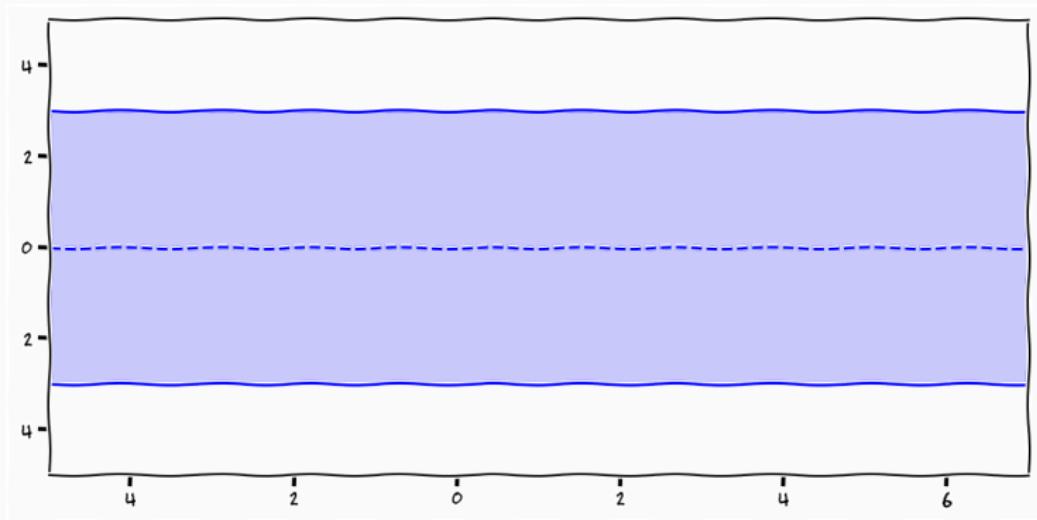
Gaussian Samples



Gaussian Samples



Gaussian Processes



Gaussian Processes: Formalism

$$p(\mathbf{f}) = \mathcal{N} \left(\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \\ \vdots \end{bmatrix} \middle| \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \\ \vdots \end{bmatrix}, \begin{bmatrix} k_{11} & k_{12} & \dots & k_{1N} & \dots \\ k_{21} & k_{22} & \dots & k_{2N} & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ k_{N1} & k_{N2} & \dots & k_{NN} & \dots \\ \vdots & \vdots & \dots & \vdots & \ddots \end{bmatrix} \right)$$

Gaussian processes

$$\begin{array}{ccc} \mathcal{GP}(\cdot, \cdot) & M \in \mathbb{R}^{\infty \times N} & \mathcal{N}(\cdot, \cdot) \\ & \rightarrow & \\ \infty & & N \end{array}$$

The Gaussian distribution is the projection of the infinite Gaussian process

Definition (Gaussian Process)

A Gaussian process is a collection of random variables who are **jointly** Gaussian distributed index by a **infinite** index set

Gaussian Processes: Formalism II

$$p(\mathbf{f}) = \mathcal{N} \left(\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \\ \vdots \end{bmatrix} \middle| \begin{bmatrix} \mu(x_1) \\ \mu(x_2) \\ \vdots \\ \mu(x_N) \\ \vdots \end{bmatrix}, \begin{bmatrix} k(x_1, x_1) & k(x_1, x_2) & \dots & k(x_1, x_N) & \dots \\ k(x_2, x_1) & k(x_2, x_2) & \dots & k(x_2, x_N) & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ k(x_N, x_1) & k(x_N, x_2) & \dots & k(x_N, x_N) & \dots \\ \vdots & \vdots & \dots & \vdots & \ddots \end{bmatrix} \right)$$

"Parametrisation"

$$k_{ij} = k(x_i, x_j)$$

- We parameterise the covariance as a function of the input

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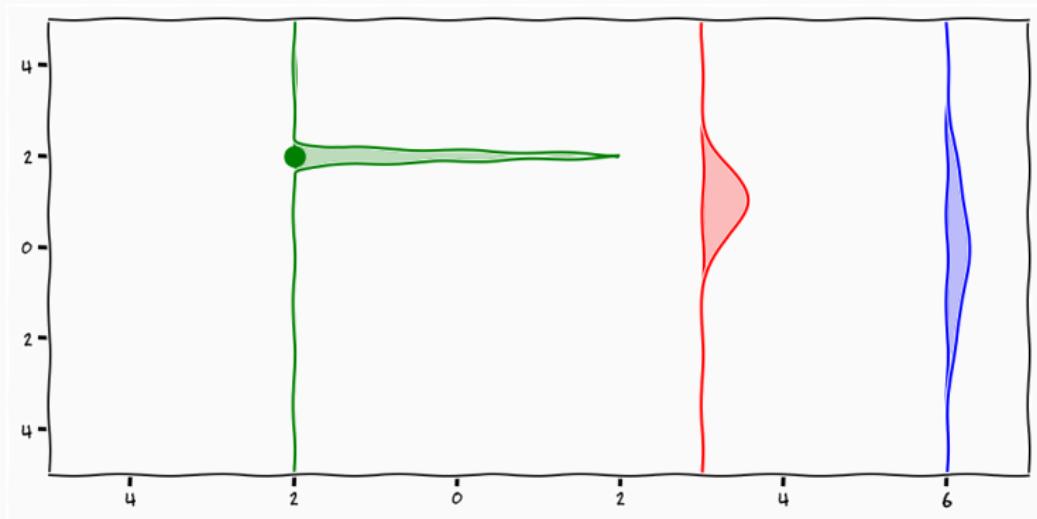
- We parameterise the covariance as a function of the input
- the index set of the measure is the uncountable infinity
- Your "handle" to input your knowledge into a GP is the covariance function
 - *you specify the degree of covariance between data-points*

"Parametrisation"

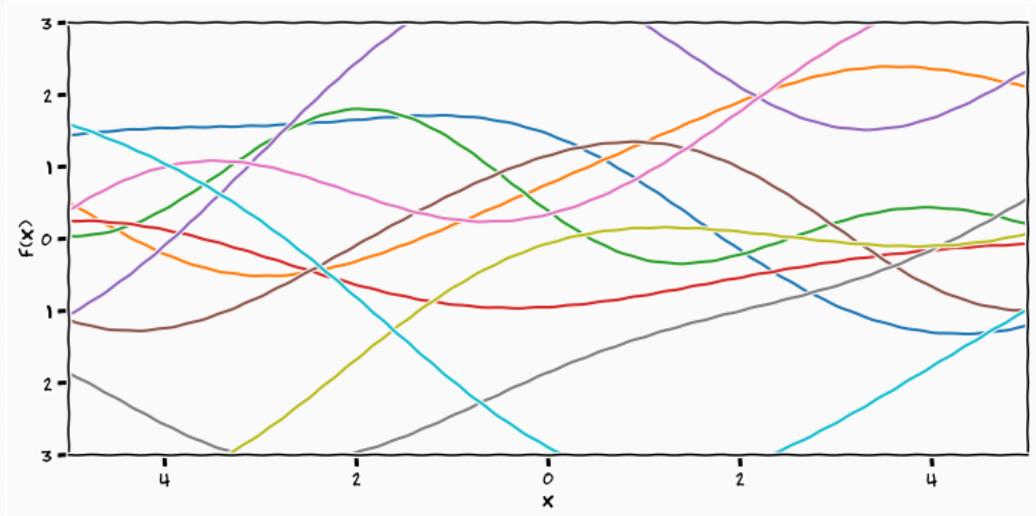
$$k_{ij} = k(x_i, x_j)$$

- We parameterise the covariance as a function of the input
- the index set of the measure is the uncountable infinity
- Your "handle" to input your knowledge into a GP is the covariance function
 - *you specify the degree of covariance between data-points*
- If this "parametrisation" aligns well with your knowledge a GP is the way forward!

