Tensors in Finite Geometry and Combinatorics

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Throughout the talk V, W $V_1, \ldots, V_m$ denote finite-dimensional vector spaces over some field $\mathbb{F}$.

Outline of this talk:

1. Tensors
2. Groups
3. Geometry
4. Tensors in Finite Geometry and Combinatorics
5. Some results on ranks
6. Some results on orbits

Tensors

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2. Groups
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Tensor spaces

A tensor space will be one of the following:

- $V_1 \otimes V_2 \otimes \ldots \otimes V_m$, tensors of format $(n_1, n_2, \ldots, n_m)$, $n_i = \dim V_i$
  Notation: $V^{\otimes m}$ if all $V_i = V$
- the space of symmetric tensors $S^d V$,
- the space of alternating tensors $\Lambda^d V$,
- partially symmetric or alternating, i.e. $S^d V \otimes W$, $\Lambda^d V \otimes W$.

Example in $V^{\otimes 3}$ with $V = \langle e_0, e_1 \rangle$

$$T = e_0 \otimes e_0 \otimes e_0 + e_0 \otimes e_1 \otimes e_1 - e_1 \otimes e_0 \otimes e_1 + e_1 \otimes e_1 \otimes e_0$$

If e_0, e_1 are linearly independent then T can be represented by the hypermatrix

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right)$$

with respect to the basis of $V \otimes V \otimes V$ consisting of the tensors $e_{ijk} = e_i \otimes e_j \otimes e_k,\ i, j, k \in \{0, 1\}$.

Tensor products are ubiquitous in science

- Matrices are tensors in $\mathbb{F}^m \otimes \mathbb{F}^n$
- Multilinear algebra
- Algebraic geometry
- Computational complexity theory
- Quantum mechanics
- Data analysis (chemistry, biology, physics, ...)
- Signal processing, source separation

There is a large amount of research activity related to tensors. See e.g. [Kolda and Bader (2009) *Tensor decompositions and applications*] and [Landsberg (2012) *Tensors: Geometry and Applications*].

Decomposition

One of the central questions concerns the decomposition of a tensor into the sum of pure tensors[1]:

$$T = \sum_i v_{1i} \otimes \ldots \otimes v_{mi} \tag{1}$$

- Generalizes Singular Value Decomposition (SVD)
- "PARAFAC", "CANDECOMP", "CP decomposition", ...
- For general tensors no efficient decomposition algorithms exist.

[1] fundamental tensors, simple tensors, tensors of rank one

Main problems

$$T = \sum_{i=1}^{r} v_{1i} \otimes \ldots \otimes v_{mi} \qquad (1)$$

Four important problems:

(1) Is there an algorithm to compute a decomposition?
(2) Uniqueness: do some tensors have a unique decomposition?
(3) Existence: given T and r, does (1) exist? $\rightarrow$ rank(T) ([Hitchcock 1927], [Kruskal 1977])
(4) Orbits: how many "different" tensors are there in a given tensor space?

These problems are known to be very hard for $m \geq 3$ (theoretical and computational). ([Håstad 1990] *Tensor rank is NP-complete*)

Two examples of tensor rank

Example 1

What is the rank of T?

$$T = e_0 \otimes e_0 \otimes e_0 + e_0 \otimes e_1 \otimes e_1 - e_1 \otimes e_0 \otimes e_1 + e_1 \otimes e_1 \otimes e_0$$

- T has rank 3 over $\mathbb{R}$:

  $$T = (e_0 - e_1) \otimes e_0 \otimes e_0 + (e_0 + e_1) \otimes e_1 \otimes e_1 + e_1 \otimes (e_0 + e_1) \otimes (e_0 - e_1)$$

- T has rank 2 over $\mathbb{C}$:

  $$T = \left(\frac{1}{2} e_0 + \frac{1}{2i} e_1\right) \otimes (e_0 + ie_1) \otimes (e_0 + ie_1)$$

  $$+ \left(\frac{1}{2} e_0 - \frac{1}{2i} e_1\right) \otimes (e_0 - ie_1) \otimes (e_0 - ie_1)$$

- T has rank 2 over $\mathbb{F}_q$ iff $q \equiv 1 \bmod 4$

Example 2

In $\mathbb{F}^4 \otimes \mathbb{F}^4 \otimes \mathbb{F}^4$, the tensor

$$M = e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_3 \otimes e_1 + e_1 \otimes e_2 \otimes e_2 + e_2 \otimes e_4 \otimes e_2$$

$$+ e_3 \otimes e_1 \otimes e_3 + e_4 \otimes e_3 \otimes e_3 + e_3 \otimes e_2 \otimes e_4 + e_4 \otimes e_4 \otimes e_4$$

has rank ≤ 7.

