Learning methods in mean field games
Part 1

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"Numerical methods for optimal transport problems, mean field games, and multi-agent dynamics"

January 8-12, 2024

Universidad Técnica Federico Santa María, Valparaíso, Chile

Survey paper: arXiv:2205.12944

Questions, comments or suggestions are most welcome.

Based on joint works with many people, including:

Andrea Angiuli, Olivier Bachem, Tamer Basar, Theophile Cabannes, René Carmona, Gökçe Dayanikli, Romuald Élie, Jean-Pierre Fouque, Matthieu Geist, Maximilien Germain, Sertan Girgin, Kenza Hamidouche, Ruimeng Hu, Ayush Jain, Alec Koppel, Raphael Marinier, Paul Muller, Rémi Munos, Julien Pérolat, Sarah Perrin, Huyên Pham, Olivier Pietquin, Georgios Piliouras, Mark Rowland, Zongjun Tan, Karl Tuyls, Muhammad Aneeq uz Zaman, . . .

as well as other people's works

Outline

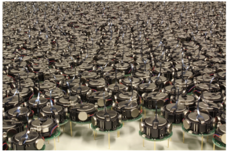
Motivations

Flocking

Crowd motion

Traffic flow

Collective AI

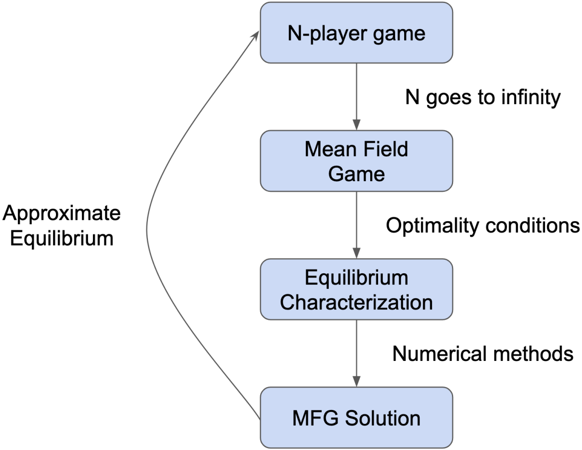


[Image credits: Unsplash, Wikimedia Commons (Kilobots)]

Some other existing approaches *("What MFGs are **not**")*

- Dynamical systems:
  - describe the dynamics of one or many agents, sometimes mean field
  - but usually **no rationality** (optimization)
- Agent based models (ABM):
  - "Agent-based models are a kind of **microscale model** that simulate the simultaneous operations and interactions of multiple agents in an attempt to re-create and predict the appearance of complex phenomena. "
  - "Individual agents are typically characterized as **boundedly rational**, presumed to be acting in what they perceive as their own interests, such as reproduction, economic benefit, or social status, using heuristics or simple decision-making rules." (Wikipedia)
- Game theory
  - optimization aspects
  - notion of Nash equilibrium, social optimum, . . .
  - but usually limited to a **finite (small) number of agents**
- Evolutionary game theory (EGT)
  - "application of game theory to evolving populations in biology"
  - "an evolutionary version of game theory **does not require players to act rationally** – only that they have a strategy" (Wikipedia)
- Non-atomic anonymous games
  - continuum of rational players; each player has her **own index** and own strategy
  - mostly limited to static games; difficulties for dynamic, stochastic games

MFG paradigm in a nutshell



Some References

- Introduction to Mean Field Games:
  - Pierre-Louis Lions' lectures at Collège de France (`https://www.college-de-france.fr/`)
  - Pierre Cardaliaguet's notes (2013):
    `https://www.ceremade.dauphine.fr/ cardaliaguet/MFG20130420.pdf`
  - Gomes, D. A., & Saúde, J. (2014). Mean field games models—a brief survey. Dynamic Games and Applications, 4, 110-154.
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  - Carmona, Delarue, Graves, Lacker, Laurière, Malhamé & Ramanan: Lecture notes of the 2020 AMS Short Course on Mean Field Games (American Mathematical Society), organized by François Delarue
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- Carmona, R., & Laurière, M. (2021). Deep Learning for Mean Field Games and Mean Field Control with Applications to Finance. arXiv preprint arXiv:2107.04568.
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- Laurière, M., Perrin, S., Geist, M., & Pietquin, O. (2022). Learning mean field games: A survey. arXiv preprint arXiv:2205.12944.

MFG & (R)L: Motivations

Main motivation: real-world applications require methods for large-scale problems

- Scaling up **population size** $\rightarrow$ **Mean Field Games**
  - Initial papers: Lasry & Lions; Caines, Huang & Malhamé (2006-2007)
  - Books: Bensoussan, Frehse & Yam; Carmona & Delarue; ...
- Scaling up **environment complexity** $\rightarrow$ (model-free) **Reinforcement Learning**
  - Book: Sutton & Barto; ...
  - Applications: Robotics, language processing, games, ...

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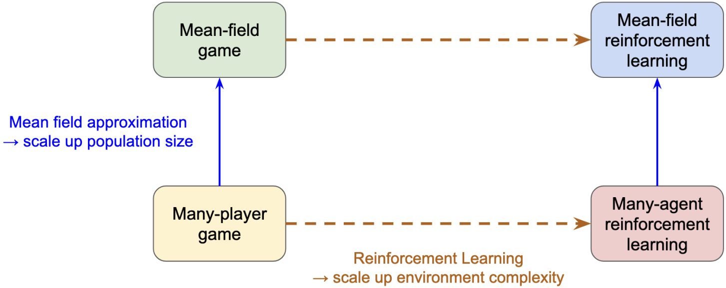

Motivations behind this overview

Rapidly growing literature

Goal: overview of the landscape & codes to make this topic more easily accessible

A few key aspects:

1. Problem setting
   $\rightarrow$ *continuous / discrete time & space, ...*

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   $\rightarrow$ *learning solution with "ideal" updates*

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4. Reinforcement learning

    $\rightarrow$ *learning solution with model-free updates*

5. Implementation

    $\rightarrow$ *code samples (OpenSpiel, ...)*

Learning

Recent successes of learning in games, e.g.:

Go [SHM+16, SSS+17, SHS+18], Chess [CHJH02], Checkers [SBB+07], Hex [ATB17], Starcraft II [VBC+19], poker games [BS17, BS19, MSB+17, BBJT15], Stratego [MLFB20], . . .

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At least **two interpretations** of "learning":

- Game theory, economics, . . . :
  Fudenberg & Levine [FL09][1]: *"The theory of learning in games [. . .] examines how, which, and what kind of equilibrium might arise as a consequence of a long-run nonequilibrium process of learning, adaptation, and/or imitation"*

- Machine Learning, Reinforcement Learning, . . . :
  Mitchell [M+97][2]: *"A computer program is said to learn from experience E with respect to some class of tasks T and performance measure P if its performance at tasks in T, as measured by P, improves with experience E."*

[1] Fudenberg, D., & Levine, D. K. (2009). Learning and equilibrium. *Annu. Rev. Econ.*, 1(1), 385-420.

