# Numerical Methods for Mean Field Games

## *Lecture 1*
## *Introduction to MFGs: Definitions and Equilibrium Conditions*

Mathieu LAURIÈRE

New York University Shanghai

UM6P Vanguard Center, Université Cadi AYYAD,
University Côte d'Azur, & GE2MI
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# About this course

- 6 lectures, 15 hours: $3 \times (3 + 2)$
- Objectives
  1. Introducing Mean Field Games
  2. Presenting the main ideas behind several numerical methods
  3. Providing sample codes to experiment with
- Feel free to ask questions
- The slides are available on my webpage: `https://mlauriere.github.io/#teaching`
- Based on the work of many contributors
- Feel free to reach out: `mathieu.lauriere@nyu.edu`

# Outline

# Many agent systems

Systems with many agents are ubiquitous in today's interconnected world

*Flocking*

*Crowd motion*

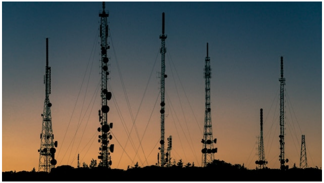
*Traffic flow*

*Collective AI*

[Image credits: Unsplash, Wikimedia Commons (Kilobots)]

# More examples

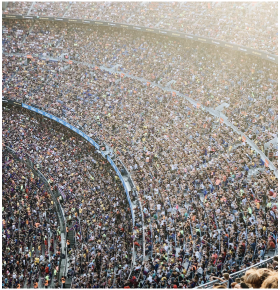
Economics & finance, energy management, telecommunications, networks, . . . .



# Groups of animals

- Flocking, schooling, herding, . . . have been extensively studied
- Predator-prey models
- Ex.: Cucker-Smale model of flocking, Lotka-Volterra system
- . . .

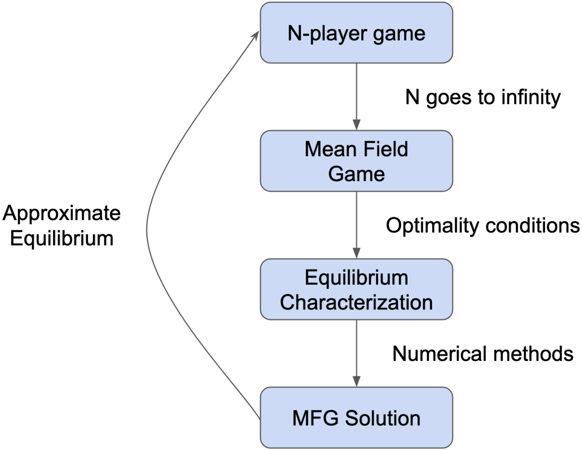
# Groups of human beings



## Some existing approaches *("What MFGs are **not**")*

- Dynamical systems:
  - describe the dynamics of one or many agents, sometimes mean field
  - but usually **no rationality** (optimization)
- Agent based models (ABM):
  - "Agent-based models are a kind of **microscale model** that simulate the simultaneous operations and interactions of multiple agents in an attempt to re-create and predict the appearance of complex phenomena. "
  - "Individual agents are typically characterized as **boundedly rational**, presumed to be acting in what they perceive as their own interests, such as reproduction, economic benefit, or social status, using heuristics or simple decision-making rules." (Wikipedia)
- Game theory
  - optimization aspects
  - notion of Nash equilibrium, social optimum, ...
  - but usually limited to a **finite (small) number of agents**
- Evolutionary game theory (EGT)
  - "application of game theory to evolving populations in biology"
  - "an evolutionary version of game theory **does not require players to act rationally** – only that they have a strategy" (Wikipedia)
- Non-atomic anonymous games
  - continuum of rational players; each player has her **own index** and own strategy
  - mostly limited to static games; difficulties for dynamic, stochastic games

# MFG paradigm in a nutshell




**Goal for this lecture:** discuss the other aspects and motivate numerical methods

Following lectures: focus on numerical methods

# Outline of this course

- Lecture 1: Introduction
  - MFG models in the static setting
  - Dynamic setting and optimality conditions
- Lectures 2 & 3: “Classical” numerical methods (Parts I & II)
- Lectures 4 & 5: Deep learning numerical methods (Parts I & II)
- Lecture 6: Reinforcement learning methods

# Some References


- Introduction to Mean Field Games:
  - Pierre-Louis Lions' lectures at Collège de France (`https://www.college-de-france.fr/`)
  - Pierre Cardaliaguet's notes (2013): `https://www.ceremade.dauphine.fr/ cardaliaguet/MFG20130420.pdf`
  - Gomes, D. A., & Saúde, J. (2014). Mean field games models—a brief survey. Dynamic Games and Applications, 4, 110-154.
  - Cardaliaguet, P., & Porretta, A. (2020). An Introduction to Mean Field Game Theory. In *Mean Field Games* (pp. 1-158). Springer, Cham.
  - Carmona, Delarue, Graves, Lacker, Laurière, Malhamé & Ramanan: Lecture notes of the 2020 AMS Short Course on Mean Field Games (American Mathematical Society), organized by François Delarue
  - Achdou, Y., Cardaliaguet, P., Delarue, F., Porretta, A., & Santambrogio, F. (2021). Mean Field Games: Cetraro, Italy 2019 (Vol. 2281). Springer Nature.
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- Monographs on Mean Field Games and Mean Field Control:
  - Bensoussan, A., Frehse, J., & Yam, P. (2013). *Mean field games and mean field type control theory* (Vol. 101). New York: Springer.
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- Surveys about numerical methods for MFGs:
  - Achdou, Y. (2013). Finite difference methods for mean field games. In *Hamilton-Jacobi equations: approximations, numerical analysis and applications* (pp. 1-47). Springer, Berlin, Heidelberg.
  - Achdou, Y., & Laurière, M. (2020). Mean Field Games and Applications: Numerical Aspects. *Mean Field Games: Cetraro, Italy 2019, 2281*, 249.
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  - Carmona, R., & Laurière, M. (2021). Deep Learning for Mean Field Games and Mean Field Control with Applications to Finance. arXiv preprint arXiv:2107.04568.
  - Hu, R., & Laurière, M. (2023). Recent developments in machine learning methods for stochastic control and games. arXiv preprint arXiv:2303.10257.
  - Laurière, M., Perrin, S., Geist, M., & Pietquin, O. (2022). Learning mean field games: A survey. arXiv preprint arXiv:2205.12944.

# Outline

# Notations

- $N$ players $[N] = \{1, \ldots, N\}$
- action space $\mathcal{A}$ (finite for simplcity)
- each player $i \in [N]$ selects an action $a^i \in \mathcal{A}$
- it induces a population profile of actions $\underline{a} = (a^1, \ldots, a^N) \in \mathcal{A}^N$
- each player pays a cost $f^i(\underline{a})$, where $f^i : \mathcal{A}^N \to \mathbb{R}$
- goal of each player: minimize her own cost $\min_{a^i} f^i(\underline{a})$

**Question:** Is there a “stable configuration” in which all the players are “satisfied”?

**Intuition:** Strategy profile such that no player is interested in deviating by herself

**Definition (Nash equilibrium (NE))**

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**Definition (Nash equilibrium (NE))**

$\underline{\hat{a}} = (\hat{a}^1, \dots, \hat{a}^N) \in \mathcal{A}^N$ is a Nash equilibrium if: for every $i \in [N]$, for every $a^i \in \mathcal{A}$

$$f^i(\underline{\hat{a}}) \leq f^i(\hat{a}^1, \dots, \hat{a}^{i-1}, a^i, \hat{a}^{i+1}, \dots, \hat{a}^N)$$

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$$f^i(\underline{\hat{a}}) \leq f^i(\hat{a}^1, \dots, \hat{a}^{i-1}, a^i, \hat{a}^{i+1}, \dots, \hat{a}^N)$$

Convenient notation: $(a^i, \underline{\hat{a}}^{-i}) = (\hat{a}^1, \dots, \hat{a}^{i-1}, a^i, \hat{a}^{i+1}, \dots, \hat{a}^N)$.

