Mean Field Games: Numerical Methods and Applications in Machine Learning Part 2

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https://mlauriere.github.io/teaching/MFG-PKU-2.pdf

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RECAP

Today

How can we characterize MFG solutions?

MFG Definition

- State space: $\mathcal{S} = \mathbb{R}^d$; action space: $\mathcal{A} = \mathbb{R}^k$
- Dynamics for typical player: initial position $X_0 \sim m_0$,

$$dX_t = b(X_t, \mu_t, v_t)dt + \sigma dW_t, \qquad t \geq 0,$$

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$$J(v; \mu) = \mathbb{E}\left[\int_0^T f(X_t, \mu_t, v_t)dt + g(X_T, \mu_T)\right]$$

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- **Mean Field Nash equilibrium:** $(\hat{v}, \hat{\mu})$ s.t. for all v

$$J(\hat{v}; \hat{\mu}) \leq J(v; \hat{\mu})$$

  where

  $\hat{\mu} =$ (mean field) population distribution if everybody uses $\hat{v}$

Many Possible Extensions

Outline

Outline

Single agent control problem

- Assuming population at equilibrium, i.e., $\hat{\mu}$, optimal control problem: min. over v

$$J(v; \hat{\mu}) = \mathbb{E}\left[\int_0^T f(X_t, \hat{\mu}_t, v_t)dt + g(X_T, \hat{\mu}_T)\right]$$

  subject to:

$$dX_t = b(X_t, \hat{\mu}_t, v_t)dt + \sigma dW_t, \quad t \geq 0, \qquad X_0 \sim m_0$$

- **Value function**: $u(T, x) = g(x, \hat{\mu}_T)$,

$$u(t, x) = \inf_v \mathbb{E}\left[\int_t^T f(X_s, \hat{\mu}_s, v_s)ds + g(X_T, \hat{\mu}_T) \,|\, X_t = x\right]$$

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- Dynamic programming

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- Dynamic programming
- **Hamilton-Jacobi-Bellman (HJB)** PDE ($\nu = \frac{1}{2}\sigma^2$):

$$0 = -\frac{\partial u}{\partial t}(t, x) - \nu\Delta u(t, x) + H(x, \hat{m}(t, \cdot), \nabla u(t, x))$$

where H is the **Hamiltonian**:

$$H(x, m, p) = \max_{v \in \mathbb{R}^k}\{-L(x, m, v, p)\},$$

where L is the **Lagrangian**, defined by

$$L(x, m, v, p) = f(x, m, v) + \langle b(x, m, v), p\rangle.$$

PDE for Population Evolution

- N particles controlled by v:

$$dX_t^i = b(X_t^i, v(t, X_t^i))dt + \sigma dW_t^i, \quad t \geq 0, \qquad X_0^i \sim m_0$$

with empirical distribution

$$\mu_t^N = \frac{1}{N}\sum_{j=1}^{N} \delta_{X_t^j}$$

[1] Sznitman, A. S. (1991). Topics in propagation of chaos. In *Ecole d'été de probabilités de Saint-Flour XIX–1989* (pp. 165-251). Springer, Berlin, Heidelberg.

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- Propagation of chaos [Kac'56; Sznitman'91][1]

$$\mu_t^N \xrightarrow[N \to +\infty]{} \mu_t = \text{MF population distribution}$$

- $\mu_t = \mathcal{L}(X_t)$ where X is a typical particle:

$$dX_t = b(X_t, v(t, X_t))dt + \sigma dW_t, \quad t \geq 0, \qquad X_0 \sim m_0$$

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$$dX_t = b(X_t, v(t, X_t))dt + \sigma dW_t, \quad t \geq 0, \qquad X_0 \sim m_0$$

- μ driven by control v solves **Kolmogorov-Fokker-Planck (KFP)** equation:

$$0 = \frac{\partial \mu}{\partial t}(t,x) - \nu \Delta \mu(t,x) + \mathrm{div}\left(\mu(t,\cdot) b(\cdot, v(t,\cdot))\right)(x), \qquad \mu_0 = m_0$$

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PDE for Population Evolution for MKV Dynamics

- N *interacting* particles controlled by v:

$$dX_t^i = b(X_t^i, \mu_t^N, v(t, X_t^i))dt + \sigma dW_t^i, \quad t \geq 0, \qquad X_0^i \sim m_0$$

with empirical distribution

$$\mu_t^N = \frac{1}{N} \sum_{j=1}^{N} \delta_{X_t^j}$$

- Propagation of chaos [Kac'56; Sznitman'91][2]

$$\mu_t^N \xrightarrow[N \to +\infty]{} \mu_t = \text{MF population distribution}$$