[2] Mitchell, T. M. (1997). *Machine Learning*. New York: McGraw-Hill. ISBN: 978-0-07-042807-2

Outline

N-Player Stochastic Differential Game

For now, continuous time and continuous space:

- N players
- Player i's state is $X_t^i \in \mathbb{R}^d$
- with dynamics:

$$dX_t^i = b(t, X_t^i, \alpha_t^i, \mu_t^N)dt + \sigma dW_t^i, \qquad X_0^i \sim m^0$$

- W^i is an idiosyncratic (individual) noise, independent from other W^j's
- The empirical state distribution is: $\mu_t^N = \frac{1}{N}\sum_{j=1}^N \delta_{X_t^j}$

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- The empirical state distribution is: $\mu_t^N = \frac{1}{N}\sum_{j=1}^N \delta_{X_t^j}$
- Instantaneous cost function f and terminal cost function g
- Goal for player i: minimize over α^i the total expected cost:

$$J(\alpha^i, \alpha^{-i}) = \mathbb{E}\left[\int_0^T f(t, X_t^i, \alpha_t^i, \mu_t^N)dt + g(X_T^i, \mu_T^N)\right]$$

Two concepts:

- **Nash equilibrium** $(\hat{\alpha}^1, \dots, \hat{\alpha}^N)$: for all $i = 1, \dots, N$ and all α^i,

$$J(\hat{\alpha}^i, \hat{\alpha}^{-i}) \le J(\alpha^i, \hat{\alpha}^{-i})$$

$\rightarrow$ no incentive for unilateral deviations

$\rightarrow$ **fixed point** problem

N-Player Stochastic Differential Game: Solution Concepts

Two concepts:

- **Nash equilibrium** $(\hat{\alpha}^1, \dots, \hat{\alpha}^N)$: for all $i = 1, \dots, N$ and all α^i,

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- **Social optimum** $(\alpha^{*1}, \dots, \alpha^{*N})$: for all $i = 1, \dots, N$ and all $(\alpha^1, \dots, \alpha^N)$,

$$J(\alpha^{*1}, \dots, \alpha^{*N}) \leq J(\alpha^1, \dots, \alpha^N)$$

  $\rightarrow$ no incentive for joint deviations

  $\rightarrow$ **optimization** problem

In general, they are different, which leads to the notion of Price of Anarchy

Mean Field Limit

Pass to the limit $N \to +\infty$?

Key assumptions:

- homogeneity: all the agents have the same f, b, σ
- symmetry/anonymity: interactions are only through the empirical distribution

Pass to the limit $N \to +\infty$?

Key assumptions:

- homogeneity: all the agents have the same f, b, σ
- symmetry/anonymity: interactions are only through the empirical distribution

In the limit, we expect to have: the cost for one representative player is:

$$J(\alpha, \mu) = \mathbb{E}\left[\int_0^T f(t, X_t, \alpha_t, \mu_t)dt + g(X_T, \mu_T)\right]$$

with the dynamics:

$$dX_t = b(t, X_t, \alpha_t, \mu_t) + \sigma dW_t$$

where

- X and α are respectively the state and the control of the representative player,
- μ is the first marginal (state-only distribution)
- we will use the notation ν for the action distribution.

Mean Field Solution Concepts

Here again, two concepts:

- **Nash equilibrium** $(\hat{\alpha}, \hat{\mu})$:
  - Optimality: for all α,
$$J(\hat{\alpha}, \hat{\mu}) \leq J(\alpha, \hat{\mu})$$
  - Consistency: $\hat{\mu}_t = \mathcal{L}(X_t^{\hat{\alpha}}, \hat{\alpha}_t)$

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$\rightarrow$ **fixed point** problem over the mean field flow μ

Mean Field Solution Concepts

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- **Social optimum** α^*: for all α,
  $$J(\alpha^*, \mu^{\alpha^*}) \leq J(\alpha, \mu^{\alpha})$$
  where $\mu_t^{\alpha} = \mathcal{L}(X_t^{\alpha}, \alpha_t)$

  $\rightarrow$ no incentive for joint deviations

  $\rightarrow$ **optimization** problem for $\alpha \mapsto J(\alpha, \mu^{\alpha})$

Optimality Conditions

Large(st) part of the MFG literature focuses on equations of the form:

INTRODUCTION

This paper is devoted to the analysis of second order mean field games systems with a local coupling. The general form of these systems is:

$$\begin{cases} (i) & -\partial_t \phi - A_{ij}\partial_{ij}\phi + H(x, D\phi) = f(x, m(x,t)) \\ (ii) & \partial_t m - \partial_{ij}(A_{ij} m) - \operatorname{div}(m D_p H(x, D\phi)) = 0 \\ (iii) & m(0) = m_0, \ \phi(x,T) = \phi_T(x) \end{cases} \tag{1}$$

Source: Cardaliaguet, P., Graber, P.J., Porretta, A. and Tonon, D., 2015. Second order mean field games with degenerate diffusion and local coupling. Nonlinear Differential Equations and Applications NoDEA, 22(5), pp.1287-1317.

In a nutshell, the probabilistic approach to the solution of the mean-field game problem results in the solution of a FBSDE of the McKean–Vlasov type

$$\begin{cases} dX_t = b\big(t, X_t, \mathbb{P}_{X_t}, \hat{\alpha}(t, X_t, \mathbb{P}_{X_t}, Y_t)\big)dt + \sigma dW_t, \\ dY_t = -\partial_x H\big(t, X_t, \mathbb{P}_{X_t}, Y_t, \hat{\alpha}(t, X_t, \mathbb{P}_{X_t}, Y_t)\big)dt + Z_t dW_t, \end{cases} \tag{3.1}$$

with the initial condition $X_0 = x_0 \in \mathbb{R}^d$, and terminal condition $Y_T = \partial_x g(X_T, \mathbb{P}_{X_T})$.

Source: Carmona, R. and Delarue, F., 2013. Probabilistic analysis of mean-field games. SIAM Journal on Control and Optimization, 51(4), pp.2705-2734.

$\rightarrow$ Theory: derivation, analysis, ...

