

Classification of low-degree ovoids

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A.D. 1308
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DI MATEMATICA E INFORMATICA

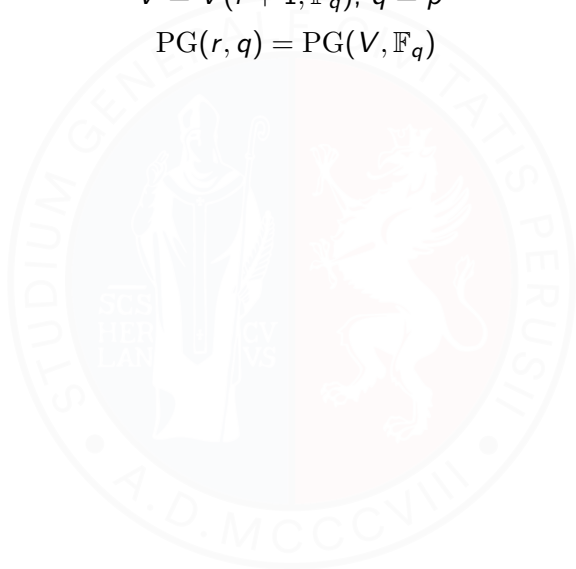
Finite Geometries 2025 Seventh Irsee Conference
31 August – 6 September 2025, Irsee, Germany

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Definition

A **quadratic form** of V is a map $\alpha : V \rightarrow \mathbb{F}_q$ such that

- $\alpha(av) = \alpha^2 f(v)$
- $\beta(u, v) = \alpha(u+v) - \alpha(u) - \alpha(v)$ is a bilinear form

for any $u, v \in V$ and $a \in \mathbb{F}_q$.

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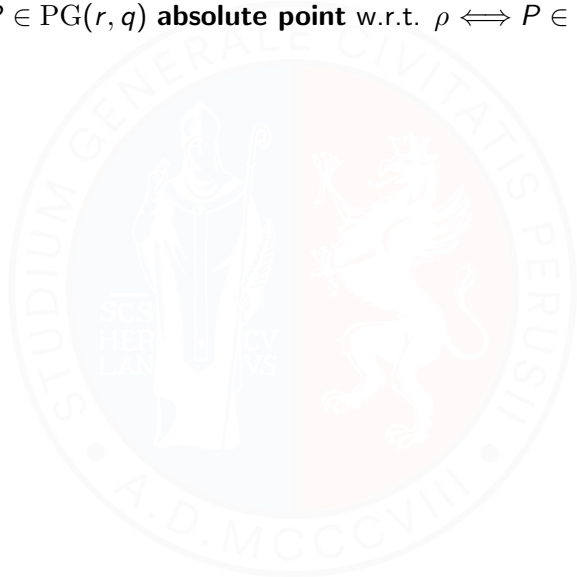
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A **polarity** ρ of $\text{PG}(r, q)$ is a collineation between $\text{PG}(r, q)$ and its dual space $\text{PG}(r, q)^*$ such that $\rho^2 = \text{id}$.

$P \in \text{PG}(r, q)$ **absolute point** w.r.t. $\rho \iff P \in P^\rho$



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An ovoid of \mathcal{P} is a set of pairwise non-collinear points on \mathcal{P}

