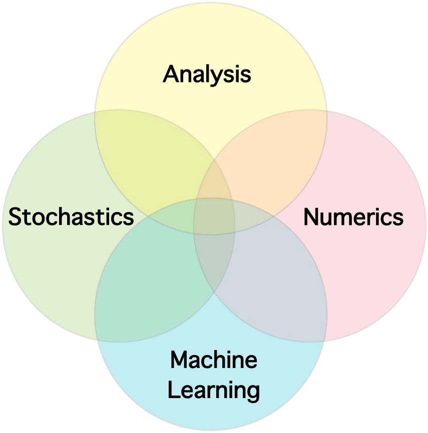
Mean Field Games: Numerical Methods and Applications in Machine Learning Part 1

Mathieu Laurière

https://mlauriere.github.io/teaching/MFG-PKU-1.pdf

August, 2021

Outline


Analysis
Stochastics
Numerics
Machine
Learning

MFGs Research Landscape

Initiated by Lasry and Lions, and Caines, Huang and Malhamé around 2006

Main research directions:

(1) Modeling: economics, crowd motion, flocking, risk management, smart grid, energy production, distributed robotics, opinion dynamics, epidemics, . . .

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- ◇ N-agent problem $\rightarrow$ mean field: convergence of equilibria / optimal control
- ◇ N-agent problem $\leftarrow$ mean field: ϵ-Nash equilibrium / ϵ-optimality

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(3) Characterization of the mean field problems solutions (**optimality conditions**):

- $\diamond$ analytical: partial differential equations (PDEs)
- $\diamond$ probabilistic: stochastic differential equations (SDEs)

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- ⋄ analytical: partial differential equations (PDEs)
- ⋄ probabilistic: stochastic differential equations (SDEs)

(4) Computation of solutions

- ⋄ crucial for applications
- ⋄ challenging (coupling between optimization & mean-field)

Optimal Control

Key ingredients:

- state
- action
- cost

Games

  

Multiple agents:

- Competition: Nash equilibrium, individual cost → "game"

Games

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Example: 2 players, 2 actions each, matrix of **costs** (to be **minimized**):

		Bob	
		b_1	b_2
Alice	a_1	$(4, 6)$	$(6, 8)$
	a_2	$(7, 5)$	

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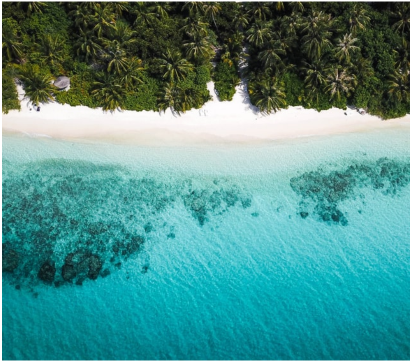
Outline

Intuition

Outline

A Static Example: Towel on the Beach



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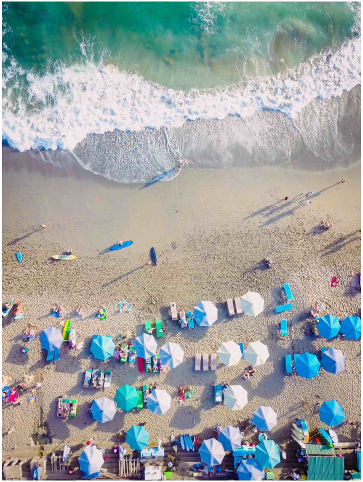

A Static Example: Towel on the Beach

- N players (people)
- State = position of the towel. Space:
  $\mathcal{S} = \{-M, -M+1, \ldots, -1, 0, 1, \ldots, M-1, M\}$

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- Each player pays a cost:
  - density of people at their location
  - distance to a point of interest
  - mean position of the population
  - ...

A Static Example: Towel on the Beach

- Infinitely many players (people)
- Simultaneously choose their location
- Population distribution μ on $\mathcal{S}$

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What if people cooperate instead of competing?

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- N players (birds)
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- Player i chooses their acceleration: $a^i \in \mathbb{R}^3$, $i = 1, \dots, N$
- Dynamics:

$$\begin{cases} x^i_{n+1} = x^i_n + v^i_n \Delta t, \\ v^i_{n+1} = v^i_n + a^i_n \Delta t + \epsilon^i_{n+1} \end{cases}$$

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- Each player pays a cost of velocity misalignment:

$$f^{\mathrm{flock},i}_{\beta}(\underline{x}, \underline{v}) = \left\| \frac{1}{N} \sum_{j=1}^{N} \frac{\left(v^i - v^j\right)}{(1 + \|x^i - x^j\|^2)^{\beta}} \right\|^2,$$

where $\beta \geq 0$ is a parameter

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  where $\beta \geq 0$ is a parameter
- Population distribution μ^N_n on $\mathcal{S}$

$$\mu^N_n = \frac{1}{N} \sum_{j=1}^N \delta_{(x^j_n, v^j_n)}$$

- New writing for $f^{\text{flock},i}_\beta$

A Dynamic Example: Flocking

Mean Field Game version [Nourian, Caines, Malhamé; . . .]:

- Infinitely many players (birds)
- Population distribution μ on $\mathcal{S}$:

$$\mu_n^N \xrightarrow[N \to \infty]{} \mu_n$$

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$$f_\beta^{\text{flock}}(x, v, \mu) = \left\| \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{(v - v')}{(1 + \|x - x'\|^2)^\beta} \, d\mu(x', v') \right\|^2,$$