“Classical” Numerical Methods for MFG: Some references

Some methods based on the deterministic approach to MFG/MFC:

- Finite difference & Newton method: [ACD10], [ACCD12], ...
- (Semi-)Lagrangian approach: [CS14, CS15], [CS18], [CCS22], ...
- Augmented Lagrangian & ADMM: [BC15], [And17a], [AL16], ...
- Primal-dual algo.: [BnAKS18], [BnAKK+19], ...
- Gradient descent based methods [LP16], [Pfe16], [LP22], ...
- Monotone operators [AFG17], [GS18], [GY20], ...
- Policy iteration [CCG21a], [CK21], [CT22], [TS22], [LST23], ...
- Finite elements [BC15], [And17b], ...
- Gaussian processes [MYZ22], ...
- Kernel-based representation [LJL+21], ...
- Fourier approximation [N+19], ...

“Classical” Numerical Methods for MFG: Some references

Some methods based on the probabilistic approach to MFG/MFC:

- Cubature [dRT15], ...
- Markov chain approximation: [BBC18], ...
- Probabilistic approach and Picard: [CCD19], [AGL$^+$19], ...
- Probabilistic approach and regression: [BHL$^+$19], ...
- ...

“Classical” Numerical Methods for MFG: Shortcomings

Many of these methods are very efficient and have been analyzed in detail

However, they are usually limited to problems with:

- (relatively) small dimension
- (relatively) simple structure

$\Rightarrow$ motivations to develop deep learning methods

Deep Learning Numerical Methods for MFG

- DL for direct approach for MFG [FZ20], [CL22], ...
- DL for McKean-Vlasov FBSDEs [FZ20], [CL22], [GMW22], ...
- DL for PDE system [AACN+19], [CL21], [ROL+20], [CGL20], ...
- DL for Master equations [GLPW22], [Lau21, Section 7.2], ...

Pros & Cons:

- Scalability in terms of dimension
- Much less understood than classical methods
  $\Rightarrow$ Lots of open questions for mathematicians!

Extensions

From the modeling viewpoint, many possible extensions:

- More settings, e.g. MFG with **ergodic** cost [CLLP12], [Fel13], [BP14], [ABC17b], [AKS23], ...
- Interactions through the **action distribution** (“extended MFGs”, “MFGs of controls”, ...): [GPV14], [GV16], [CL18], [AK20], [LT22], [Kob22], ...
- **Common noise**: in the continuous space case see [CD18] and references therein; in the finite state case, see e.g. [BLL19], [BCCD21], ...
- **Several populations** MFGs: [HMC$^+$06b], [Fel13], [Cir15], [ABC17a], [BHL18], ...
- **Mean field type games**: [DTT17], [BGT21] and references therein; [MP19], [CP19], [CLT19], ...
- **Mean field control games**: [ADF$^+$22b], [ADF$^+$22a]

Extensions

- **Major player**: [CZ16], [CK16], [CW17], [LL18], [CCP20], [CD21], [CDL22], ...
- **Stackelberg** MFGs [BCY15], [MB18], [EMP19], [FSJ21], [ACDL22b], [VB22], [GHZ22], [DL23],...
- **Graphon** games [PO19], [CH19], [CH21], [LS22], [GTC20], [VMV21], [CCGL22], [ACL22], [ACDL22a], [BWZ23], ...
- **Correlated** equilibria [CF22], [MRE$^+$21], [MER$^+$22], ...
- ...

These lectures

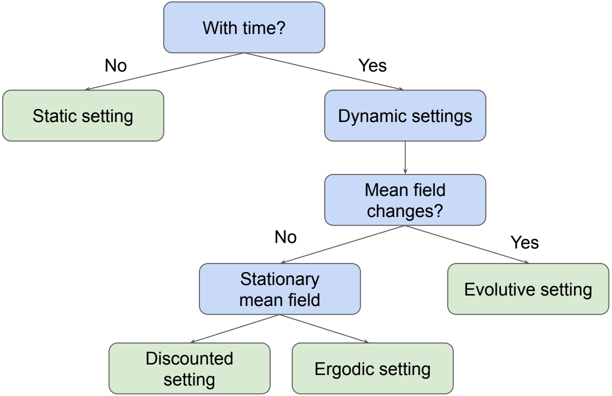
For simplicity, in most of the presentation, we will consider

- "plain" MFGs/MFCs,
- with discrete time and spaces

but many ideas can be extended in a (more or less) straightforward way.

Outline

Settings: Intuition



Settings

4 different settings:

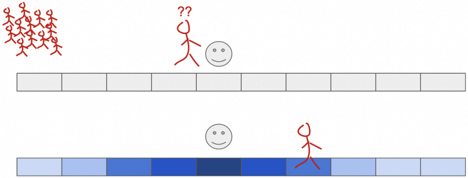
- **Static:**
  - **No states** (normal-form game): each player chooses an **action** $a \sim \pi(\cdot)$
  - Reward: depends on own action & population's action distribution
  - Examples: towel on the beach, urban settlement, ...
- **Evolutive:**
  - One-step reward: depends on own state, action & population's (state,action) distribution.
  - Fixed initial state distribution; finite or infinite time horizon.
  - Policy: **time-dependant policy** $\pi_n(\cdot|x)$
  - Examples: crowd motion, traffic routing, ...
- **Infinite horizon discounted & stationary:**
  - One-step reward: similar to Evolutive case.
  - Total reward: infinite horizon discounted sum.
  - Initial state distribution = stationary distribution induced by the population's policy.
  - Policy: **stationary policy** $\pi(\cdot|x)$
  - Examples: player joining a crowd already in a steady state
- **Ergodic:**
  - Similar to infinite horizon discounted & stationary.
  - But: Total reward = long time average.
- Other settings: asymptotic, γ-discounted, ...

Outline

Static game

Example: Population distribution (towel on the beach, . . .)

- action: choice of position
- reward: depends on my position and on the density of people



Source: unsplash

Static setting

- Finite action set A *(e.g., beach = possible towels' positions)*
- Player's behavior $\pi \in \Delta_A = \mathcal{P}(A)$
- Population's behavior $\xi \in \Delta_A$
- Player's reward: for player policy $\pi \in \Delta_A$ and population behavior $\xi \in \Delta_A$,

$$J(\pi; \xi) = \mathbb{E}_{a \sim \pi} \left[r(a, \xi) \right]$$

(e.g., crowd aversion, ice cream stall attraction, ...)