- $\mu_t = \mathcal{L}(X_t)$ where X is a typical particle with **McKean-Vlasov (MKV)** dynamics:

$$dX_t = b(X_t, \mathcal{L}(X_t), v(t, X_t))dt + \sigma dW_t, \quad t \geq 0, \qquad X_0 \sim m_0$$

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MFG PDE system

It can be shown (see e.g., [BFY'13, §3.1][3]) that a necessary condition for $\hat{v}$ to be an equilibrium control for MFG is that:

$$\hat{v}(t,x) = \operatorname*{argmax}_{v \in \mathbb{R}^k} \Big\{ -L(x, m(t,\cdot), v, \nabla u(t,x)) \Big\},$$

where (u, m) solves the following forward-backward PDE system:

$$\begin{cases} 0 = -\dfrac{\partial u}{\partial t}(t,x) - \nu \Delta u(t,x) + H(x, m(t,\cdot), \nabla\, u(t,x)), \\ 0 = \dfrac{\partial m}{\partial t}(t,x) - \nu \Delta m(t,x) - \operatorname{div}\left(m(t,\cdot) \partial_p H(\cdot, m(t), \nabla\, u(t,\cdot)) \right)(x), \\ \quad u(T,x) = g(x, m(T,\cdot)), \qquad m(0,x) = m_0(x) \end{cases}$$

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Coupling:

- Hamilton-Jacobi-Bellman (HJB) PDE for the value function
- Kolmogorov-Fokker-Planck (KFP) PDE for the population distribution (density)

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- Hamilton-Jacobi-Bellman (HJB) PDE for the value function
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Notation: $\mathrm{v}^*(x,m,p) = \operatorname{argmax}_{v\in\mathbb{R}^k} \Big\{ -L(x,m,v,p)\Big\}$
So: $\hat{v}(t,x) = \mathrm{v}^*(x, m(t,\cdot), \nabla\, u(t,x))$

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LQ MFG

- Setting: $d = 1$,

$$
\begin{aligned}
b(x, \mu, v) &= b(x, \overline{\mu}, v) = Ax + \bar{A}\overline{\mu} + Bv \\
f(x, \mu, v) &= f(x, \overline{\mu}, v) = \frac{1}{2}\left[Qx^2 + \bar{Q}\left(x - S\overline{\mu}\right)^2 + Cv^2\right] \\
g(x, \mu) &= g(x, \overline{\mu}) = \frac{1}{2}\left[Q_T x^2 + \bar{Q}_T\left(x - S_T\overline{\mu}\right)^2\right]
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- Lagrangian:

$$
L(x, \mu, v, p) = L(x, \overline{\mu}, v, p) = f(x, \overline{\mu}, v) + b(x, \overline{\mu}, v)p
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- Hamiltonian:

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H(x, \mu, p) = H(x, \overline{\mu}, p) = \max_{v \in \mathbb{R}^k}\{-L(x, \overline{\mu}, v, p)\}
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$$H(x, \mu, p) = H(x, \overline{\mu}, p) = \max_{v \in \mathbb{R}^k}\{-L(x, \overline{\mu}, v, p)\}$$

- Optimal control:

$$\hat{v}(t, x) = \dots$$

LQ Case – ODE system

- **Mean process:** integrate KFP on $\mathcal{S}$

$$0 = \frac{\partial m}{\partial t}(t,x) - \nu \Delta m(t,x) - \operatorname{div}\left(m(t,\cdot)\partial_p H(\cdot, m(t), \nabla u(t,\cdot))\right)(x)$$

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- **Value function:** plug the following ansatz

$$u(t,x) = \frac{1}{2} p_t x^2 + r_t x + s_t$$

in the HJB equation:

$$0 = -\frac{\partial u}{\partial t}(t,x) - \nu \Delta u(t,x) + H(x, m(t,\cdot), \nabla u(t,x)).$$

Then, identify terms

(see e.g., [BFY'13, §6.2])

Stochastic Optimal Control – Bellman viewpoint

- Consider X_t following: $X_0 \sim m_0, \quad dX_t = b(X_t, \hat{\mu}_t, \hat{v}_t)dt + \sigma dW_t$
- Let $Y_t = u(t, X_t)$
- It solves the backward stochastic differential equation (BSDE):

$$\begin{cases} Y_T = g(X_T, \hat{\mu}_T), \\ dY_t = -f(X_t, \hat{\mu}_t, \hat{v}_t)dt + Z_t dW_t \end{cases}$$

[4] Carmona, R., & Delarue, F. (2018). Probabilistic Theory of Mean Field Games with Applications I: Mean Field FBSDEs, Control, and Games (Vol. 83). Springer.