The above condition rewrites: $f^i(\hat{a}^i, \underline{\hat{a}}^{-i}) \leq f^i(a^i, \underline{\hat{a}}^{-i})$

# Example: Population distribution

## Example (Target position; no interactions)

Cost:

$$f^i(\underline{a}) = -|a^i - a_{target}|^2$$

Nash equilibrium:

## Example (Attraction to the group; interaction through the mean)

Cost:

$$f^i(\underline{a}) = |a^i - \frac{1}{N}\sum_{j=1}^{N} a^j|$$

Nash equilibrium:

### Example (Group aversion)

For simplicity, assume $\mathcal{A} = \{1, 2, \dots, d\}$ is a finite set.

Assume there are $N = k \times d$ players for some integer $k$.

Cost:

$$f^i(\underline{a}) = \sum_{j=1}^{N} 1_{\{a^j = a^i\}}$$

which is the number of players who choose the same action.

Nash equilibrium?

# Example: Population distribution

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$a^i = \lfloor i/k \rfloor + 1$ form a Nash equilibrium with uniform distribution over actions.

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Nash equilibrium?

$a^i = \lfloor i/k \rfloor + 1$ form a Nash equilibrium with uniform distribution over actions.

Remark: Player $i$ does not know the other players' actions before choosing her actions

$\Rightarrow$ Need to anticipate

# Example: Population distribution

## Exercise

Consider the following example with spatial preferences + group aversion.

Cost:

$$f^i(\underline{a}) = -|a^i - a_{target}| + \sum_{j=1}^{N} 1_{\{a^j = a^i\}}$$

Nash equilibrium?

# Example: Population distribution

**Example (Example without NE)**

Rock-Paper-Scissor game.

Number of player: $N = 2$.

Action set: $\mathcal{A} = \{R, P, S\}$.

Cost:

$$f^i(\underline{a}) = \begin{cases} 1 & \text{if } (a^i, a^{-i}) \in \{(P,R),(R,S),(S,P)\} \\ 0 & \text{if } a^i = a^{-i} \\ -1 & \text{otherwise} \end{cases}$$

No Nash equilibrium (in the sense defined previously).

# Nash Theorem

**Warning:** a pure NE does not always exist

**Nash theorem:** existence of mixed NE

THEOREM 1. *Every finite game has an equilibrium point.*

Von Neumann and Morgenstern have developed a very fruitful theory of two-person zero-sum games in their book *Theory of Games and Economic Behavior*. This book also contains a theory of $n$-person games of a type which we would call cooperative. This theory is based on an analysis of the interrelationships of the various coalitions which can be formed by the players of the game.

Our theory, in contradistinction, is based on the *absence* of coalitions in that it is assumed that each participant acts independently, without collaboration or communication with any of the others.

The notion of an *equilibrium point* is the basic ingredient in our theory. This notion yields a generalization of the concept of the solution of a two-person zero-sum game. It turns out that the set of equilibrium points of a two-person zero-sum game is simply the set of all pairs of opposing "good strategies."

In the immediately following sections we shall define equilibrium points and prove that a finite non-cooperative game always has at least one equilibrium point.

Source: [Nash, 1951]

# Mixed strategies

- $N$ players $[N] = \{1, \ldots, N\}$
- action space $\mathcal{A}$ (finite for simplicity)
- each player chooses a mixed strategy $\pi^i \in \mathcal{P}(\mathcal{A})$ = probability measures on $\mathcal{A}$
- each player (independently) picks an action according to her strategy $a^i \sim \pi^i$
- it induces population profiles of strategies $\underline{\pi}$ and actions $\underline{a}$
- each player pays a cost $f^i(\underline{a})$
- goal for each player: minimize her own *expected* cost

$$J^i(\underline{\pi}) = \mathbb{E}_{a^j \sim \pi^j, j=1,\ldots,N}[f^i(\underline{a})]$$

# Mixed strategies

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$$J^i(\underline{\pi}) = \mathbb{E}_{a^j \sim \pi^j, j=1,\dots,N}[f^i(\underline{a})]$$

- Remark: the distribution $\frac{1}{N}\sum_j \delta_{a_j}$ is *random*. But less and less as $N \to +\infty$.

# Mixed Nash equilibrium

**Definition (Mixed Nash equilibrium in N-player game)**

$\underline{\hat{\pi}} \in \mathcal{P}(\mathcal{A})^N$ is a mixed Nash equilibrium for the $N$-player game if:

$$J^i(\hat{\pi}^i, \underline{\hat{\pi}}^{-i}) \leq J^i(\pi^i, \underline{\hat{\pi}}^{-i}), \qquad \forall i, \forall \pi^i$$

**Exercise**

Revisit the examples which had a solution.

For each example, compute the mixed Nash equilibria.

**Question:** What happens to the Rock-Paper-Scissor example without a solution?

**Answer:** $(\pi^1, \pi^2) = (\hat{\pi}, \hat{\pi})$, with $\hat{\pi} = (1/3, 1/3, 1/3)$ is a Nash equilibrium.

# Large populations

**Question:** *What if $N$ is very large?*

- In many applications, the number of players is extremely large
- Intuitively,
  - each player has a **negligible impact** on the rest of the population
  - the population distribution of actions becomes **deterministic**
- This should simplify the analysis
- Can we formalize this intuition?

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- This should simplify the analysis
- Can we formalize this intuition?
- Idea: let $N$ go to infinity and study the problem we obtain in the limit
- Key assumptions: homogeneity and anonymity
- "Mean field game" paradigm [Lasry, Lions; Caines, Huang, Malhamé 2006]

# Outline

# Notations

We assume homogeneity and anonymity:

$$f^i(\underline{a}) = f\left(a^i, \frac{1}{N}\sum_{j=1}^{N}\delta_{a^j}\right), \qquad i = 1, \ldots, N$$

# Notations

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**Passing to the limit** (formally) as $N \to +\infty$, we have the following setting:

- "Infinitely many" players
- representative player chooses a (mixed) strategy $\pi \in \mathcal{P}(\mathcal{A})$
- player picks an action according to the strategy $a \sim \pi$
- the empirical distribution $\frac{1}{N}\sum_{j=1}^{N}\delta_{a^j}$ converges to a population distribution of actions: $\pi' \in \mathcal{P}(\mathcal{A})$
- representative player pays a cost $f(a, \pi')$
- goal for each player: minimize her own average cost $J(\pi, \pi') = \mathbb{E}_{a\sim\pi}[f(a, \pi')]$

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- goal for each player: minimize her own average cost $J(\pi, \pi') = \mathbb{E}_{a\sim\pi}[f(a, \pi')]$

**Key points:**

- it is enough to understand the behavior of one representative player
- each player has no influence on the rest of the population $\pi'$
- "mean-field interactions" is more general than "interactions through the mean"

# Mean field Nash equilibrium

The notion of solution in a MFG is:

**Definition (Mean field Nash equilibrium (MFNE))**

$\hat{\pi} \in \mathcal{P}(\mathcal{A})$ is a mean field Nash equilibrium strategy if:

- $\hat{\pi}$ is an optimal strategy (best response) for a representative player, given the population distribution
- and the population distribution is $\hat{\pi}$

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Fixed point formulation:

$$\hat{\pi} \in \operatorname*{argmin}_{\pi} J(\pi, \hat{\pi}) = \mathrm{BR}(\hat{\pi})$$

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Fixed point formulation:

$$\hat{\pi} \in \operatorname*{argmin}_{\pi} J(\pi, \hat{\pi}) = \mathrm{BR}(\hat{\pi})$$

This yields a first algorithm: fixed point iterations $\pi^k \mapsto \pi^{k+1}$.

Simple to implement, but fails to converge on many examples. (More details later.)

# Example: Population distribution

## Exercise

Revisit the previous finite-population examples in the MFG setting.

- Attraction to the group
- Aversion to the group

## From MFG to finite player game

**Key motivation for MFG**

An MFG equilibrium strategy provides an **approximate** (and usually **decentralized**) Nash equilibrium in the corresponding finite-population game.