Static game: Solution concepts

- **Static MFG Nash equilibrium:** $(\hat{\pi}, \hat{\xi}) \in \Delta_A \times \Delta_A$ s.t.
  1. Best response: $\hat{\pi} \in \mathrm{BR}(\hat{\xi}) := \mathrm{argmax}_\pi J(\pi; \hat{\xi})$
  2. Consistency: $\hat{\xi} = \hat{\pi}$

- **Static MFC Social optimum:** $\pi^* \in \Delta_A$ s.t.
  - Optimality: $\pi^* \in \mathrm{argmax}_\pi J(\pi; \pi)$

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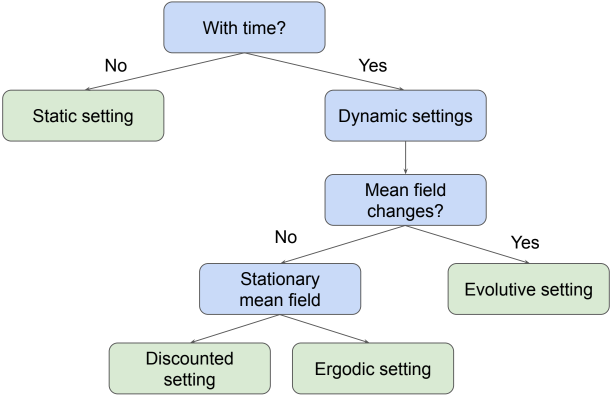
- **Static MFC Social optimum:** $\pi^* \in \Delta_A$ s.t.
  - Optimality: $\pi^* \in \mathrm{argmax}_\pi \, J(\pi; \pi)$
- Note: at social optimum, the population distribution is $\xi^* = \pi^*$
- But in general $\pi^* \neq \hat{\pi}$ so $\hat{\xi} \neq \xi^*$

Settings: Intuition – Reminder



Outline

Notation for Dynamic Settings

- State $x \in S$, action $a \in A$ (S, A finite for most of this presentation)
- Mean field state $\mu \in \Delta_S = \mathcal{P}(S)$ (extensions: state-action distrib.)
- Discrete time $n \in \mathbb{N}$
- Player's transition probability: $p(\cdot|x, a, \mu)$
- Player's reward: $r(x, a, \mu)$
- One-step policy: $\pi \in \Pi := (\Delta_A)^S$, functions $S \to \Delta_A$
- One-step mean field transition matrix: $P_{\mu,\pi}(x, y) = \sum_{a \in A} \pi(a|x) p(y|x, a, \mu)$

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- One-step mean field transition matrix: $P_{\mu,\pi}(x, y) = \sum_{a \in A} \pi(a|x)p(y|x, a, \mu)$
- What happens in one time step?
  - "Each" player selects an action (we focus on one "representative" player)
  - "Each" player gets a reward
  - "Each" player state is updated
  - Mean field is updated
- Mathematically: with policy π_n and mean field μ_n

$$
\begin{aligned}
&a_n \sim \pi_n(\cdot|x_n) \\
&r(x_n, a_n, \mu_n) \\
&x_{n+1} \sim p(\cdot|x_n, a_n, \mu_n) \\
&\mu_{n+1} = P^\top_{\mu_n,\pi_n}\mu_n = \sum_{y \in S} \mu_n(y) \sum_{a \in A} \pi_n(a|y)p(\cdot|y, a, \mu_n)
\end{aligned}
$$

Stationary setting

Stationary game

Example: joining a population in a stationary regime (flocking, economics, ...)

- the population is at equilibrium $\rightarrow$ MF distribution is stationary
- a player wants to join $\rightarrow$ optimal control problem
- but the distribution is the result of the agents' decisions $\rightarrow$ fixed point problem

Source: unsplash

Stationary setting

- Stationary setting: $N_T = \infty$
- No fixed initial m_0 but a stationary distribution
- Notation: $\mathrm{MF}(\hat{\pi}) :=$ stationary distribution when using policy $\hat{\pi}$:

$$\hat{\mu} = P_{\hat{\mu},\hat{\pi}}^{\top} \hat{\mu} =: \mathcal{P}^{\hat{\pi}}(\hat{\mu})$$

- Player's reward: for player's policy $\pi \in \Delta_A$ and mean field $\mu \in \Delta_S$,

$$J(\pi; \mu) = \mathbb{E}\left[\sum_{n=0}^{\infty} \gamma^n r(x_n, a_n, \mu)\right]$$

where $\gamma \in [0,1]$ is a discount parameter, and

$$a_n \sim \pi(\cdot|x_n), \qquad x_0 \sim \mu, \quad x_{n+1} \sim p(\cdot|x_n, a_n, \mu), n \geq 0$$

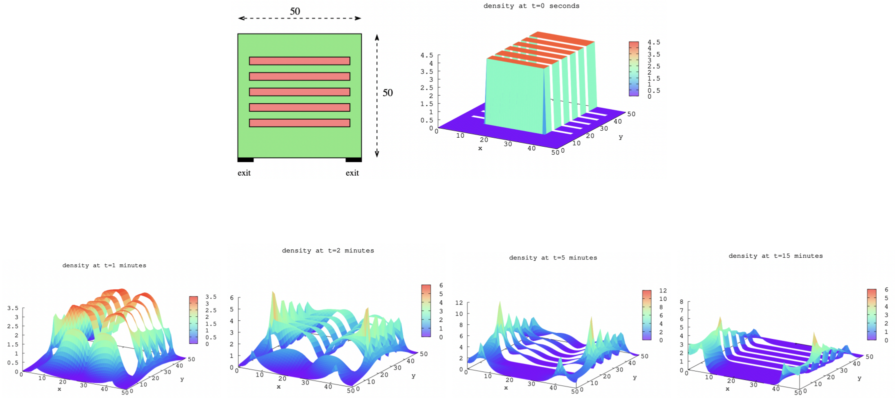
Stationary setting: Solution concepts

- **Stationary MFG Nash equilibrium:** $(\hat{\pi}, \hat{\mu}) \in \Pi \times \Delta_{S\times A}$ s.t.
  1. Best response: $\hat{\pi} \in \mathrm{BR}(\hat{\mu}) := \mathrm{argmax}_{\pi} J(\pi; \hat{\mu})$
  2. Mean field state: $\hat{\mu} = \mathrm{MF}(\hat{\pi})$
- Fixed point: $\hat{\mu} \in \mathrm{MF}(\mathrm{BR}(\hat{\mu}))$

- **Stationary MFC Social optimum:** $\pi^* \in \Pi$ s.t.
  - Optimality: $\pi^* \in \mathrm{argmax}_{\pi^*} J(\pi^*; \mu^{\pi^*})$ where $\mu^{\pi^*} = \mathrm{MF}(\pi^*)$

Evolutive setting

Example: Crowd exiting a room [AL15]



Evolutive setting

- Horizon: $N_T \in \mathbb{N}$ (extensions: p, r depending on n; infinite horizon)
- Fixed initial state distribution: $m_0 \in \Delta_S$
- The MF evolves in time: $\boldsymbol{\mu} = (\boldsymbol{\mu}_n)_{n=0,\dots,N_T} \in \Delta_S^{N_T}$
- Notation $\mathrm{MF}_{m_0,N_T}(\boldsymbol{\pi}) :=$ generated by policy $\boldsymbol{\pi}$ starting from m_0:

$$\begin{cases} \boldsymbol{\mu}_0 = m_0, \\ \boldsymbol{\mu}_{n+1} = P^\top_{\boldsymbol{\mu}_n, \boldsymbol{\pi}_n} \boldsymbol{\mu}_n, \quad n \geq 0 \end{cases}$$

