Introduction to Mean Field Games

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Motivations: Example 1 – Crowd Motion

Source: CGTN, Youtube

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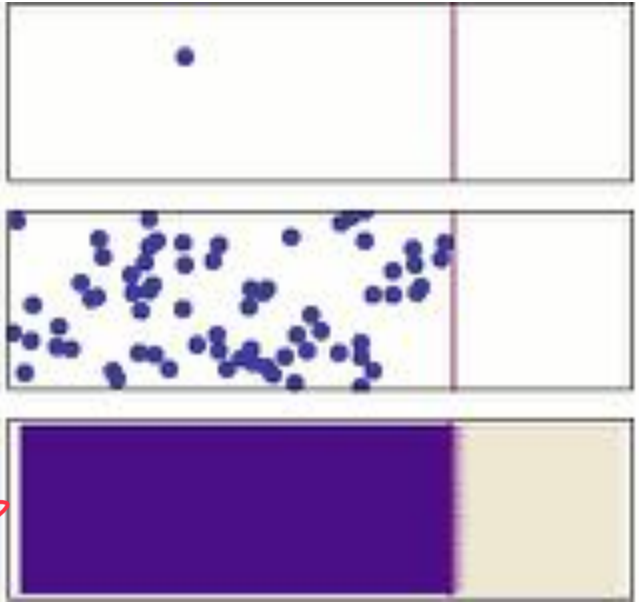
Source: CGTN, Youtube