Proof. Verify that

$$(e_1 + e_4) \otimes (e_1 + e_4) \otimes (e_1 + e_4) + (e_2 + e_4) \otimes e_1 \otimes (e_2 - e_4)$$

$$+ e_1 \otimes (e_3 - e_4) \otimes (e_3 + e_4) + e_4 \otimes (-e_1 + e_2) \otimes (e_1 + e_3)$$

$$+ (e_1 + e_3) \otimes e_4 \otimes (-e_1 + e_3) + (-e_1 + e_2) \otimes (e_1 + e_3) \otimes e_4$$

$$+ (e_3 - e_4) \otimes (e_2 + e_4) \otimes e_1.$$

is equal to M. $\square$

Connection to complexity theory

$V^\vee$ denotes the dual of V.

In general, given a tensor T in $V \otimes V \otimes V^\vee$ we can turn T into an algebra A. For example, for a pure tensor $u \otimes v \otimes w \in V \otimes V \otimes V^\vee$ define the multiplication

$$\Psi_T \ : \ V^\vee \times V^\vee \to V^\vee \ : \ (a, b) \mapsto a(u)b(v)w.$$

This connection is a well-studied and important topic in complexity theory: the rank of the tensor T corresponds to the complexity of multiplication in the algebra A.

[Bürgisser - Clausen - Shokrollahi (1997) *Algebraic Complexity Theory*]

[von zur Gathen - Gerhard (2013) *Modern Computer Algebra*]

Example 2 continued

Observing that

$$M = (e_1 \otimes e_1 + e_2 \otimes e_3) \otimes e_1 + (e_1 \otimes e_2 + e_2 \otimes e_4) \otimes e_2$$

$$+(e_3 \otimes e_1 + e_4 \otimes e_3) \otimes e_3 + (e_3 \otimes e_2 + e_4 \otimes e_4) \otimes e_4$$

can be rewritten as (where $e_1 = e_{11}$, $e_2 = e_{12}$, $e_3 = e_{21}$, $e_4 = e_{22}$)

$$M = (e_{11} \otimes e_{11} + e_{12} \otimes e_{21}) \otimes e_{11} + (e_{11} \otimes e_{12} + e_{12} \otimes e_{22}) \otimes e_{12}$$

$$+(e_{21} \otimes e_{11} + e_{22} \otimes e_{21}) \otimes e_{21} + (e_{21} \otimes e_{12} + e_{22} \otimes e_{22}) \otimes e_{22}$$

hints at the multiplication of 2×2-matrices.

Indeed, the tensor $M \in (\mathbb{F}^{2\times 2})^{\otimes 3}$ represents the matrix algebra $\mathbb{F}^{2\times 2}$.

Example 2 continued

$$M = (e_{11} \otimes e_{11} + e_{12} \otimes e_{21}) \otimes e_{11} + (e_{11} \otimes e_{12} + e_{12} \otimes e_{22}) \otimes e_{12}$$

$$+(e_{21} \otimes e_{11} + e_{22} \otimes e_{21}) \otimes e_{21} + (e_{21} \otimes e_{12} + e_{22} \otimes e_{22}) \otimes e_{22}$$

Then for $T = (e_{11} \otimes e_{11} + e_{12} \otimes e_{21}) \otimes e_{11}$ we get

$$\Psi_T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} u & v \\ w & z \end{bmatrix}\right) = \begin{bmatrix} au + bw & 0 \\ 0 & 0 \end{bmatrix}, \ etc.$$

Hence

$$\Psi_M\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} u & v \\ w & z \end{bmatrix}\right) = \begin{bmatrix} au + bw & av + bz \\ cu + dw & cv + dz \end{bmatrix}$$

Rank$(M) = 7$

The fact that M has rank ≤ 7 implies that two 2×2-matrices can be multiplied, by performing ≤ 7 instead of 8 multiplications: one multiplication for each pure tensor in the decomposition of M.

This was discovered by [Strassen 1969]. Soon after that [Hopcroft and Kerr 1971] and [Winograd 1971] proved that M has rank 7.

This decomposition can also be used to multiply $n \times n$-matrices (by adding zeros and block decomposition of $2^k \times 2^k$-matrices). ($\rightarrow$ exponent ω of matrix multiplication, see [Bürgisser et al., Chapter 15]).

Algebraic varieties related to tensors

The problems of decomposition and rank have natural geometric interpretations, and the following connections with algebraic geometry are well-known.