- Player's reward: for player's policy $\boldsymbol{\pi} \in \Pi^{N_T}$ and mean field $\boldsymbol{\mu} \in \Delta_S^{N_T}$,

$$J(\boldsymbol{\pi}; \boldsymbol{\mu}) = \mathbb{E}\left[\sum_{n=0}^{N_T} r(x_n, a_n, \boldsymbol{\mu}_n)\right]$$

where

$$a_n \sim \boldsymbol{\pi}_n(\cdot|x_n), \qquad x_0 \sim m_0, \quad x_{n+1} \sim p(\cdot|x_n, a_n, \boldsymbol{\mu}_n), n \geq 0$$

Evolutive setting: Solution concepts

- **Evolutive MFG Nash equilibrium:** $(\hat{\boldsymbol{\pi}}, \hat{\boldsymbol{\mu}}) \in \Pi^{N_T} \times \Delta_S^{N_T}$ s.t.
  1. Best response: $\hat{\boldsymbol{\pi}} \in \mathrm{BR}(\hat{\boldsymbol{\mu}}) := \mathrm{argmax}_{\boldsymbol{\pi}} J(\boldsymbol{\pi}; \hat{\boldsymbol{\mu}})$
  2. Mean field flow: $\hat{\boldsymbol{\mu}} = \mathrm{MF}_{m_0, N_T}(\hat{\boldsymbol{\pi}})$
- Fixed point: $\hat{\boldsymbol{\mu}} \in \mathrm{MF}_{m_0, N_T}(\mathrm{BR}(\hat{\boldsymbol{\mu}}))$

- **Evolutive MFC Social optimum:** $\boldsymbol{\pi}^* \in \Pi^{N_T}$ s.t.
  - Optimality: $\boldsymbol{\pi}^* \in \mathrm{argmax}_{\boldsymbol{\pi}} J(\boldsymbol{\pi}; \boldsymbol{\mu}^{\boldsymbol{\pi}})$ where $\boldsymbol{\mu}^{\boldsymbol{\pi}} = \mathrm{MF}_{m_0, N_T}(\boldsymbol{\pi})$

Outline

Value function: stationary case

- Value function of a (stationary) policy π given a (stationary) mean field μ:

$$V^{\mu,\pi}(x) = \mathbb{E}_{a\sim\pi(\cdot|x)}\left[\underbrace{r(x,a,\mu) + \gamma\mathbb{E}_{x'\sim p(\cdot|x,a,\mu)}[V^{\mu,\pi}(x')]}_{Q^{\mu,\pi}(x,a)}\right]$$

$$V^{\mu,\pi} = \mathcal{T}^{\mu,\pi}V^{\mu,\pi}$$

$$Q^{\mu,\pi} = \mathcal{B}^{\mu,\pi}Q^{\mu,\pi}$$

- Optimal value function given a mean field μ: $V^{\mu,*}(x) = \max_\pi V^{\mu,\pi}(x)$:

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$$\mathrm{supp}(\pi^*(\cdot|x)) \subseteq \operatorname*{argmax}_{a\in A} Q^{\mu,*}(x,a)$$

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- Bellman equations are **fixed point equations**

Value function: finite horizon evolutive case

Finite horizon evolutive case ($N_T < +\infty$):

- Value function of a policy $\boldsymbol{\pi}$ given a mean field $\boldsymbol{\mu}$:

$$\begin{cases} V^{\boldsymbol{\mu},\boldsymbol{\pi}}_{N_T+1}(x) = 0 \\ V^{\boldsymbol{\mu},\boldsymbol{\pi}}_n(x) = \mathbb{E}_{a\sim\boldsymbol{\pi}_n(\cdot|x)}\left[\underbrace{r(x,a,\boldsymbol{\mu}_n) + \mathbb{E}_{x'\sim p(\cdot|x,a,\boldsymbol{\mu}_n)}[V^{\boldsymbol{\mu},\boldsymbol{\pi}}_{n+1}(x')]}_{Q^{\boldsymbol{\mu},\boldsymbol{\pi}}_n(x,a)}\right], \\ \qquad\qquad\qquad\qquad\qquad\qquad n = N_T - 1, \dots, 0 \end{cases}$$

- Optimal value function given a mean field $\boldsymbol{\mu}$:

$$V^{\boldsymbol{\mu},*}_n(x) = \max_{\boldsymbol{\pi}} V^{\boldsymbol{\mu},\boldsymbol{\pi}}_n(x)$$

$$Q^{\boldsymbol{\mu},*}_n(x,a) = \max_{\boldsymbol{\pi}} Q^{\boldsymbol{\mu},\boldsymbol{\pi}}_n(x,a)$$

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$$\mathrm{supp}(\boldsymbol{\pi}^*_n(\cdot|x)) \subseteq \operatorname*{argmax}_{a\in A} Q^{\boldsymbol{\mu},*}_n(x,a)$$

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- Bellman equations are **backward induction equations**

Variations

Many possible variations, for example:

- Settings: γ-discounted setting ($\mu_\gamma = \sum_n \gamma^n \mu_n$), ...
- Interactions: action distribution, state-action distribution, ...
- Solution notion: correlated equilibrium, ...
- Policy type: distribution-dependent, ...

Outline

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We are going to focus mostly on MFG Nash equilibria computation

MFG Equilibrium Computation: General Principles

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Two main objects: policy π and population distribution μ

Most basic idea: alternate

1. Update of the policy
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Many other possibilities using optimality conditions, e.g.

- traditional methods such as Newton's method for the PDE system [ACCD12]
- deep learning methods for PDE/FBSDE system, see [HL22]

But cannot be directly adapted to the model-free RL setting.