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A Dynamic Example: Flocking

Another Dynamic Example: Price Impact

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- N players (traders)
- State of player i: $(S^i, X^i, K^i) \in \mathbb{R}^3$
  - Price process:

$$dS_t = \sigma_0 dW_t^0$$

  - Inventory: action = trading speed v_t^i

$$dX_t^i = v_t^i dt + \sigma dW_t^i$$

  - Wealth:

$$dK_t^i = -\left(v_t^i S_t + |v_t^i|^2\right) dt$$

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$$J^i(v^i) = \mathbb{E}\Big[V_T^i - \int_0^T |X_t^i|^2 dt - |X_T^i|^2\Big]$$

where $V_t^i = K_t^i + X_t^i S_t =$ portfolio value

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Another Dynamic Example: Price Impact

Mean Field Game version [Carmona, Lacker; Carmona, Delarue; . . .]:

- Infinitely many players (traders)
- State of a typical player: $(S, X, K) \in \mathbb{R}^3$
  - Price process: with $\nu =$ traders' distribution of actions,

$$dS_t = \sigma_0 dW_t^0 + \gamma \int_{\mathbb{R}} a d\nu_t(a) dt$$

  - Inventory: Typical agent's inventory:

$$dX_t^v = v_t dt + \sigma dW_t$$

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Another Dynamic Example: Price Impact

- Simpler rewriting:
  - By the self-financing condition,

$$dV_t^{v} = \left[-|v_t|^2 + \gamma X_t^{v} \int_{\mathbb{R}} a d\nu_t(a) \right] dt + \sigma S_t dW_t + \sigma_0 X_t^{v} dW_t^0$$

  - Hence: maximize

$$J(v, \nu) = \mathbb{E}\left[\int_0^T \left(\gamma X_t^{v} \int_{\mathbb{R}} a d\nu_t(a) - |v_t|^2 - |X_t^{v}|^2 \right) dt + |X_T^{v}|^2 \right]$$

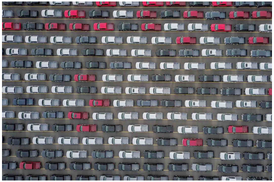
    subject to inventory dynamics:

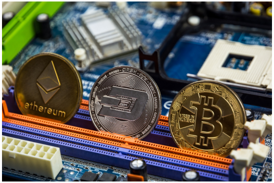
$$dX_t^{v} = v_t dt + \sigma dW_t$$

- Linear-Quadratic (LQ) structure

Another Dynamic Example: Price Impact

More Examples





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Linear-Quadratic N-Player Game

- N players
- State space: $\mathcal{S} = \mathbb{R}^d$; action space: $\mathcal{A} = \mathbb{R}^k$
- Dynamics for player i: initial position $X_0^i \sim \mathcal{N}(\bar{x}_0, \sigma_0^2)$,

$$dX_t^i = b(X_t^i, \overline{\mu}_t^N, v_t^i)dt + \sigma dW_t^i, \qquad t \geq 0,$$

with $\overline{\mu}_t^N$ = mean position at time t and

$$b(x, m, v) = Ax + \bar{A}m + Bv$$

where X_0^i and W^i are i.i.d.

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- Cost for player i:

$$J^i(v^1, \dots, v^N) = \mathbb{E}\left[\int_0^T f(X_t^i, \overline{\mu}_t^N, v_t^i)dt + g(X_T^i, \overline{\mu}_T^N)\right]$$

  with

$$f(x, m, v) = \frac{1}{2}\left[Qx^2 + \bar{Q}\,(x - Sm)^2 + Cv^2\right]$$
$$g(x, m) = \frac{1}{2}\left[Q_T x^2 + \bar{Q}_T\,(x - S_T m)^2\right]$$

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- **Nash equilibrium:** $\underline{\hat{v}} = (\hat{v}^1, \dots, \hat{v}^N)$ s.t. for all i, for all v^i

$$J^i(\hat{v}^1, \dots, \hat{v}^{i-1}, \hat{v}^i, \hat{v}^{i+1}, \dots, \hat{v}^N) \leq J^i(\hat{v}^1, \dots, \hat{v}^{i-1}, v^i, \hat{v}^{i+1}, \dots, \hat{v}^N)$$

Linear-Quadratic Mean Field Game

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- By symmetry & homogeneity, we can write $J^i(v^1, \dots, v^N) = J^{MFNE}(v^i, \overline{\mu}^N)$

Linear-Quadratic Mean Field Game

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- By symmetry & homogeneity, we can write $J^i(v^1, \dots, v^N) = J^{MFNE}(v^i, \overline{\mu}^N)$
- Reformulation: $\underline{\hat{v}} = \hat{v}^1, \dots, \hat{v}^N$ s.t. for all i, for all v^i

$$J^{MFNE}(\hat{v}^i, \overline{\mu}^N) \leq J^{MFNE}(v^i, \widetilde{\mu}^N)$$

where

$$\begin{cases} \overline{\mu}^N = \text{ mean process with } (\hat{v}^1, \dots, \hat{v}^{i-1}, \hat{v}^i, \hat{v}^{i+1}, \dots, \hat{v}^N) \\ \widetilde{\mu}^N = \text{ mean process with } (\hat{v}^1, \dots, \hat{v}^{i-1}, v^i, \hat{v}^{i+1}, \dots, \hat{v}^N) \end{cases}$$

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- **Mean Field Nash equilibrium:** $(\hat{v}, \overline{\mu})$ s.t. for all v

$$J^{MFNE}(\hat{v}, \overline{\mu}) \leq J^{MFNE}(v, \overline{\mu})$$

where

$$\overline{\mu} = \text{ mean process if everybody uses } \hat{v}$$

What does it mean to “solve” this MFG?