Motivations: Example 1 – Crowd Motion

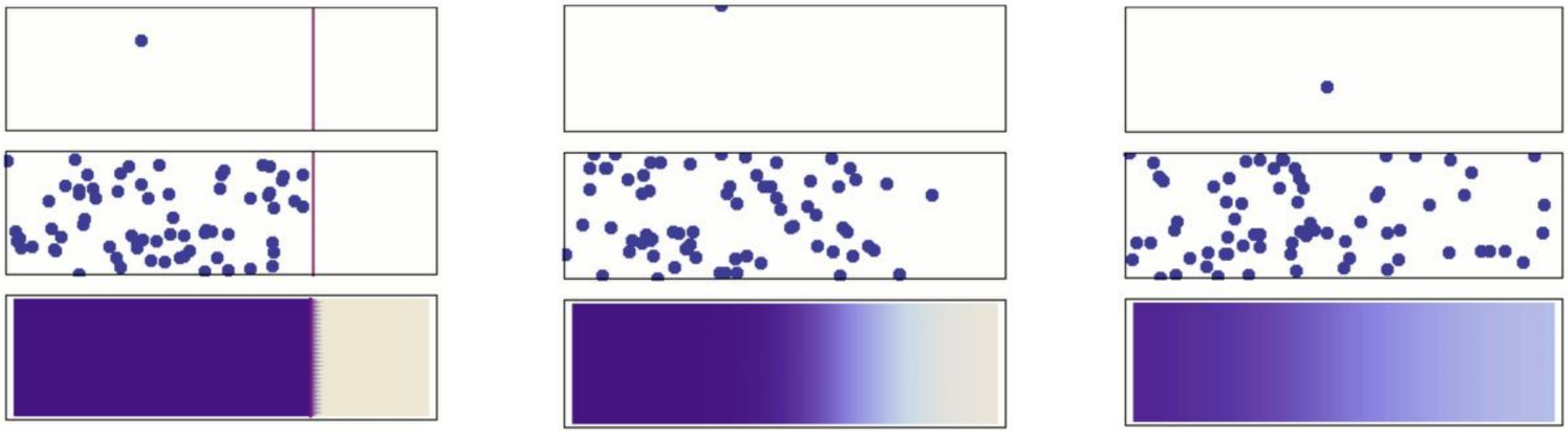
Macroscopic approximation



Source: Wikipedia

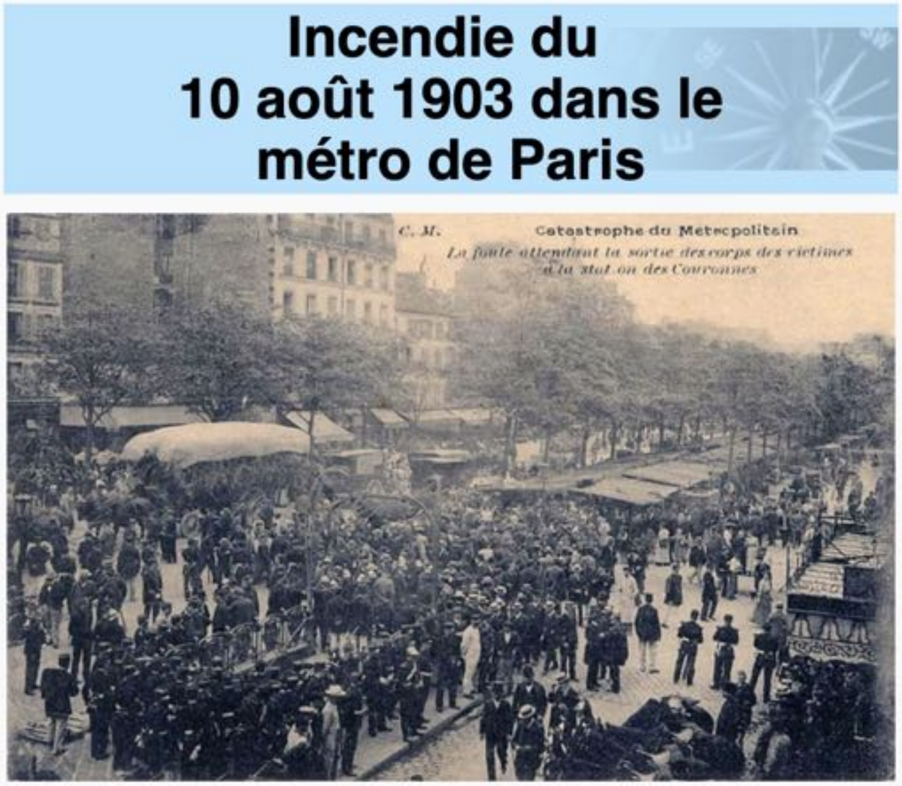
Motivations: Example 1 – Crowd Motion

Macroscopic approximation



Source: Wikipedia

Motivations: Example 1 – Crowd Motion



rational
selfish

Source: Wikipedia

Motivations: Example 2 – Economic Market

Source: Unsplash

Motivations: Example 3 – Climate Change

Source: Unsplash

Outline of the mini-course

1. MFG Models

2. Optimality Conditions

3. Numerical Methods

1. MFG models

Outline

1. MFG Models

    1.1 Static Setting

    1.2 Social Optimum

    1.3 Dynamic Setting

Outline

$\inf_x f(x)$

time = special dimension

↳ special way to write optimality conditions

Optimization problem	Optimal control problem
Static game	Dynamic game

$\sup_y \inf_x f(x,y)$

or N-player, general-sum, game

Outline

Optimization problem

Optimal control problem

Static game

Dynamic game

$N < \infty$

Number of agents

$N = \infty$

1.1 Static Setting

1.1.1 Static Setting: Finite-Population Game

Notation

- N players : $i = 1 \dots N$
- action space A (finite)
- each player selects an action $a^i \in A$
- it induces a population profile of actions $\underline{a} = (a^1, \dots, a^N)$
- each player pays a cost $f^i(\underline{a})$, $f^i : A^N \to \mathbb{R}$
- goal of each player: minimize her own cost

$$\min_a f^i(a^1, \dots, a^{i-1}, \overset{\downarrow}{a}, a^{i+1}, \dots a^N)$$

Nash Equilibrium

Notation: $\underline{a}^{-i} = (a^1, \ldots, a^{i-1}, a^{i+1}, \ldots, a^N)$

Main question: Is there a “stable configuration”?

Definition: Nash equilibrium (NE): a strategy profile $\hat{\underline{a}}$ such that:

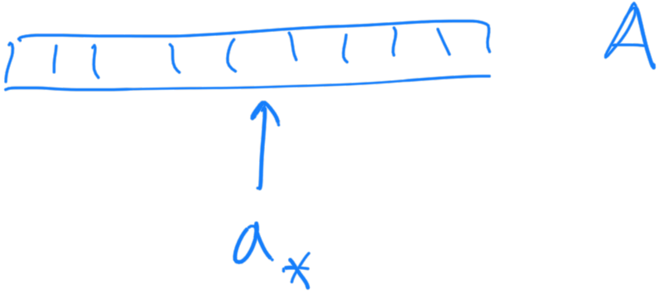
for all i, for all $a \in A$,

$$f^i(\hat{a}^i, \underbrace{\hat{\underline{a}}^{-i}}_{(\hat{a}^1, \ldots, \hat{a}^{i-1}, \hat{a}^{i+1}, \ldots, \hat{a}^N)}) \leq f^i(a, \hat{\underline{a}}^{-i})$$

An Example: Population Distribution

Population distribution game

Version 1: target position



If $f^i(\underline{a}) = |a^i - a_*|$ for all i,

then $\underline{\hat{a}} = (a_*, \dots, a_*)$ is a NE.

Exercise: is it the unique NE?

An Example: Population Distribution

Population distribution game

Version 2: attraction to the group

$$\bar{a} := \frac{1}{N} \sum_{j=1}^{N} a^j$$

If $f^i(\underline{a}) = |a^i - \bar{a}|$,

then for any $a \in A$, the strategy profile $\underline{\hat{a}} = (a, \dots, a)$ is a NE.

So we have at least as many NE as a's in A.

An Example: Population Distribution

Population distribution game

Version 3: repulsion from the group

If $f^i(\underline{a}) = -|a^i - \bar{a}|$, then?

Example without Nash equilibrium?

Let $A = \{1,2,3\}$. Let $N = 3$. Let $\underline{\hat{a}} = (\hat{a}^1, \hat{a}^2, \hat{a}^3) \in A^3$.

If $\underline{\hat{a}} = (1,2,3)$, then $\bar{a} = 2$ and $f^2(\underline{\hat{a}}) = 0 > f^2(1, \underline{\hat{a}}^{-2})$. So $\hat{a}$ is not a NE.

If $\underline{\hat{a}} = (1,1,2)$, then player 1 (or 2) can be better off by playing action 3.

[...]

An Example: Population Distribution

Population distribution game

Version 4: $\underbrace{\text{spatial preferences}}_{\text{e.g. } |a^i - a_*|}$ + attraction to / $\underbrace{\text{repulsion from the group}}_{\text{\# of players at the same location.}}$

e.g. $|a^i - a_*|$

\# of players at the same location.

e.g. $\sum_{\substack{j=1 \\ j \neq i}}^{N} \delta_{a^i}(a^j)$

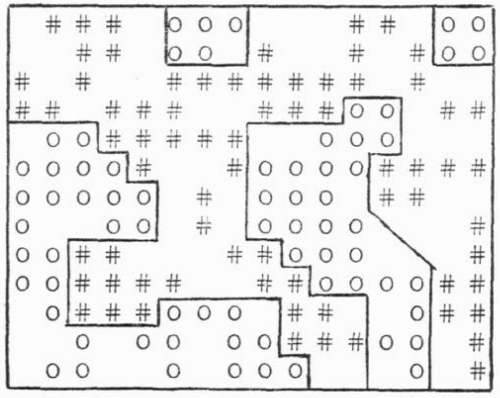
An Example: Population Distribution

Schelling's model of segregation

Goal: explain how segregation arises "naturally" in cities

- 2 types of agents
- each agent desires a fraction *B* of their neighbors to be of the same type
- repeat at each round:
  - check if the fraction is good
  - if not, relocate to a free location with a good fraction
- Schelling's result: threshold value $B_{seg} \cong 1/3$ such that

  If $B > B_{seg}$, then the iterations lead to *segregation*



Source: Schelling, T.C., 1971. Dynamic models of segregation. Journal of mathematical sociology, 1(2), pp.143-186.