Updating the policy

For standard MDPs:

- Bellman operators
  - Optimal Bellman operator:

$$\mathcal{B}^* : (Q(x,a))_{x,a} \mapsto \mathcal{B}^* Q = \Big(r(x,a) + \gamma \mathbb{E}_{x' \sim p(\cdot|x,a)} [\max_{a'} Q(x',a')] \Big)_{x,a}$$

  - Bellman operator associated to a policy π:

$$\mathcal{B}^{\pi} : (Q(x,a))_{x,a} \mapsto \mathcal{B}^{\pi} Q = \Big(r(x,a) + \gamma \mathbb{E}_{x' \sim p(\cdot|x,a), a' \sim \pi} [Q(x,a')] \Big)_{x,a}$$

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- Iterative learning methods:
  - **Value iteration:**

$$Q^{k+1} = \mathcal{B}^* Q^k$$

  - **Policy iteration:**

$$\begin{cases} Q^{k+1} = Q^{\pi^k} & \text{(policy evaluation)} \\ \pi^{k+1} \in \operatorname{argmax} Q^{k+1} & \text{(policy improvement)} \end{cases}$$

where the policy evaluation can be done by applying $\mathcal{B}^{\pi^k}$ many times

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$\rightarrow$ For MFG: intertwine applications of $\mathcal{B}^{\mu,*}$ or $\mathcal{B}^{\mu,\pi}$ with MF updates

Iterative methods for MFG: Stationary case

Goal: find MFG Nash equilibirum $(\hat{\pi}, \hat{\mu}) \in \Pi \times \Delta_S$

- Iterations based on **Best response computation**:
  1. Compute best response: $\pi^{k+1} = \mathrm{BR}(\mu^k)$:
     - 1.1 Compute the optimal value function: $Q^{\mu^k,*} = \mathcal{B}^{\mu^k,*} Q^{\mu^k,*}$
     - 1.2 Let: $\pi^{k+1}(\cdot|x) \in \mathrm{argmax}_a\, Q^{\mu^k,*}(x,a)$
  2. Compute stationary MF: $\mu^{k+1} = \mathrm{MF}(\pi^{k+1})$: $\mu^{k+1} = \mathcal{P}^{\pi^{k+1}} \mu^{k+1}$
- Iterations based on **Policy evaluation** ("policy iteration"):
  1. Update policy:
     - 1.1 Evaluate policy: $Q^{\mu^k,\pi^k} = \mathcal{B}^{\mu^k,\pi^k} Q^{\mu^k,\pi^k}$
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Sometimes: one application of fixed point operator instead of true fixed point:

- $\mu^{k+1} = \mathcal{P}^{\pi^{k+1}}(\mu^k)$ instead of μ^{k+1} s.t. $\mu^{k+1} = \mathcal{P}^{\pi^{k+1}}(\mu^{k+1})$
- Learning step $\approx$ time step in the game

Iterative methods for MFG: Evolutive case

Goal: find MFG Nash equilibrium $(\hat{\boldsymbol{\pi}}, \hat{\boldsymbol{\mu}}) \in \Pi^{N_T} \times \Delta_S^{N_T}$

- Iterations based on **Best response computation**:
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     1.2 Let: $\boldsymbol{\pi}_n^{k+1}(\cdot|x) \in \mathrm{argmax}_a\, Q_n^{\boldsymbol{\mu}^k,*}(x,a)$
  2. Compute MF flow: $\boldsymbol{\mu}^{k+1} = \mathrm{MF}_{m_0,N_T}(\boldsymbol{\pi}^{k+1})$
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Backward equations instead of fixed point equations as in stationary case

Challenges

Potential issues (for both stationary and evolutive settings):

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  $\Rightarrow$ Oscillations / instabilities

Several variations / improvements have been studied

Outline

Damping / Averaging

Damping / smoothing:

- for policies: instead of:

$$\pi^{k+1} = \mathrm{BR}(\mu^k)$$

  use:

$$\bar{\pi}^{k+1} = \sum_{i=1}^{k} \alpha_i \mathrm{BR}(\mu^i)$$

  for some coefficients $(\alpha_i)_i$, and then:

$$\mu^{k+1} = \mathrm{MF}(\bar{\pi}^{k+1})$$

- and/or average mean fields, value functions, ...

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$\rightarrow$ Encompasses many possible variants such as:

- Fixed point iteration / value iteration (no damping): e.g. [HMC06a, GHXZ19, AKS20] ...
- Fictitious Play: e.g. [CH17, Had17, MJMdC18, PPL$^+$20, MH21, DV21] ...
- Policy Iteration: e.g. [CCG21b, CT21, LST21] ...
- Online Mirror Descent (OMD): e.g. [Had17, Had18, PPE$^+$21a] ...

Smooth policies

Class of smooth(er) policies:

- E.g. softmax/Botzmann policies: instead of

$$\pi^{k+1}(\cdot|x) \in \operatorname{argmax} Q^k(x,\cdot)$$

use:

$$\pi^{k+1}(\cdot|x) = \mathrm{softmax}_\tau Q^k(x,\cdot) = \frac{e^{\frac{1}{\tau}Q(x,\cdot)}}{\sum_a e^{\frac{1}{\tau}Q(x,a)}}$$

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- forces to play every action with a positive probability
- temperature τ can be decreased progressively if needed
- solves the problem of ambiguity among possible elements of argmax
- but the equilibrium policy $\hat{\pi}$ is not necessarily of $\mathrm{softmax}$ form!

Reward regularization

Reward regularization:

- Modify the reward with a regularizing penalty
- For instance, entropy penalty: instead of:

$$r(x, a, \mu)$$

use:

$$r(x, a, \mu) - \eta \log \left(\frac{\pi(a|x)}{\tilde{\pi}(a|x)} \right)$$

where $\tilde{\pi}$ is a reference policy (e.g., uniform)

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  where $\tilde{\pi}$ is a reference policy (e.g., uniform)
- it depends on the whole policy $\pi(\cdot|x)$ and not just on the action played
- helps to ensure uniqueness of the equilibrium and the BR
- but only for the *modified* game $\neq$ original game

Some Canonical Examples

Algorithm: Fixed point iter.

input : Initial policy π^0

$\mu^0 := \mu^{\pi^0}$;
for $k = 1, \dots, K$ **: do**
  $\pi^k :=$ BR against μ^{k-1};
  $\mu^k := \mu^{\pi^k}$;
return π^K, μ^K

$\downarrow$

Algorithm: Fictitious Play

input : Initial policy π^0

$\bar{\pi}^0 := \pi^0$;
$\bar{\mu}^0 := \mu^{\bar{\pi}^0}$;
for $k = 1, \dots, K$ **: do**
  $\pi^k :=$ BR against $\bar{\mu}^{k-1}$;
  $\bar{\mu}^k := \frac{k}{k+1}\bar{\mu}^{k-1} + \frac{1}{k+1}\mu^{\pi^k}$;
  $\bar{\pi}^k :=$ policy giving $\bar{\mu}^k$;
return $\bar{\pi}^K$, $\bar{\mu}^K$

Algorithm: Policy iter.

input : Initial policy π^0

$\mu^0 := \mu^{\pi^0}$;
for $k = 1, \dots, K$ **: do**
  $Q^k :=$ Q-func. for π^{k-1} given μ^{k-1};
  $\pi^k := \mathrm{argmax}\ Q^k$;
  $\mu^k := \mu^{\pi^k}$;
return π^K, μ^K

$\downarrow$

Algorithm: OMD

input : Initial policy π^0

$\mu^0 := \mu^{\pi^0}$;
for $k = 1, \dots, K$ **: do**
  $Q^k :=$ Q-func. for π^{k-1} given μ^{k-1};
  $\bar{Q}^k := \bar{Q}^{k-1} + \alpha Q^k$;
  $\pi^k := \mathrm{softmax}_\tau\ \bar{Q}^k$;
  $\mu^k := \mu^{\pi^k}$;
return π^K, μ^K

Assumptions and convergence guarantees

Several classes of assumptions to guarantee convergence of the iterations:

1. "**Quantitative**" assumptions:
   - small Lipschitz constants / short time
   - proof by strict contraction
   - Ex: [HMC06a, GHXZ19, AKS20, LST21] ...
2. "**Qualitative**/structural" assumptions:
   - potential structure / monotonicity
   - proof by Lyapunov stability
   - Ex: [CH17, Had17, Had18, MJMdC18, PPL+20, PPE+21a] ...

