Math189 SP20
Homework 5
Monday, Mar 2, 2019

Feel free to work with other students, but make sure you write up the homework and code on your own (no copying homework *or* code; no pair programming). Feel free to ask students or instructors for help debugging code or whatever else, though.

1 (Murphy 12.5 - Deriving the Residual Error for PCA) It may be helpful to reference section 12.2.2 of Murphy.

(a) Prove that

$$\left\| \mathbf{x}_i - \sum_{j=1}^{k} z_{ij}\mathbf{v}_j \right\|^2 = \mathbf{x}_i^\top \mathbf{x}_i - \sum_{j=1}^{k} \mathbf{v}_j^\top \mathbf{x}_i \mathbf{x}_i^\top \mathbf{v}_j.$$

Hint: first consider the case when $k = 2$. Use the fact that $\mathbf{v}_i^\top \mathbf{v}_j$ is 1 if $i = j$ and 0 otherwise. Recall that $z_{ij} = \mathbf{x}_i^\top \mathbf{v}_j$.

(b) Now show that

$$J_k = \frac{1}{n}\sum_{i=1}^{n}\left(\mathbf{x}_i^\top \mathbf{x}_i - \sum_{j=1}^{k} \mathbf{v}_j^\top \mathbf{x}_i \mathbf{x}_i^\top \mathbf{v}_j\right) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i^\top \mathbf{x}_i - \sum_{j=1}^{k} \lambda_j.$$

Hint: recall that $\mathbf{v}_j^\top \boldsymbol{\Sigma} \mathbf{v}_j = \lambda_j \mathbf{v}_j^\top \mathbf{v}_j = \lambda_j$.

(c) If $k = d$ there is no truncation, so $J_d = 0$. Use this to show that the error from only using $k < d$ terms is given by

$$J_k = \sum_{j=k+1}^{d} \lambda_j.$$

Hint: partition the sum $\sum_{j=1}^{d} \lambda_j$ into $\sum_{j=1}^{k} \lambda_j$ and $\sum_{j=k+1}^{d} \lambda_j$.

(a) We have

$$\begin{aligned}
(\mathbf{x}_i - \sum_{j=1}^{k} z_{ij}\mathbf{v}_j)^\top(\mathbf{x}_i - \sum_{j=1}^{k} z_{ij}\mathbf{v}_j) &= \mathbf{x}_i^\top \mathbf{x}_i - 2\sum_{j=1}^{k} z_{ij}\mathbf{v}_j^\top \mathbf{x}_i + \sum_{j=1}^{K} \mathbf{v}_j^\top z_{ij}^\top z_{ij}\mathbf{v}_j \text{ (since } \mathbf{v}_j\mathbf{v}_j = 1) \\
&= \mathbf{x}_i^\top \mathbf{x}_i - 2\sum_{j=1}^{k} \mathbf{x}_i^\top \mathbf{v}_j \mathbf{v}_j^\top \mathbf{x}_i + \sum_{j=1}^{k} \mathbf{v}_j^\top \mathbf{x}_i \mathbf{x}_i^\top \mathbf{v}_j \text{ (since } \mathbf{x}_i^\top \mathbf{v}_j \text{ is a scalar)} \\
&= \mathbf{x}_i^\top \mathbf{x}_i - \sum_{j=1}^{k} \mathbf{v}_j^\top \mathbf{x}_i \mathbf{x}_i^\top \mathbf{v}_j
\end{aligned}$$

(b) We have $\boldsymbol{\Sigma} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i^\top$ if $\mathbf{X}$ is standardized (shifted by mean)

$$
\begin{aligned}
J_k &= \frac{1}{n}\sum_{i=1}^{n}(\mathbf{x}_i^\top \mathbf{x}_i - \sum_{j=1}^{k}\mathbf{v}_j^\top \mathbf{x}_i \mathbf{x}_i^\top \mathbf{v}_j) \\
&= \frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i^\top \mathbf{x}_i - \frac{1}{n}\sum_{j=1}^{k}\mathbf{v}_j^\top \mathbf{\Sigma} \cdot n \cdot \mathbf{v}_j \\
&= \frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i^\top \mathbf{x}_i - \sum_{j=1}^{k}\lambda_j
\end{aligned}
$$

(c) We have $J_d = 0$ so $\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i^\top \mathbf{x}_i = \sum_{j=1}^{d}\lambda_j$.