Mixed Strategies

Warning: a pure NE does not always exist

- N players
- each player chooses a mixed strategy
- each player picks an action according to her strategy
- it induces population profiles of strategies and actions
- each player pays a cost
- goal for each player: minimize her own average cost

$\pi^i \in \Pi = \mathcal{P}(A)$

$\underline{\pi} = (\pi^1, \ldots, \pi^N)$

$\underline{a} = (a^1, \ldots, a^N)$

pick $a^i \sim \pi^i$, $i = 1 \ldots N$ iid

$f^i(\underline{a})$

$\underbrace{\mathbb{E}_{\underline{a} \sim \underline{\pi}}\left[f^i(\underline{a})\right]}_{J^i(\underline{\pi})}$

Note: the population distribution of actions is random

Note: average cost against other players' actions
$\neq$ cost against "average actions"

Mixed Strategies

Warning: a pure NE does not always exist

Nash theorem: existence of mixed NE

THEOREM 1. *Every finite game has an equilibrium point.*

Von Neumann and Morgenstern have developed a very fruitful theory of two-person zero-sum games in their book *Theory of Games and Economic Behavior*. This book also contains a theory of n-person games of a type which we would call cooperative. This theory is based on an analysis of the interrelationships of the various coalitions which can be formed by the players of the game.

Our theory, in contradistinction, is based on the *absence* of coalitions in that it is assumed that each participant acts independently, without collaboration or communication with any of the others.

The notion of an *equilibrium point* is the basic ingredient in our theory. This notion yields a generalization of the concept of the solution of a two-person zero-sum game. It turns out that the set of equilibrium points of a two-person zero-sum game is simply the set of all pairs of opposing "good strategies."

In the immediately following sections we shall define equilibrium points and prove that a finite non-cooperative game always has at least one equilibrium point.

Source: Nash, J., 1951. Non-cooperative games. Annals of Mathematics, 54(2).

Mixed Strategies

$\pi \xrightarrow{BR} \pi'$

$\underline{BR} = (BR^1, \dots, BR^N)$, $BR^i : \Pi^N \to 2^{\Pi}$ (set-valued)

$\underline{\pi} \mapsto \underbrace{\arg\min_{\pi} J^i(\pi, \underline{\pi}^{-i})}_{\text{set of optimal strategies for player } i \text{ against } \underline{\pi}}$

Warning: a pure NE does not always exist

Nash theorem: existence of mixed NE

Proof based on fixed point theorem:

- Kakutani's fixed point theorem: based on the best-response mapping, which is in general a multi-valued mapping

> THEOREM 1. *If $x \to \Phi(x)$ is an upper semi-continuous point-to-set mapping of an r-dimensional closed simplex S into $\mathfrak{R}(S)$, then there exists an $x_0 \in S$ such that $x_0 \in \Phi(x_0)$.*

- Brouwer's fixed point theorem: *"Every continuous function from a convex compact subset K of a Euclidean space to K itself has a fixed point."*

can be applied if players optimize over $\mathcal{P}(A)$ but not if they optimize over A (finite set)

Source: Kakutani, S., 1941. A generalization of Brouwer's fixed point theorem. Duke mathematical journal, 8(3), pp.457-459.

Population Distribution Example: Mixed Strategies

Population distribution game

Version 1: target position

$$\hat{\pi} = \delta_{a_*}$$

Population Distribution Example: Mixed Strategies

Population distribution game

Version 2: attraction to the group

Population Distribution Example: Mixed Strategies

Population distribution game

Version 3: repulsion from the group

What happens to the example without Nash equilibrium?

$$\hat{\pi} = U_A$$

$$\underline{\hat{\pi}} = (\hat{\pi}, \ldots, \hat{\pi})$$

Population Distribution Example: Mixed Strategies

Population distribution game

Version 4: spatial preferences + attraction to / repulsion from the group

Large Population Games: $N \to +\infty$

- In many applications, the number of players is extremely large
- Intuitively,
    - each player has a negligible impact on the rest of the population
    - the population distribution of actions becomes deterministic
- This should simplify the analysis
- Can we formalize this intuition?

$$\underbrace{\frac{1}{N}\sum_{j=1}^{N} \delta_{a^j}}_{} \in \mathcal{P}(A)$$

$$\underbrace{\frac{1}{N}\delta_{a^i}}_{\text{negligible when } N\to\infty} + \underbrace{\frac{1}{N}\sum_{j\neq i}\delta_{a^j}}_{\text{deterministic when } N\to\infty}$$

where $a^j \sim \pi^j$

this is a probability distribution but it is stochastic!