- population behavior $\overline{\mu} = (\overline{\mu}_t)_{t\in[0,T]}$
- individual behavior $\hat{v} = (\hat{v}_t)_{t\in[0,T]}$
- individual value function u

Value function:

$$u(t,x) = \text{ optimal cost-to-go}$$

for a player starting at x at time t while the population flow is at equilibrium

Explicit Solution

Taking $d = 1$ to alleviate notation, it can be shown:

$$\begin{cases} \overline{\mu}_t = z_t, \\ \hat{v}(t,x) = -B(p_t x + r_t)/C, \\ u(t,x) = \frac{1}{2} p_t x^2 + r_t x + s_t, \end{cases}$$

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where (z, p, r, s) solve the following system of ordinary differential equations (ODEs):

$$\begin{cases} \frac{dz}{dt} = (A + \bar{A} - B^2 C^{-1} p_t) z_t - B^2 C^{-1} r_t, & z_0 = \overline{\mu}_0, \\ -\frac{dp}{dt} = 2A p_t - B^2 C^{-1} p_t^2 + Q + \bar{Q}, & p_T = Q_T + \bar{Q}_T, \\ -\frac{dr}{dt} = (A - B^2 C^{-1} p_t) r_t + (p_t \bar{A} - \bar{Q} S) z_t, & r_T = -\bar{Q}_T S_T z_T, \\ -\frac{ds}{dt} = \nu p_t - \frac{1}{2} B^2 C^{-1} r_t^2 + r_t \bar{A} z_t + \frac{1}{2} S^2 \bar{Q} z_t^2, & s_T = \frac{1}{2} \bar{Q}_T S_T^2 z_T^2. \end{cases}$$

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$$\begin{cases} \overline{\mu}_t = z_t, \\ \hat{v}(t,x) = -B(p_t x + r_t)/C, \\ u(t,x) = \frac{1}{2} p_t x^2 + r_t x + s_t, \end{cases}$$

where (z, p, r, s) solve the following system of ordinary differential equations (ODEs):

$$\begin{cases} \frac{dz}{dt} = (A + \bar{A} - B^2 C^{-1} p_t) z_t - B^2 C^{-1} r_t, & z_0 = \overline{\mu}_0, \\ -\frac{dp}{dt} = 2A p_t - B^2 C^{-1} p_t^2 + Q + \bar{Q}, & p_T = Q_T + \bar{Q}_T, \\ -\frac{dr}{dt} = (A - B^2 C^{-1} p_t) r_t + (p_t \bar{A} - \bar{Q} S) z_t, & r_T = -\bar{Q}_T S_T z_T, \\ -\frac{ds}{dt} = \nu p_t - \frac{1}{2} B^2 C^{-1} r_t^2 + r_t \bar{A} z_t + \frac{1}{2} S^2 \bar{Q} z_t^2, & s_T = \frac{1}{2} \bar{Q}_T S_T^2 z_T^2. \end{cases}$$

Key points:

- coupling between z and r
- **forward-backward** structure

Outline

Algorithm 1: Banach-Picard Iterations

Input: Initial guess $(\tilde{z}, \tilde{r})$; number of iterations $\mathtt{K}$
Output: Approximation of $(\hat{z}, \hat{r})$

1 Initialize $z^{(0)} = \tilde{z}, r^{(0)} = \tilde{r}$
2 **for** $\mathtt{k} = 0, 1, 2, \ldots, \mathtt{K} - 1$ **do**
3 Let $r^{(\mathtt{k}+1)}$ be the solution to:

$$-\frac{dr}{dt} = (A - P_t B^2 C^{-1}) r_t + (P_t \bar{A} - \bar{Q} S) z_t^{(\mathtt{k})}, \qquad r_T = -\bar{Q}_T S_T z_T^{(\mathtt{k})}$$

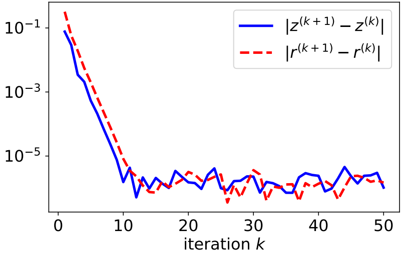
4 Let $z^{(\mathtt{k}+1)}$ be the solution to:

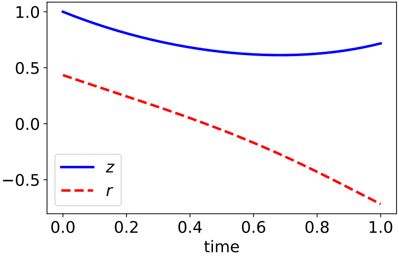
$$\frac{dz}{dt} = (A + \bar{A} - B^2 C^{-1}) z_t - B^2 C^{-1} r_t^{(\mathtt{k}+1)}, \qquad z_0 = \bar{x}_0$$