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- **Optimality condition** (from **Bellman** dynamic programming principle):

$$\hat{v}_t = \mathrm{v}^*(X_t, \mathcal{L}(X_t), \sigma^{-1} Z_t)$$

where (X, Y, Z) solves the **McKean-Vlasov (MKV) FBSDE** system:

$$\begin{cases} dX_t = b(X_t, \mathcal{L}(X_t), \mathrm{v}^*(X_t, \mathcal{L}(X_t), \sigma^{-1} Z_t))dt + \sigma dW_t \\ dY_t = -f(X_t, \mathcal{L}(X_t), \mathrm{v}^*(X_t, \mathcal{L}(X_t), \sigma^{-1} Z_t))dt + Z_t dW_t \\ X_0 \sim m_0, \qquad Y_T = g(X_T, \mathcal{L}(X_T)) \end{cases}$$

(see e.g., [CD'18, Vol. I, §4.4][4])

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Stochastic Optimal Control – Pontryagin viewpoint

- Consider X_t following: $X_0 \sim m_0, \quad dX_t = b(X_t, \hat{\mu}_t, \hat{v}_t)dt + \sigma dW_t$
- Let $Y_t = \partial_x u(t, X_t)$
- It solves the backward stochastic differential equation (BSDE):

$$\begin{cases} Y_T = \partial_x g(X_T, \hat{\mu}_T), \\ dY_t = -\partial_x H(X_t, \hat{\mu}_t, \hat{v}_t)dt + Z_t dW_t \end{cases}$$

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- **Optimality condition** (from **Pontryagin** stochastic maximum principle):

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(see e.g., [BFY'13, §3.2; CD'18, Vol. I, §4.5])

McKean-Vlasov FBSDE systems

Summary: two possible MKV FBSDE systems:

$$\begin{cases} dX_t = b(X_t, \mathcal{L}(X_t), \mathrm{v}^*(X_t, \mathcal{L}(X_t), \sigma^{-1} Z_t))dt + \sigma dW_t \\ dY_t = -f(X_t, \mathcal{L}(X_t), \mathrm{v}^*(X_t, \mathcal{L}(X_t), \sigma^{-1} Z_t))dt + Z_t dW_t \\ X_0 \sim m_0, \qquad Y_T = g(X_T, \mathcal{L}(X_T)) \end{cases}$$

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⚠Same notation (X, Y, Z) but different meaning for Y (and Z)!

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Generic form of a **MKV FBSDE** system:

$$\begin{cases} dX_t = B(X_t, \mathcal{L}(X_t), Y_t, Z_t)dt + \sigma dW_t \\ dY_t = -F(X_t, \mathcal{L}(X_t), Y_t, Z_t)dt + Z_t dW_t \\ X_0 \sim m_0, \qquad Y_T = G(X_T, \mathcal{L}(X_T)) \end{cases}$$

McKean-Vlasov FBSDE systems

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Rich theory; in particular: **existence** of solution:

- Banach fixed point theorem (short time)
- Schauder's fixed point theorem (see e.g., [CD'18, Vol. I, §4.3])

LQ MFG

Outline

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1. Equilibrium conditions for MFG

2. Optimality conditions for MFC
   - PDE viewpoint
   - SDE viewpoint

3. Example: Crowd Motion with Congestion

4. Example: Systemic Risk

5. Toward Algorithms

Distribution control problem

- Mean field control problem: minimize

$$J(v) = \mathbb{E}\left[\int_0^T f(X_t, \mu_t^v, v_t)dt + g(X_T, \mu_T^v)\right]$$

subject to:

$$dX_t = b(X_t, \mu_t^v, v_t)dt + \sigma dW_t, \quad t \geq 0, \qquad X_0 \sim m_0$$

- Population distribution μ^v driven by control v:

$$0 = \frac{\partial m^v}{\partial t}(t,x) - \nu \Delta m^v(t,x) - \operatorname{div}\left(m^v(t,\cdot)b(\cdot, m^v(t), v(t,\cdot))\right)(x)$$

Distribution control problem

- Mean field control problem: minimize

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- Value function?
- Dynamic programming?