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An MFG equilibrium strategy provides an **approximate** (and usually **decentralized**) Nash equilibrium in the corresponding finite-population game.

**Definition (Approximate Nash equilibrium in N-player game)**

Let $\epsilon > 0$. $\underline{\hat{\pi}} \in \mathcal{P}(\mathcal{A})^N$ is an $\epsilon$-Nash equilibrium for the $N$-player game if:

$$J^i(\hat{\pi}^i, \underline{\hat{\pi}}^{-i}) \leq J^i(\pi^i, \underline{\hat{\pi}}^{-i}) + \epsilon, \qquad \forall i, \forall \pi^i$$

**Theorem (Informal statement)**

*Consider an $N$-player game and the corresponding MFG.*

*Let $\hat{\pi}$ be a mean field NE.*

*Then, in the $N$-player game, $\hat{\pi}$ is an $\epsilon$-NE, with $\epsilon \to 0$ as $N \to +\infty$.*

Interpretation: If everyone was using a MFG equilibrium policy, then anyone could be better off by at most $\epsilon$ by unilateral deviations.

# Approximate NE: Example

**Example (Interaction through the mean)**

Consider a cost:

$$f(a,\nu) = \varphi(a,\bar{\nu}), \qquad \bar{\nu} = \mathbb{E}_{a' \sim \nu}[a']$$

with $\varphi$ Lipschitz in $\nu$ uniformly in $a$.

Assumption: Mean field Nash equilibrium property: $\hat{\pi}$ such that

$$J(\hat{\pi},\hat{\pi}) \le J(\pi,\hat{\pi}), \qquad \forall \pi$$

Goal: $\epsilon$-Nash equilibrium for N-player game:

$$J^i(\hat{\pi},\hat{\pi}^{-i}) \le J^i(\pi,\hat{\pi}^{-i}) + \epsilon, \qquad \forall \pi$$

where $\hat{\pi}^{-i} = (\hat{\pi},\dots,\hat{\pi}) \in \mathcal{P}(\mathcal{A})^{N-1}$.

# Approximate NE: Example

**Proof sketch:**

- Idea: compare $N$-player cost with MF cost:

$$J^i(\hat{\pi}, \hat{\pi}^{-i}) - J^i(\pi, \hat{\pi}^{-i})$$

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- Idea: compare $N$-player cost with MF cost:

$$J^i(\hat{\pi}, \hat{\pi}^{-i}) - J^i(\pi, \hat{\pi}^{-i})$$
$$= \underbrace{J^i(\hat{\pi}, \hat{\pi}^{-i}) - J(\hat{\pi}, \hat{\pi})}_{\textbf{(i)}} + \underbrace{J(\hat{\pi}, \hat{\pi}) - J(\pi, \hat{\pi})}_{\leq 0} + \underbrace{J(\pi, \hat{\pi}) - J^i(\pi, \hat{\pi}^{-i})}_{\textbf{(ii)}}$$

# Approximate NE: Example

**Proof sketch:**

- Idea: compare $N$-player cost with MF cost:

$$
\begin{aligned}
&J^i(\hat\pi, \hat\pi^{-i}) - J^i(\pi, \hat\pi^{-i}) \\
&= \underbrace{J^i(\hat\pi, \hat\pi^{-i}) - J(\hat\pi, \hat\pi)}_{\textbf{(i)}} + \underbrace{J(\hat\pi, \hat\pi) - J(\pi, \hat\pi)}_{\leq 0} + \underbrace{J(\pi, \hat\pi) - J^i(\pi, \hat\pi^{-i})}_{\textbf{(ii)}}
\end{aligned}
$$

- $J^i(\hat\pi, \hat\pi^{-i}) = \mathbb{E}_{\hat a^i \sim \hat\pi} \mathbb{E}_{\hat a^j \sim \hat\pi, j \neq i} \left[\varphi(\hat a^i, \bar{\hat a})\right]$, where $\bar{\hat a} := \frac{1}{N} \sum_{j=1}^N \hat a^j$

- and $J(\hat\pi, \hat\pi) = \mathbb{E}_{\hat a \sim \hat\pi} \left[\varphi(\hat a, \bar{\hat\pi})\right]$, where $\bar{\hat\pi} := \mathbb{E}_{a \sim \hat\pi}[a]$

# Approximate NE: Example

**Proof sketch:**

- Idea: compare $N$-player cost with MF cost:
$$J^i(\hat\pi, \hat\pi^{-i}) - J^i(\pi, \hat\pi^{-i})$$
$$= \underbrace{J^i(\hat\pi, \hat\pi^{-i}) - J(\hat\pi, \hat\pi)}_{(\mathbf{i})} + \underbrace{J(\hat\pi, \hat\pi) - J(\pi, \hat\pi)}_{\leq 0} + \underbrace{J(\pi, \hat\pi) - J^i(\pi, \hat\pi^{-i})}_{(\mathbf{ii})}$$

- $J^i(\hat\pi, \hat\pi^{-i}) = \mathbb{E}_{\hat a^i \sim \hat\pi} \mathbb{E}_{\hat a^j \sim \hat\pi, j \neq i} \left[\varphi(\hat a^i, \bar{\hat a})\right]$, where $\bar{\hat a} := \frac{1}{N} \sum_{j=1}^N \hat a^j$

- and $J(\hat\pi, \hat\pi) = \mathbb{E}_{\hat a \sim \hat\pi} \left[\varphi(\hat a, \bar{\hat\pi})\right]$, where $\bar{\hat\pi} := \mathbb{E}_{a \sim \hat\pi}[a]$

- We have $\varphi$ is Lipschitz and $\bar{\hat a} \approx \bar{\hat a}^{-i} := \frac{1}{N} \sum_{j \neq i} \hat a^j \approx \bar{\hat\pi}$, so:
$$\left|\varphi(\hat a^i, \bar{\hat a}) - \varphi(\hat a^i, \bar{\hat\pi})\right| \leq C|\bar{\hat a} - \bar{\hat\pi}| \leq C \underbrace{|\bar{\hat a} - \bar{\hat a}^{-i}|}_{= \frac{1}{N}|\hat a^i|} + C|\bar{\hat a}^{-i} - \bar{\hat\pi}|$$

- Hence:
$$(\mathbf{i}) \leq \frac{C}{N} \mathbb{E}_{\hat a^i \sim \hat\pi} |\hat a^i| + C \underbrace{\mathbb{E}_{\hat a^j \sim \hat\pi, j \neq i} |\bar{\hat a}^{-i} - \bar{\hat\pi}|}_{\to 0 \text{ as } N \to \infty \text{ by LLN}}$$

- Similarly for $(\mathbf{ii})$

# Outline

# Definition

- Goal: minimize the social cost = average cost for the agents in the population

- $N$-agent social cost:

$$J^{soc}(\underline{\pi}) = \frac{1}{N} \sum_{i=1}^{N} J^i(\pi, \underline{\pi}^{-i})$$

- Mean field social cost:

$$J^{soc}(\pi) = J(\pi, \pi)$$

- Optimization problem and not fixed point problem anymore

# Nash Equilibrium vs Social Optimum

- In general the two notions are different
- i.e., the socially optimal strategy is different from the Nash equilibrium policy
- Price of Anarchy [Koutsoupias & Papadimitriou, 1999]:

ABSTRACT

In a system where noncooperative agents share a common resource, we propose the price of anarchy, which is the ratio between the worst possible Nash equilibrium and the social optimum, as a measure of the effectiveness of the system. Deriving upper and lower bounds for this ratio in a model where several agents share a very simple network leads to some interesting mathematics, results, and open problems.[2]

- More on this later (see lectures 2 and 3, LQ setting and crowd motion)

# Nash Equilibrium vs Social Optimum

- In *some cases*, the two notions coincide.
- Example: Potential MFG with cost: $f(a, \nu) = \nabla F(\nu)(a)$, $\mathcal{A}$ finite for simplicity
- The average cost is: $J(\pi, \nu) = \mathbb{E}_{a \sim \pi}[f(a, \nu)] = \sum_a \pi(a) \nabla F(\nu)(a) = \pi \cdot \nabla F(\nu)$
- Assuming the potential $F$ convex, we have the equivalence:

$$
\begin{aligned}
\hat{\pi} \text{ is a NE} &\Leftrightarrow J(\pi, \hat{\pi}) - J(\hat{\pi}, \hat{\pi}) \geq 0, \quad \forall \pi \\
&\Leftrightarrow (\pi - \hat{\pi}) \cdot \nabla F(\hat{\pi}) \geq 0, \quad \forall \pi \\
&\Leftrightarrow \nabla F(\hat{\pi}) = 0 \\
&\Leftrightarrow \hat{\pi} \text{ is a minimizer of } F \\
&\Leftrightarrow \hat{\pi} \text{ is a SO}
\end{aligned}
$$

- Example: entropy: $F(\nu) = \sum_a \nu(a) \log(\nu(a))$
- More on this later (see lecture 3, optimization methods for variational MFGs)

# Exercises

**Exercise**

Design a static MFG with exactly 2 pure NE. How many mixed NE are there?