Gaussian Processes

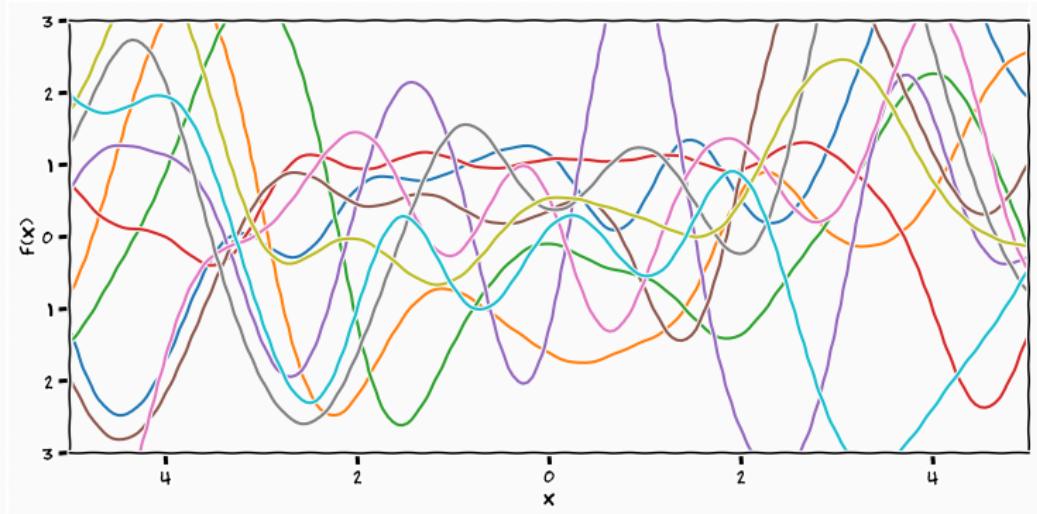


Gaussian Processes Samples



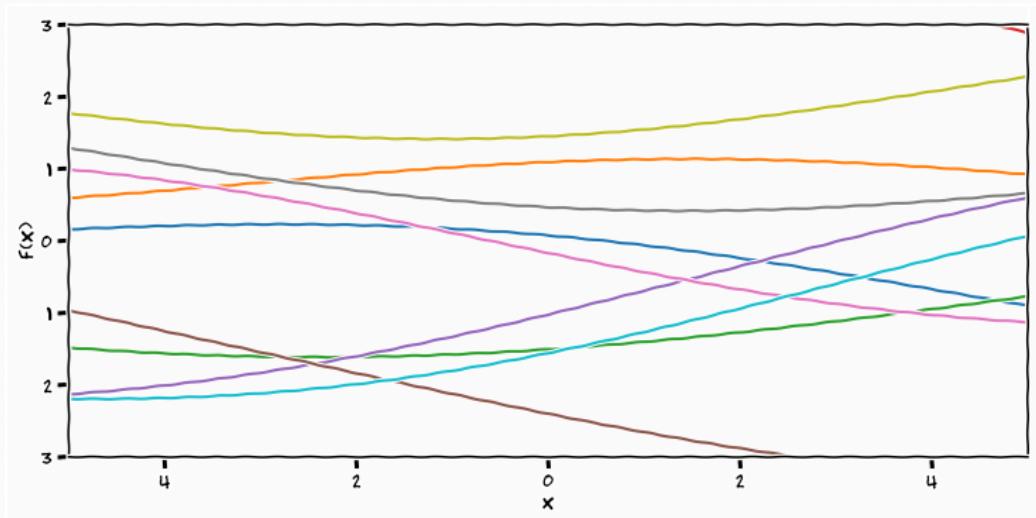
$$k(x_i, x_j) = 3 \cdot e^{-\frac{(x_i - x_j)^2}{15}}$$

Gaussian Processes Samples



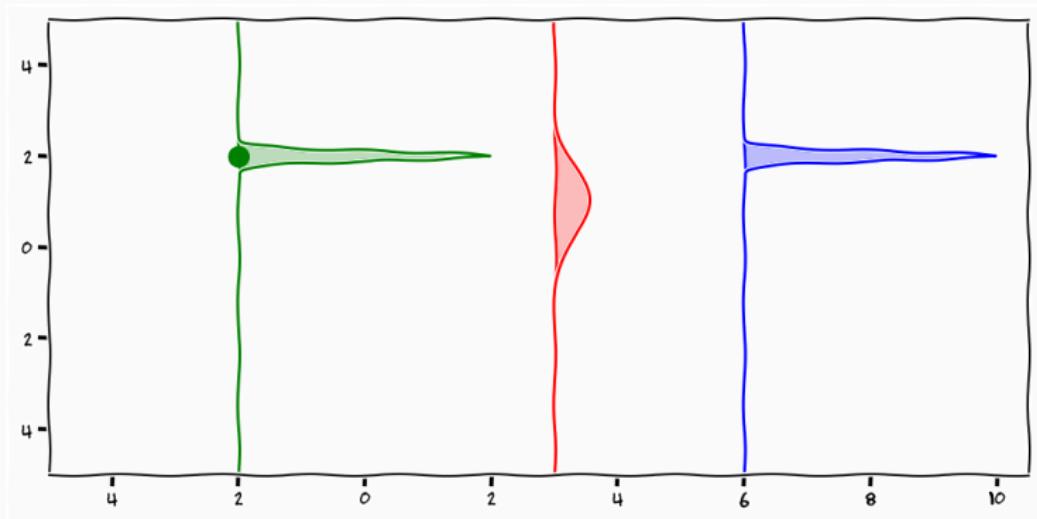
$$k(x_i, x_j) = 3 \cdot e^{-\frac{(x_i - x_j)^2}{1}}$$

Gaussian Processes Samples



$$k(x_i, x_j) = 3 \cdot e^{-\frac{(x_i - x_j)^2}{150}}$$

Gaussian Processes



Bayesian Inference

Bayes' Rule

$$p(\mathbf{f}_* \mid \mathbf{f}) = \frac{p(\mathbf{f}, \mathbf{f}_*)}{p(\mathbf{f})} = \frac{p(\mathbf{f}, \mathbf{f}_*)}{\int p(\mathbf{f}, \mathbf{f}_*) d\mathbf{f}_*}$$

Marginal Likelihood

$$\int p(\mathbf{f}, \mathbf{f}_*) d\mathbf{f}_* = \int p(\mathbf{f} \mid \mathbf{f}_*) p(\mathbf{f}_*) d\mathbf{f}_*$$

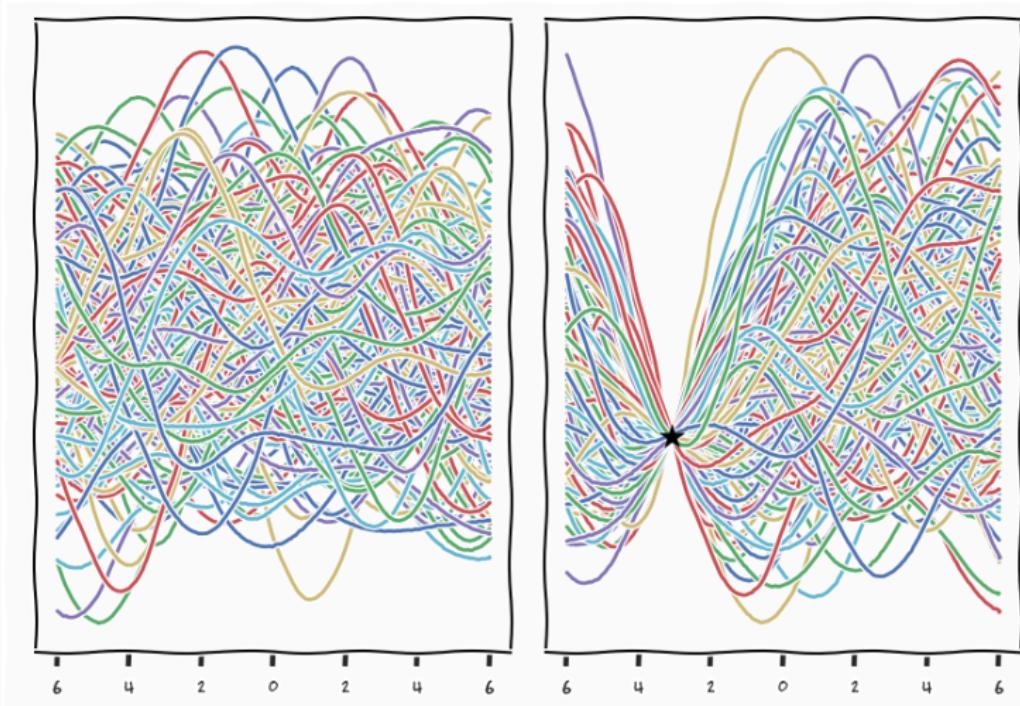
- Take every possible function value/marginal \mathbf{f}_* at location \mathbf{x}_* according to their probability

Marginal Likelihood

$$\int p(\mathbf{f}, \mathbf{f}_*) d\mathbf{f}_* = \int p(\mathbf{f} \mid \mathbf{f}_*) p(\mathbf{f}_*) d\mathbf{f}_*$$

- Take every possible function value/marginal \mathbf{f}_* at location \mathbf{x}_* according to their probability
- Check if these marginals are **consistent** with the marginals we observe \mathbf{f} at location \mathbf{x}

Gaussian Processes: Posterior Samples



Gaussian Process: "Predictive Posterior"

$$p(\mathbf{f}, \mathbf{f}_*) = p(\mathbf{f}_* \mid \mathbf{f})p(\mathbf{f})$$

- We have defined $p(\mathbf{f}, \mathbf{f}_*)$

Gaussian Process: "Predictive Posterior"

$$p(\mathbf{f}, \mathbf{f}_*) = p(\mathbf{f}_* \mid \mathbf{f})p(\mathbf{f})$$

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- We have defined $p(\mathbf{f}, \mathbf{f}_*)$
- We know through the marginal property of the Gaussian that $p(\mathbf{f})$ is consistent
- We know that $p(\mathbf{f}_* \mid \mathbf{f})$ is Gaussian
- \Rightarrow We can just solve for $p(\mathbf{f}_* \mid \mathbf{f})$

Gaussian Process: "Predictive Posterior"