5 **return** $(z^{(\mathtt{K})}, r^{(\mathtt{K})})$

Algorithm 1: Banach-Picard Iterations – Illustration 1

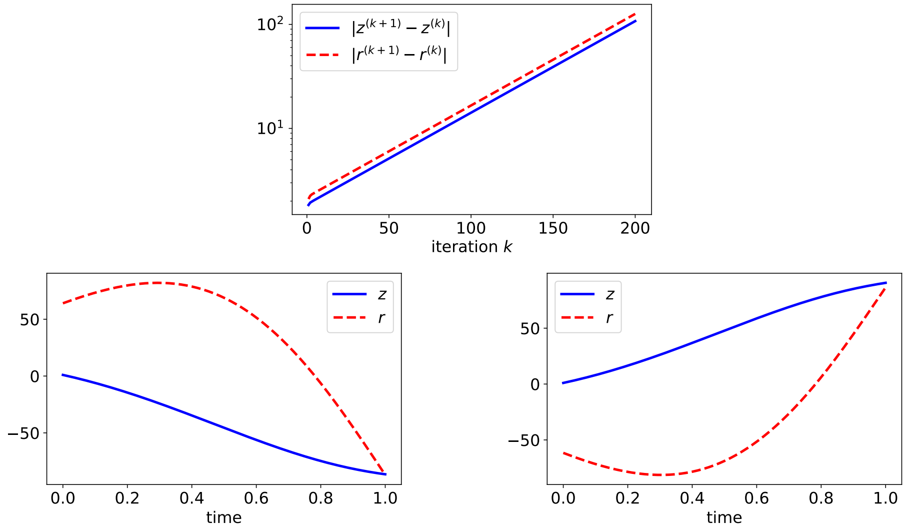
Test case 1 *(see [L., AMS notes'21])*





Algorithm 1: Banach-Picard Iterations – Illustration 2

Test case 2 *(see [L., AMS notes'21])*



Note: Banach-Picard Iterations with Damping

Input: Initial guess $(\tilde{z}, \tilde{r})$; damping $\delta \in [0, 1)$; number of iterations $\mathtt{K}$
Output: Approximation of $(\hat{z}, \hat{r})$

Initialize $z^{(0)} = \tilde{z}^{(0)} = \tilde{z}, r^{(0)} = \tilde{r}$
for $\mathtt{k} = 0, 1, 2, \ldots, \mathtt{K} - 1$ **do**
  Let $r^{(\mathtt{k}+1)}$ be the solution to:

$$-\frac{dr}{dt} = (A - P_t B^2 C^{-1}) r_t + (P_t \bar{A} - \bar{Q} S) \tilde{z}_t^{(\mathtt{k})}, \qquad r_T = -\bar{Q}_T S_T \tilde{z}_T^{(\mathtt{k})}$$

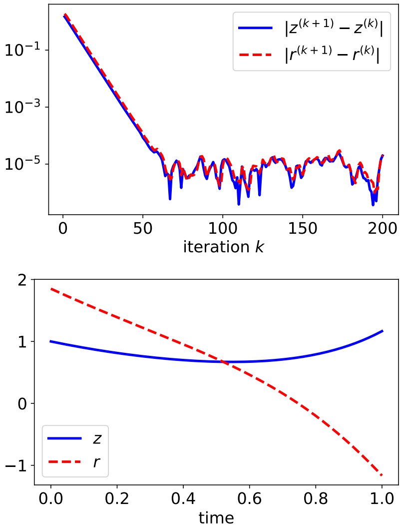
  Let $z^{(\mathtt{k}+1)}$ be the solution to:

$$\frac{dz}{dt} = (A + \bar{A} - B^2 C^{-1}) z_t - B^2 C^{-1} r_t^{(\mathtt{k}+1)}, \qquad z_0 = \bar{x}_0$$

  Let $\tilde{z}^{(\mathtt{k}+1)} = \delta \tilde{z}^{(\mathtt{k})} + (1 - \delta) z^{(\mathtt{k}+1)}$
return $(z^{(\mathtt{K})}, r^{(\mathtt{K})})$

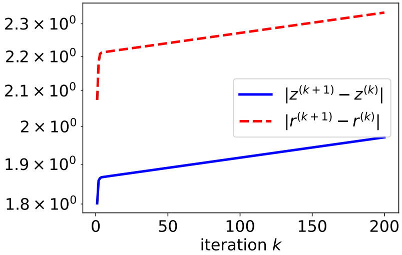
Algorithm 1’: Banach-Picard Iterations with Damping – Illustration 1

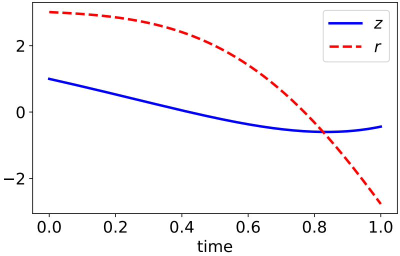
Test case 2
Damping = 0.1

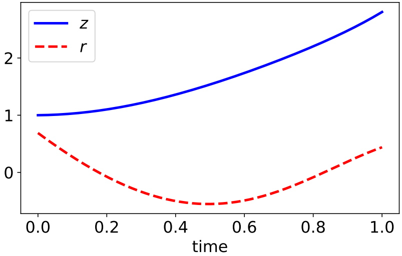


Algorithm 1’: Banach-Picard Iterations with Damping – Illustration 2

Test case 2
Damping = 0.01







Algorithm 2: Fictitious Play

Input: Initial guess $(\tilde{z}, \tilde{r})$; number of iterations $\mathtt{K}$

Output: Approximation of $(\hat{z}, \hat{r})$

Initialize $z^{(0)} = \tilde{z}^{(0)} = \tilde{z}, r^{(0)} = \tilde{r}$

for $\mathtt{k} = 0, 1, 2, \ldots, \mathtt{K} - 1$ **do**

Let $r^{(\mathtt{k}+1)}$ be the solution to:

$$-\frac{dr}{dt} = (A - P_t B^2 C^{-1}) r_t + (P_t \bar{A} - \bar{Q} S) \tilde{z}_t^{(\mathtt{k})}, \qquad r_T = -\bar{Q}_T S_T \tilde{z}_T^{(\mathtt{k})}$$