**Exercise**

Design a static MFG with exactly 2 mixed social optima.

**Exercise**

Design a static MFG with a unique mixed NE and a unique mixed SO, such that their values are different (price of anarchy different from 1).

Same question with “such that their values are the same”.

## Some references

- As far as I know, the static MFGs have not been studied extensively
- In fact a static MFG can be recast as a dynamic MFG with a single time step and and a single state
- Normal form MFGs (Coop/Betray/Punish, Rock/Paper/Scissor) [Muller et al., 2022b, Muller et al., 2022a]
- Static MFGs Section 2.1 in the survey [Laurière et al., 2022a]

# Outline

1. Motivations

2. MFG Models: Static Setting

3. MFG Models: Dynamic setting
- Finite Population Games
- Mean Field Games
- Continuous Time & Spaces

4. Optimality & Equilibrium Conditions

5. Conclusion

# Overview

| Space \ Time | Discrete | Continuous |
|---|---|---|
| Discrete | | |
| Continuous | | |

# Overview

| Space \ Time | Discrete | Continuous |
|---|---|---|
| Discrete | | |
| Continuous | | |

For simplicity of presentation, we start with the “discrete & discrete” case.

# Outline

# Notations

For simplicity of presentation, we start with discrete time & discrete (finite) space.

- Time $T < +\infty$, $t \in [T] = \{0, 1, \dots, T\}$
- State space $\mathcal{X}$ finite (for now)
- Action space $\mathcal{A}$ finite (for now)
- Player's state $X_t^i \in \mathcal{X}$
- Population's state $m_t^N = \frac{1}{N} \sum_{i=1}^N \delta_{X_t^i} \in \mathcal{P}(\mathcal{X})$
- One-step strategy (deterministic or mixed)
- "Control" or "policy": let us for now focus on the Markovian case:
  - Control (deterministic): $\alpha^i : [T] \times \mathcal{X} \to \mathcal{A}$
  - Policy (mixed): $\pi^i : [T] \times \mathcal{X} \to \mathcal{P}(\mathcal{A})$
  - Other choices (open-loop, ...)

# Notations

- We assume homogeneity and anonymity, meaning:
  - same transition rules and same cost functions
  - interactions only through aggregate quantities

- Player's dynamics:

$$X^i_{t+1} \sim P(\cdot | X^i_t, A^i_t, m^N_t)$$

where $A^i_t = \alpha^i(t, X^i_t)$ or $A^i_t \sim \pi^i(t, X^i_t)$

- For instance:

$$X^i_{t+1} = F(X^i_t, A^i_t, m_t) + \epsilon^i_{t+1}$$

where $\epsilon^i_{t+1}$ is a random perturbation

- Population's dynamics:

$$m^N_{t+1} = \frac{1}{N} \sum_{i=1}^{N} \delta_{X^i_t}$$

which is *random* if the players' states are.

# Notations and Definition

- Running cost $f : \mathcal{X} \times \mathcal{A} \times \mathcal{P}(\mathcal{X}) \to \mathbb{R}$

- Terminal cost $g : \mathcal{X} \times \mathcal{P}(\mathcal{X}) \to \mathbb{R}$

- Total cost when player $i$ uses policy $\pi^i$ and the rest of the population uses $\underline{\pi}^{-i}$:

$$J^i(\pi^i, \underline{\pi}^{-i}) = \mathbb{E}\left[\sum_{t=0}^{T-1} f(X_t^i, A_t^i, m_t^N) + g(X_T^i, m_T^N)\right]$$

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**Definition (Nash equilibrium in dynamic N-player game)**

$\underline{\hat{\pi}} \in \mathcal{P}(\mathcal{A})^N$ is a Nash equilibrium for the $N$-player game if:

$$J^i(\hat{\pi}^i, \underline{\hat{\pi}}^{-i}) \leq J^i(\pi^i, \underline{\hat{\pi}}^{-i}), \qquad \forall i, \forall \pi^i$$

The definition of Nash equilibrium is exactly the same, but the definition of $J^i$ is more involved than in the static case.

# Outline

# Notations

- Time $T < +\infty$, $t \in [T] = \{0, 1, \dots, T\}$
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  - Deterministic is a special case of mixed: $\pi_t(a|x) = \delta_{\alpha_t(x)}(a)$
  - Other choices (open-loop, ...)
- Remark: decentralized policies/controls are enough (in this setting at least) i.e., we do not need to add $m_t$ as an input to $\pi_t$ or $\alpha_t$

# Notations

- Player's dynamics given a mean-field sequence $m = (m_t)_{t=0,\dots,T}$:

$$X_{t+1} \sim P(\cdot|X_t, A_t, m_t)$$

where $A_t = \alpha(t, X_t)$ or $A_t \sim \pi(t, X_t)$

- For instance:

$$X_{t+1} = F(X_t, A_t, m_t) + \epsilon_{t+1}$$

- Population distribution dynamics associated to a policy $\pi$:

$$m^\pi_{t+1} = P^\pi_t m^\pi_t$$

where $P^\pi_t$ is the transition matrix

$$P^\pi_t(x, x') = \sum_{a \in \mathcal{A}} \pi_t(a|x) P(x'|x, a, m^\pi_t)$$

- $P^\pi_t$ depends implicitly on $m^\pi_t$ $\Rightarrow$ non linear Markov chain

# Notations

- Running cost $f : \mathcal{X} \times \mathcal{A} \times \mathcal{P}(\mathcal{X}) \to \mathbb{R}$
- Terminal cost $g : \mathcal{X} \times \mathcal{P}(\mathcal{X}) \to \mathbb{R}$
- Total cost for a representative player using policy $\pi$ and the rest of the population uses $\pi'$:

$$J(\pi, \pi') = \mathbb{E}\left[\sum_{t=0}^{T-1} f(X_t, A_t, m_t^{\pi'}) + g(X_T, m_T^{\pi'})\right]$$

# Notations

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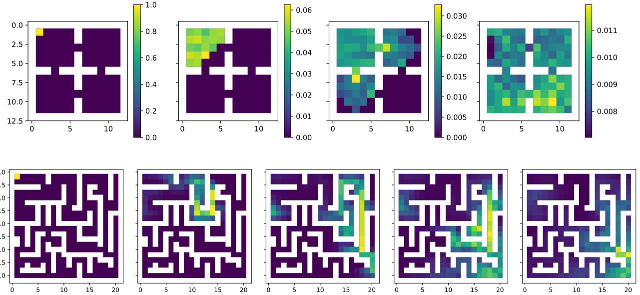
- Actually $J(\pi, \pi') = J(\pi, m^{\pi'})$, and we can define $J(\pi, m)$ more generally
- Best response (BR) to a mean field $m \in [T] \times \mathcal{P}(\mathcal{X})$:

$$\mathrm{BR}(m) = \operatorname*{argmin}_{\pi} J(\pi, m)$$

- In general, $\mathrm{BR}(m)$ is a set.