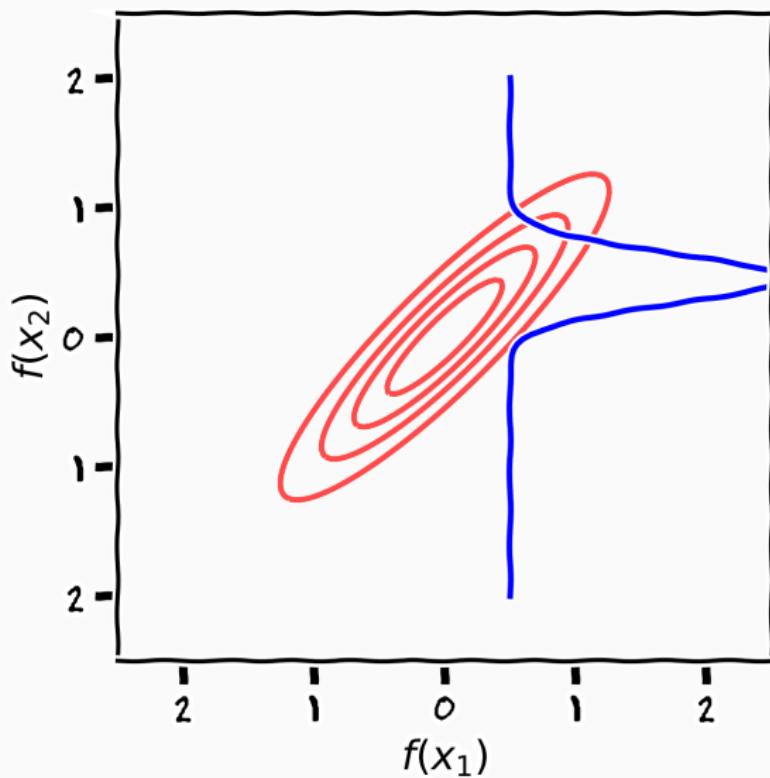
- All instantiations are jointly Gaussian

$$\begin{bmatrix} \mathbf{f} \\ \mathbf{f}_* \end{bmatrix} \sim \mathcal{N} \left(\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} k(\mathbf{x}, \mathbf{x}) & k(\mathbf{x}, \mathbf{x}_*) \\ k(\mathbf{x}_*, \mathbf{x}) & k(\mathbf{x}_*, \mathbf{x}_*) \end{bmatrix} \right)$$

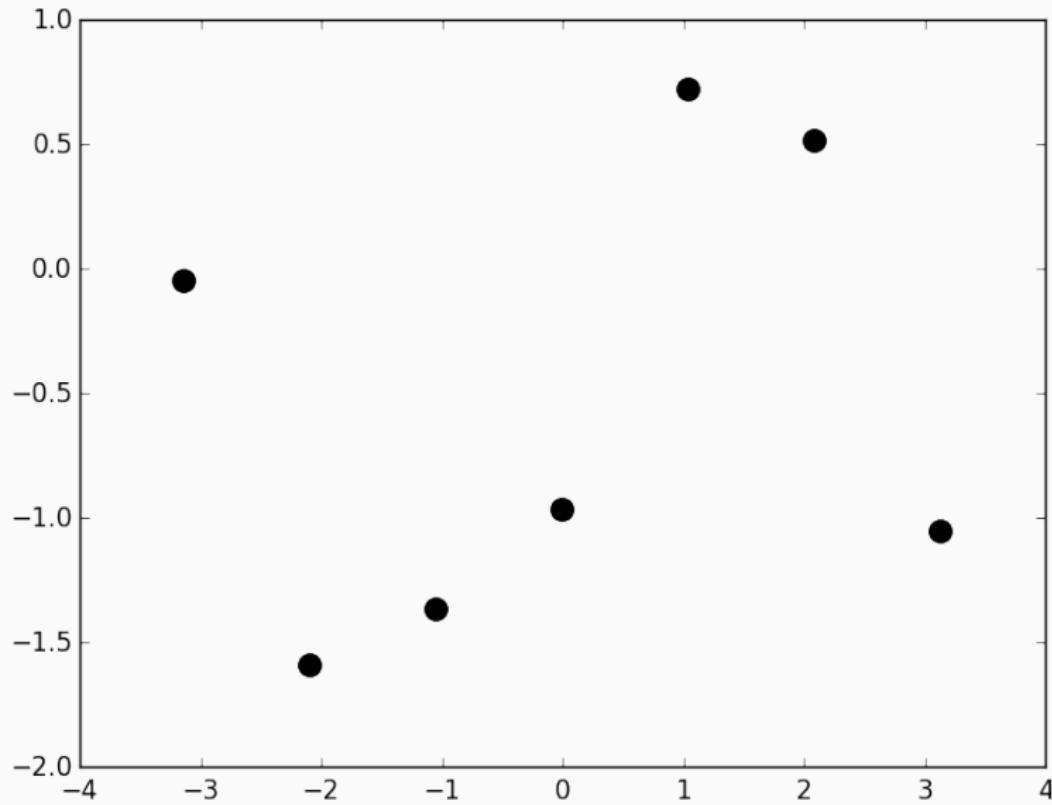
- Conditional Gaussian

$$p(f_* | \mathbf{x}_*, \mathbf{x}, \mathbf{f}) = \mathcal{N}(k(\mathbf{x}_*, \mathbf{x})^T k(\mathbf{x}, \mathbf{x})^{-1} \mathbf{f}, k(\mathbf{x}_*, \mathbf{x}_*) - k(\mathbf{x}_*, \mathbf{x})^T k(\mathbf{x}, \mathbf{x})^{-1} k(\mathbf{x}, \mathbf{x}_*))$$

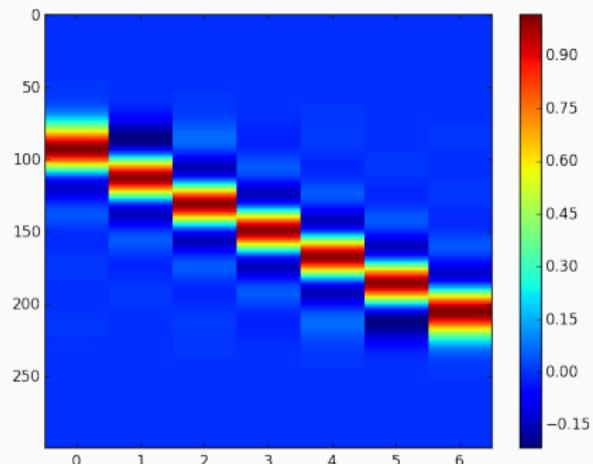
Conditional Gaussians



Gaussian Process: "Predictive Posterior"

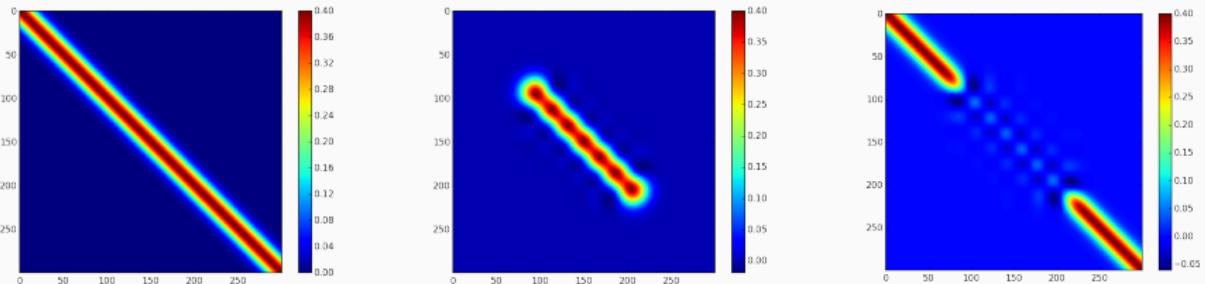


Does it make sense: Mean



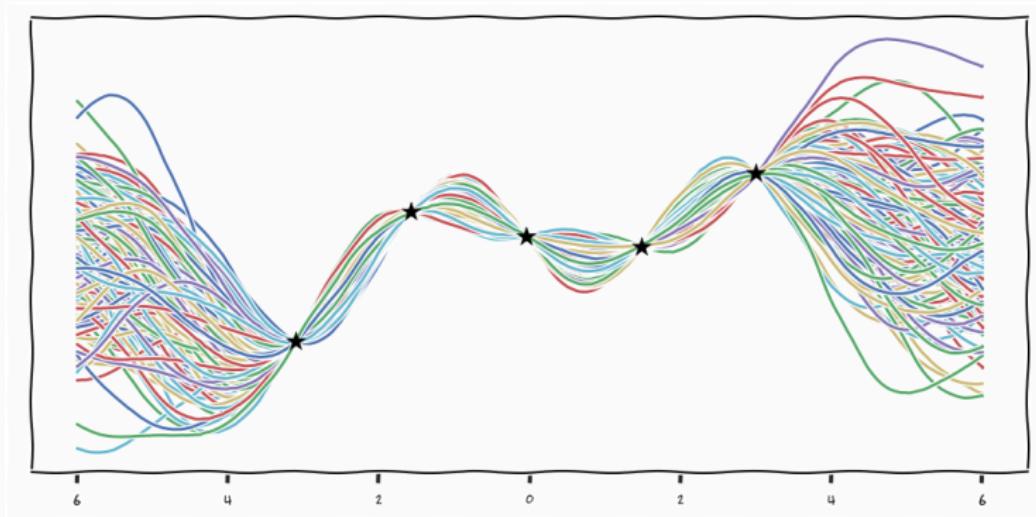
$$k(\mathbf{x}_*, \mathbf{X})^T k(\mathbf{X}, \mathbf{X})^{-1} \mathbf{f}$$