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How can we check whether the algorithm has converged?

Beware:

- Total reward of a player is not a good indicator of convergence
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$\rightarrow$ **Exploitability:**

- Evaluates the quality of a policy in a game [ZJBP07, LWZB09]
- *How "far"* π *is from being a Nash equilibrium policy?*

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- *How "far" π is from being a Nash equilibrium policy?*

In the context of MFGs:

- Definition: The **exploitability** $\mathcal{E}(\pi)$ of a policy π is defined as:

$$\mathcal{E}(\pi) := \max_{\pi'} J(\pi', \mu^\pi) - J(\pi, \mu^\pi)$$

- Interpretation: $\mathcal{E}(\pi)$ quantifies the average gain for a representative player to replace its policy by a best response, while the rest of the population plays with policy π.
- If $\mathcal{E}(\pi) = 0$, then π is a Nash equilibrium policy.

Outline

OpenSpiel

- Open source framework for research in learning in games
- Main motivation: multi-agent reinforcement learning (MARL)
- Marc Lanctot (Google DeepMind) + many contributors
- Mostly in C++ and Python; APIs in Julia, . . .
- Various games including zero-sum games, N-player games, imperfect information, . . .
- Chess, Blackjack, Atari, Kuhn poker, Go, . . .
- And also: Mean field games

OpenSpiel

Introduction to OpenSpiel:

- https://openspiel.readthedocs.io/en/latest/intro.html
- Python notebook:
  https://colab.research.google.com/github/deepmind/open_spiel/blob/master/open_spiel/colabs/OpenSpielTutorial.ipynb
- Tutorial by Marc Lanctot available online:
  https://www.youtube.com/watch?v=8NCPqtPwlFQ
- Paper [LLL+19]
- Two big components:
  - Games
  - Algorithms

- Julien Pérolat, Raphael Marinier, Sertan Girgin & growing number of contributors Théophille Cabannes, Sarah Perrin, Paul Muller, ...
- For today, three main questions:
  - How to use the existing material?
  - How to define a new MFG model (environment/game)?
  - How to define a new algorithm to learn the MFG solution?

Existing codes for MFG in OpenSpiel

- MFG models in C++: https://github.com/deepmind/open_spiel/tree/master/open_spiel/games/mfg
- MFG models in Python: https://github.com/deepmind/open_spiel/tree/master/open_spiel/python/mfg/games
  - Crowd modeling 1D illustrated in [PPL+20]
  - Crowd modeling 2D illustrated in [PPL+20, GPL+22]
  - Dynamic routing illustrated in [CLP+22]
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  - Deep fictitious play [LPG+22]
  - Boltzmann policy iteration [CK21]
  - Fictitious play [PPL+20], . . .
  - Fixed point
  - Mirror descent [PPE+21b]
  - Munchausen deep mirror descent [LPG+22]
  - Munchausen mirror descent

as well as codes for policies and an evaluation metric: exploitability (nash_conv)

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  as well as codes for policies and an evaluation metric: exploitability (nash_conv)
- Some examples: `https://github.com/deepmind/open_spiel/tree/master/open_spiel/python/mfg/examples`

More to come soon. Contributions are welcome!

MFG in OpenSpiel: Usecase

Q1. *How to use existing material?*

- Install & imports
- Creating a game (e.g., grid world)
- Running a learning algorithm (e.g., fictitious play)
- Plotting the results (e.g., exploitability and distribution)

Tutorial 1: Introduction to MFG in OpenSpiel

Code

Sample code to illustrate: IPython notebook

`https://colab.research.google.com/drive/16p95oXZGdhzCAX9MTPlcMNnsD3dyW9ur?usp=sharing`

- Installation and imports
- Creating a game
- Running an algorithm
- Visualizing the results

* Special thanks to Marc Lanctot, Julien Pérolat, Raphael Marinier, Sertan Girgin, Sarah Perrin and Kai Shao for this notebook

MFG model in OpenSpiel: State

Q2. *How to define a new MFG model?*

- State of the game = all the information required to describe the current stage
- In an MFG: representative player's state and mean field state
- Evolution of the state:
  - Players play in turn
  - Every change to the state occurs through a node
  - Each node has a set of possible actions and a probability to pick each action

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  - and the "noise" is viewed as a node too

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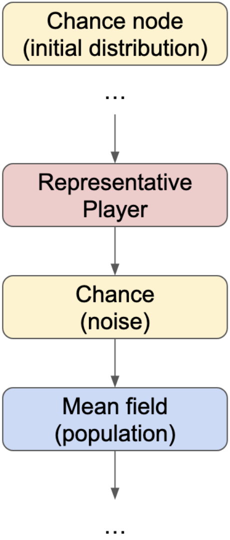
  - So: the representative player is a node
  - the "mean field" is viewed as a node
  - and the "noise" is viewed as a node too
  - Time is part of the state: (t, x)
- The state evolves along a tree of possibilities

MFG model in OpenSpiel: State evolution



MFG model in OpenSpiel: State types

- Initial **chance** node:
  - actions: possible states
  - probabilities: given by the initial state distribution

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  - probabilities: distribution of the noise values
- **Mean field:** no actions

MFG in OpenSpiel: Distribution

- The distribution is something specific to MFGs (compared with other games in OpenSpiel)
- Remember that time is part of the state object. Evaluating the distribution at a given state means evaluating the distribution at (t, x).
- `master/open_spiel/python/mfg/algorithms/distribution.py`
  - Computes the distribution of a policy
  - `DistributionPolicy`
    - `evaluate`: based on the logic behind nodes
    - `_one_forward_step`
- `master/open_spiel/python/mfg/distribution.py`
  - Representation of a distribution for a game
  - `Distribution`
- `master/open_spiel/python/mfg/tabular_distribution.py`
  - Tabular representation of a distribution for a game
  - `TabularDistribution`