Large Population Games: $N \to +\infty$

- In many applications, the number of players is extremely large
- Intuitively,
  - each player has a negligible impact on the rest of the population
  - the population distribution of actions becomes deterministic
- This should simplify the analysis
- Can we formalize this intuition?
- Idea: let N go to infinity and study the problem we obtain in the limit
- Key assumptions: **homogeneity** and **anonymity**
- "**Mean field game**" paradigm [Lasry, Lions; Caines, Huang, Malhamé ~2006]

1.1.2 Static Setting: Mean Field Game

Key Assumptions

To pass to the mean field limit, we assume **homogeneity** and **anonymity**

$$f^i(\underline{a}) = f(a^i; \cancel{\underline{a}^{-i}})$$

$$\frac{1}{N}\sum_{j=1}^{N} \delta_{a^j}$$

changes if a^i changes

We will consider that the cost for player i is of the form $f\left(a^i; \frac{1}{N}\sum_{j=1}^{N}\delta_{a^j}\right)$ with $f: A \times \mathcal{P}(A) \longrightarrow \mathbb{R}$

Mean Field Game: Notation

But we are not going to write the players' indices

- "Infinitely many" players
- each player chooses a (mixed) strategy
- each player picks an action according to the strategy
- it induces a $\boxed{\text{population distribution of actions}}$ → $\nu \in \mathcal{P}(A)$
- each player pays a $\boxed{\text{cost}}$ → $f(a,\nu)$ when playing action a while the rest of population distribution is ν
- goal for each player: minimize her own average cost

"Unilateral deviation": this is fixed when we optimize over π

$$J(\pi, \nu) = \underbrace{\mathbb{E}_{a \sim \pi}}_{\text{only the randomness of the representative player}} \left[f(a, \nu) \right]$$

representative player's strategy

distribution over actions induced by the population

Mean Field Game: Notation

- "Infinitely many" players
- each player chooses a (mixed) strategy
- each player picks an action according to the strategy
- it induces a population distribution of actions
- each player pays a cost
- goal for each player: minimize her own average cost

Key points:

- it is enough to understand the behavior of one representative player
- each player has no influence on the rest of the population

Mean Field Nash Equilibrium

Definition: mean field Nash equilibrium (MFNE) $\hat{\pi} \in \mathcal{P}(A)$

for all π, $J(\hat{\pi}, \hat{\pi}) \leq J(\pi, \hat{\pi})$

$\hat{\nu}$ resulting from everyone playing $\hat{\pi}$

Fixed point formulation

$\hookrightarrow \hat{\pi} \in \underbrace{BR(\hat{\pi})}$ = set of minimizers of $\pi \mapsto J(\pi, \hat{\pi})$

Population Distribution Example: MFG Viewpoint

Population distribution game

Version 1: target position

$$f(a, \nu) = |a - a_*|$$

$$J(\pi, \nu) = \mathbb{E}_{a \sim \nu}\left[\,|a - a_*|\,\right]$$

(independent of ν)

(no interactions)

Population Distribution Example: MFG Viewpoint

Population distribution game

Version 2: $\underbrace{\text{attraction to the group}}$

mean $\quad \bar{a} = \frac{1}{N} \sum_{j=1}^{N} a^j$

$N \to \infty \quad \downarrow$

$\bar{\nu} = \mathbb{E}_{a \sim \nu} [a]$

$f(a, \nu) = |a - \bar{\nu}|$

$J(\pi, \nu) = \mathbb{E}_{a \sim \pi} [\, |a - \bar{\nu}| \,]$

Population Distribution Example: MFG Viewpoint

Population distribution game

Version 3: repulsion from the group

Population Distribution Example: MFG Viewpoint

Population distribution game

Version 4: spatial preferences + attraction to / repulsion from the group

Examples: interact with the proportion of players using the same action

$f(a,\nu) = \nu(a)$: cost increases linearly with this proportion

$f(a,\nu) = \log \nu(a)$. $J(\hat{\pi},\hat{\pi}) = \mathbb{E}_{a\sim\hat{\pi}}[\log(\hat{\pi}(a))] = -$ entropy of $\hat{\pi}$

㊟ "mean-field interactions" is more general than "interactions through the mean"

Approximate Nash Equilibrium

Definition: Approximate Nash equilibrium in N-player game : $\hat{\pi} \in \Pi$ such that :

$$\exists \varepsilon > 0, \forall \pi \in \Pi, \quad J^i(\hat{\pi}, \hat{\pi}^{-i}) \leq \underbrace{J^i(\pi, \hat{\pi}^{-i}) + \varepsilon}_{\text{By deviating unilaterally, you can be better off by at most } \varepsilon.}$$

Intuition: *An MFG equilibrium strategy provides an* approximate Nash equilibrium *in the corresponding finite-player game*

$\hat{\pi}$

$\to N$

$\to \varepsilon(N)$

the quality of the approximate NE depends on N

(it improves with N

i.e. $\varepsilon(N) \to 0$ as $N \to \infty$)

Approximate Nash Equilibrium: Example

Example: interaction through the mean

Model: $f(a,\nu) = \varphi(a,\bar{\nu})$ with $\bar{\nu} = \mathbb{E}_{a\sim\nu}[a]$. E.g. $\varphi(a,m) = |a-m|$

Assume φ Lipschitz in m uniformly in a: $\exists C\ \forall a\ \forall m,m'\ |\varphi(a,m)-\varphi(a,m')| \leq C|m-m'|$

Assumption: Mean field Nash equilibrium property

$\hat{\pi}$ such that: $\forall \pi,\ J(\hat{\pi},\hat{\pi}) \leq J(\pi,\hat{\pi})$

Goal: ε-Nash equilibrium for N-player game

$\exists \varepsilon$ s.t.: $\forall \pi,\ J^i(\hat{\pi},\hat{\pi}^{-i}) \leq J(\underline{\pi},\hat{\pi}^{-i}) + \varepsilon$

Approximate Nash Equilibrium: Example

Example: interaction through the mean

Proof sketch: $J^i(\hat\pi, \hat\pi^{-i}) = \mathbb{E}_{\hat a^i \sim \hat\pi} \mathbb{E}_{\hat a^j \sim \hat\pi, j \neq i} \left[f(\hat a^i, \bar{\hat a}) \right]$

⚠ Can't swap $\mathbb{E}$ and f !

whereas $J(\hat\pi, \hat\pi) = \mathbb{E}_{\hat a \sim \hat\pi} \left[f(\hat a, \bar{\hat\pi}) \right]$