MFC PDE system

It can be shown (see e.g., [BFY'13, §3.1]) that a necessary condition for v^* to be an optimal control for MFC is that:

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where (u, m) solves the following forward-backward PDE system:

$$\begin{cases} 0 = -\dfrac{\partial u}{\partial t}(t,x) - \nu \Delta u(t,x) + H(x, m(t,\cdot), \nabla u(t,x)) \\ \qquad\qquad + \displaystyle\int_{\mathcal{S}} \dfrac{\partial H}{\partial m}(\xi, m(t,\cdot), \nabla u(t,\xi))(x) m(t,\xi) d\xi, \\ 0 = \dfrac{\partial m}{\partial t}(t,x) - \nu \Delta m(t,x) - \operatorname{div}\left(m(t,\cdot) \partial_p H(\cdot, m(t), \nabla u(t,\cdot)) \right)(x), \\ u(T,x) = g(x, m(T,\cdot)), \qquad m(0,x) = m_0(x) \end{cases}$$

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where $\partial H / \partial m$:

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- L-derivative if measure: see e.g., [CD'18, Vol. I, §5 and §6]

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Coupling:

- Hamilton-Jacobi-Bellman (HJB) PDE for u ⚠
- Kolmogorov-Fokker-Planck (KFP) PDE for the population distribution (density)

Outline

Stochastic Optimal Control – Bellman viewpoint

- Consider X_t following

$$\begin{cases} X_0 \sim m_0, \\ dX_t = b(X_t, \mu_t^*, v_t^*)dt + \sigma dW_t \end{cases}$$

- Let

$$Y_t = u(t, X_t)$$

- It solves the backward stochastic differential equation (BSDE):

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- **Optimality condition** (from **Bellman** dynamic programming principle):

$$v_t^* = \mathrm{v}^*(X_t, \mathcal{L}(X_t), \sigma^{-1} Z_t)$$

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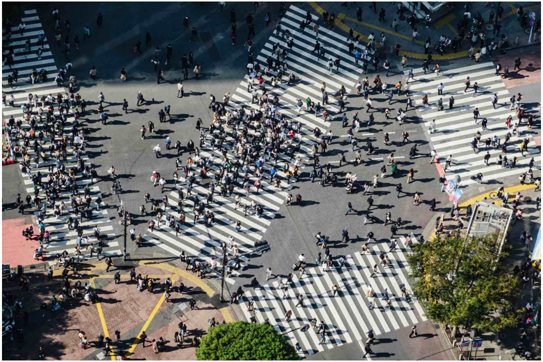
(see e.g., [BFY'13, §4.3; CD'18, Vol. I, §6.2])

Outline

Crowd models with Congestion effects



- Agents = people (pedestrians, . . .)
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- Geometry: possibly complex (building, . . .)

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Crowd models with Congestion effects

- Agents = people (pedestrians, . . .)
- Dynamics / decision, planning
- Geometry: possibly complex (building, . . .)
- **Crowd aversion:** not comfortable when density is high
- **Congestion:** difficult to move quickly when the density is high
  - slower movement $\rightarrow$ drift function
  - more effort ("soft" congestion) $\rightarrow$ cost function
  - maximum density ("hard" congestion) $\rightarrow$ density constraint

Mean Field Model

- Given population density flow $m = (m_t)_{t \in [0,T]}$, minimize over v:

$$J(v;\mu) = \mathbb{E}\left[\int_0^T f(X_t, m(t,x), v_t)dt + g(X_T, m(T,x))\right]$$

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- Players directly control their velocity: $b(x, m, v) = v$
  and pay a running cost:

$$f(x, m, v) = C_\beta (1+m)^\gamma |v|^{\beta^*} + \ell(x, m), \qquad (x, m, v) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^d$$

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- Remarks:
  - **local** dependence on m: $m(t,x)$
  - non-local variant: $(1+\rho\star m_t(x))^\gamma$, $\rho =$ regularizing kernel

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  - **local** dependence on m: $m(t,x)$
  - non-local variant: $(1 + \rho \star m_t(x))^\gamma$, $\rho =$ regularizing kernel
  - **congestion** VS **aversion** $\rightarrow$ roles of α VS ℓ
  - case $\beta = 2, \gamma = 1$: $f(x,m,v) = C_\beta(1+m)|v|^2 + \ell(x,m)$