Does it make sense: Covariance

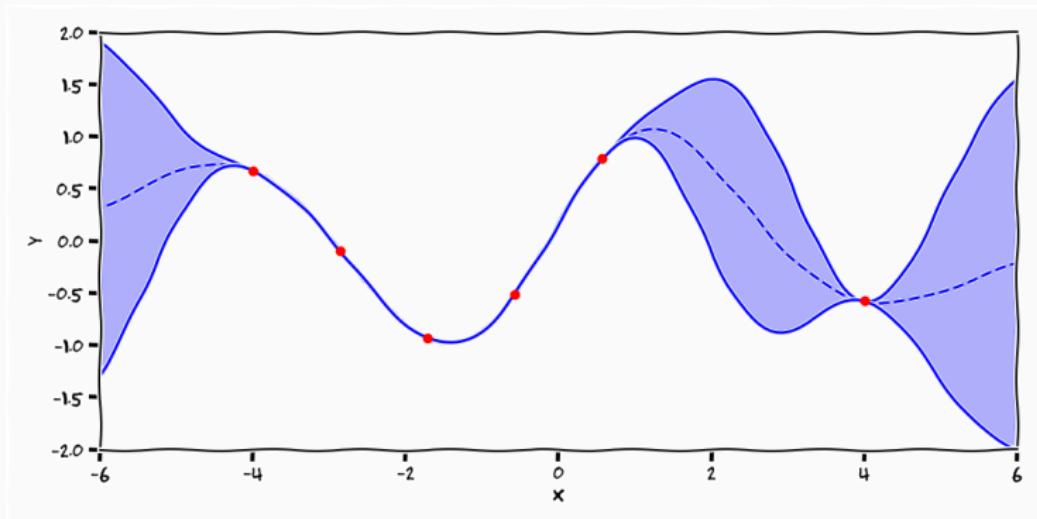


$$k(\mathbf{x}_*, \mathbf{x}_*) - k(\mathbf{x}_*, \mathbf{x})^T k(\mathbf{x}, \mathbf{x})^{-1} k(\mathbf{x}, \mathbf{x}_*)$$

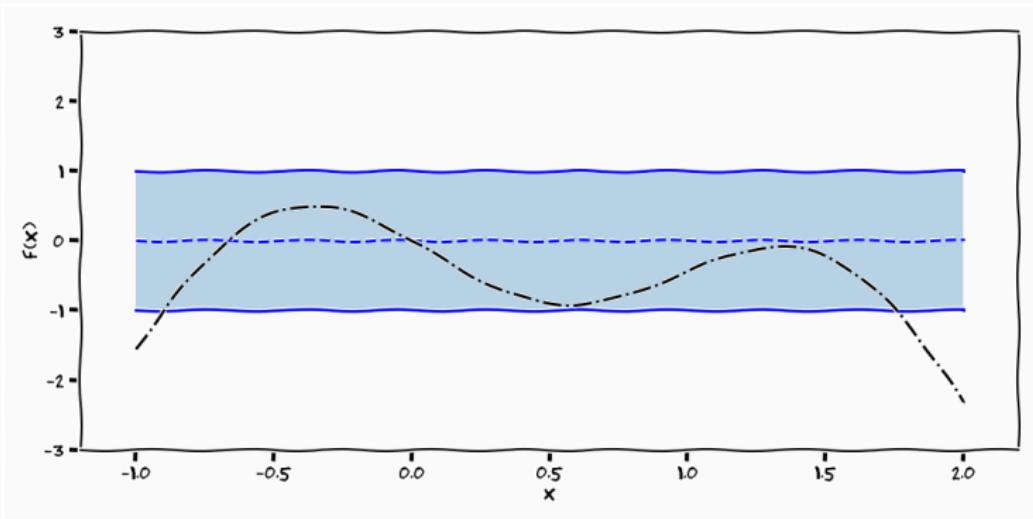
Gaussian Processes: "Predictive Posterior Samples"



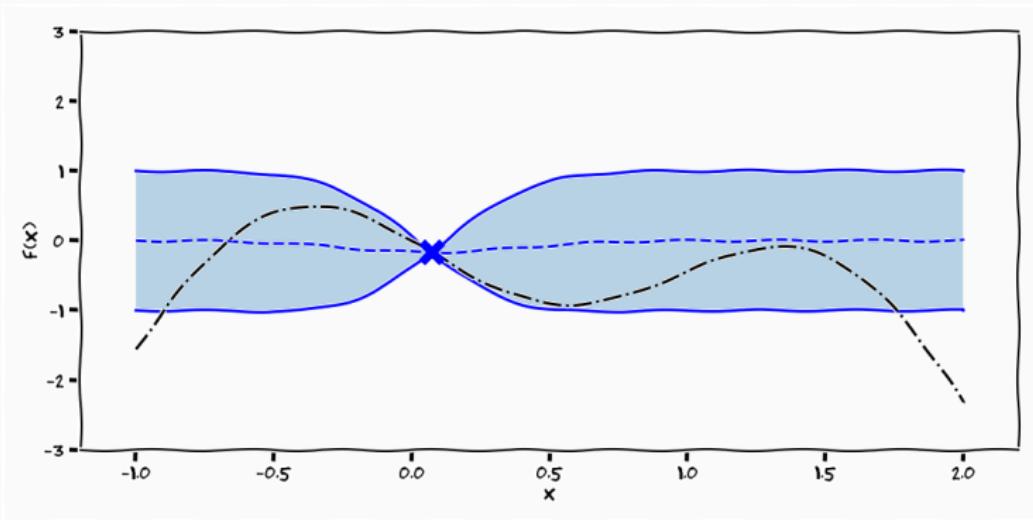
Gaussian Processes: "Predictive Posterior Process"