We have $\bar{\hat a} \simeq \tilde a := \frac{1}{N} \sum_{j \neq i} \hat a^j \simeq \bar{\hat\pi}$ and f is Lipschitz so:

$$\left| f(\hat a^i, \bar{\hat a}) - f(\hat a^i, \bar{\hat\pi}) \right| \leq C \left| \bar{\hat a} - \tilde a \right| + C \left| \tilde a - \bar{\hat\pi} \right| = C \frac{1}{N} |\hat a^i| + C |\tilde a - \bar{\hat\pi}|$$

Hence: $\left| J^i(\hat\pi, \hat\pi^{-i}) - J(\hat\pi, \hat\pi) \right| \leq C\, \mathbb{E}_{\hat a^i \sim \hat\pi} \mathbb{E}_{\hat a^j \sim \hat\pi, j \neq i} \left[\frac{1}{N} |\hat a^i| + |\tilde a - \bar{\hat\pi}| \right]$

$$= \underbrace{\frac{C}{N} \mathbb{E}_{\hat a \sim \hat\pi} |\hat a|}_{\xrightarrow{N \to \infty} 0} + \underbrace{\mathbb{E}_{\hat a^j \sim \hat\pi, j \neq i} \left[|\tilde a - \bar{\hat\pi}| \right]}_{\xrightarrow{N \to \infty} 0}$$ by Law of Large Numbers

$\boxed{\text{We can proceed similarly for the right hand side.}}$

Remarks

- Deterministic vs randomized decisions
- Discrete vs continuous action spaces
- Non-atomic anonymous games (continuum of players)

$i \in I = [0, 1]$

$\int_I f^i \, di$

Summary

Main takeaways so far

1.2 Social optimum

Social Optimum: Static Setting

- Goal: minimize the social cost = average cost for the agents in the population
- N-agent social cost:
- Mean field social cost:

Social Optimum vs Nash Equilibrium

- In general the two notions are different
- The socially optimal strategy is different from the Nash equilibrium policy
- Price of Anarchy

ABSTRACT

In a system where noncooperative agents share a common resource, we propose the price of anarchy, which is the ratio between the worst possible Nash equilibrium and the social optimum, as a measure of the effectiveness of the system. Deriving upper and lower bounds for this ratio in a model where several agents share a very simple network leads to some interesting mathematics, results, and open problems.[2]

Source: Koutsoupias, E. and Papadimitriou, C., 1999, March. Worst-case equilibria. In Annual symposium on theoretical aspects of computer science (pp. 404-413). Springer, Berlin, Heidelberg.

Social Optimum = Nash equilibrium: Example

In *some cases*, the two notions coincide.

Example: Potential MFG with cost: $f(x, \nu) = \nabla F(\nu)(a)$

The average cost is: $J(\pi, \nu) = \mathbb{E}_{a \sim \pi}[f(a, \nu)] = \sum_a \pi(a) \nabla F(\nu)(a) = \pi \cdot \nabla F(\nu)$

Assuming the potential convex, we have the equivalence:

$$
\begin{aligned}
\hat{\pi} \text{ is a NE } &\Leftrightarrow J(\pi, \hat{\pi}) - J(\hat{\pi}, \hat{\pi}) \geq 0, \quad \forall \pi \\
&\Leftrightarrow (\pi - \hat{\pi}) \cdot \nabla F(\nu) \geq 0, \quad \forall \pi \\
&\Leftrightarrow F(\pi) - F(\hat{\pi}) \geq 0, \quad \forall \pi \\
&\Leftrightarrow \hat{\pi} \text{ is a minimizer of } F
\end{aligned}
$$

Example: entropy: $F(\nu) = \sum_a \nu(a) \log(\nu(a))$

Exercises

Ex. 1: Find a static MFG with exactly 2 pure NE. How many mixed NE are there?

Ex. 2: Find a static MFG with exactly 2 mixed social optima.

Ex. 3: Find a static MFG with a unique mixed NE and a unique mixed SO, such that their values are different. Same question with "such that their values are the same".

1.3 Dynamic Setting

1.2.1 Dynamic Setting: Finite-Population Game

Dynamic N-player Game

Main difference with static case: each player has a state which evolves in time

"Static" game	"Dynamic" game

Dynamic N-player Game: Notation

- Time
- State space
- Action space
- One-step strategy (deterministic or mixed)
- Control or policy
- Player's state
- Population's state

Dynamic N-player Game: Notation

We assume **homogeneity** and **anonymity**

- Player’s dynamics

- Population’s dynamics

Dynamic N-player Game: Notation

- Running cost
- Terminal cost
- Total cost

Nash Equilibrium

Definition: Nash equilibrium in dynamic N-player game

Example: Crowd Motion

- Dynamics:

- Cost:
  - Running cost:
    - cost to move (congestion):

    - discomfort (aversion):

  - Terminal cost:
    - spatial preference:

1.2.2 Dynamic Setting: Mean Field Game

Dynamic Mean Field Game: Notation

- Time
- State space
- Action space
- One-step strategy (deterministic or mixed)
- Control or policy
- Player’s state
- Population’s state

Dynamic Mean Field Game: Notation

- Player's dynamics:

- Population distribution dynamics:

- Mean field (MF) induced by a policy:

Dynamic Mean Field Game: Notation

- Running cost
- Terminal cost
- Total cost

- Best response (BR) to a mean field:

Mean Field Nash Equilibrium

Definition: Mean field Nash equilibrium (MFNE) in a dynamic MFG

Fixed point formulation:

Exercises

Ex. 1: Find a dynamic MFG such that:

(1) there is a unique NE

(2) given the equilibrium mean field sequence, there are multiple BR

1.2.3 Dynamic Setting: Continuous time & space

MFG in Continuous Time and Space

For a (the) large(st) part, the MFG literature starts like this:

INTRODUCTION

This paper is devoted to the analysis of second order mean field games systems with a local coupling. The general form of these systems is:

$$\begin{cases} (i) & -\partial_t\phi - A_{ij}\partial_{ij}\phi + H(x, D\phi) = f(x, m(x,t)) \\ (ii) & \partial_t m - \partial_{ij}(A_{ij}m) - \mathrm{div}(mD_pH(x, D\phi)) = 0 \\ (iii) & m(0) = m_0,\ \phi(x,T) = \phi_T(x) \end{cases} \tag{1}$$

Source: Cardaliaguet, P., Graber, P.J., Porretta, A. and Tonon, D., 2015. Second order mean field games with degenerate diffusion and local coupling. Nonlinear Differential Equations and Applications NoDEA, 22(5), pp.1287-1317.