N-Player Model

PDE system

- Hamiltonian:

$$H(x, m, p) = \max_{v \in \mathbb{R}^k}\{-L(x, m, v, p)\} = \frac{|p|^{\beta}}{(1+m)^{\alpha}} - \ell(x, m)$$

PDE system

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$$H(x, m, p) = \max_{v \in \mathbb{R}^k} \{-L(x, m, v, p)\} = \frac{|p|^\beta}{(1+m)^\alpha} - \ell(x, m)$$

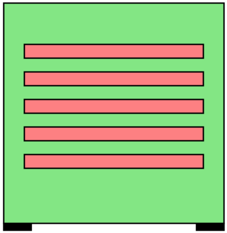
- Take $\beta = 2$ for simplicity
- MFG PDE system:

$$\begin{cases} 0 = -\dfrac{\partial u}{\partial t}(t,x) - \nu \Delta u(t,x) + \dfrac{|\nabla u(t,x)|^2}{(1+m(t,x))^\alpha} - \ell(x,m), \\ 0 = \dfrac{\partial m}{\partial t}(t,x) - \nu \Delta m(t,x) - 2\,\mathrm{div}\left(m(t,\cdot)(1+m(t,\cdot))^{-\alpha} \nabla u(t,\cdot)\right)(x), \\ \quad u(T,x) = g(x, m(T,\cdot)), \qquad m(0,x) = m_0(x) \end{cases}$$

- MFC PDE system: analogous but with an extra term

Example: Exit of a Room – Distribution

Example: evacuation of a room with obstacles and congestion [Achdou, L.'15][5]



Geometry of the room

[5] Achdou, Y., & Laurière, M. (2015). On the system of partial differential equations arising in mean field type control. *Discrete & Continuous Dynamical Systems*, 35(9), 3879.

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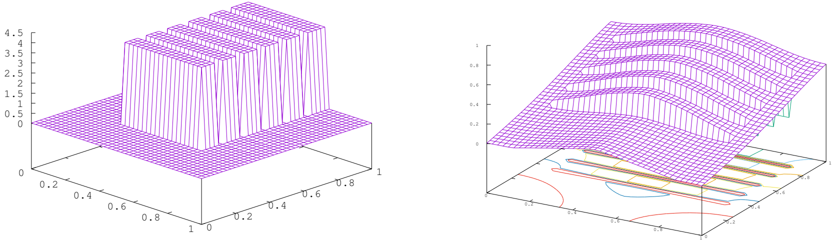

Initial density (left) and final cost (right)

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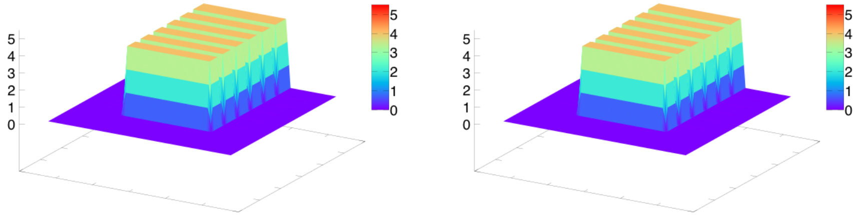

Density in MFGame (left) and MFControl (right)

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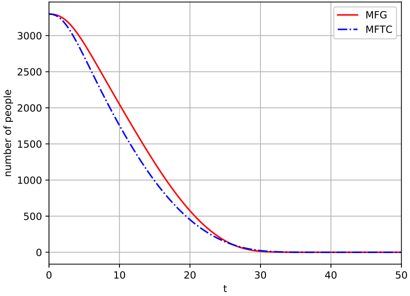