- Pure tensors corresponds to the set of points on a Segre variety in $\mathbb{P}^N$ where $N = \prod \dim V_i - 1$.
- Pure symmetric tensors in $S^d V$ corresponds to the points of a Veronese variety in $\mathbb{P}^N$ where $N = \binom{d+\dim V-1}{d} - 1$.
- Pure alternating tensors in $\Lambda^r V$ corresponds to the points of a Grassmann variety $\mathbb{P}^N$ where $N = \binom{\dim V}{r}$ (Plücker embedding)
- Tensors in $S^d V \otimes W$ corresponds to linear systems of hypersurfaces of degree d in $\mathbb{P}^{\dim V-1}$.

Groups

The main problems from before

(1) decomposition
(2) rank
(3) uniqueness
(4) orbits

are invariant under certain natural group actions.

Group actions I

- An element $(g_1, g_2, \ldots g_m)$ of $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \times \ldots \times \mathrm{GL}(V_m)$ acts on $V_1 \otimes V_2 \otimes \ldots \otimes V_m$ as follows:

$$v_1 \otimes v_2 \otimes \ldots \otimes v_m \mapsto v_1^{g_1} \otimes v_2^{g_2} \otimes \ldots \otimes v_m^{g_m}.$$

- If $V_i = V$ for k of the i's $(1 \leq k \leq m)$, then we also have a natural action of $Sym(k)$ on $V_1 \otimes V_2 \otimes \ldots \otimes V_m$. For example, for $k = m$ we have the action of $\pi \in Sym(m)$ defined by

$$\pi : v_1 \otimes v_2 \otimes \ldots \otimes v_m \mapsto v_{\pi(1)} \otimes v_{\pi(2)} \otimes \ldots \otimes v_{\pi(m)}.$$

- A combination of these gives the action of a wreath product ($\mathrm{GL}(V) \wr Sym(m)$ in the above example) on $V_1 \otimes V_2 \otimes \ldots \otimes V_m$.

Group actions II

Recall: $V^{\otimes d} = V \otimes \ldots \otimes V$ (d factors)

- The space $S^d V$ of symmetric tensors in $V^{\otimes d}$ consists of all fixed points of the action of $Sym(d)$ on $V^{\otimes d}$.
- The space $\Lambda^d V$ of alternating tensors in $V^{\otimes d}$ consists of all $T \in V^{\otimes d}$ for which $X^\pi = sgn(\pi)X, \ \forall \pi \in Sym(d)$.
- An element g of $\mathrm{GL}(V)$ acts on $V^{\otimes d}$ as follows:

$$g : v_1 \otimes \ldots \otimes v_d \mapsto v_1^g \otimes \ldots \otimes v_d^g$$

- Action of $\mathrm{GL}(V)$ on both subspaces $S^d V$ and $\Lambda^d V$.
- Action of $\mathrm{GL}(V) \times \mathrm{GL}(W)$ on $S^d V \otimes W$ and $\Lambda^d V \otimes W$.

For the purpose of this talk the acting group will usually be denoted by G or K.

Tensor spaces with a finite number of G-orbits over $\mathbb{C}$

These spaces have been classified by [Victor Kac 1980].

(1) $V_1 \otimes \ldots \otimes V_m$ and $G = \mathrm{GL}(V_1) \times \ldots \times \mathrm{GL}(V_m)$ $(m \geq 2)$

(a) $m = 2$
(b) $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^a$
(c) $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^a$

(2) $S^d V$ with $G = \mathrm{GL}(V)$

(a) $S^2\mathbb{C}^a$
(b) $S^3\mathbb{C}^2$

(3) $S^d V \otimes W$ with $G = \mathrm{GL}(V) \times \mathrm{GL}(W)$

(a) $S^2\mathbb{C}^2 \otimes \mathbb{C}^a$
(b) $S^2\mathbb{C}^3 \otimes \mathbb{C}^2$

Further results

1. There is a list of tensor spaces over $\mathbb{C}$ whose G-orbits can be parametrized. These cases correspond to group actions (G, V) where the GIT[2] quotient $G//V$ is such that each fiber $\varphi^{-1}(w)$, $w \in V//G$, consists of finitely many G-orbits.

[Schwarz 1978] [Kac 1980] [Kráskiewicz, Weyman 2009]

2. There are many results known on G-orbits on $\Lambda^r(\mathbb{F}^n)$, for various $\mathbb{F}$, e.g. algebraically closed fields $\mathbb{F} = \overline{\mathbb{F}}$, finite fields $\mathbb{F} = \mathbb{F}_q$, and fields of cohomological dimension ≤ 1.