- $Q^-(2r+1, q)$ **elliptic quadric** of $\text{PG}(2r+1, q)$

$$\alpha(x) = X_0X_{2r+1} + \dots + X_rX_{r+3} + h(X_{r+1}, X_{r+2})$$

h is a homogeneous irreducible polynomial of degree 2 over \mathbb{F}_q

- $Q^+(2r+1, q)$ **hyperbolic quadric** of $\text{PG}(2r+1, q)$

$$\alpha(x) = X_0X_{2r+1} + \dots + X_rX_{r+3} + X_{r+1}X_{r+2}$$

- $Q(2r, q)$ **parabolic quadric** of $\text{PG}(2r, q)$

$$\alpha(x) = X_0X_{2r} + \dots + X_{r-1}X_{r+1} + X_r^2$$

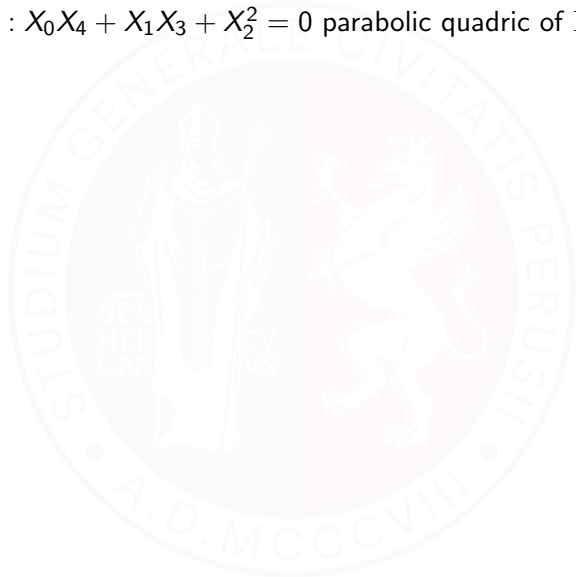
- **symplectic polar space** of $\text{PG}(2r+1, q)$

$W(2r+1, q)$ full pointset of $\text{PG}(2r+1, q)$

- **Hermitian variety** of $\text{PG}(r, q^2)$

$$H(r, q^2) : X_0^{q+1} + \dots + X_r^{q+1} = 0$$

$Q(4, q) : X_0X_4 + X_1X_3 + X_2^2 = 0$ parabolic quadric of $\text{PG}(4, q)$



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Any ovoid of $Q(4, q)$ is equivalent to one of the form:

$$\mathcal{O}(f) = \{(1, x, y, f(x, y), -y^2 - xf(x, y))\}_{x, y \in \mathbb{F}_q} \cup \{(0, 0, 0, 0, 1)\}$$

for some polynomial f s.t. $f(0, 0) = 0$

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Known ovoids of $Q(4, q)$

| Name | $f(x, y)$ | Restrictions |
|--------------------------|-----------------------------------|---|
| <i>Elliptic quadric</i> | $-nx$ | q odd, $n \notin \square_q$ |
| <i>Elliptic quadric</i> | $ax + y$ | q even, $a \neq 1, \text{Tr}_{q/2}(a) = 1$ |
| <i>Kantor</i> | $-nx^\sigma$ | $q = p^h$ odd, $h > 1$ $n \notin \square_q, \sigma \neq 1$ |
| <i>Penttila-Williams</i> | $-x^9 - y^{81}$ | $q = 3^5$ |
| <i>Thas-Payne</i> | $-nx - (n^{-1}x)^{1/9} - y^{1/3}$ | $q = 3^h, h > 2, n \notin \square_q$ |
| <i>Ree-Tits slice</i> | $-x^{2\sigma+3} - y^\sigma$ | $q = 3^{2h+1}, \sigma = 3^{h+1}$ |
| <i>Tits</i> | $x^{\sigma+1} + y^\sigma$ | $q = 2^{2h+1}, \sigma = 2^{h+1}$ |

Theorem (Bartoli, Durante - 2022)

If $q > 6.3(\deg(f) + 1)^{13/3}$ and $\mathcal{O}(f)$ is an ovoid of $Q(4, q)$, then $\mathcal{O}(f)$ is either an elliptic quadric or a Kantor ovoid.

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β symmetric form associated to the quadratic form of $Q(4, q)$

$\mathcal{O}(f)$ is an ovoid $\iff \beta(P_1, P_2) \neq 0$ for any $P_1 \neq P_2$ in $\mathcal{O}(f)$ \iff
 $(y_1 - y_2)^2 + (x_1 - x_2)(f(x_2, y_2) - f(x_1, y_1)) \neq 0$
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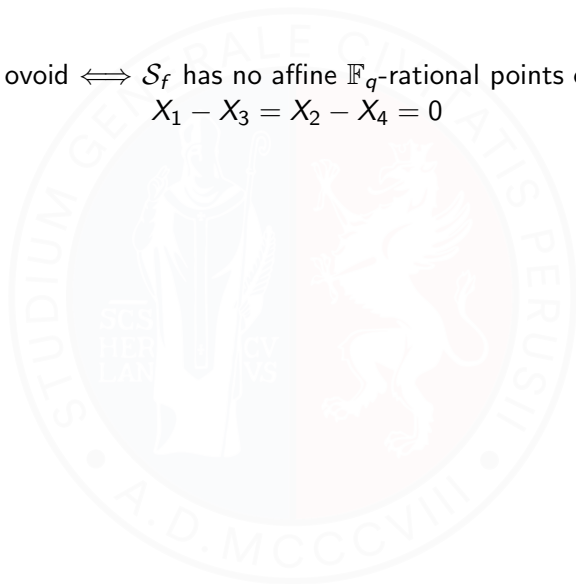
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S_f hypersurface of $PG(4, q)$ with equation

$$(X_2 - X_4)^2 X_0^{d-1} + (X_1 - X_3) \left(\tilde{f}(X_1, X_2, X_0) - \tilde{f}(X_4, X_3, X_0) \right) = 0$$