Remaining mass inside the room

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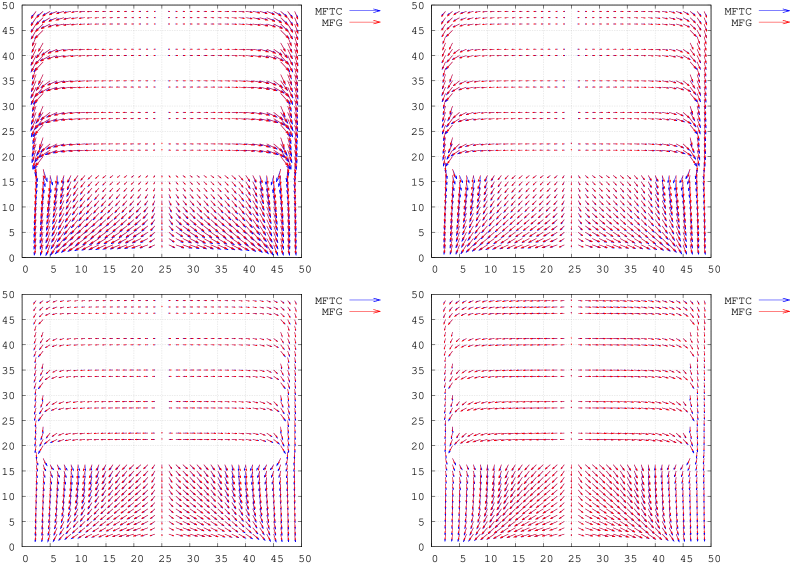
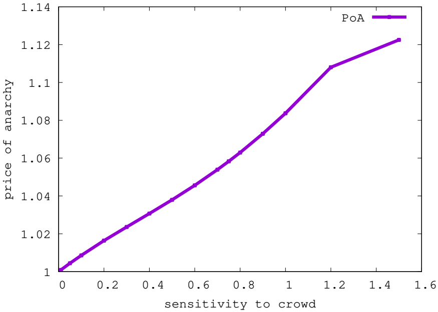
Example: Exit of a Room – Velocity



MFG & MFC velocity fields (controls) at 4 time steps

Example: Exit of a Room – Price of Anarchy



Price of Anarchy: $\frac{\text{Nash Eq.}}{\text{Social Opt.}} = \frac{\text{MFG cost}}{\text{MFC cost}}$

Outline

MFG for Systemic risk

MFG for inter-bank borrowing/lending [Carmona, Fouque, Sun'13][6]

- State X = log-monetary reserve $\in \mathbb{R}$,
- Control v = rate of borrowing (> 0) or lending (< 0) to central bank $\in \mathbb{R}$

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- Dynamics:

$$dX_t = [a(\overline{\mu}_t - X_t) + v_t]dt + \sigma dW_t$$

where $\overline{\mu} = (\overline{\mu}_t)_{t \geq 0}$ is the mean log-reserve

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- Cost:

$$J(v; \overline{\mu}) = \mathbb{E}\left[\int_0^T \left[\frac{1}{2}{v_t}^2 - q v_t(\overline{\mu}_t - X_t) + \frac{\epsilon}{2}(\overline{\mu}_t - X_t)^2\right] dt + \frac{c}{2}(\overline{\mu}_T - X_T)^2\right]$$

- Interpretation:
  - $a(\overline{\mu}_t - X_t)$ with $a > 0$: borrowing or lending between banks
  - $q v_t(\overline{\mu}_t - X_t)$ with $q > 0$: incentive to borrow if X_t is below the mean $\overline{\mu}_t$
  - q can be viewed as chosen by the regulator (q large $\Rightarrow$ low fees)
  - $(\overline{\mu}_t - X_t)^2$: penalizes departure from the average
  - running cost is convex in v provided $q^2 \leq \epsilon$

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FBSDE system

- Hamiltonian:

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so

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- MKV FBSDE from Pontryagin principle for MFG:

$$\begin{cases} dX_t = \Big[(a+q)(\mathbb{E}[X_t] - X_t) - Y_t \Big] dt + \sigma dW_t \\ dY_t = \Big[(a+q) Y_t + (\epsilon - q^2)(\mathbb{E}[X_t] - X_t) \Big] dt + Z_t dW_t \\ X_0 \sim m_0, \qquad Y_T = c(X_T - \mathbb{E}[X_t]) \end{cases}$$

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- Or Bellman principle: MKV FBSDE with $Y_t =$ value function

Outline

1. Equilibrium conditions for MFG

2. Optimality conditions for MFC

3. Example: Crowd Motion with Congestion

4. Example: Systemic Risk

5. Toward Algorithms

Reminder: Forward-Backward system of equations

Numerical Aspects: Algorithms

Reminder: Forward-Backward system of equations

Based on the LQ examples seen in Part I, we can think about using:

- Fixed point iterations
  - pure Banach-Picard iterations
  - damped version
  - Fictitious Play
- Newton's method

Numerical Aspects: Main Challenges

- Backward equation
  - HJB PDE
  - BSDE

Numerical Aspects: Main Challenges

- Backward equation
  - HJB PDE
  - BSDE
- Discretization of time and space