# Example: Crowd motion in a grid world

- State space: grid world
- Dynamics: move to a neighboring cell
- Running cost:
  - cost to move:
  - discomfort if crowded:
- Terminal cost:
  - spatial preference:
- Illustrations: [Geist et al., 2022, Laurière et al., 2022b]



The notion of solution in a MFG is:

**Definition (Mean field Nash equilibrium (MFNE))**

$\hat{\pi}$ is a mean field Nash equilibrium policy if:

- $\hat{\pi}$ is an optimal policy for a representative player, given the population distribution
- and the population distribution is *generated* by $\hat{\pi}$

# Mean field Nash equilibrium (dynamic)

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Fixed point formulation:

$$\hat{\pi} \in \operatorname*{argmin}_{\pi} J(\pi, m^{\hat{\pi}})$$

# Mean field Nash equilibrium (dynamic)

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- and the population distribution is *generated* by $\hat{\pi}$

Fixed point formulation:

$$\hat{\pi} \in \operatorname*{argmin}_{\pi} J(\pi, m^{\hat{\pi}})$$

This yields a first algorithm: fixed point iterations $\pi^k \mapsto m^k \mapsto \pi^{k+1}$.

Simple to implement, but fails to converge on many examples. (See lectures 2 and 6.)

# Existence

**Existence** of an equilibrium is generally based on the fixed point formulation

Typically:

- Banach/Picard fixed point theorem (requires strict contraction)
- Brouwer/Schauder fixed point theorem (requires only continuity)

# Uniqueness

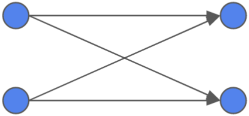
**Uniqueness** is typically ensured through two types of arguments:

- **Strict contractivity:** uniqueness is obtained as a consequence of Banach fixed point theorem
- **Monotonicity:** uniqueness is a consequence of the monotonicity of the cost
  - Typical setting: $b(x,a,m) = b(x,a)$, $f(x,a,m) = \tilde{f}(x,a) + V(x,m)$
  - $V$ is monotone in $L^2$ if: $m_1, m_2 \in L^2(\mathbb{R}^d)$,
    $$\int (V(x,m_1) - V(x,m_2))(m_1(x) - m_2(x))dx \geq 0$$
  - Example: crowd aversion

# Example of existence proof

Sketch of existence proof: look for a fixed point of $\Phi : \pi \xrightarrow{\text{MF}} \tilde{m} \xrightarrow{\text{BR}} \tilde{\pi}$

A simple model:



- $\mathcal{X} = \{0, 1\}$, $\mathcal{A} = \{-1, 0, 1\}$, $T = 1$
- $X_{t+1} = X_t + A_t$ with walls at $x = -1, 2$
- $f(x, a, m) = 0, g(x, m) = |x - \bar{m}|, \bar{m} =$ mean of $m$
- $m_0 = (\frac{1}{2}, \frac{1}{2})$

# Example of existence proof

**Step 1:** Convexity and compactness

**Step 2:** Continuity of $\Phi$

**Step 2.a:** Continuity of MF?

**Step 2.b:** Continuity of BR?

# Exercises

**Exercise**

Complete the proof of existence in the previous example (2 states, 1 time step).

**Exercise**

Define a dynamic MFG such that the following two conditions hold:

1. there is a unique NE
2. given the equilibrium mean field sequence $m$, $\mathrm{BR}(m)$ is not a singleton (there are multiple optimal policies)

# Some references

Some references on discrete time, finite state space MFGs:

- [Gomes et al., 2010]
- Link with continuous MFGs [Hadikhanloo and Silva, 2019]
- Reinforcement learning for MFGs is often studied in this setting: [Chen et al., 2022], [Guo et al., 2019], [Subramanian and Mahajan, 2019], [Elie et al., 2020b], ...
- See lecture 6 for more references on RL for MFGs; survey: [Laurière et al., 2022a]

# Outline

# Motivations

Why do we care about continuous time & space?

- More natural for many applications
- Discretizing a continuous time/space process is not trivial
- We can use calculus

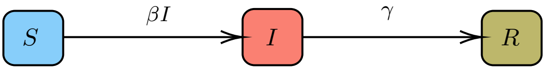
## Example: Epidemics (continuous time, discrete space)

### Example (SIR model)

3 possible states: Susceptible of infection, Infected, Recovered. Mean field dynamics:

$$\begin{cases} \dot{S}(t) = -\beta I(t) S(t) \\ \dot{I}(t) = \beta I(t) S(t) - \gamma I(t) \\ \dot{R}(t) = \gamma I(t) \end{cases}$$

Basic reproduction number: $R_0 = \beta/\gamma$.



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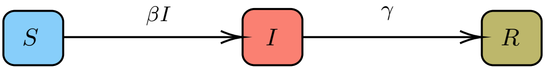
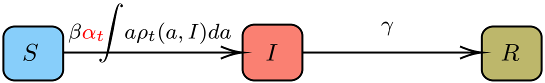



MFG for epidemics: [Laguzet and Turinici, 2015], [Hubert and Turinici, 2018], [Elie et al., 2020a], [Lee et al., 2021], [Olmez et al., 2022], [Aurell et al., 2022b], [Doncel et al., 2022], [Aurell et al., 2022a] . . .

# Example: MFG for epidemics (continuous time, discrete space)

## Example (MFG extension of SIR; borrowed from [Aurell et al., 2022b])

- Action: $\alpha_t$ = "contact factor"
- Action and state distribution $\rho(a, x)$
- Individual's transition rate from $S$ to $I$: $\beta\alpha_t \int a\rho_t(a, I)da$



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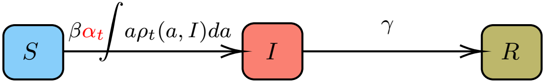


## Exercise

Write the mean field dynamics corresponding to the above model, for a fixed state-action distribution flow $\rho = (\rho_t)_{t\in[0,T]}, \rho_t \in \mathcal{P}(\mathcal{A} \times \mathcal{X})$.

# Example: MFG for epidemics (continuous time, discrete space)

**Example (MFG extension of SIR; borrowed from [Aurell et al., 2022b])**

- Action: $\alpha_t =$ "contact factor"
- Action and state distribution $\rho(a, x)$
- Individual's transition rate from $S$ to $I$: $\beta\alpha_t \int a\rho_t(a, I)da$

$$S \xrightarrow{\beta\alpha_t \int a\rho_t(a, I)da} I \xrightarrow{\gamma} R$$

**Exercise**

Write the mean field dynamics corresponding to the above model, for a fixed state-action distribution flow $\rho = (\rho_t)_{t\in[0,T]}, \rho_t \in \mathcal{P}(\mathcal{A} \times \mathcal{X})$.

Missing ingredient: to have game, we need to define a cost function.

Other MFG models in discrete space & continuous time: [Gomes et al., 2013, Kolokoltsov and Bensoussan, 2016, Bayraktar et al., 2021], . . .

# Example: Flocking (continuous time, continuous space)

**Example (Cucker-Smale model [Cucker and Smale, 2007])**

Position and velocity:

$$\begin{cases} \dot{x}^i(t) = v^i(t) \\ \dot{v}^i(t) = \displaystyle\sum_{j=1}^{N} \frac{v_j(t) - v^i(t)}{(\epsilon + |x^j - x^i|)^\beta} \end{cases}$$

# Example: Flocking (continuous time, continuous space)

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MFG for flocking & acceleration control: [Nourian et al., 2011], [Grover et al., 2018], [Achdou et al., 2020], [Bardi and Cardaliaguet, 2021], [Santambrogio and Shim, 2021], [Perrin et al., 2021], ...