$\tilde{f}(x, y, t)$ homogenization of $f(x, y)$

$\mathcal{O}(f)$ is an ovoid $\iff \mathcal{S}_f$ has no affine \mathbb{F}_q -rational points off the plane
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Theorem (Cafure, Matera - 2006)

Let $\mathcal{V} \subset \text{AG}(n, q)$ be an absolutely irreducible \mathbb{F}_q -variety of dimension $r > 0$ and degree δ . If $q > 2(r + 1)\delta^2$, then the following estimate holds:

$$|\#(\mathcal{V} \cap \mathbb{F}_q^n) - q^r| \leq (\delta - 1)(\delta - 2)q^{r-1/2} + 5\delta^{13/3}q^{r-1}.$$

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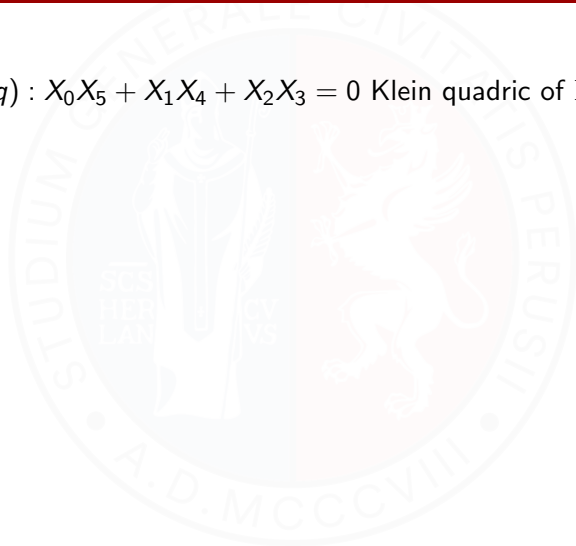
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Ovoids of $Q^+(5, q)$

$Q^+(5, q) : X_0X_5 + X_1X_4 + X_2X_3 = 0$ Klein quadric of $PG(5, q)$



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\mathcal{S}_{f_1, f_2} hypersurface of $\text{PG}(4, q)$ with equation

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We get classification results regarding ovoids associated to a flock of the
quadratic cone of $\text{PG}(3, q)$:

in this case $f_1(x, y) = y + g(x)$

Known ovoids of $Q^+(5, q)$ associated to flocks of the quadratic cone of $PG(3, q)$

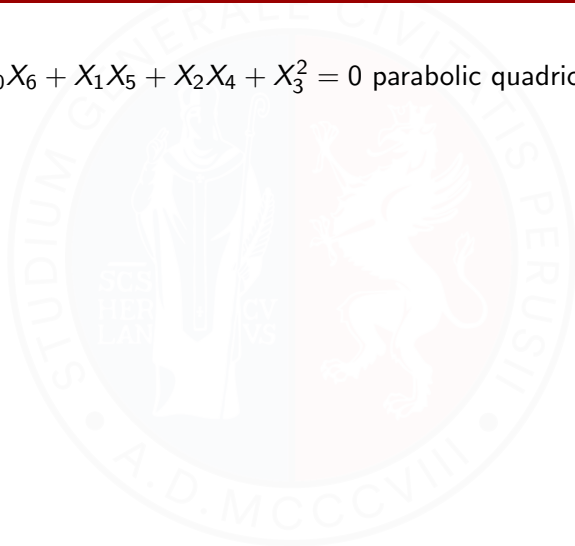
| Name | $f_1(x, y)$ | $f_2(x, y)$ | Restrictions |
|---------------------------|------------------|--|--|
| <i>Fisher-Thas-Walker</i> | $y - x^2$ | $x^3/3$ | $q \equiv -1 \pmod{3}$ |
| <i>Kantor-Payne</i> | $y - \beta x^3$ | γx^5 | $q = p^{2h+1}$ $p \equiv \pm 2 \pmod{5}$ $\beta^2 = 5\gamma$ |
| <i>Law-Penttila</i> | $y - x^4 - nx^2$ | $-n^{-1}x^9 + x^7$ $+n^2x^3 - n^3x$ | $q = 3^h, n \notin \square_q$ |
| <i>Ganley</i> | $y - dx^3$ | $-n_1x^9 - n_1n_2^2x$ | $q = 3^h, h > 2$ $n_1, n_2 \notin \square_q, d^2 = n_1n_2$ |
| <i>Kantor</i> | $y - x^2$ | $\frac{1}{3}x^3 - nx^5 - n^{-1}x$ | $q = 5^h, n \notin \square_q$ |
| <i>Penttila-Williams</i> | $y + x^{27}$ | $-x^9$ | $q = 3^5$ |