In a nutshell, the probabilistic approach to the solution of the mean-field game problem results in the solution of a FBSDE of the McKean–Vlasov type

$$\begin{cases} dX_t = b(t, X_t, \mathbb{P}_{X_t}, \hat{\alpha}(t, X_t, \mathbb{P}_{X_t}, Y_t))dt + \sigma dW_t, \\ dY_t = -\partial_x H(t, X_t, \mathbb{P}_{X_t}, Y_t, \hat{\alpha}(t, X_t, \mathbb{P}_{X_t}, Y_t))dt + Z_t dW_t, \end{cases} \tag{3.1}$$

with the initial condition $X_0 = x_0 \in \mathbb{R}^d$, and terminal condition $Y_T = \partial_x g(X_T, \mathbb{P}_{X_T})$.

Source: Carmona, R. and Delarue, F., 2013. Probabilistic analysis of mean-field games. SIAM Journal on Control and Optimization, 51(4), pp.2705-2734.

Continuous Setting

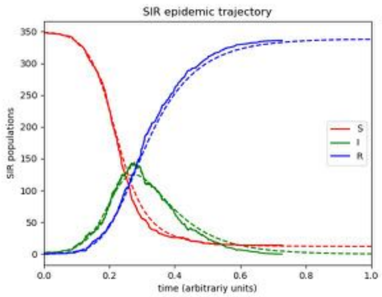
Why do we care about continuous time & space?

Continuous Setting

Why do we care about continuous time & space?

- Calculus!
- More natural for many applications
- Discretizing a continuous time/space process is not trivial

Example 4 - Epidemics



SIR model

$$\text{Susceptible} \xrightarrow{\beta SI/N} \text{Infectious} \xrightarrow{\gamma I} \text{Recovered}$$

ODE system:

$$\begin{cases} \dfrac{dS}{dt} = -\dfrac{\beta I S}{N}, \\ \dfrac{dI}{dt} = \dfrac{\beta I S}{N} - \gamma I, \\ \dfrac{dR}{dt} = \gamma I, \end{cases}$$

Basic reproduction number: $R_0 = \dfrac{\beta}{\gamma}$

Source: Wikipedia
Source: Kermack WO, McKendrick AG (1927). "A Contribution to the Mathematical Theory of Epidemics". Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character. 115 (772): 700–721.

$$S \xrightarrow{\beta \alpha_t \int a \rho_t(a, I)} I \xrightarrow{\gamma} R$$

where the control α_t is a “contact” factor that depends on the individual’s behavior (e.g., socialization, wearing mask, …)

The running cost encodes the individual’s preferences

It is also possible to include other aspects: vaccination, incentives, age structure, spatial movement, …

Source: Aurell, A., Carmona, R., Dayanikli, G. and Lauriere, M., 2022. Optimal incentives to mitigate epidemics: a Stackelberg mean field game approach. SIAM Journal on Control and Optimization, 60(2), pp.S294-S322.
See also: Turinici, Hubert, et al.

Example 5 - Flocking

Cucker-Smale model

Position and velocity:

$$\begin{cases} x_i(t+1) = x_i(t) + v_i(t)\Delta t \\ v_i(t+1) = v_i(t) + \sum_j a_{i,j}(v_j(t) - v_i(t)) \end{cases}$$

with a matrix of interactions based on the positions:

$$a_{ij} = \eta\left(\|x_i - x_j\|^2\right) \qquad \eta(y) = \frac{K}{(\sigma^2 + y)^\beta}$$

Source: Unsplash
Source: Cucker, F. and Smale, S., 2007. Emergent behavior in flocks. IEEE Transactions on automatic control, 52(5), pp.852-862.

Nourian, Caines & Malhamé'10:

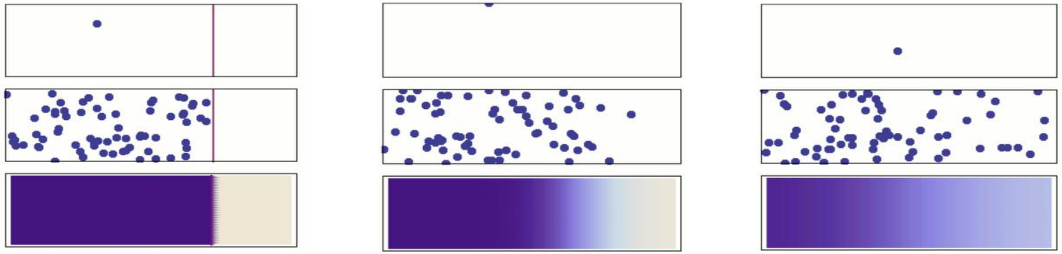
> Our aim in this work is to synthesize the collective behaviour of the set of agents from fundamental principles rather than to analyze this behaviour resulting from ad-hoc feedback laws. Hence the model in this paper may be regarded as a controlled game theoretic formulation of the uncontrolled C-S flocking model in which each agent, instead of responding to an ad-hoc algorithm, obtains its control law from a game theoretic Nash equilibrium depending upon its individual cost function and those of all other agents.