MFG model in OpenSpiel: Example

We take a concrete example: crowd modeling in 1D with a grid world

`master/open_spiel/python/mfg/games/crowd_modelling.py`

3 main classes

- `MFGCrowdModellingGame`:
  - `__init__`: initialization
  - `new_initial_state`: generate new initial state
- `MFGCrowdModellingState`:
  - `__init__`: initialization
  - `_legal_actions`: actions that are valid
  - `chance_outcomes`: distribution over values of the noise in the dynamics
  - `_apply_action`: will be called at each node to modify the state based on the action
  - `_rewards`: representative player's reward
- `Observer`:
  - defines an observation, here basically t and x

Tutorial 2: Comparing Learning Algorithms

Code

Sample code to illustrate: IPython notebook

https://colab.research.google.com/drive/1L1MIVba_2Wm534TDcGL35W2D5vxCsFeo?usp=sharing

- Four room grid world
- Running multiple pre-defined algorithms
- Comparing their exploitabilities

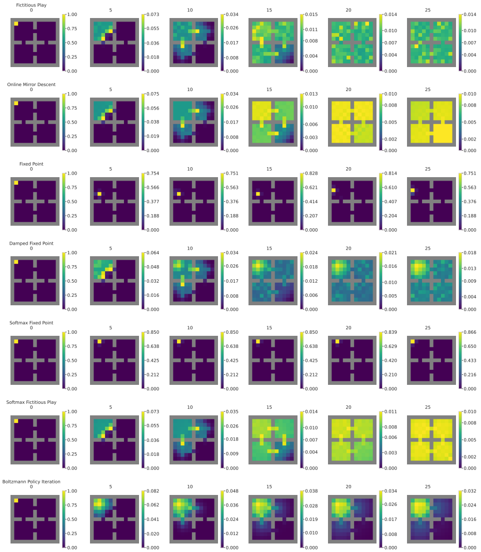
* Special thanks to Marc Lanctot, Julien Pérolat, Raphael Marinier, Sertan Girgin, Sarah Perrin and Kai Shao for this notebook

Comparing Learning Algorithms – Results

Game: crowd aversion in a four-room grid world

Test case 1: Noise level = 0.2

State distribution at different time steps (columns) for different algorithms (rows):



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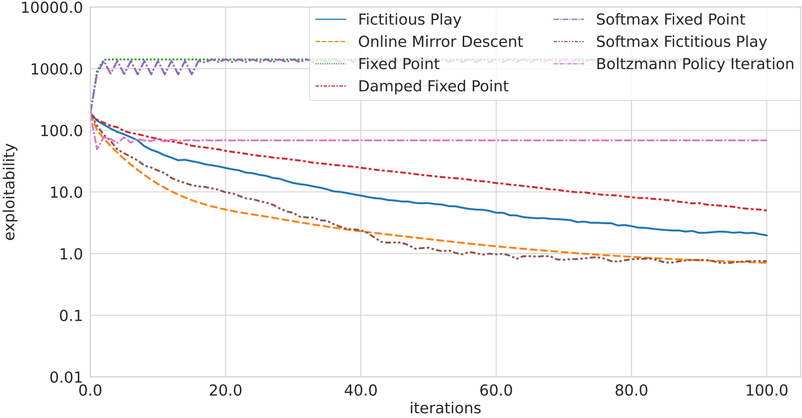
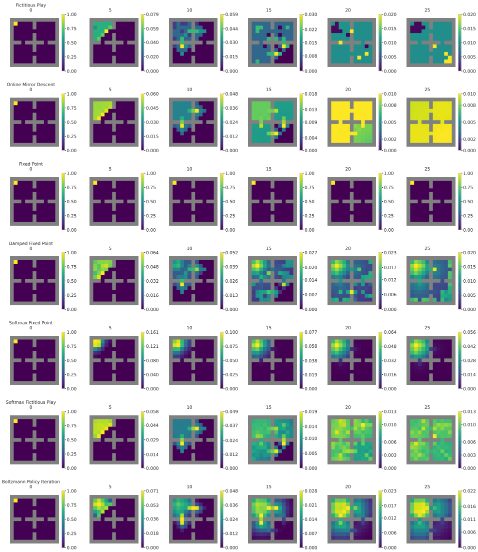
Exploitability vs number of steps:



Comparing Learning Algorithms – Results

Game: crowd aversion in a four-room grid world
Test case 2: Noise level = 0
State distribution at different time steps (columns) for different algorithms (rows):



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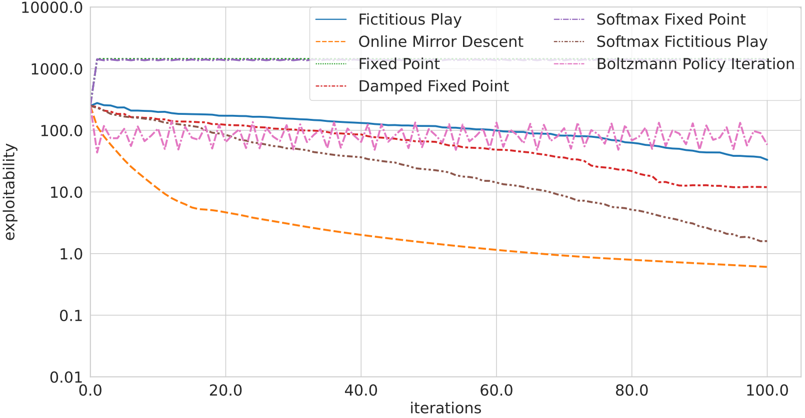

MFG algorithms in OpenSpiel: Principles

Q3. *How to define a new algorithm?*

Simplest one: Fixed point
`master/open_spiel/python/mfg/algorithms/fixed_point.py`

A bit more involved: Fictitious play
`master/open_spiel/python/mfg/algorithms/fictitious_play.py`

- Main class `FictitiousPlay`
- Main method `iteration`
  - Compute the distribution (sequence) associated to the current policy
  - Update the policy (using fictitious play rule); this uses an auxiliary class `MergedPolicy` to mix the previous policy and the new one
- `get_policy`: returns the current policy

Tutorial 3: Game construction

Code

Sample code to illustrate: IPython notebook

https://colab.research.google.com/drive/1uIcDYxQ9f7ngqIOo7ittZ4jEmXFOOZs9?usp=sharing

- Details of the definition of an MFG game in OpenSpiel
- Modification of an existing game
- Reward function, transitions, ...

* Special thanks to Marc Lanctot, Julien Pérolat, Raphael Marinier, Sertan Girgin, Sarah Perrin and Kai Shao for this notebook

Next: Reinforcement Learning

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