# Example: Flocking (continuous time, continuous space)

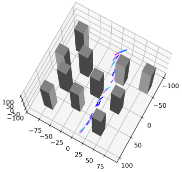
## Example (MFG model of flocking [Nourian et al., 2011])

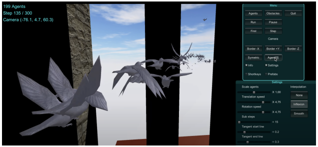
- Action: $\alpha_t =$ acceleration. Dynamics:

$$\begin{cases} \dot{x}(t) = v(t) \\ \dot{v}(t) = \alpha(t) \end{cases}$$

- Running cost: penalizes deviation from neighbors' velocity:

$$f((x,v),m) = \left| \int \frac{v' - v}{(\epsilon + |x' - x|)^\beta} m(dx', dv') \right|^2$$

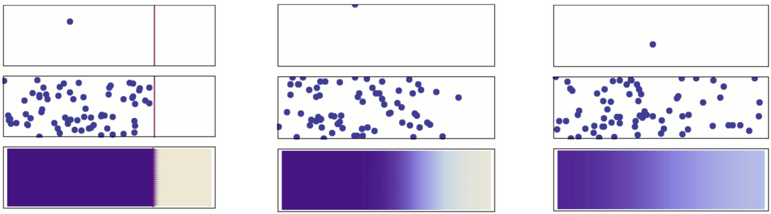





[Perrin et al., 2021], video: `https://www.youtube.com/watch?v=TdXysW_FA3k`

# Interacting particle systems

Diffusion (source: Wikipedia):



- Particle's dynamics: $dX_t = \sigma dW_t$, with $W$ a Brownian motion
- Macroscopic distribution dynamics: $\partial_t m(t,x) - \frac{\sigma^2}{2}\Delta m(t,x) = 0$
- Link with $N$-particle system: propagation of chaos [Kac, 1956, Sznitman, 1991]

# Interacting particle systems

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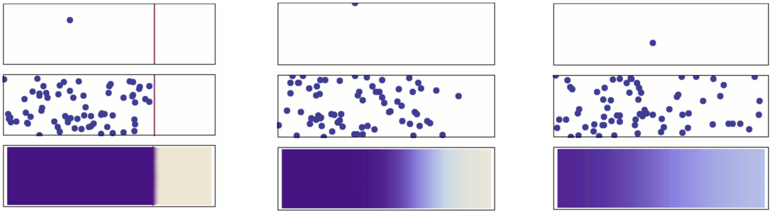

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- Macroscopic distribution dynamics: $\partial_t m(t,x) - \frac{\sigma^2}{2}\Delta m(t,x) = 0$
- Link with $N$-particle system: propagation of chaos [Kac, 1956, Sznitman, 1991]
- Note: We can also add a transport term (convection–diffusion equation):
  - $dX_t = b(t, X_t)dt + \sigma dW_t$
  - $\partial_t m(t,x) - \frac{\sigma^2}{2}\Delta m(t,x) + \mathrm{div}(b(t,x)m(t,x)) = 0$

# Continuous time, continuous space MFG

- Player $i$'s state $X_t^i \in \mathbb{R}^d$

- with dynamics:

$$dX_t^i = b(t, X_t^i, \alpha^i(t, X_t^i), m_t^N)dt + \sigma dW_t^i, \qquad X_0^i \sim m^0$$

- $W^i$ is an idiosyncratic (individual) noise, independent from other $W^j$'s

- $W$ is a noise for the representative player

- The population empirical distribution is:

$$m_t^N = \frac{1}{N} \sum_{j=1}^{N} \delta_{X_t^j}$$

- Here again, it is stochastic ... but less and less as $N \to +\infty$

- Passing rigorously to the limit in the MFG framework: see e.g. [Cardaliaguet et al., 2019], Lacker's lecture notes [Lacker, 2018, Delarue, 2021] and the references therein

# Continuous time, continuous space MFG

- Time horizon $T < +\infty$, $t \in [0, T]$
- Player's state $X_t \in \mathbb{R}^d$
- Player's control (deterministic) $\alpha_t$, typically:
  - most often focus on deterministic controls
  - closed-loop Markovian: $\alpha_t = \alpha(t, X_t)$
  - open-loop: $\alpha_t = \alpha(t, \omega)$ progressively measurable
- Player's dynamics:

$$dX_t = b(t, X_t, \alpha(t, X_t), m_t)dt + \sigma dW_t, \qquad X_0 \sim m^0$$

- Population dynamics: Kolmogorov-Fokker-Planck equation

$$\partial_t m(t,x) - \frac{\sigma^2}{2}\Delta m(t,x) + \mathrm{div}(b(t,x,\alpha(t,x))m(t,x)) = 0, \qquad m_{|t=0} = m^0$$

# Continuous time, continuous space MFG

Cost: dependence on the mean field

- non-local (typically “regularizing” operator)

$$f(t, X_t, \alpha_t, m_t)$$

- local (if the population distribution has a density, still denoted by $m$)

$$f(t, X_t, \alpha_t, m(t, X_t))$$

# Next steps

Main question for the rest of this course:

*How can we characterize and compute mean field Nash equilibria?*

## Outline

# Literature

Many papers on MFGs start like this:

INTRODUCTION

This paper is devoted to the analysis of second order mean field games systems with a local coupling. The general form of these systems is:

$$\begin{cases} (i) & \boxed{-\partial_t \phi - A_{ij}\partial_{ij}\phi + H(x, D\phi) = f(x, m(x,t))} \\ (ii) & \partial_t m - \partial_{ij}(A_{ij} m) - \operatorname{div}(m D_p H(x, D\phi)) = 0 \\ (iii) & m(0) = m_0,\ \phi(x,T) = \phi_T(x) \end{cases} \tag{1}$$

Source: Cardaliaguet, P., Graber, P.J., Porretta, A. and Tonon, D., 2015. Second order mean field games with degenerate diffusion and local coupling. Nonlinear Differential Equations and Applications NoDEA, 22(5), pp.1287-1317.

*What do these equations mean?*

In a nutshell, the probabilistic approach to the solution of the mean-field game problem results in the solution of a FBSDE of the McKean–Vlasov type

$$(3.1) \qquad \begin{cases} dX_t = b\big(t, X_t, \mathbb{P}_{X_t}, \hat{\alpha}(t, X_t, \mathbb{P}_{X_t}, Y_t)\big)dt + \sigma dW_t, \\ \boxed{dY_t = -\partial_x H\big(t, X_t, \mathbb{P}_{X_t}, Y_t, \hat{\alpha}(t, X_t, \mathbb{P}_{X_t}, Y_t)\big)dt + Z_t dW_t,} \end{cases}$$

with the initial condition $X_0 = x_0 \in \mathbb{R}^d$, and terminal condition $Y_T = \partial_x g(X_T, \mathbb{P}_{X_T})$.

Source: Carmona, R. and Delarue, F., 2013. Probabilistic analysis of mean-field games. SIAM Journal on Control and Optimization, 51(4), pp.2705-2734.

# Outline

# Value function

- Value of a state = sum of future costs, when starting from this state

- Value function of a representative player given a mean field sequence

# Bellman equation

- Bellman equation for the value function (Dynamic Programming Principle):
  - Terminal time: $V_T(x) = g(X_T, m_T)$
  - Backward induction:
    $V_t(x) = \min_a \mathbb{E}\left[f(X_t, A_t, m_t) + V_{t+1}(X_{t+1}) | X_t = x, A_t = a\right]$
- Recovering the optimal control from the value function: using $\operatorname{argmin}$

# Forward-backward system

**Coupled system:**

- Forward equation for the mean field:

$$\begin{cases} m_{t+1}(x) = \sum_{x'} m_t(x') \sum_a \pi_t(a|x) p(x|x', a, m_t), \\ m_0 \text{ given} \end{cases}$$

- Backward equation for the value function:

$$\begin{cases} V_t(x) = \min_a \mathbb{E}\left[f(X_t, A_t, m_t) + V_{t+1}(X_{t+1}) | X_t = x, A_t = a\right], \\ V_T(x) = g(X_T, m_T) \end{cases}$$

- Equilibrium policy: $\pi$ satisfies: (1) is optimal against $m$ and (2) generates $m$

Challenge: We cannot (fully) solve one equation before the other!

# Outline

# Value function

- Value of a state = sum of future costs, when starting from this state
- Value function of a representative player given a mean field flow
- Dynamic Programming Principle?