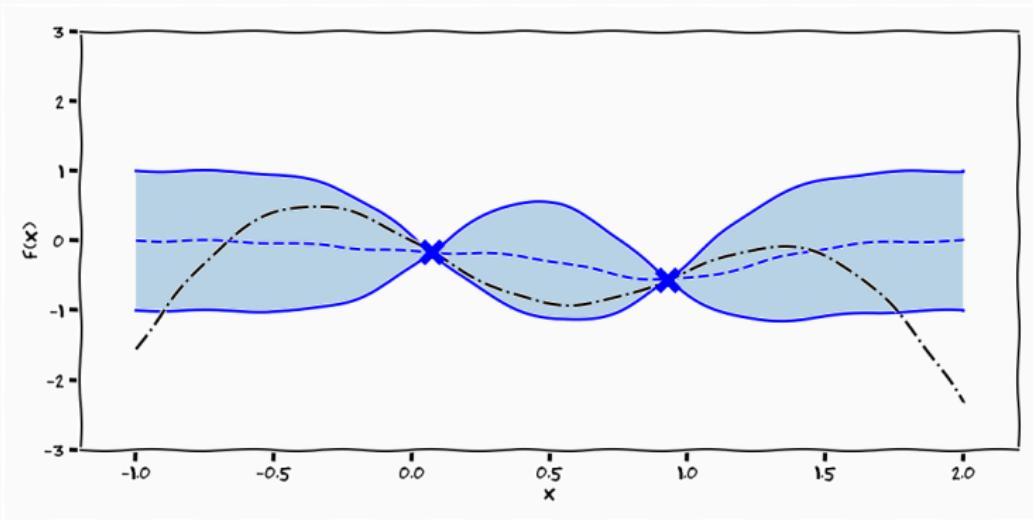
Posterior Processes



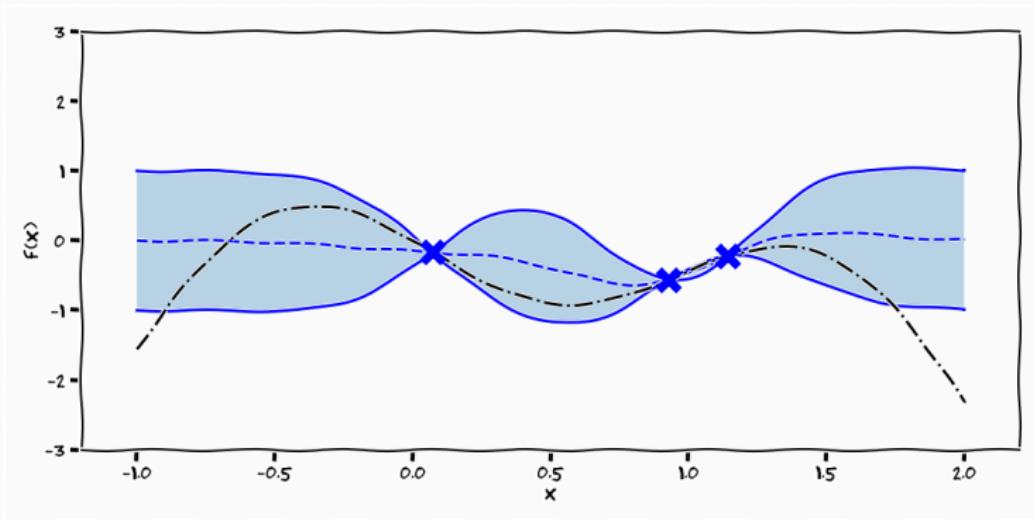
Posterior Processes



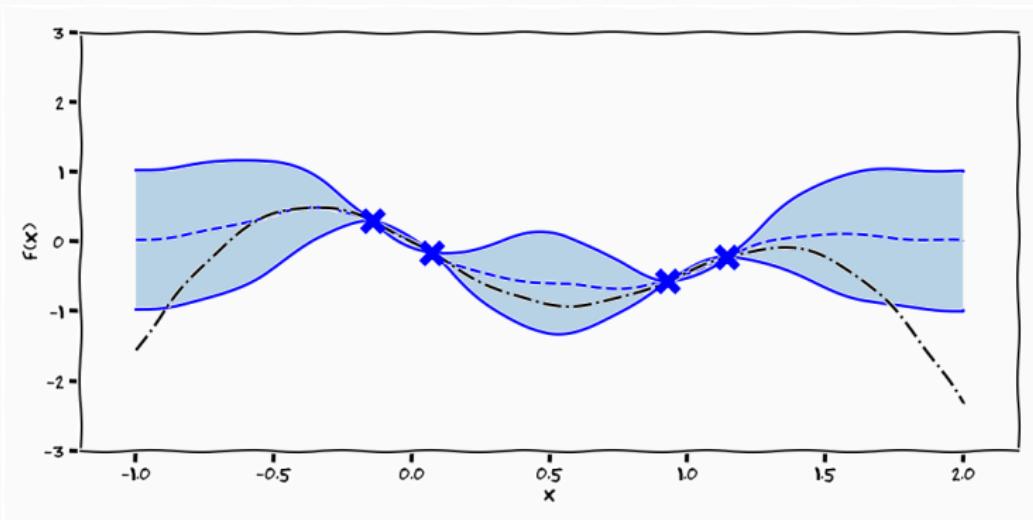
Posterior Processes



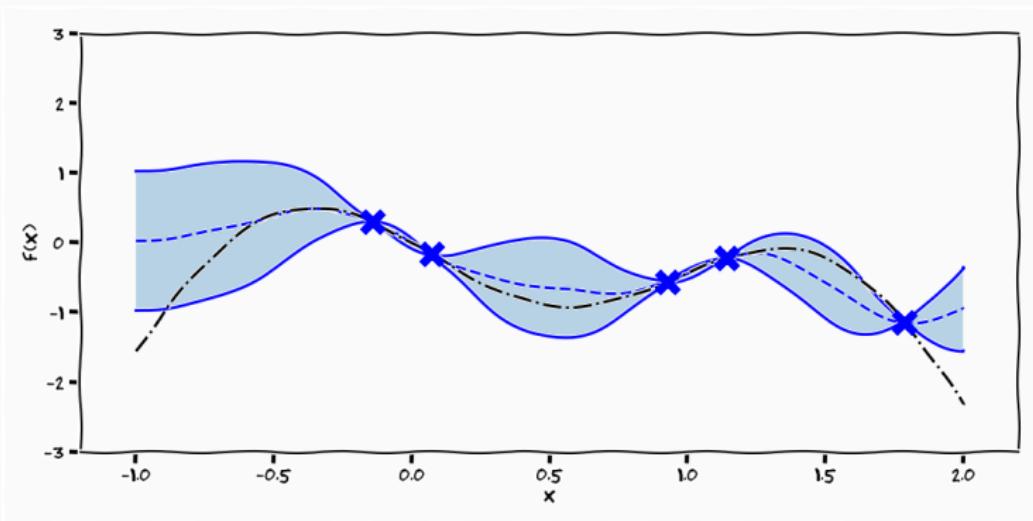
Posterior Processes



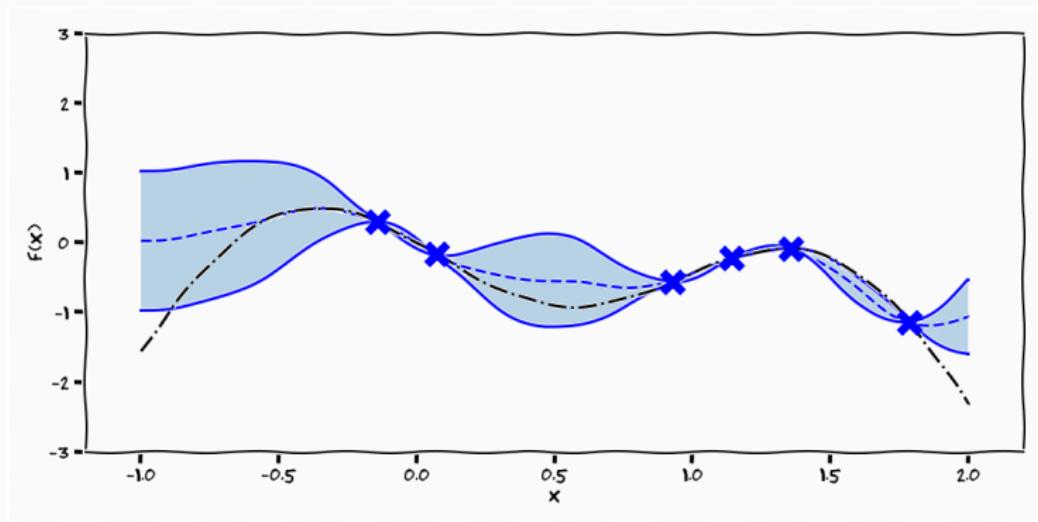
Posterior Processes



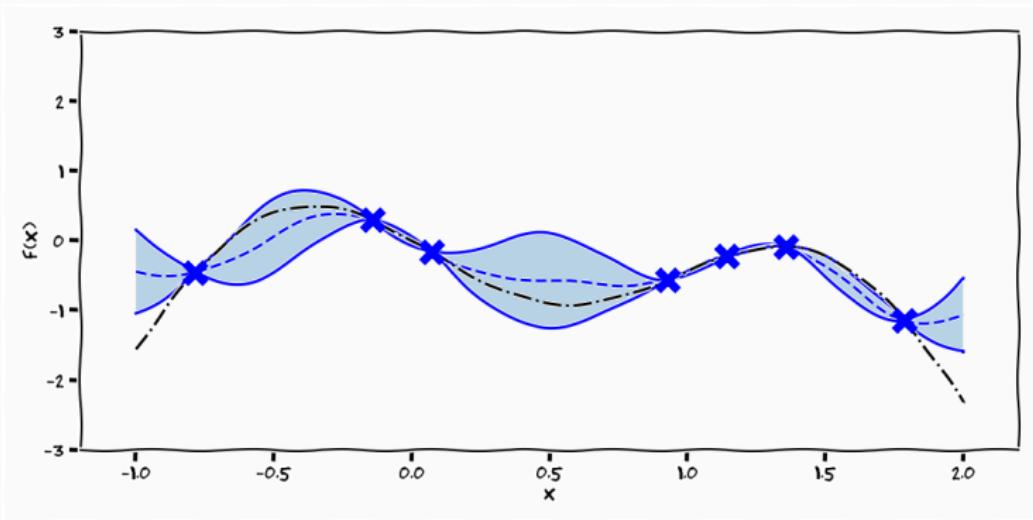
Posterior Processes



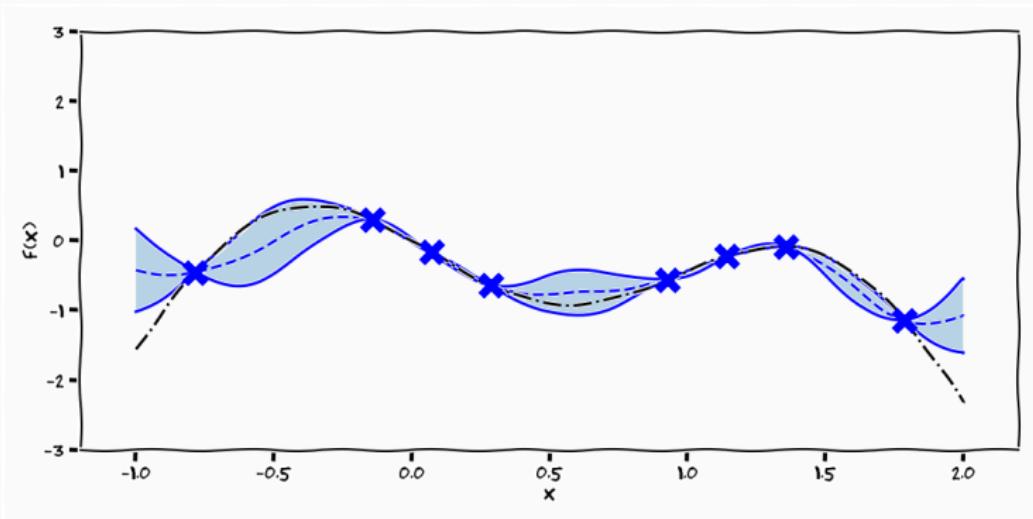
Posterior Processes



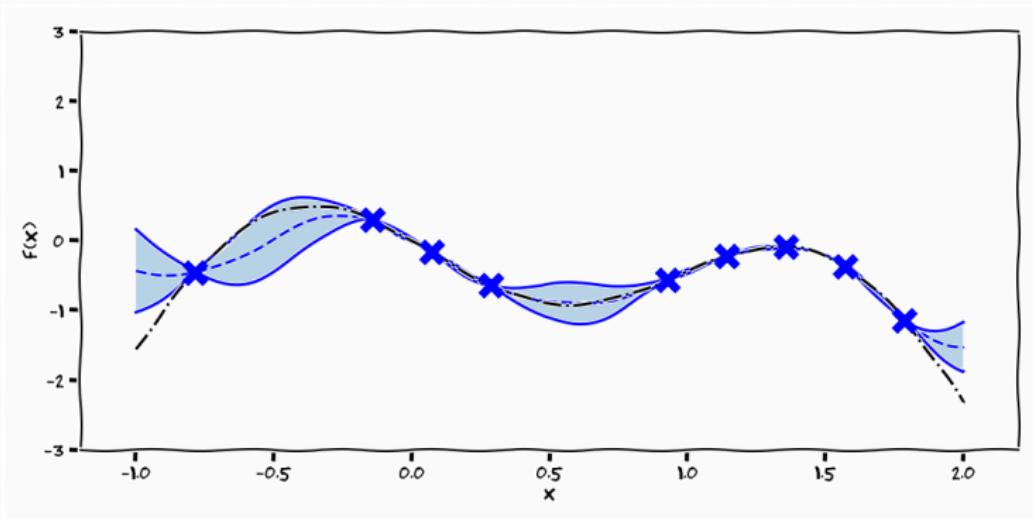
Posterior Processes



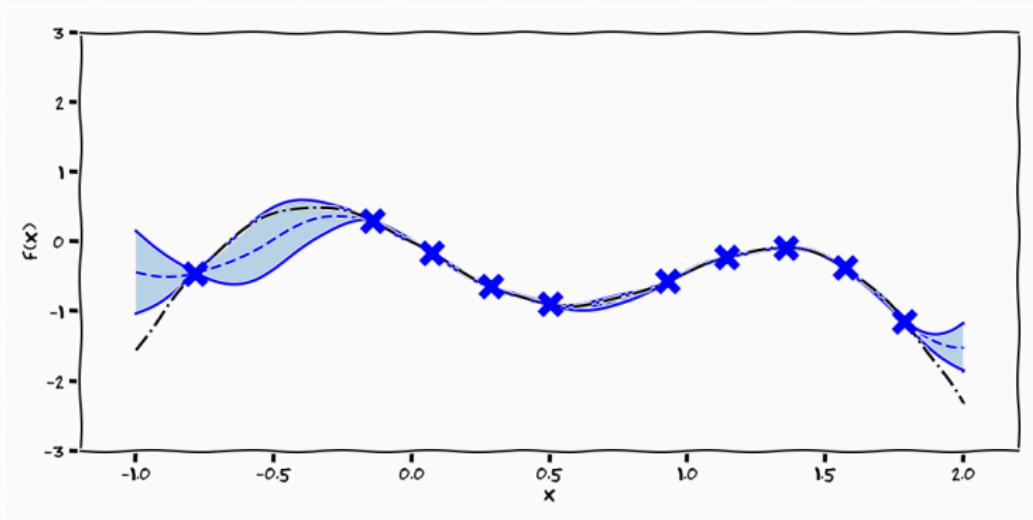
Posterior Processes



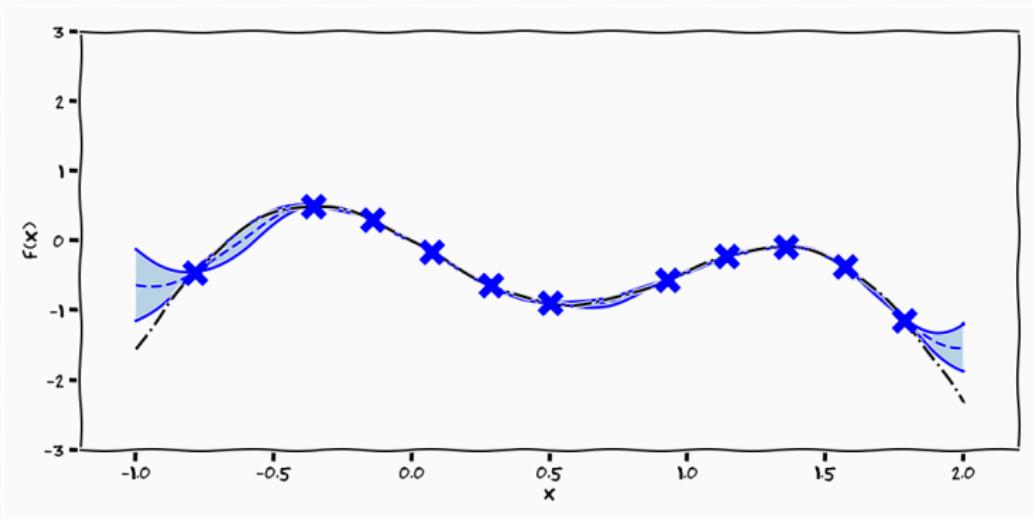