Since $\sum_{j=1}^{d}\lambda_j = \sum_{j=1}^{k}\lambda_j + \sum_{j=k+1}^{d}\lambda_j$, then

$\sum_{j=1}^{k}\lambda_j = \sum_{j=1}^{d}\lambda_j - \sum_{j=k+1}^{d}\lambda_j$.

Then $J_k = \frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i^T \mathbf{x}_i - (\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i^\top \mathbf{x}_i - \sum_{j=k+1}^{d}\lambda_j) = \sum_{j=k+1}^{d}\lambda_j$

■

2 (ℓ_1-Regularization) Consider the ℓ_1 norm of a vector $\mathbf{x} \in \mathbb{R}^n$:

$$\|\mathbf{x}\|_1 = \sum_i |\mathbf{x}_i|.$$

Draw the norm-ball $B_k = \{\mathbf{x} : \|\mathbf{x}\|_1 \leq k\}$ for $k = 1$. On the same graph, draw the Euclidean norm-ball $A_k = \{\mathbf{x} : \|\mathbf{x}\|_2 \leq k\}$ for $k = 1$ behind the first plot. (Do not need to write any code, draw the graph by hand).

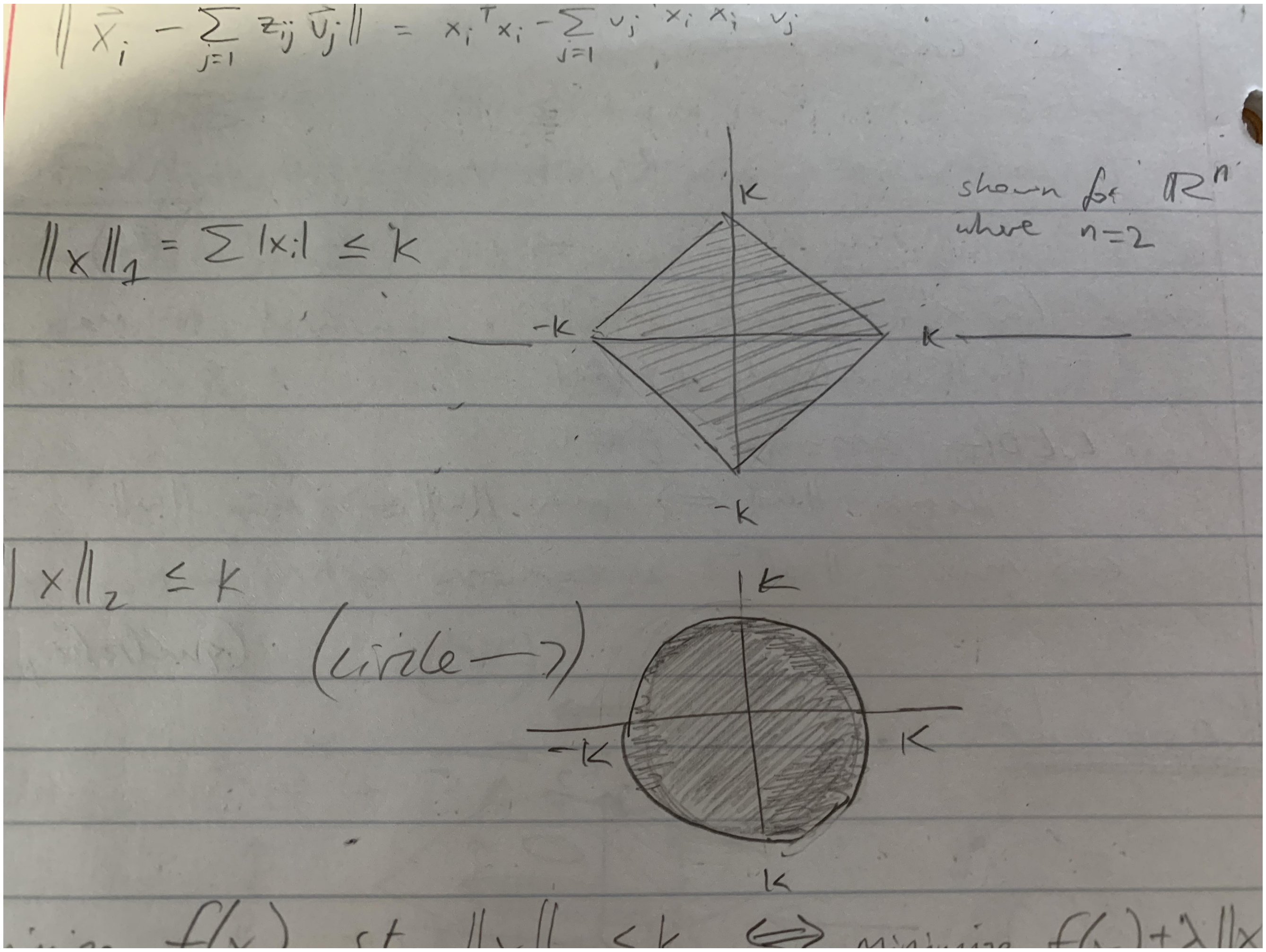
Show that the optimization problem

$$\begin{aligned} \text{minimize: } & f(\mathbf{x}) \\ \text{subj. to: } & \|\mathbf{x}\|_p \leq k \end{aligned}$$

is equivalent to

$$\text{minimize: } f(\mathbf{x}) + \lambda \|\mathbf{x}\|_p$$

(hint: create the Lagrangian). With this knowledge, and the plots given above, argue why using ℓ_1 regularization (adding a $\lambda\|\mathbf{x}\|_1$ term to the objective) will give sparser solutions than using ℓ_2 regularization for suitably large λ.



With the constraint $\|\mathbf{x}\|_p \leq k$, the Lagrangian is $L(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda\|\mathbf{x}\|_p$