# HJB equation

- Hamiltonian:

$$H(x,m,p) = \max_a -L(x,a,m,p), \quad L(x,a,m,p) = f(x,a,m) + b(x,a,m) \cdot p$$

- Hamilton-Jacobi-Bellman equation, given the mean field flow:

$$\begin{cases} -\partial_t u(t,x) - \frac{\sigma^2}{2}\Delta u(t,x) + H(x, m(t), \nabla u(t,x))) = 0, \\ u(T,x) = g(x, m(T)) \end{cases}$$

- Recovering the optimal control: optimizer of the Hamiltonian

- Unique action minimizes $H$ under strict convexity assumptions

# HJB equation

- Hamiltonian:

$$H(x,m,p) = \max_a -L(x,a,m,p), \quad L(x,a,m,p) = f(x,a,m) + b(x,a,m) \cdot p$$

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- Recovering the optimal control: optimizer of the Hamiltonian
- Unique action minimizes $H$ under strict convexity assumptions
- **Warning:** Another convention: $H(x,m,p) = \min_a L(x,a,m,p) \Rightarrow -H$ in HJB.

# Forward-backward PDE system for MFG

The equilibrium control minimizes the Hamiltonian:

$$\hat{\alpha}(t,x) = \underset{a}{\operatorname{argmax}} -L(t,x,a,\nabla u(t,x))$$

where $(m,u)$ solve the forward-backward PDE system:

- Forward equation for the mean field:

$$\begin{cases} \partial_t m(t,x) - \frac{\sigma^2}{2}\Delta m(t,x) + \operatorname{div}(m(t,x)H_p(x,m(t),\nabla u(t,x))) = 0, \\ m(0,x) = m_0(x) \end{cases}$$

- Backward equation for the value function:

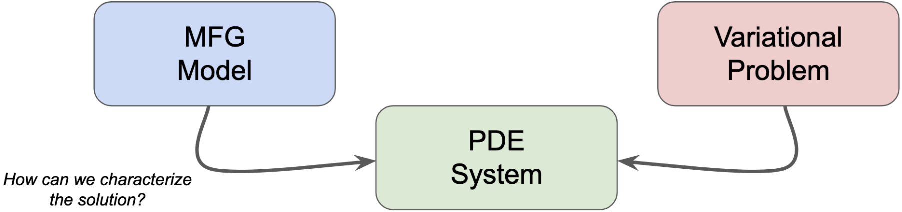
$$\begin{cases} -\partial_t u(t,x) - \frac{\sigma^2}{2}\Delta u(t,x) + H(x,m(t),\nabla u(t,x))) = 0, \\ u(T,x) = g(x,m(T)) \end{cases}$$

Challenge: We cannot (fully) solve one equation before the other!

## Existence and uniqueness of MFNE

- **Existence:** generally obtained by applying a fixed point theorem, such as:
  - Banach fixed point theorem: typically applicable under "smallness" conditions (small time or small Lipschitz constants); gives uniqueness too
  - Schauder fixed point theorem: applicable more generally; does not yield uniqueness
  - Compactness can be challenging
- **Uniqueness:**
  - Contractivity (application of Banach fixed point theorem; "smallness" assumptions)
  - Monotonicity condition (Lasry & Lions; "structural" assumption)

# Remark: Variational MFGs




In some cases, the MFG PDE system can be interpreted as the optimality conditions for a variational problem

See e.g. [Lasry and Lions, 2007], [Cardaliaguet and Graber, 2015], . . .

This can also inspire numerical methods. (More on this in lecture 3.)

# Remark: MFG with common noise

- Common noise: randomness affecting the whole population
- Example: extra Brownian motion common to all the players
- Then the two PDEs become stochastic PDEs

$$
\begin{cases}
d_t u_t = \left\{-(1+\beta)\Delta u_t + H(x, Du_t) - F(x, m_t) - \sqrt{2\beta}\,\mathrm{div}(v_t)\right\}dt + v_t \cdot \sqrt{2\beta}\,dW_t \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } [0,T] \times \mathbb{T}^d, \\
d_t m_t = \left[(1+\beta)\Delta m_t + \mathrm{div}\big(m_t D_p H(m_t, Du_t)\big)\right]dt - \mathrm{div}\big(m_t \sqrt{2\beta}\,dW_t\big), \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } [0,T] \times \mathbb{T}^d, \\
u_T(x) = G(x, m_T),\ m_0 = m_{(0)}, \qquad \text{in } \mathbb{T}^d
\end{cases}
$$

Source: [Cardaliaguet et al., 2019]

# Remark: Master equation

- Common noise: randomness affecting the whole population
- Example: extra Brownian motion common to all the players
- Convergence analysis (as $N \to \infty$) based on the Master equation

$$\begin{cases} -\partial_t U - (1+\beta)\Delta_x U + H(x, D_x U) \\ \qquad -(1+\beta)\displaystyle\int_{\mathbb{R}^d} \mathrm{div}_y \left[D_m U\right] \, dm(y) + \int_{\mathbb{R}^d} D_m U \cdot D_p H(y, D_x U) \, dm(y) \\ \qquad -2\beta \displaystyle\int_{\mathbb{R}^d} \mathrm{div}_x \left[D_m U\right] dm(y) - \beta \int_{\mathbb{R}^{2d}} \mathrm{Tr}\left[D^2_{mm} U\right] \, dm \otimes dm = F(x, m) \\ \qquad\qquad \text{in } [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \\ U(T, x, m) = G(x, m) \qquad \text{in } \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \end{cases}$$

Source: [Cardaliaguet et al., 2019]

# Exercises

### Exercise

For the following drift and running cost functions ($d = 1$ to simplicity), write the KFP equation, the Hamiltonian and the HJB equation:

- Linear-quadratic (LQ):

  $$b(x,a,m) = Ax + Ba + \bar{A}\bar{m}^2, f(x,a,m) = Qx^2 + Ra^2 + \bar{Q}\bar{m}^2, g(x,m) = Q_T x^2 + \bar{Q}_T \bar{m}^2$$

  with $\bar{m} = \int \xi m(\xi) d\xi$
- Congestion: $b(x,a,m) = a$, $f(x,a,m) = m(x)|a|^2$
- Aversion: $b(x,a,m) = a$, $f(x,a,m) = |a|^2 + m(x)$

### Exercise

Derive optimality conditions for the social optimum problem.

## Exercise [Bogachev, Krylov, Röckner, Shaposhnikov; Thm 9.8.41]

Consider the MFG PDE system:

$$\begin{cases} -\partial_t u - \Delta u + \frac{1}{2}|\nabla u|^2 = F(x, m_t), & \mathbb{R}^d \times [0,T) \\ \partial_t m - \Delta m - \operatorname{div}(\nabla u m) = 0, & \mathbb{R}^d \times (0,T] \end{cases}$$

with $u(T,x) = G(x, m(T))$ and $m_0 = \nu$.