Posterior Processes



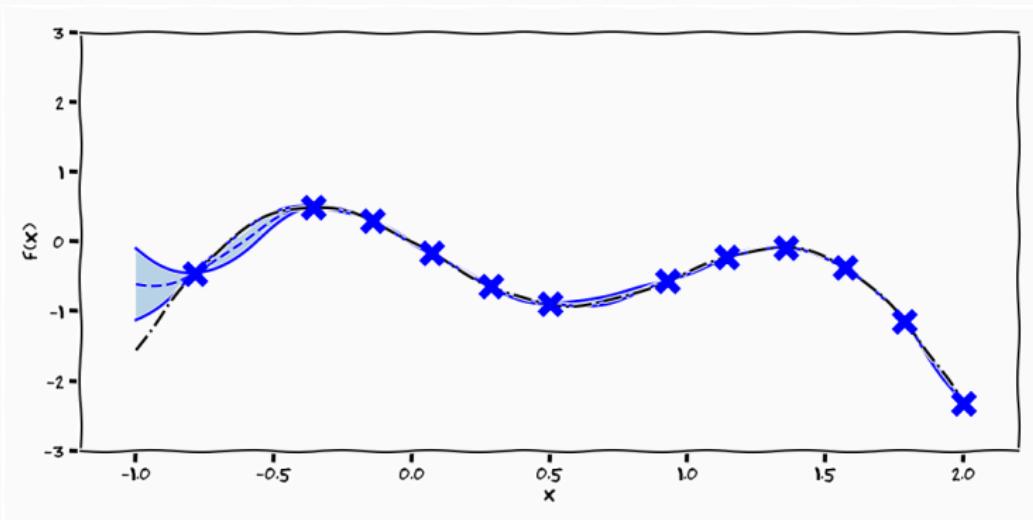
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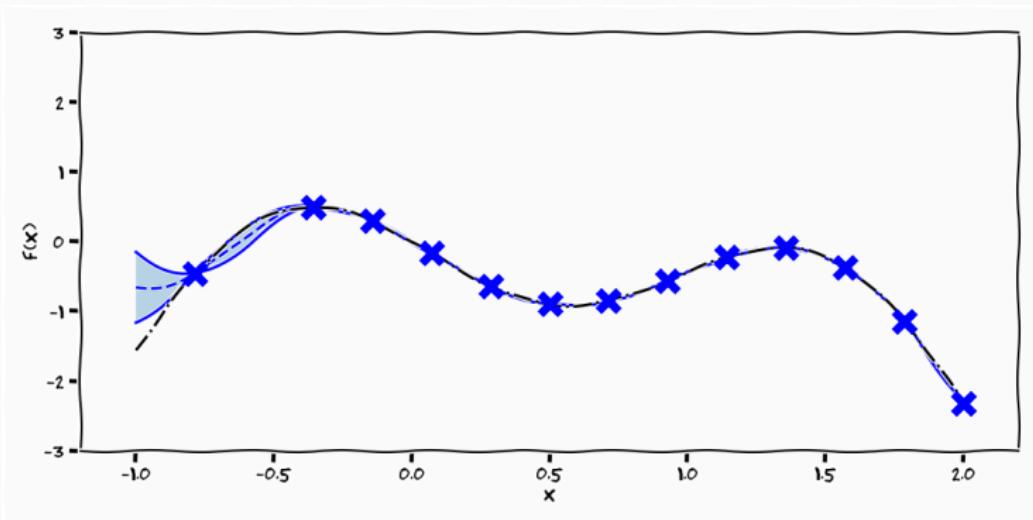
Posterior Processes



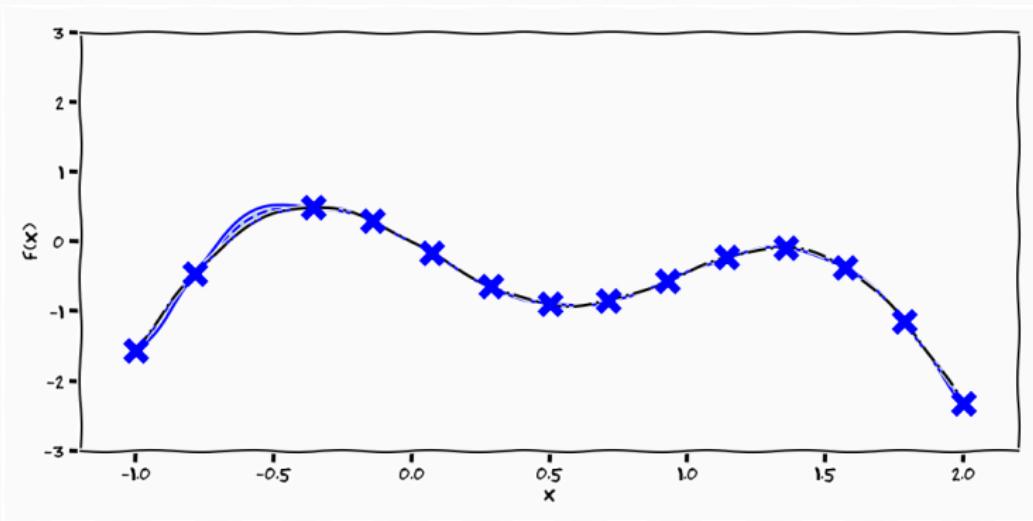
Posterior Processes



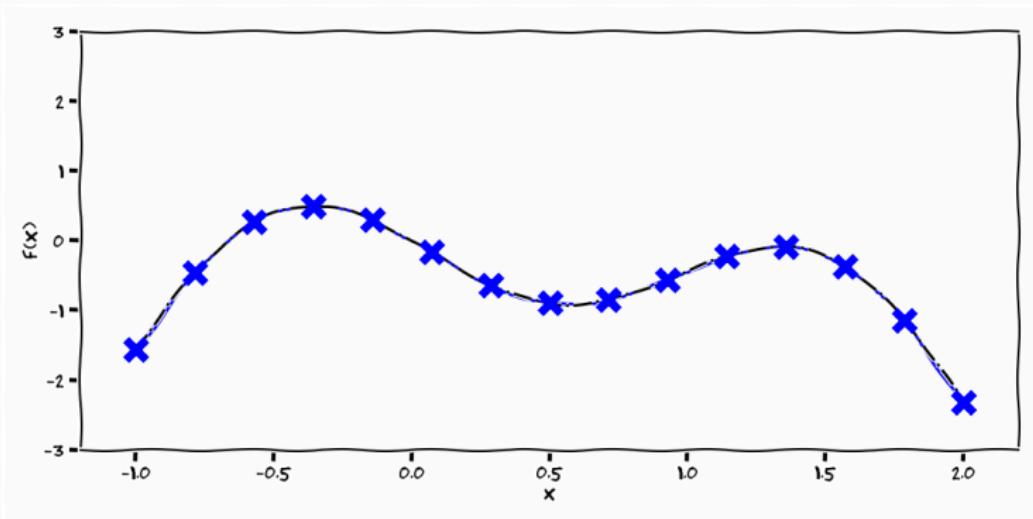
Posterior Processes



Posterior Processes



Posterior Processes



- So far we have only looked at the prior
 - the same as when we sampled from $p(\mathbf{w})$ in the previous lecture
 - the "predictive posterior" has really been the "conditional prior"
- Now lets introduce a likelihood