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If $q > 6.3(\max\{\deg(f_1), \deg(f_2)\} + 1)^{13/3}$ and $\mathcal{O}(f_1, f_2)$ is an ovoid of $Q^+(5, q)$, then $\mathcal{O}(f_1, f_2)$ is either a Fisher-Thas-Walker ovoid or a Kantor-Payne ovoid or a Law-Penttila ovoid or a Ganley ovoid or a Kantor ovoid.

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β symmetric form associated to the quadratic form of $Q(6, q)$

$$\mathcal{O}(f_1, f_2) \text{ is an ovoid} \iff \beta(P_1, P_2) \neq 0 \text{ for any } P_1 \neq P_2 \text{ in } \mathcal{O}(f_1, f_2) \iff \\ (z_1 - z_2)^2 + (x_2 - x_1)(f_2(x_2, y_2, z_2) - f_2(x_1, y_1, z_1)) + \\ (y_2 - y_1)(f_1(x_2, y_2, z_2) - f_1(x_1, y_1, z_1)) \neq 0 \\ \text{for any } (x_1, y_1, z_1) \neq (x_2, y_2, z_2) \text{ in } \mathbb{F}_q^3$$

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$$(X_3 - X_6)^2 X_0^{d-1} + (X_4 - X_1) \left(\tilde{f}_2(X_4, X_5, X_6, X_0) - \tilde{f}_2(X_1, X_2, X_3, X_0) \right) \\ + (X_5 - X_2) \left(\tilde{f}_1(X_4, X_5, X_6, X_0) - \tilde{f}_1(X_1, X_2, X_3, X_0) \right) = 0$$

$\tilde{f}_i(x, y, z, t)$ homogenization of $f_i(x, y, z)$, $i \in \{1, 2\}$

$\mathcal{O}(f_1, f_2)$ is an ovoid $\iff \mathcal{W}_{f_1, f_2}$ has no affine \mathbb{F}_q -rational points off the solid $X_1 - X_4 = X_2 - X_5 = X_3 - X_6 = 0$

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Theorem (Bartoli, Durante, G. - 2024)

If $q > 6.3(\max\{\deg(f_1), \deg(f_2)\} + 1)^{13/3}$ and \mathcal{W}_{f_1, f_2} contains an absolutely irreducible component \mathcal{V} over \mathbb{F}_q , then $\mathcal{O}(f_1, f_2)$ is not an ovoid of $Q(6, q)$.

Known ovoids of $Q(6, q)$

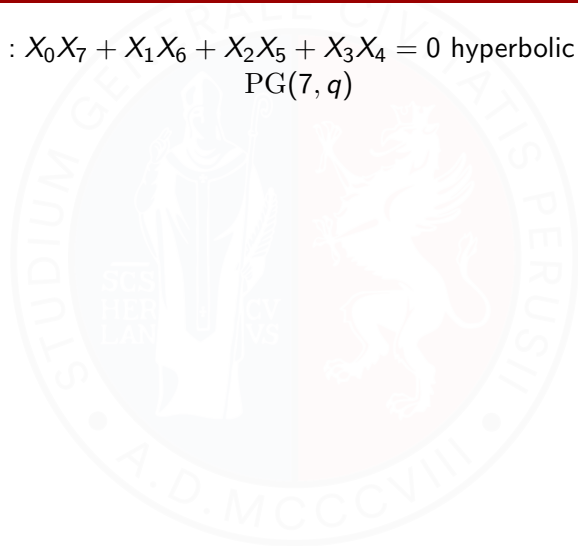
| Name | $f_1(x, y, z)$ | $f_2(x, y, z)$ | Restrictions |
|-------------|--|---|----------------------------------|
| Thas-Kantor | $-ny^3 + x^2y - xz$ | $-1/nx^3 + xy^2 + yz$ | $q = 3^h, n \notin \square_q$ |
| Ree-Tits | $-x^{\sigma+3} + y^\sigma + x^2y - xz$ | $-x^{2\sigma+3} + x^\sigma y^\sigma - z^\sigma + xy^2 + yz$ | $q = 3^{2h+1}, \sigma = 3^{h+1}$ |