Velocity change = acceleration = control
Running cost penalizes deviation from neighbors' velocity:

$$\phi_i^{(N)}((x_i, v_i); (x, v)_{-i}) \triangleq \left\| \frac{1}{N} \sum_{j=1}^{N} w(\|x_i - x_j\|)(v_j - v_i) \right\|_Q^2$$

Source: Nourian, M., Caines, P.E. and Malhamé, R.P., 2010, September. Synthesis of Cucker-Smale type flocking via mean field stochastic control theory: Nash equilibria. In 2010 48th Annual Allerton Conference on Communication, Control, and Computing (Allerton) (pp. 814-819). IEEE.

Diffusion Model



- Particle's dynamics: $dX_t = \sigma dW_t$
- Macroscopic distribution dynamics: $\partial_t m(t,x) - \frac{\sigma^2}{2}\Delta m(t,x) = 0$
- Link with *N*-particle system: propagation of chaos [Kac'76]

- Note: We can also add a transport term (convection–diffusion equation)

Source: Wikipedia

MFG in Continuous Time and Space

- Time
- Player's control (deterministic)
- Player's dynamics: $dX_t = b(X_t, v(t, X_t), m(t))dt + \sigma dW_t, \quad X_0 \sim \mu_0$
- Population dynamics: Kolmogorov-Fokker-Planck equation

$$\partial_t m(t,x) - \frac{\sigma^2}{2}\Delta m(t,x) + \mathrm{div}\Big(m(t,x)b(x, v(t,x), m(t))\Big) = 0, \quad m(0,x) = m_0(x)$$

See e.g. Bogachev, V.I., Krylov, N.V., Röckner, M. and Shaposhnikov, S.V., 2022. Fokker–Planck–Kolmogorov Equations (Vol. 207). American Mathematical Society.

MFG in Continuous Time and Space

- Cost: dependence on the mean field
  - non-local (typically “regularizing” operator)
  - local (if the distribution has a density)

Summary of Different Settings

Space \ Time	Discrete	Continuous
Discrete		
Continuous		

Remarks and Extensions

- Discrete vs continuous: time, action and state spaces
- Controls:
  - Open-loop vs closed-loop
  - Deterministic vs randomized (pure vs mixed)
  - Interaction through the distribution of controls ("extended" MFG)
- Noise/perturbations:
  - With or without idiosyncratic noise ("first order" MFG)
  - With or without common noise
- Homogeneity: extension with multiple groups, major-minor, Stackelberg, …
- Anonymity: multiple groups, graphon, …

Characterization of MFNE

Question:

How can we characterize and compute mean field Nash equilibria?

Characterization of MFNE

Question:

How can we characterize and compute mean field Nash equilibria?

Answer:

… in the rest of the mini-course.

2. Optimality Conditions

Outline

2. Optimality conditions

   2.1 Introduction

   2.2 Deterministic viewpoint

   2.3 Stochastic viewpoint

2.1 Introduction

Reminder: Dynamic MFNE

In the dynamic setting:

- Definition of MFNE
- Characterization?
  - Fixed point formulation
  - Population behavior
  - Best response characterization?

Optimality conditions

What do these equations mean?

Large(st) part of the MFG literature starts like this:

INTRODUCTION

This paper is devoted to the analysis of second order mean field games systems with a local coupling. The general form of these systems is:

$$\begin{cases} (i) & \boxed{-\partial_t \phi - A_{ij}\partial_{ij}\phi + H(x, D\phi) = f(x, m(x,t))} \\ (ii) & \partial_t m - \partial_{ij}(A_{ij}m) - \operatorname{div}(m D_p H(x, D\phi)) = 0 \\ (iii) & m(0) = m_0, \ \phi(x,T) = \phi_T(x) \end{cases} \tag{1}$$

Source: Cardaliaguet, P., Graber, P.J., Porretta, A. and Tonon, D., 2015. Second order mean field games with degenerate diffusion and local coupling. Nonlinear Differential Equations and Applications NoDEA, 22(5), pp.1287-1317.

In a nutshell, the probabilistic approach to the solution of the mean-field game problem results in the solution of a FBSDE of the McKean–Vlasov type

$$\begin{cases} dX_t = b(t, X_t, \mathbb{P}_{X_t}, \hat{\alpha}(t, X_t, \mathbb{P}_{X_t}, Y_t))dt + \sigma dW_t, \\ \boxed{dY_t = -\partial_x H(t, X_t, \mathbb{P}_{X_t}, Y_t, \hat{\alpha}(t, X_t, \mathbb{P}_{X_t}, Y_t))dt + Z_t dW_t,} \end{cases} \tag{3.1}$$

with the initial condition $X_0 = x_0 \in \mathbb{R}^d$, and terminal condition $Y_T = \partial_x g(X_T, \mathbb{P}_{X_T})$.

Source: Carmona, R. and Delarue, F., 2013. Probabilistic analysis of mean-field games. SIAM Journal on Control and Optimization, 51(4), pp.2705-2734.

2.2 Deterministic viewpoint

2.2.1 Deterministic Viewpoint: Discrete Setting

Discrete Setting: Value Function

Definition: Value function of a representative player given a mean field sequence

Value of a state = sum of future costs, when starting from this state

Discrete Setting: Bellman Equation

Bellman equation for the value function (Dynamic Programming Principle):

- Terminal time:
- Backward induction:

Recovering the optimal control from the value function:

Discrete Setting: Forward-Backward System

Coupled system:

- Forward equation for the mean field:

$$\mu_{t+1}(x) = \sum_{x'} \mu_t(x') \sum_a \pi_t(a|x) p(x|x', a, \mu_t), \quad \mu_0 \text{ given}$$

- Backward equation for the value function:

$$V_t(x) = \min_a \mathbb{E}[f(X_t, A_t, \mu_t) + V_{t+1}(X_{t+1}) | X_t = x, A_t = a], \quad V_T(x) = g(x, \mu_T)$$

- Equilibrium policy: π satisfies: (1) is optimal against μ and (2) generates μ

Challenge: *We cannot (fully) solve one equation before the other!*

Discrete Setting: Existence and uniqueness?