Model

$$p(\mathbf{y}, \mathbf{f}) = p(\mathbf{y} \mid \mathbf{f})p(\mathbf{f}) = p(\mathbf{f}) \prod_{i=1} \mathcal{N}(p(y_i \mid f_i))$$

$$p(\mathbf{y} \mid \mathbf{f}) = \mathcal{N}(\mathbf{y} \mid \mathbf{f}, \sigma^2 \mathbf{I})$$

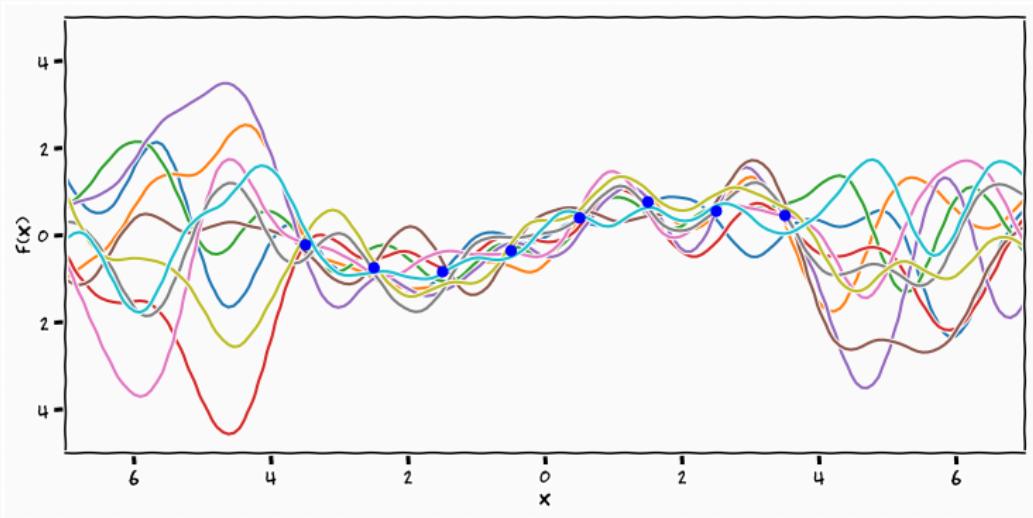
- Same motivation as with linear regression
- we want to **explain away** our ignorance from the data using σ

Predictive Posterior Process

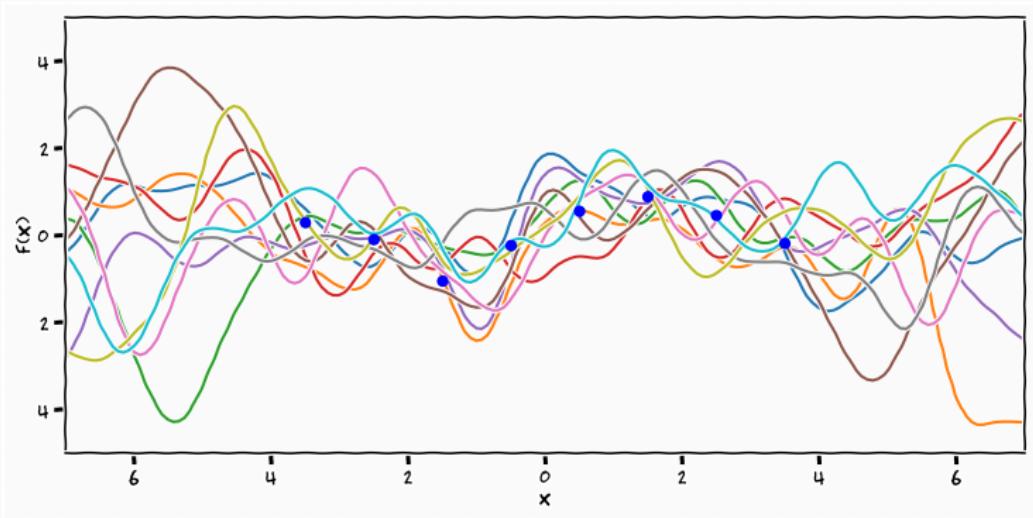
$$\begin{bmatrix} \mathbf{y} \\ \mathbf{f}_* \end{bmatrix} \sim \mathcal{N} \left(\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} k(\mathbf{x}, \mathbf{x}) + \sigma^2 \mathbf{I} & k(\mathbf{x}, \mathbf{x}_*) \\ k(\mathbf{x}_*, \mathbf{x}) & k(\mathbf{x}_*, \mathbf{x}_*) \end{bmatrix} \right)$$

$$p(f_* | \mathbf{x}_*, \mathbf{x}, \mathbf{y}, \boldsymbol{\theta}) = \mathcal{N}(k(\mathbf{x}_*, \mathbf{x})^T(K(\mathbf{x}, \mathbf{x}) + \sigma^2 \mathbf{I}))^{-1} \mathbf{y},$$
$$k(\mathbf{x}_*, \mathbf{x}_*) - k(\mathbf{x}_*, \mathbf{x})^T(K(\mathbf{x}, \mathbf{x}) + \sigma^2 \mathbf{I})^{-1} K(\mathbf{x}, \mathbf{x}_*))$$