([Schouten 1933] [Dieudonné 1955] [Gurevich 1964] [Cohen and Helminck 1988] [DeBruyn and Kwiatkowski 2013] [Cardinali, Giuzzi, and Pasini 2017] [Borovoi, de Graaf, Lê 2022])

[2] The Geometric Invariant Theory quotient: the image of V under the map $\varphi : v \mapsto (f_1(v), \ldots, f_m(v))$ where $f_1, \ldots, f_m \in \mathbb{C}[V]$ are generators for the ring of invariants $\mathbb{C}[V]^G$

Classification - Finite Fields - Combinatorics

Since we are primarily interested in tensors over finite fields, with a classification of the G-orbits we mean

a complete list of representatives for the G-orbits.

(And of course we want a complete proof!)

If possible we would also like to have

- the stabiliser group of each representative,
- the size of each orbit,
- an algorithm to determine the orbit of a given tensor,
- Schreier elements for constructive recognition,
- combinatorial invariants for the orbits (preferably complete).

Geometry

Geometry

The geometric setting is as follows.

The pure tensors correspond to points of classical algebraic varieties in projective space $\mathbb{P}^N$

- $V_1 \otimes V_2 \otimes \ldots \otimes V_m \to$ Segre variety with $N = \prod n_i - 1$
- $S^d V \to$ Veronese variety with $N = \binom{n+d-1}{d} - 1$
- $\Lambda^d V \to$ Grassmann variety with $N = \binom{n}{d} - 1$

In each case the group G induces a subgroup K of $\mathrm{PGL}(\mathbb{P}^N)$ leaving the relevant variety invariant.

Since K leaves the tensor rank invariant one can speak of the rank of a K-orbit.

Finite Geometry and Combinatorics

Semifields are tensors

Inspired by [Knuth 1965] and [Liebler1981]

An n-dimensional semifield $\mathbb{S}$ gives $T_{\mathbb{S}} \in V_1 \otimes V_2 \otimes V_3,\ V_i \cong \mathbb{F}_q^n$

A semifield is a (finite) possibly non-associative division algebra.

- An n-dimensional semifield $\mathbb{S}$ gives $T_{\mathbb{S}} \in V_1 \otimes V_2 \otimes V_3,\ V_i \cong \mathbb{F}_q^n$
- The tensor $T_{\mathbb{S}} \in V_1 \otimes V_2 \otimes V_3$ is nonsingular (each non-trivial double contraction gives a nonzero vector).
- To every nonsingular tensor $T \in V_1 \otimes V_2 \otimes V_3$ there corresponds a (pre-)semifield $\mathbb{S}$ for which $T = T_{\mathbb{S}}$.
- The map $\mathbb{S} \mapsto T_{\mathbb{S}}$ is injective.
- Semifields $\leftrightarrow$ projective planes (with a lot of symmetry)

Orbits and isotopism (isomorphism)

- The isomorphism classes (planes) $\leftrightarrow$ isotopism classes (semifields)
- The Knuth orbit of a semifield $\mathbb{S}$ is represented in the projective space $\mathbb{P}^{n^3-1}$ as the orbit of $\langle T_{\mathbb{S}} \rangle$ under the group $\mathrm{GL} \wr S_3$.
- The tensor rank of a semifield is an invariant for the Knuth orbit of a semifield

Moral of the story: the tensor $T_{\mathbb{S}}$ is what we, geometers, should be looking at.

[ML 2013] *Finite semifields and non-singular tensors* (Irsee 5, 2011)

Theorem

A tensor $\tau \in \mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n$ is singular if and only if

$$\langle \tau \rangle \subset \langle x_1, \ldots, x_j, S_{k_1,k_2,k_3} \rangle$$

for some $j < n$ points and a S_{k_1,k_2,k_3} properly contained in $S_{n,n,n}$.

Some updates

- Summer school on Finite Geometry in Istanbul (August 8-12 2022)
- Irsee 6 Finite Geometries (28 August - 2 September, 2022)
- Aart and Combinatorics conference (20-24 February, 2023)

RD codes are tensors

An n-dimensional RD code C gives $T_C \in \mathbb{F}_q^k \otimes \mathbb{F}_q^n \otimes \mathbb{F}_q^m$, $V_i \cong \mathbb{F}_q^{n_i}$

[Byrne, Neri, Ravagnani, Sheekey 2019], [Byrne-Cotardo 202*]