Theorem (Bartoli, Durante, G. - 2024)

If $q > 6.3(\max\{\deg(f_1), \deg(f_2)\} + 1)^{13/3}$ and $\mathcal{O}(f_1, f_2)$ is an ovoid of $Q(6, q)$, then $\mathcal{O}(f_1, f_2)$ is a Thas-Kantor ovoid.

Ovoids of $Q^+(7, q)$

$Q^+(7, q) : X_0X_7 + X_1X_6 + X_2X_5 + X_3X_4 = 0$ hyperbolic quadric of
 $PG(7, q)$



Ovoids of $Q^+(7, q)$

$Q^+(7, q) : X_0X_7 + X_1X_6 + X_2X_5 + X_3X_4 = 0$ hyperbolic quadric of $\text{PG}(7, q)$

Any ovoid of $Q^+(7, q)$ is equivalent to one of the form:

$$\mathcal{O}(f_1, f_2, f_3) = \{P_{x,y,z}\}_{x,y,z \in \mathbb{F}_q} \cup \{(0, 0, 0, 0, 0, 0, 0, 1)\}$$

$$P_{x,y,z} =$$

$$(1, x, y, z, f_1(x, y, z), f_2(x, y, z), f_3(x, y, z), -zf_1(x, y, z) - yf_2(x, y, z) - xf_3(x, y, z))$$

for some polynomials f_1, f_2, f_3 s.t. $f_i(0, 0, 0) = 0$, $i = 1, 2, 3$

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for some polynomials f_1, f_2, f_3 s.t. $f_i(0, 0, 0) = 0$, $i = 1, 2, 3$

β symmetric form associated to the quadratic form of $Q^+(7, q)$

$$\begin{aligned} \mathcal{O}(f_1, f_2, f_3) \text{ is an ovoid} &\iff \beta(P_1, P_2) \neq 0 \text{ for any } P_1 \neq P_2 \text{ in } \mathcal{O}(f_1, f_2, f_3) \\ &\iff (x_1 - x_2)(f_3(x_2, y_2, z_2) - f_3(x_1, y_1, z_1)) \\ &\quad + (y_1 - y_2)(f_2(x_2, y_2, z_2) - f_2(x_1, y_1, z_1)) \\ &\quad + (z_1 - z_2)(f_1(x_2, y_2, z_2) - f_1(x_1, y_1, z_1)) \neq 0 \\ &\quad \text{for any } (x_1, y_1, z_1) \neq (x_2, y_2, z_2) \text{ in } \mathbb{F}_q^3 \end{aligned}$$

$\mathcal{S}_{f_1, f_2, f_3}$ hypersurface of $\text{PG}(6, q)$ with equation

$$\begin{aligned} & (X_1 - X_4) \left(\tilde{f}_3(X_4, X_5, X_6, X_0) - \tilde{f}_3(X_1, X_2, X_3, X_0) \right) + \\ & (X_2 - X_5) \left(\tilde{f}_2(X_4, X_5, X_6, X_0) - \tilde{f}_2(X_1, X_2, X_3, X_0) \right) + \\ & (X_3 - X_6) \left(\tilde{f}_1(X_4, X_5, X_6, X_0) - \tilde{f}_1(X_1, X_2, X_3, X_0) \right) = 0 \end{aligned}$$

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$\mathcal{O}(f_1, f_2, f_3)$ is an ovoid $\iff \mathcal{S}_{f_1, f_2, f_3}$ has no affine \mathbb{F}_q -rational points off the solid $X_1 - X_4 = X_2 - X_5 = X_3 - X_6 = 0$

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Theorem (Bartoli, Durante, G., Timpanella - 202x)

If $q > 6.3(\max\{\deg(f_1), \deg(f_2), \deg(f_3)\} + 1)^{13/3}$ and $\mathcal{S}_{f_1, f_2, f_3}$ contains an absolutely irreducible component \mathcal{V} over \mathbb{F}_q , then $\mathcal{O}(f_1, f_2, f_3)$ is not an ovoid of $Q^+(7, q)$.