**Part 1:** Write the player's dynamics and the cost function.

**Part 2:** Show existence of a classical solution, assuming:

- $\nu$ is a probability distribution on $\mathbb{R}^d$ with finite second moment
- $F, G : \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}$ are bounded and Lipschitz

Source: [Bogachev et al., 2022]

# Outline

# From Bellman equation

- Value function of a representative player given a mean field sequence

- Hamilton-Jacobi-Bellman equation, given equilibrium mean field flow $\hat{m}_t$:

$$-\partial_t u(t,x) - \frac{\sigma^2}{2}\Delta u(t,x) + H(x, \hat{m}(t), \nabla u(t,x))) = 0, \qquad u(T,x) = g(x, m(T))$$

# From Bellman equation

- Value function of a representative player given a mean field sequence
- Hamilton-Jacobi-Bellman equation, given equilibrium mean field flow $\hat{m}_t$:

$$-\partial_t u(t,x) - \frac{\sigma^2}{2}\Delta u(t,x) + H(x, \hat{m}(t), \nabla u(t,x))) = 0, \qquad u(T,x) = g(x, m(T))$$

- Actually in practice, we do not really need to know $u$ *everywhere*
- Motivation for probabilistic numerical methods (see lectures 3, 4, 5)
- Two approaches, based respectively on Bellman & Pontryagin principles

# From Bellman equation

- We want to know $u$ and the control along the path of $X$
- Introduce $Y_t = u(t, X_t)$ where $u$ is the value function, solution to HJB
- Dynamics of $Y$:

$$
\begin{aligned}
dY_t &= \frac{d}{dt} u(t, X_t) \\
&= \left[\partial_t u(t, X_t) + \frac{\sigma^2}{2} \Delta u(t, X_t)\right] dt + \nabla u(t, X_t) dX_t && \text{(by Itô's lemma)} \\
&= -f(X_t, A_t^*, \hat{m}_t) dt + Z_t dW_t && \text{(by HJB equation)}
\end{aligned}
$$

where $A_t^*$ is the optimal action when at state $X_t$

# Pontryagin's maximum principle: ODE

- Assume $X$ has a deterministic evolution:

$$\dot{x}_t = b(x_t, a_t), \quad x_0 \text{ given}$$

- Hamiltonian:

$$H(x,p) = \max_a -L(x,a,p), \quad L(x,a,p) = f(x,a) + b(x,a) \cdot p$$

- Pontryagin's maximum principle:

$$\begin{cases} \dot{x}_t^* = b(x_t^*, a_t^*), & x_0^* \text{ given} \\ \dot{y}_t^* = -\nabla L(x_t^*, a_t^*, y_t^*), & y_T^* = \nabla g(x_T^*) \\ a_t^* = \operatorname{argmax}_a -L(x_t^*, a, y_t^*) & \end{cases}$$

- In fact, $y_t^*$ can be interpreted as $\nabla u(t, x_t^*)$

# Pontryagin's maximum principle: SDE

- If $X$ satisfies an SDE:

$$dX_t = b(X_t, A_t)dt + \sigma dW_t, \quad X_0 \sim m_0$$

- Hamiltonian:

$$H(x,p) = \max_a -L(x,a,p), \quad L(x,a,p) = f(x,a) + b(x,a)\cdot p$$

- Stochastic Pontryagin maximum principle:

$$\begin{cases} dX_t^* = b(X_t^*, A_t^*)dt + \sigma dW_t, & X_0^* \sim m_0 \\ dY_t^* = -\nabla L(X_t^*, A_t^*, Y_t^*)dt + Z_t^* dW_t, & Y_T^* = \nabla g(X_T^*) \\ A_t^* = \mathrm{argmax}_a -L(X_t^*, a, Y_t^*) \end{cases}$$

- In fact, $Y_t^*$ can be interpreted as $\nabla u(t, X_t^*)$

# Pontryagin's maximum principle: MKV SDE

- If $X$ satisfies a mean field SDE:

$$dX_t = b(X_t, A_t, \hat{m}_t)dt + \sigma dW_t, \quad X_0 \sim m_0$$

- Hamiltonian:

$$H(x,p) = \max_a -L(x,a,p), \quad L(x,a,p) = f(x,a) + b(x,a)\cdot p$$

- Stochastic Pontryagin maximum principle with mean field interactions:

$$\begin{cases} dX_t^* = b(X_t^*, A_t^*, \hat{m}_t)dt + \sigma dW_t, & X_0^* \sim m_0 \\ dY_t^* = -\nabla L(X_t^*, A_t^*, \hat{m}_t, Y_t^*)dt + Z_t^* dW_t, & Y_T^* = \nabla g(X_T^*, \hat{m}_T) \\ A_t^* = \mathrm{argmax}_a -L(X_t^*, a, \hat{m}_t, Y_t^*) \end{cases}$$

- In fact, $Y_t^*$ can be interpreted as $\nabla u(t, X_t^*)$
- For the equilibrium, we need to include the consistency condition for the MF
- $\hat{m}_t = \mathcal{L}(X_t^*)$

# MKV FBSDE systems

In both cases (from Bellman or Pontryagin's principles), we get an instance of a McKean-Vlasov forward-backward SDEs (MKV-FBSDE):

$$\begin{cases} dX_t = B(X_t, Y_t, Z_t, m_t)dt + \sigma dW_t, & X_0 \sim m_0 \\ dY_t = F(X_t, Y_t, Z_t, m_t)dt + Z_t dW_t, & Y_T = G(X_T, m_T) \\ m_t = \mathcal{L}(X_t) \end{cases}$$

- Analysis: existence, uniqueness, ...
- Extensions (common noise, ...)
- Link with Master equation
- See book [Carmona and Delarue, 2018a, Carmona and Delarue, 2018b] for (many) more details

# Exercises

## Exercise

For the following drift and running cost functions ($d = 1$ to simplicity), write the MKV FBSDE system:

- Linear-quadratic (LQ):

  $$b(x,a,m) = Ax + Ba + \bar{A}\bar{m}^2, f(x,a,m) = Qx^2 + Ra^2 + \bar{Q}\bar{m}^2, g(x,m) = Q_T x^2 + \bar{Q}_T\bar{m}^2$$

  with $\bar{m} = \int \xi m(\xi) d\xi$
- Congestion: $b(x,a,m) = a$, $f(x,a,m) = m(x)|a|^2$
- Aversion: $b(x,a,m) = a$, $f(x,a,m) = |a|^2 + m(x)$

## Exercise

Derive an FBSDE system for the social optimum problem.

# Outline

# Summary

- $N$-player games
- Mean field games
- Connection in two directions
- Several settings (static, dynamics discrete/continuous)
- Optimality conditions

# Extensions

From the modeling viewpoint, many possible extensions:

- More settings, e.g. MFG with **ergodic** cost [Cardaliaguet et al., 2012], [Feleqi, 2013], [Bardi and Priuli, 2014], [Arapostathis et al., 2017], [Anahtarci et al., 2023], . . .
- Interactions through the **action distribution** (“extended MFGs”, “MFGs of controls”, . . . ): [Gomes et al., 2014], [Gomes and Voskanyan, 2016], [Cardaliaguet and Lehalle, 2018], [Achdou and Kobeissi, 2020], [Laurière and Tangpi, 2022], [Kobeissi, 2022], . . .
- **Common noise**: in the continuous space case see [Carmona and Delarue, 2018b] and references therein; in the finite state case, see e.g. [Bertucci et al., 2019], [Bayraktar et al., 2021], . . .
- **Several populations** MFGs: [Huang et al., 2006], [Feleqi, 2013], [Cirant, 2015], [Achdou et al., 2017], [Bensoussan et al., 2018], . . .
- **Mean field type games**: [Djehiche et al., 2017], [Barreiro-Gomez and Tembine, 2021] and references therein; [Miller and Pham, 2019], [Cosso and Pham, 2019], [Carmona et al., 2019], . . .
- **Mean field control games**: [Angiuli et al., 2022b], [Angiuli et al., 2022a]

# Extensions

- **Major player**: [Carmona and Zhu, 2016], [Caines and Kizilkale, 2016], [Carmona and Wang, 2017], [Lasry and Lions, 2018], [Cardaliaguet et al., 2020], [Carmona and Dayanıklı, 2021], [Carmona et al., 2022b], . . .
- **Stackelberg** MFGs [Bensoussan et al., 2015], [Moon and Başar, 2018], [Elie et al., 2019], [Firoozi et al., 2021], [Aurell et al., 2022b], [Vasal and Berry, 2022], [Guo et al., 2022], [Dayanikli and Lauriere, 2023],. . .
- **Graphon** games [Parise and Ozdaglar, 2019], [Caines and Huang, 2019], [Caines and Huang, 2021], [Lacker and Soret, 2022], [Gao et al., 2020], [Vasal et al., 2021], [Carmona et al., 2022a], [Aurell et al., 2022c], [Aurell et al., 2022a], [Bayraktar et al., 2023], . . .
- . . .

For simplicity, in these lectures, we will mostly focus on “plain” MFGs, although many ideas can be extended.

Thank you for your attention

Questions?

Feel free to reach out: `mathieu.lauriere@nyu.edu`

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