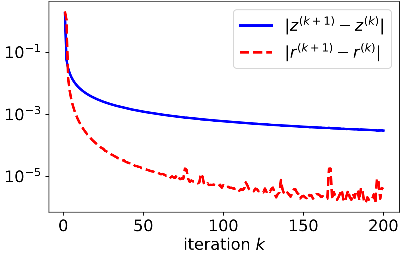
Let $z^{(\mathtt{k}+1)}$ be the solution to:

$$\frac{dz}{dt} = (A + \bar{A} - B^2 C^{-1}) z_t - B^2 C^{-1} r_t^{(\mathtt{k}+1)}, \qquad z_0 = \bar{x}_0$$

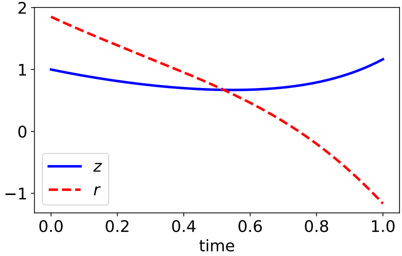
Let $\tilde{z}^{(\mathtt{k}+1)} = \frac{\mathtt{k}}{\mathtt{k}+1} \tilde{z}^{(\mathtt{k})} + \frac{1}{\mathtt{k}+1} z^{(\mathtt{k}+1)}$

return $(z^{(\mathtt{K})}, r^{(\mathtt{K})})$

Test case 2



10^{-1}
10^{-3}
10^{-5}
0
50
100
150
200
iteration k
$|z^{(k+1)} - z^{(k)}|$
$|r^{(k+1)} - r^{(k)}|$



2
1
0
−1
0.0
0.2
0.4
0.6
0.8
1.0
time
z
r

Algorithms 1, 1' & 2: Common Framework

Input: Initial guess $(\tilde{z}, \tilde{r})$; damping $\delta(\cdot)$; number of iterations $\mathtt{K}$
Output: Approximation of $(\hat{z}, \hat{r})$

Initialize $z^{(0)} = \tilde{z}^{(0)} = \tilde{z}, r^{(0)} = \tilde{r}$
for $\mathtt{k} = 0, 1, 2, \ldots, \mathtt{K} - 1$ **do**

Let $r^{(\mathtt{k}+1)}$ be the solution to:

$$-\frac{dr}{dt} = (A - P_t B^2 C^{-1}) r_t + (P_t \bar{A} - \bar{Q} S) \tilde{z}_t^{(\mathtt{k})}, \qquad r_T = -\bar{Q}_T S_T \tilde{z}_T^{(\mathtt{k})}$$

Let $z^{(\mathtt{k}+1)}$ be the solution to:

$$\frac{dz}{dt} = (A + \bar{A} - B^2 C^{-1}) z_t - B^2 C^{-1} r_t^{(\mathtt{k}+1)}, \qquad z_0 = \bar{x}_0$$

Let $\tilde{z}^{(\mathtt{k}+1)} = \delta(\mathtt{k}) \tilde{z}^{(\mathtt{k})} + (1 - \delta(\mathtt{k})) z^{(\mathtt{k}+1)}$

return $(z^{(\mathtt{K})}, r^{(\mathtt{K})})$

Algorithm 3: Shooting Method

- Intuition: *instead of solving a backward equation, choose a starting point and try to shoot for the right terminal point*
- Concretely: replace the forward-backward system

$$\begin{cases} \dfrac{dz}{dt} = (A + \bar{A} - B^2 C^{-1} p_t) z_t - B^2 C^{-1} r_t, & z_0 = \overline{\mu}_0, \\ -\dfrac{dr}{dt} = (A - B^2 C^{-1} p_t) r_t + (p_t \bar{A} - \bar{Q} S) z_t, & r_T = -\bar{Q}_T S_T z_T \end{cases}$$

by the forward-forward system

$$\begin{cases} \dfrac{d\zeta}{dt} = (A + \bar{A} - B^2 C^{-1} p_t) \zeta_t - B^2 C^{-1} \rho_t, & z_0 = \overline{\mu}_0, \\ -\dfrac{d\rho}{dt} = (A - B^2 C^{-1} p_t) \rho_t + (p_t \bar{A} - \bar{Q} S) \zeta_t, & \rho_0 = \text{ chosen} \end{cases}$$

and try to ensure: $\rho_T = -\bar{Q}_T S_T \zeta_T$

Algorithm 4: Newton Method – Intuition

- Look for x^* such that: $\mathtt{f}(x^*) = 0$
- Start from initial guess x_0
- Repeat:

$$x_{k+1} = x_k - \frac{\mathtt{f}(x_k)}{\mathtt{f}'(x_k)}$$

Aside: (Time) Discretization

- Uniform grid on $[0, T]$, step Δt
- Discrete ODE system:

$$\begin{cases} \dfrac{Z^{n+1} - Z^n}{\Delta t} = (A + \bar{A} - B^2 C^{-1} P^n) Z^{n+1} - B^2 C^{-1} R^n, \\ Z^0 = \bar{x}_0, \\ -\dfrac{R^{n+1} - R^n}{\Delta t} = (A - B^2 C^{-1} P^n) R^n + (P^n \bar{A} - \bar{Q} S) Z^{n+1}, \\ R^{N_T} = -\bar{Q}_T S_T Z^{N_T}. \end{cases}$$

Algorithm 4: Newton Method – Implementation

- Recast the problem:

$$(Z, R) \text{ solve forward-forward discrete system} \Leftrightarrow \mathcal{F}(Z, R) = 0.$$