Known ovoids of $Q^+(7, q)$

| Name | Restrictions |
|--------------------|---|
| <i>Thas-Kantor</i> | $q = 3^h, h > 0$ |
| <i>Ree-Tits</i> | $q = 3^{2h+1}, h > 0$ |
| <i>Kantor (1)</i> | $q = p^h, p \equiv 2 \pmod{3}$ (prime), h odd |
| <i>Kantor (2)</i> | $q = 2^h, h \geq 1$ |
| <i>Dye</i> | $q = 8$ |

Thas-Kantor and Ree-Tits are ovoids of $Q(6, q)$

Dye ovoid:

$$f_1(x, y, z) = x + y + z + x^2y + x^4y^2 + xy^2 + x^2y^4 + x^4y^4$$

$$f_2(x, y, z) = y + x^2z + x^4z^2 + xz^2 + x^2z^4 + x^4z^4$$

$$f_3(x, y, z) = x + y + y^2z + y^4z^2 + yz^2 + y^2z^4 + y^4z^4$$

Kantor (1) ovoid, $q \equiv 2 \pmod{3}$

$$V = \left\{ M = \begin{pmatrix} \alpha & \beta & c \\ \gamma & a & \beta^q \\ b & \gamma^q & \alpha^q \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{F}_{q^2}, a, b, c \in \mathbb{F}_q, a + \alpha + \alpha^q = 0 \right\}$$

$$Q(M) = \text{Tr}_{q^2/q}(\alpha)^2 - N_{q^2/q}(\alpha) + \text{Tr}_{q^2/q}(\beta\gamma) + bc$$

$$\mathcal{O} = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} \mu^q & \mu^q \rho & \mu^{q+1} \\ \rho^q & \rho^{q+1} & \mu \rho^q \\ 1 & \rho & \mu \end{pmatrix} : \text{Tr}_{q^2/q}(\mu) + N_{q^2/2}(\rho) = 0 \right\}$$

Theorem (Bartoli, Durante, G., Timpanella - 202x)

The set \mathcal{O} corresponds to the ovoid $\mathcal{O}(f_1, f_2, f_3)$, where if q is odd

$$f_1(x, y, z) = -6xy - 3y^3 - 9z^3 - z^2y - 3y^2z$$

$$f_2(x, y, z) = -y^3 - 3yz^2 + 6xz + 3y^2z + 9z^3$$

$$f_3(x, y, z) = -3x - 3y^2 - 9z^2,$$

if q is even

$$f_1(x, y, z) = z^3 + z^2y + zy^2 + xy$$

$$f_2(x, y, z) = z^3 + y^3 + xz$$

$$f_3(x, y, z) = x + z^2 + zy + y^2.$$

Kantor (2) ovoid, $q = 2^h, h \geq 1$

$$V = \mathbb{F}_q \oplus \mathbb{F}_{q^3} \oplus \mathbb{F}_{q^3} \oplus \mathbb{F}_q$$

$$Q((a, \gamma, \delta, d)) = ad + \text{Tr}_{q^3/q}(\gamma\delta)$$

$$\mathcal{O} = \{(0, 0, 0, 1)\} \cup \{(1, t, t^{q+q^2}, N_{q^3/q}(t)) : t \in \mathbb{F}_{q^3}\}$$

Theorem (Bartoli, Durante, G., Timpanella - 202x)