The derivative $\frac{d\|\mathbf{x}\|_p}{d\mathbf{x}} = \frac{\sum |\mathbf{x}_i|^{p-1} \cdot \operatorname{sgn} \mathbf{x}_i}{\|\mathbf{x}\|_p^{p-1}}$.

When $p = 1$, if $x_i = 0$ then the contribution to the derivative is 0. If $x_i < 0$, the contribution is negative, and if $x_i > 0$, the contribution is positive, so the x_i will be 'squeezed' towards 0 when minimizing $\frac{d\|\mathbf{x}\|_p}{d\mathbf{x}}$.

When $p = 2$, the contributions are proportional to both the sign of x_i and their magnitude so the minimizing $\mathbf{x}$ is less sparse.

(got help from https://stats.stackexchange.com/questions/45643/why-l1-norm-for-sparse-models) ∎

Extra Credit (Lasso) Show that placing an equal zero-mean Laplace prior on each element of the weights $\boldsymbol{\theta}$ of a model is equivelent to ℓ_1 regularization in the Maximum-a-Posteriori estimate

$$\text{maximize: } \mathbb{P}(\boldsymbol{\theta}|\mathcal{D}) = \frac{\mathbb{P}(\mathcal{D}|\boldsymbol{\theta})\mathbb{P}(\boldsymbol{\theta})}{\mathbb{P}(\mathcal{D})}.$$

Note the form of the Laplace distribution is

$$\text{Lap}(x|\mu, b) = \frac{1}{2b}\exp\left(-\frac{|x-\mu|}{b}\right)$$

where μ is the location parameter and $b > 0$ controls the variance. Draw (by hand) and compare the density $\text{Lap}(x|0,1)$ and the standard normal $\mathcal{N}(x|0,1)$ and suggest why this would lead to sparser solutions than a Gaussian prior on each elements of the weights (which correspond to ℓ_2 regularization).

■