Existence: generally based on fixed point formulation

Typically:

- Banach/Picard fixed point theorem
- Brouwer/Schauder fixed point theorem

Discrete Setting: Existence and uniqueness?

Uniqueness: two cases:

- Contractivity: uniqueness is a consequence of Banach fixed point theorem
- Monotonicity: V is monotone in L^2 if: $\int (V(x, m_1) - V(x, m_2))(m_1 - m_2)(x)dx \geq 0$
  - Typical setting: $b(x, a, \mu) = b(x, a), \quad f(x, a, \mu) = \tilde{f}(x, a) + V(x, \mu)$
  - Example: crowd motion (control = velocity) with cost = movement + crowd aversion

Source: Lasry, J.M. and Lions, P.L., 2007. Mean field games. Japanese journal of mathematics, 2(1), pp.229-260.

Discrete Setting: Example of Existence Proof

Sketch of existence proof: $\Phi : \mu \xrightarrow{\mathrm{BR}} \tilde{\pi} \xrightarrow{\mathrm{MF}} \tilde{\mu}$

A simple model:

- $\mathcal{X} = \{-1, 0, 1\}, \mathcal{A} = \{-2, -1, 0, 1, 2\}$
- $X_{t+1} = X_t + A_t$ with walls at $x = -2, 2$
- $f(x, a, \mu) = g(x, \mu) = |x - \bar{\mu}|, \bar{\mu}$ mean of μ
- $\mu_0 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

Discrete Setting: Example of Existence Proof

Step 1: Convexity and compactness

Step 2: Continuity of Φ

Step 2.a: Continuity of MF

Step 2.b: Continuity of BR

2.2.1 Deterministic Viewpoint: Continuous Setting

Continuous Setting: Value Function

Definition: Value function of a representative player given a mean field flow

Value of a state = sum of future costs, when starting from this state

Dynamic Programming Principle?

Continuous Setting: HJB Equation

Hamiltonian: $H(x,m,p) = \max_a -L(x,a,m,p), \quad L(x,a,m,p) = f(x,a,m) + b(x,a,m) \cdot p$

Hamilton-Jacobi-Bellman equation:

$$-\partial_t u(t,x) - \frac{\sigma^2}{2}\Delta u(t,x) + H(x,m(t),\nabla u(t,x)) = 0, \qquad u(T,x) = g(x,m(T))$$

Recovering the optimal control:

Continuous Setting: Forward-Backward System

Coupled system:

- Forward equation for the mean field:

$$\partial_t m(t,x) - \frac{\sigma^2}{2}\Delta m(t,x) - \operatorname{div}(m(t,x)H_p(x,m(t),\nabla u(t,x))) = 0, \quad m(0,x) = m_0(x)$$

- Backward equation for the value function:

$$-\partial_t u(t,x) - \frac{\sigma^2}{2}\Delta u(t,x) + H(x,m(t),\nabla u(t,x)) = 0, \qquad u(T,x) = g(x,m(T))$$

Challenge: *We cannot (fully) solve one equation before the other!*

Existence and Uniqueness of MFNE

- Existence: generally obtained by applying a fixed point theorem, such as:
  - Banach fixed point theorem: typically applicable under “smallness” conditions (small time or small Lipschitz constants); gives uniqueness too
  - Schauder fixed point theorem: applicable more generally; does not yield uniqueness
  - Compactness can be challenging
- Uniqueness:
  - Contractivity (application of Banach fixed point theorem; “smallness” assumptions)
  - Monotonicity condition (Lasry & Lions; “structural” assumption)

Exercises

Ex. 1: For the following drift and running cost function, write the KFP equation, the Hamiltonian and the HJB equation:

$$\begin{aligned}
&\text{LQ}: && b(x,a,m) = Ax + Ba + \bar{A}\bar{m}^2, \quad f(x,a,m) = Qx^2 + Ra^2 + \bar{Q}\bar{m}^2, \quad \bar{m} = \int \xi m(\xi) d\xi \\
&\text{Congestion} && b(x,a,m) = a, \quad f(x,a,m) = m(x)^\gamma |a|^2 \\
&\text{Aversion}: && b(x,a,m) = a, \quad f(x,a,m) = |a|^2 + m(x)
\end{aligned}$$

Ex. 2: Derive optimality conditions for the social optimum problem.

Exercises

Ex. 3 [Bogachev, Krylov, Röckner, Shaposhnikov; Thm 9.8.41]:

Consider the MFG PDE system:

$$\begin{cases} -\partial_t u - \Delta u + \frac{1}{2}|\nabla u|^2 = F(x, \mu_t), & \mathbb{R}^d \times [0,T), \\ \partial_t \mu_t - \Delta \mu_t - \operatorname{div}(\mu_t \nabla u) = 0, & \mathbb{R}^d \times (0,T], \end{cases}$$

with $u(x,T) = G(x, \mu_T), \mu_0 = \nu.$

Part 1: Write the player's dynamics and the cost function.

Part 2: Show existence of a classical solution, assuming:

- ν is a probability distribution on R^d with finite second moment
- $F, G\colon \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}$ are bounded and Lipschitz

Source: Bogachev, V.I., Krylov, N.V., Röckner, M. and Shaposhnikov, S.V., 2022. Fokker–Planck–Kolmogorov Equations (Vol. 207). American Mathematical Society.

2.3 Stochastic viewpoint

Some Extra References

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