Let $\{1, \alpha, \beta\}$ be an \mathbb{F}_q -basis of \mathbb{F}_{q^3} , and put $t = x + y\alpha + z\beta$ with $x, y, z \in \mathbb{F}_q$. Then \mathcal{O} corresponds to the ovoid $\mathcal{O}(f_1, f_2, f_3)$, where

$$f_1(x, y, z) = \text{Tr}_{q^3/q}(\alpha\beta^2 + \alpha\beta^{q^2})xy + \text{Tr}_{q^3/q}(\beta)x^2 + \text{Tr}_{q^3/q}(\alpha^{q+1}\beta^{q^2})y^2 + N_{q^3/q}(\beta)z^2$$

$$f_2(x, y, z) = \text{Tr}_{q^3/q}(\alpha\beta^2 + \alpha\beta^{q^2})xz + \text{Tr}_{q^3/q}(\alpha)x^2 + N_{q^3/q}(\alpha\beta^{q^2})y^2 + \text{Tr}_{q^3/q}(\alpha\beta^{q^2+q})z^2$$

$$f_3(x, y, z) = \text{Tr}_{q^3/q}(\alpha\beta^2 + \alpha\beta^{q^2})yz + x^2 + \text{Tr}_{q^3/q}(\alpha^{q+1}\beta^{q^2})y^2 + \text{Tr}_{q^3/q}(\beta^{q+1})z^2.$$

CASE $d = 2$

Theorem (Bartoli, Durante, G., Timpanella - 202x)

Let $q > 6.3 \cdot 3^{13/3}$. If $\mathcal{O}(f_1, f_2, f_3)$ is an ovoid of $Q^+(7, q)$ and f_1, f_2, f_3 have degree 2, then q is even and $\mathcal{O}(f_1, f_2, f_3)$ is the Kantor (2) ovoid.

CASE $d = 2$

Theorem (Bartoli, Durante, G., Timpanella - 202x)

Let $q > 6.3 \cdot 3^{13/3}$. If $\mathcal{O}(f_1, f_2, f_3)$ is an ovoid of $Q^+(7, q)$ and f_1, f_2, f_3 have degree 2, then q is even and $\mathcal{O}(f_1, f_2, f_3)$ is the Kantor (2) ovoid.

CASE $d = 3$

If $\mathcal{O}(f_1, f_2, f_3)$ is an ovoid of $Q^+(7, q)$ and $q > 6.3 \cdot 4^{13/3}$, then $\mathcal{S}_{f_1, f_2, f_3}$ does not contain any absolutely irreducible component defined over \mathbb{F}_q . Since $\mathcal{S}_{f_1, f_2, f_3}$ is of degree 4, it must split into either four hyperplanes or two quadrics.

CASE $d = 2$

Theorem (Bartoli, Durante, G., Timpanella - 202x)

Let $q > 6.3 \cdot 3^{13/3}$. If $\mathcal{O}(f_1, f_2, f_3)$ is an ovoid of $Q^+(7, q)$ and f_1, f_2, f_3 have degree 2, then q is even and $\mathcal{O}(f_1, f_2, f_3)$ is the Kantor (2) ovoid.

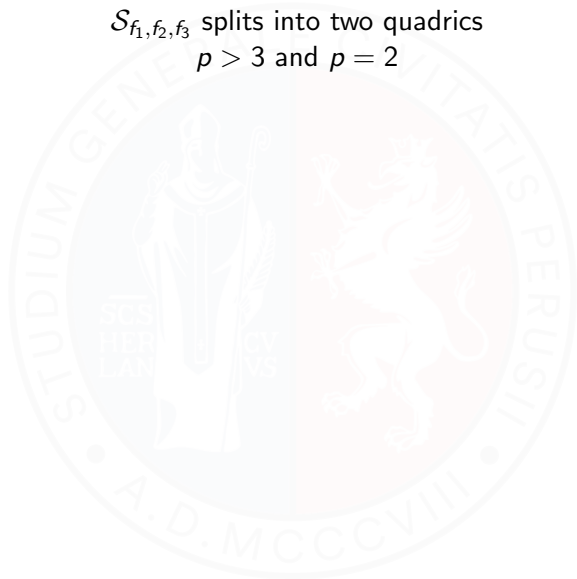
CASE $d = 3$

If $\mathcal{O}(f_1, f_2, f_3)$ is an ovoid of $Q^+(7, q)$ and $q > 6.3 \cdot 4^{13/3}$, then $\mathcal{S}_{f_1, f_2, f_3}$ does not contain any absolutely irreducible component defined over \mathbb{F}_q . Since $\mathcal{S}_{f_1, f_2, f_3}$ is of degree 4, it must split into either four hyperplanes or two quadrics.