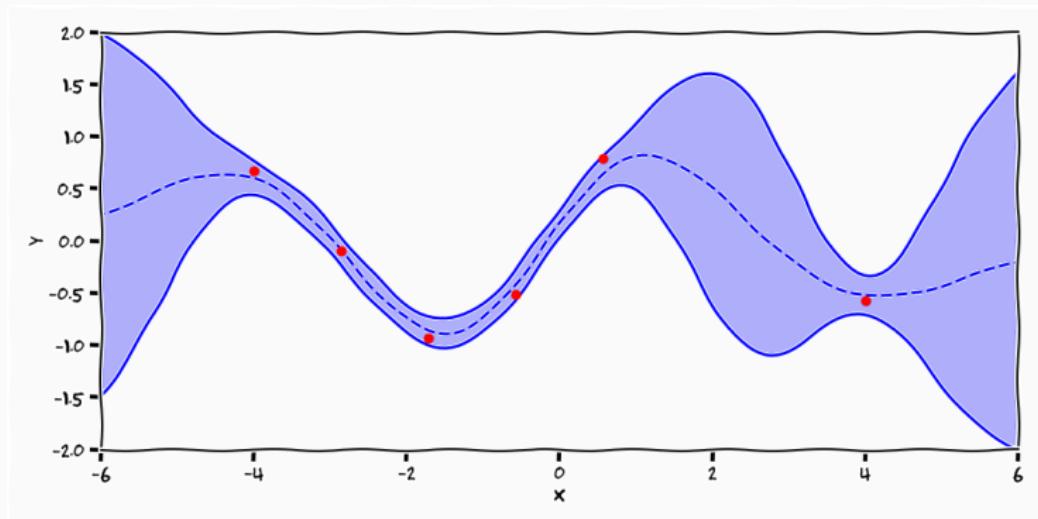
Posterior Samples



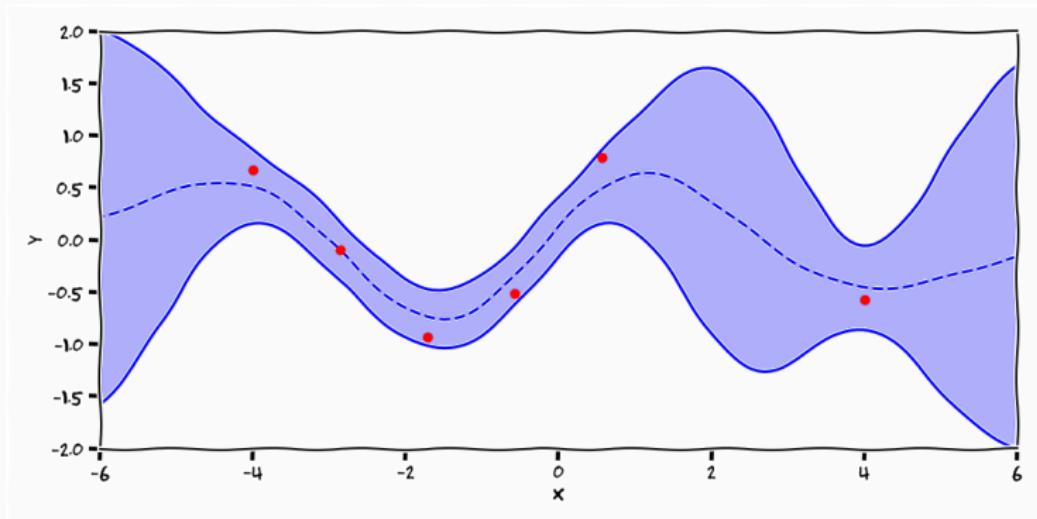
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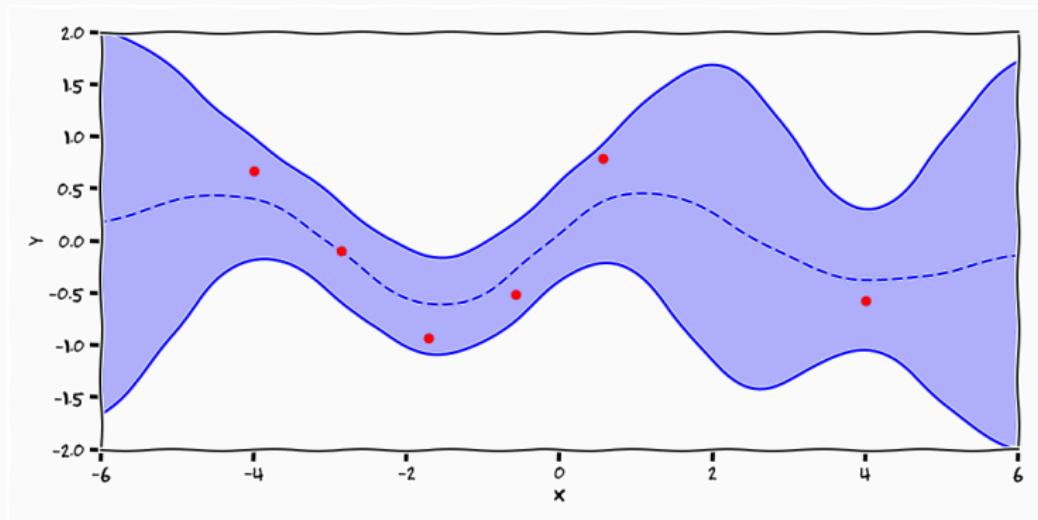
Gaussian Processes: Predictive Posterior Process



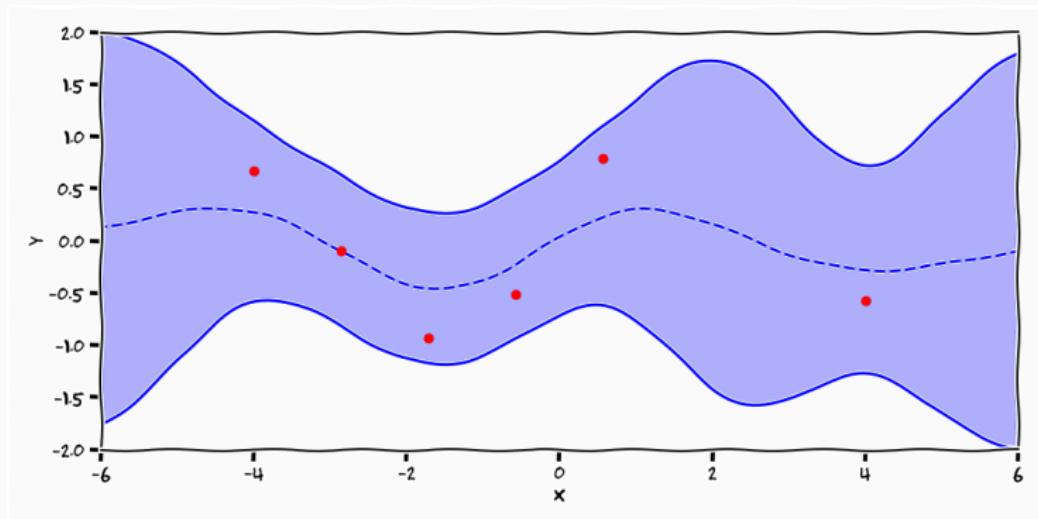
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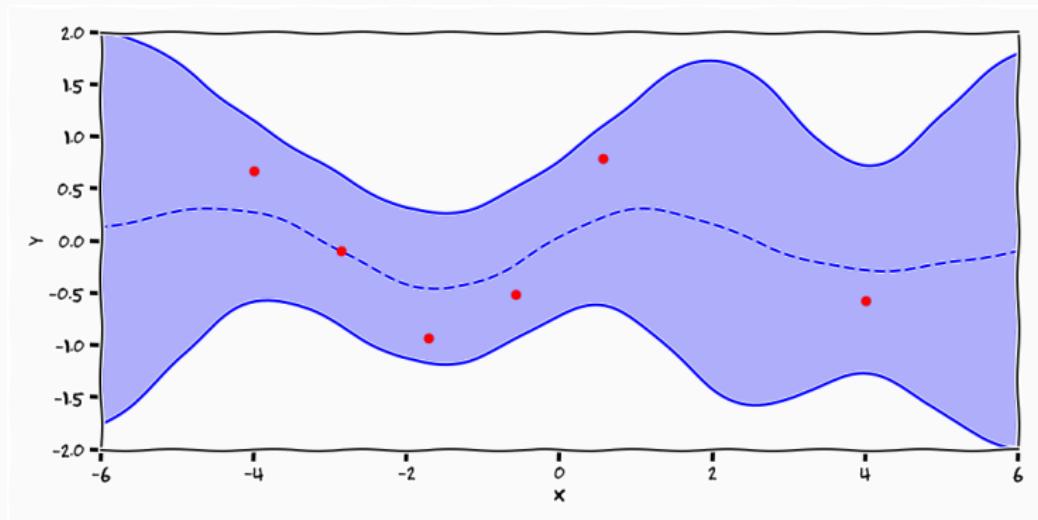
Gaussian Processes: Predictive Posterior Process



Gaussian Processes: Predictive Posterior Process



Gaussian Processes: Predictive Posterior Process



Predictive Posteriors

- Gaussian Process - *Non-parametric formulation*

$$p(y_* \mid \mathbf{y}, \mathbf{x}_*, \mathbf{X}, \theta) = \mathcal{N}(y_* \mid \mu_*, K_*)$$

$$\mu_* = k(x_*, x)k(x, x)^{-1}\mathbf{y}$$

$$K_* = k(x_*, x_*) - k(x_*, x)k(x, x)^{-1}k(x, x_*)$$

Predictive Posteriors

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- Linear Regression - *Parametric formulation*

$$\begin{aligned} p(y_* | \mathbf{y}, \mathbf{x}_*, \mathbf{X}, \alpha, \beta) &= \int p(y_* | \mathbf{x}_*, \mathbf{w}, \beta) p(\mathbf{w} | \mathbf{y}, \mathbf{X}, \alpha, \beta) d\mathbf{w} \\ &= \mathcal{N}((y_* \mid \mu_*(x_*), \Sigma_*(x_*))) \end{aligned}$$

$$\mu_* = (\beta(\alpha\mathbf{I} + \beta\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}\mathbf{y})^T \mathbf{x}_*$$

$$= \mathbf{x}_*^T \mathbf{x} (\beta(\alpha\mathbf{I} + \beta\mathbf{x}^T\mathbf{x}))^{-1} \mathbf{y}$$

$$\Sigma_* = \frac{1}{\beta} + \mathbf{x}_*^T (\alpha\mathbf{I} + \beta\mathbf{x}^T\mathbf{x})^{-1} \mathbf{x}$$

Regression Models: Mean

$$k(x_*, x)k(x, x)^{-1}\mathbf{y}$$

$$\mathbf{x}_*^T \mathbf{x} \left(\beta(\alpha \mathbf{I} + \beta \mathbf{x}^T \mathbf{x}) \right)^{-1} \mathbf{y}$$

- both means are linear with respect to the output data

Regression Models: Mean

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- both means are linear with respect to the output data
- how about if we take a basis function approach such that

$$\mathbf{x}_*^T \mathbf{x} = \Phi(\mathbf{x}_*)^T \Phi(\mathbf{x})$$

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- if we could parametrise $\Phi(\mathbf{x}_*)^T \Phi(\mathbf{x}) = k(\mathbf{x}_*, \mathbf{x})$ as a function
- this leads to the interpretation of GPs as infinite basis functions

Summary

Summary

- Non-parametrics
 - parametrise relationship between data

Summary

- Non-parametrics
 - parametrise relationship between data
- Gaussian processes
 - **Implementation:** "just a big big Gaussian"
 - **Theory:** projection of an infinite stochastic process

Kolmogrovs Extension Theorem

For all permutations π , measurable sets $F_i \subseteq \mathbb{R}^n$ and probability measure ν

1. Exchangeable

$$\nu_{t_{\pi(1)} \dots t_{\pi(k)}} (F_{\pi(1)} \times \dots \times F_{\pi(k)}) = \nu_{t_1 \dots t_k} (F_1 \times \dots \times F_k)$$

2. Marginal

$$\nu_{t_1 \dots t_k} (F_1 \times \dots \times F_k) = \nu_{t_1 \dots t_k, t_{k+1} \dots t_{k+m}} (F_1 \times \dots \times F_k \times \mathbb{R}^n \times \dots \times \mathbb{R}^n)$$

In this case the finite dimensional probability measure is a realisation of an underlying stochastic process

eof