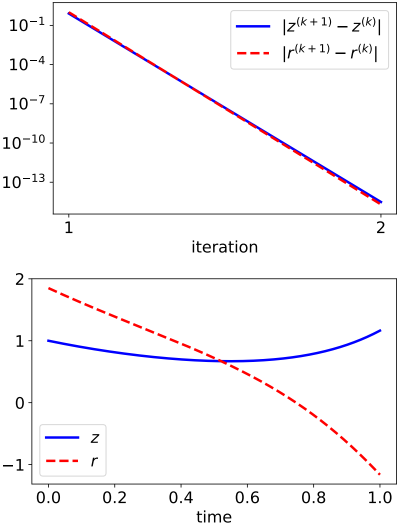
- $\mathcal{F}$ takes into account the initial and terminal conditions.
- $D\mathcal{F} =$ differential of this operator

Input: Initial guess $(\tilde{Z}, \tilde{R})$; number of iterations $\mathtt{K}$
Output: Approximation of $(\hat{z}, \hat{r})$
Initialize $(Z^{(0)}, R^{(0)}) = (\tilde{Z}, \tilde{R})$
for $\mathtt{k} = 0, 1, 2, \ldots, \mathtt{K} - 1$ **do**
  Let $(\tilde{Z}^{(\mathtt{k}+1)}, \tilde{R}^{(\mathtt{k}+1)})$ solve

$$D\mathcal{F}(Z^{(\mathtt{k})}, R^{(\mathtt{k})})(\tilde{Z}^{(\mathtt{k}+1)}, \tilde{R}^{(\mathtt{k}+1)}) = \mathcal{F}(Z^{(\mathtt{k})}, R^{(\mathtt{k})})$$

  Let $(Z^{(\mathtt{k}+1)}, R^{(\mathtt{k}+1)}) = (\tilde{Z}^{(\mathtt{k}+1)}, \tilde{R}^{(\mathtt{k}+1)}) + (Z^{(\mathtt{k})}, R^{(\mathtt{k})})$
  return $(Z^{(\mathtt{K})}, R^{(\mathtt{K})})$

Test case 2



10^{-1}
10^{-4}
10^{-7}
10^{-10}
10^{-13}
1
2
iteration
$|z^{(k+1)} - z^{(k)}|$
$|r^{(k+1)} - r^{(k)}|$
2
1
0
−1
0.0
0.2
0.4
0.6
0.8
1.0
time
z
r

Algorithm 4: Newton Method – Explanation

Reminder: Discrete ODE system:

$$\begin{cases} \dfrac{Z^{n+1}-Z^n}{\Delta t} = (A+\bar{A}-B^2C^{-1}P^n)Z^{n+1} - B^2C^{-1}R^n, \\ Z^0 = \bar{x}_0, \\ -\dfrac{R^{n+1}-R^n}{\Delta t} = (A-B^2C^{-1}P^n)R^n + (P^n\bar{A}-\bar{Q}S)Z^{n+1}, \\ R^{N_T} = -\bar{Q}_T S_T Z^{N_T}. \end{cases}$$

Algorithm 4: Newton Method – Explanation

Reminder: Discrete ODE system:

$$\begin{cases} \dfrac{Z^{n+1}-Z^n}{\Delta t} = (A+\bar{A}-B^2C^{-1}P^n)Z^{n+1} - B^2C^{-1}R^n, \\ Z^0 = \bar{x}_0, \\ -\dfrac{R^{n+1}-R^n}{\Delta t} = (A-B^2C^{-1}P^n)R^n + (P^n\bar{A}-\bar{Q}S)Z^{n+1}, \\ R^{N_T} = -\bar{Q}_T S_T Z^{N_T}. \end{cases}$$

Can be rewritten as a linear system:

$$\mathbf{M}\begin{pmatrix} Z \\ R \end{pmatrix} + \mathbf{B} = 0$$

Outline

Linear-Quadratic N-Agent Control

- N agents
- State space: $\mathcal{S} = \mathbb{R}^d$; action space: $\mathcal{A} = \mathbb{R}^k$
- Dynamics for player i: initial position $X_0^i \sim \mathcal{N}(\bar{x}_0, \sigma_0^2)$,

$$dX_t^i = b(X_t^i, \overline{\mu}_t^N, v_t^i)dt + \sigma dW_t^i, \qquad t \geq 0,$$

with $\overline{\mu}_t^N$ = mean position at time t and same $b(\cdot,\cdot,\cdot)$ as in MFG where X_0^i and W^i are i.i.d.

Linear-Quadratic N-Agent Control

- N agents
- State space: $\mathcal{S} = \mathbb{R}^d$; action space: $\mathcal{A} = \mathbb{R}^k$
- Dynamics for player i: initial position $X_0^i \sim \mathcal{N}(\bar{x}_0, \sigma_0^2)$,

$$dX_t^i = b(X_t^i, \overline{\mu}_t^N, v_t^i)dt + \sigma dW_t^i, \qquad t \geq 0,$$

  with $\overline{\mu}_t^N$ = mean position at time t and same $b(\cdot,\cdot,\cdot)$ as in MFG
  where X_0^i and W^i are i.i.d.
- Cost for player i:

$$J^i(v^1, \dots, v^N) = \mathbb{E}\left[\int_0^T f(X_t^i, \overline{\mu}_t^N, v_t^i)dt + g(X_T^i, \overline{\mu}_T^N)\right]$$

  with same $f(\cdot,\cdot,\cdot)$ and $g(\cdot,\cdot)$ as in MFG

Linear-Quadratic N-Agent Control

- N agents
- State space: $\mathcal{S} = \mathbb{R}^d$; action space: $\mathcal{A} = \mathbb{R}^k$
- Dynamics for player i: initial position $X_0^i \sim \mathcal{N}(\bar{x}_0, \sigma_0^2)$,

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  with $\overline{\mu}_t^N$ = mean position at time t and same $b(\cdot,\cdot,\cdot)$ as in MFG
  where X_0^i and W^i are i.i.d.