Theorem (Bartoli, Durante, G., Timpanella - 202x)

Assume $q > 6.3 \cdot 4^{13/3}$ and f_1, f_2, f_3 of degree 3. If $\mathcal{S}_{f_1, f_2, f_3}$ splits into four hyperplanes, then the set $\mathcal{O}(f_1, f_2, f_3)$ is not an ovoid of $Q^+(7, q)$.

$\mathcal{S}_{f_1, f_2, f_3}$ splits into two quadrics
 $p > 3$ and $p = 2$



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 $p > 3$ and $p = 2$

Theorem (Bartoli, Durante, G., Timpanella - 202x)

Let $q > 6 \cdot 3 \cdot 4^{13/3}$, $q \equiv 1 \pmod{3}$, $p > 3$, f_1, f_2, f_3 of degree 3, and $\mathcal{S}_{f_1, f_2, f_3}$ split into two quadrics. The set $\mathcal{O}(f_1, f_2, f_3)$ is not an ovoid of $Q^+(7, q)$.

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Theorem (Bartoli, Durante, G., Timpanella - 202x)

Let $q > 6.3 \cdot 4^{13/3}$, $q \equiv 2 \pmod{3}$, $p = 2$, and f_1, f_2, f_3 of degree 3. Assume that $\mathcal{S}_{f_1, f_2, f_3}$ splits into two quadrics. If $\mathcal{O}(f_1, f_2, f_3)$ is an ovoid of $Q^+(7, q)$, then

$$\begin{aligned} f_1(X, Y, Z) = & Z^3 + Y^2Z + YZ^2 + (A + B)Y^2 + BZ^2 + XY \\ & + CX + DY + B^2Z, \end{aligned}$$

$$f_2(X, Y, Z) = Y^3 + Z^3 + AY^2 + (A + B)Z^2 + XZ + EX + A^2Y + DZ,$$

$$f_3(X, Y, Z) = Y^2 + Z^2 + YZ + X + EY + CZ.$$

Theorem (Bartoli, Durante, G., Timpanella - 202x)

Let $q > 6 \cdot 3 \cdot 4^{13/3}$, $q \equiv 2 \pmod{3}$, $p > 3$, and f_1, f_2, f_3 of degree 3. Assume that S_{f_1, f_2, f_3} splits into two quadrics. If $\mathcal{O}(f_1, f_2, f_3)$ is an ovoid of $Q^+(7, q)$, then

$$f_1(X, Y, Z) = -\frac{4}{27}Z^3 + \frac{4\epsilon}{3}Y^3 - \frac{4}{9}Y^2Z + \frac{4\epsilon}{9}YZ^2 + \left(\frac{4\epsilon}{3}A - 2B\right)Y^2 - \frac{2}{3}BZ^2 + \frac{4\epsilon}{3}XY + \frac{4\epsilon}{3}BZ + CY - B^2Z,$$

$$f_2(X, Y, Z) = -\frac{4}{3}Y^3 - \frac{4\epsilon}{9}Z^3 - \frac{4}{9}YZ^2 + \frac{4\epsilon}{3}Y^2Z - 2C_4Y^2 - \left(\frac{4\epsilon}{3}A - \frac{2}{3}B\right)Z^2 - \frac{4\epsilon}{3}XZ + \frac{4\epsilon}{3}AYZ + DX - A^2Y - (2AB + C)Z,$$

$$f_3(X, Y, Z) = -2Y^2 - \frac{2}{3}Z^2 - X - (2A + D)Y - (2B + E)Z,$$

where $\epsilon = \pm 1$.

Open Problem


Determine whether for f_1, f_2, f_3 defined as in the previous statements, the ovoid $\mathcal{O}(f_1, f_2, f_3)$ is equivalent to the Kantor (1) ovoid.

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Open Problem

Obtain a full classification for ovoids $\mathcal{O}(f_1, f_2, f_3)$ such that f_1, f_2, f_3 have degree 3 when $p = 3$.

The background features a large, faint, circular watermark of the University of Perugia seal. The seal is divided into four quadrants, each containing a different heraldic symbol. The text "STUDII GENERALE CIVITATIS PERUSINAE" is inscribed around the top inner edge, and "A.D. MCCCXVIII" is at the bottom. The text "SCS HER LAN" and "CV VS" are also visible within the seal's design.

Thanks for your attention!