- Cost for player i:

$$J^i(v^1, \dots, v^N) = \mathbb{E}\left[\int_0^T f(X_t^i, \overline{\mu}_t^N, v_t^i)dt + g(X_T^i, \overline{\mu}_T^N)\right]$$

  with same $f(\cdot,\cdot,\cdot)$ and $g(\cdot,\cdot)$ as in MFG

- Social cost for the population:

$$J^{Soc}(\underline{v}) = \frac{1}{N}\sum_{i=1}^N J^i(\underline{v})$$

Linear-Quadratic N-Agent Control

- N agents
- State space: $\mathcal{S} = \mathbb{R}^d$; action space: $\mathcal{A} = \mathbb{R}^k$
- Dynamics for player i: initial position $X_0^i \sim \mathcal{N}(\bar{x}_0, \sigma_0^2)$,

$$dX_t^i = b(X_t^i, \overline{\mu}_t^N, v_t^i)dt + \sigma dW_t^i, \qquad t \geq 0,$$

  with $\overline{\mu}_t^N$ = mean position at time t and same $b(\cdot,\cdot,\cdot)$ as in MFG
  where X_0^i and W^i are i.i.d.
- Cost for player i:

$$J^i(v^1, \ldots, v^N) = \mathbb{E}\left[\int_0^T f(X_t^i, \overline{\mu}_t^N, v_t^i)dt + g(X_T^i, \overline{\mu}_T^N)\right]$$

  with same $f(\cdot,\cdot,\cdot)$ and $g(\cdot,\cdot)$ as in MFG
- Social cost for the population:

$$J^{Soc}(\underline{v}) = \frac{1}{N}\sum_{i=1}^{N} J^i(\underline{v})$$

- **Social optimum:** $\underline{v}^* = (v^{*,1}, \ldots, v^{*,N})$ s.t. for all i, all $\underline{v} = (v^1, \ldots, v^N)$

$$J^{Soc}(\underline{v}^*) \leq J^{Soc}(\underline{v})$$

Linear-Quadratic Mean Field Control

- Infinitely many agents
- Mean field social cost:

$$J^{MFSoc}(v) = \mathbb{E}\left[\int_0^T f(X_t, \overline{\mu}_t, v_t)dt + g(X_T, \overline{\mu}_T)\right]$$

where

$$dX_t = b(X_t, \overline{\mu}_t, v_t)dt + \sigma dW_t, \qquad t \geq 0,$$

and

$\overline{\mu} = \overline{\mu}^v =$ mean process if everybody uses v

Linear-Quadratic Mean Field Control

- Infinitely many agents
- Mean field social cost:

$$J^{MFSoc}(v) = \mathbb{E}\left[\int_0^T f(X_t, \overline{\mu}_t, v_t)dt + g(X_T, \overline{\mu}_T)\right]$$

where

$$dX_t = b(X_t, \overline{\mu}_t, v_t)dt + \sigma dW_t, \qquad t \geq 0,$$

and

$\overline{\mu} = \overline{\mu}^v =$ mean process if everybody uses $v = \mathbb{E}[X_t]$

Linear-Quadratic Mean Field Control

- Infinitely many agents
- Mean field social cost:

$$J^{MFSoc}(v) = \mathbb{E}\left[\int_0^T f(X_t, \overline{\mu}_t, v_t)dt + g(X_T, \overline{\mu}_T)\right]$$

where

$$dX_t = b(X_t, \overline{\mu}_t, v_t)dt + \sigma dW_t, \qquad t \geq 0,$$

and

$$\overline{\mu} = \overline{\mu}^v = \text{ mean process if everybody uses } v = \mathbb{E}[X_t]$$

- **Mean field social optimum:** v^*, s.t. for all v

$$J^{MFSoc}(v^*) \leq J^{MFSoc}(v)$$

Linear-Quadratic Mean Field Control

- Infinitely many agents
- Mean field social cost:

$$J^{MFSoc}(v) = \mathbb{E}\left[\int_0^T f(X_t, \overline{\mu}_t, v_t)dt + g(X_T, \overline{\mu}_T)\right]$$

  where

$$dX_t = b(X_t, \overline{\mu}_t, v_t)dt + \sigma dW_t, \qquad t \geq 0,$$

  and

$$\overline{\mu} = \overline{\mu}^v = \text{ mean process if everybody uses } v = \mathbb{E}[X_t]$$

- **Mean field social optimum:** v^*, s.t. for all v

$$J^{MFSoc}(v^*) \leq J^{MFSoc}(v)$$

- Key point: v changes $\Rightarrow \overline{\mu}^v$ changes

Price of Anarchy

- MFG solution: mean field Nash equilibrium: $(\hat{v}, \overline{\mu})$ s.t. for all v

$$J^{MFNE}(\hat{v}, \overline{\mu}^{\hat{v}}) \leq J^{MFNE}(v, \overline{\mu}^{\hat{v}})$$

- MFC solution: mean field social optimum: v^* s.t. for all v

$$J^{MFSoc}(v^*) \leq J^{MFSoc}(v)$$

Price of Anarchy

- MFG solution: mean field Nash equilibrium: $(\hat{v}, \overline{\mu})$ s.t. for all v

$$J^{MFNE}(\hat{v}, \overline{\mu}^{\hat{v}}) \leq J^{MFNE}(v, \overline{\mu}^{\hat{v}})$$

- MFC solution: mean field social optimum: v^* s.t. for all v

$$J^{MFSoc}(v^*) \leq J^{MFSoc}(v)$$

- For any v,

$$J^{MFSoc}(v) = J^{MFNE}(v, \overline{\mu}^{v})$$

Price of Anarchy

- MFG solution: mean field Nash equilibrium: $(\hat{v}, \overline{\mu})$ s.t. for all v

$$J^{MFNE}(\hat{v}, \overline{\mu}^{\hat{v}}) \leq J^{MFNE}(v, \overline{\mu}^{\hat{v}})$$

- MFC solution: mean field social optimum: v^* s.t. for all v

$$J^{MFSoc}(v^*) \leq J^{MFSoc}(v)$$

- For any v,

$$J^{MFSoc}(v) = J^{MFNE}(v, \overline{\mu}^{v})$$

- In general:

$$\hat{v} \neq v^*$$

$$\overline{\mu}^{\hat{v}} \neq \overline{\mu}^{v^*}$$

$$J^{MFNE}(\hat{v}, \overline{\mu}^{\hat{v}}) \neq J^{MFSoc}(v^*)$$

Price of Anarchy

- MFG solution: mean field Nash equilibrium: $(\hat{v}, \overline{\mu})$ s.t. for all v

$$J^{MFNE}(\hat{v}, \overline{\mu}^{\hat{v}}) \leq J^{MFNE}(v, \overline{\mu}^{\hat{v}})$$

- MFC solution: mean field social optimum: v^* s.t. for all v

$$J^{MFSoc}(v^*) \leq J^{MFSoc}(v)$$

- For any v,

$$J^{MFSoc}(v) = J^{MFNE}(v, \overline{\mu}^{v})$$

- In general:

$$\hat{v} \neq v^*$$

$$\overline{\mu}^{\hat{v}} \neq \overline{\mu}^{v^*}$$

$$J^{MFNE}(\hat{v}, \overline{\mu}^{\hat{v}}) \neq J^{MFSoc}(v^*)$$

- **Price of Anarcy (PoA)**:

$$PoA = \frac{J^{MFNE}(\hat{v}, \overline{\mu}^{\hat{v}})}{J^{MFSoc}(v^*)}$$

Explicit Solution

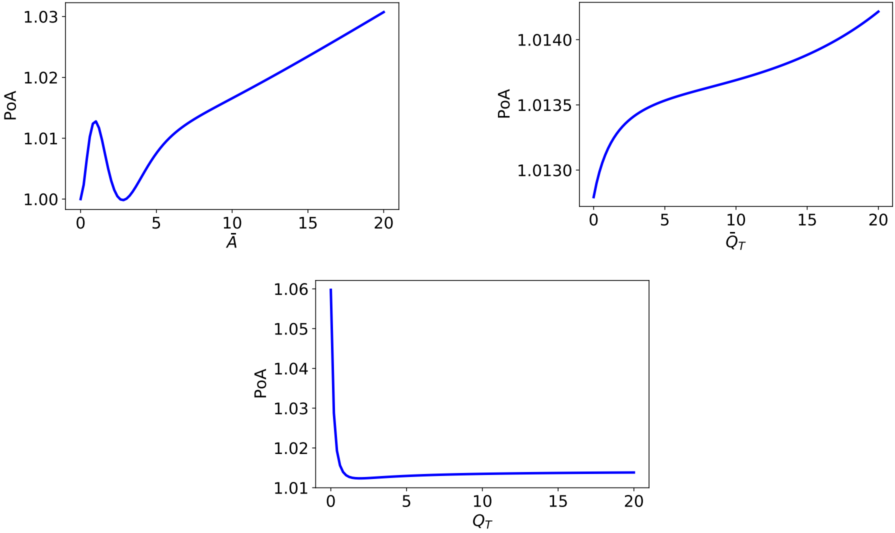
Mean field social optimum:

$$\begin{cases} \overline{\mu}_t^{v^*} = \check{z}_t, \\ v^*(t,x) = -B(\check{p}_t x + \check{r}_t)/C, \end{cases}$$

where $(\check{z}, \check{p}, \check{r}, \check{s})$ solve the following system of ODEs:

$$\begin{cases} \dfrac{d\check{z}}{dt} = (A + \bar{A} - B^2C^{-1})\check{z}_t - B^2C^{-1}\check{r}_t, & \check{z}_0 = \bar{x}_0, \\ -\dfrac{d\check{p}}{dt} = 2A\check{p}_t - B^2C^{-1}\check{p}_t^2 + Q + \bar{Q}, & \check{p}_T = Q_T + \bar{Q}_T, \\ -\dfrac{d\check{r}}{dt} = (A + \bar{A} - \check{p}_t B^2 C^{-1})\check{r}_t + (2\check{p}_t\bar{A} - 2\bar{Q}S + \bar{Q}S^2)\check{z}_t, & \check{r}_T = -\bar{Q}_T S_T \check{z}_T, \\ -\dfrac{ds}{dt} = \nu\check{p}_t - \dfrac{1}{2}B^2C^{-1}\check{r}_t^2 + \check{r}_t\bar{A}\check{z}_t + \dfrac{1}{2}S^2\bar{Q}\check{z}_t^2, & \check{s}_T = \dfrac{1}{2}\bar{Q}_T S_T^2 \check{z}_T^2. \end{cases}$$

Price of Anarchy – Illustration



Preview of Next Lectures