Some techniques

A contraction of a tensor

$$T \in V_1 \otimes \ldots \otimes V_m$$

is a tensor in

$$V_1 \otimes \ldots \otimes \hat{V}_i \otimes \ldots \otimes V_m$$

obtained as the image of T under $u_i^\vee \in V_i^\vee$, defined by

$$u_i^\vee(v_1 \otimes \ldots \otimes v_m) = u_i^\vee(v_i) v_1 \otimes \ldots v_{i-1} \otimes v_{i+1} \otimes \ldots v_m.$$

The i-th contraction space of T is the subspace

$$C_i(T) := \{u_i^\vee(T) \ : \in u_i^\vee \in V_i^\vee\} \leq V_1 \otimes \ldots \otimes \hat{V}_i \otimes \ldots \otimes V_m.$$

Contraction spaces and more ranks

A tensor $T \in V_1 \otimes \ldots \otimes V_m$ is called *i-concise* if $\dim C_i(T) = n_i$ and concise if T is i-concise for all $i \in \{1, \ldots, m\}$.

Each contraction space admits a natural action of the group G_i.

The rank of a subspace U of $V_1 \otimes \ldots \otimes V_m$ is defined as the minimal number of pure tensors needed to span a subspace containing U.

The generalized tensor ranks of U is the list of minimum tensor ranks of subspaces of U.

Two useful lemma's

Lemma

Two tensors $V_1 \otimes \ldots \otimes V_m$ are G-equivalent if and only if the their i-th contraction spaces $C_i(T)$ and $C_i(S)$ are G_i-equivalent.

In particular, two G-equivalent tensors have contraction spaces of the same dimension, giving us a combinatorial invariant $[\dim C_i(T)]_i$ for the G-orbits.

Lemma

The rank of a tensor T is equal to the rank of the i-th contraction space of T, i.e.

$$rank(T) = rank(C_i(T))$$

for $i \in \{1, \ldots, m\}$.

Recent results

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Some results on tensor rank over finite fields

(i) The maximum rank in $\mathbb{F}^3 \otimes \mathbb{F}^3 \otimes \mathbb{F}^3$ is 6 [ML-Pavan-Zanella 2012] Since it is known that $\mathbb{F}_{q^n}$ has $\mathbb{F}_q$-tensor rank $2n-1$ if $q \geq 2n-2$, and $> 2n-1$ if $q < 2n-2$ [Winograd 1979], [de Groote 1983], this shows that $\mathbb{F}_{27}$ has $\mathbb{F}_3$-tensor rank 6.

(ii) All semifields of order 27 have $\mathbb{F}_3$-tensor rank equal to 6 [ML-Pavan-Zanella 2012].

(iii) The tensor rank of a semifield of order 16 is at least nine. In particular, the tensor rank of $\mathbb{F}_{16}$ over $\mathbb{F}_2$ is nine Chudnovsky-Chudnovsky (1988).

(iv) All semifields of order 16 have $\mathbb{F}_2$-tensor rank equal to 9. [ML-Sheekey 2022]

Tensor rank of semifields of order 81

Using connections to coding theory, and the classification of semifields of order 81, we were able to determine the tensor ranks of all semifields of order 81.

Theorem

The field and GTF of order 81 have $\mathbb{F}_3$-tensor rank 9, all other semifields of order 81 have $\mathbb{F}_3$-tensor rank 8.

This establishes the tensor rank of a semifield as a non-trivial invariant of the Knuth orbit of semifields.

This also shows that there exist semifields with lower multiplicative complexity than the field of the same order.

[ML-Sheekey 2022]

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G-orbits - results over $\mathbb{F}_q$

Besides the previously mentioned studies of G-orbits on $\Lambda^r(\mathbb{F}_q^n)$, the G-orbits have been studied in the following cases:

(i) $\mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^r$ [ML-Sheekey 2015], [ML-Sheekey 2017], [Alnajjarine-ML 2020]

(ii) $S^2\mathbb{F}_q^3 \otimes \mathbb{F}_q^2$ [ML-Popiel 2019]

(iii) $S^2\mathbb{F}_q^3 \otimes \mathbb{F}_q^3$ [ML-Popiel-Sheekey 2020, 2021], [Alnajjarine-ML 202*]

(iv) $S^2\mathbb{F}_q^3 \otimes \mathbb{F}_q^4$ [Alnajjarine-ML-Popiel 2022]

(v) $S^3\mathbb{F}_q^2 \otimes \mathbb{F}_q^2$ [Günay-ML 2022], [Davydov-Marcugini-Pambianco 2021], [Blokhuis-Pellikaan-Szőnyi 2022], [Ceria-Pavese 202*]

G-orbits - computational results for small values of q

There are computational classification results for the G-orbits in the following cases:

1. $\mathbb{F}_q^2 \otimes \mathbb{F}_q^2 \otimes \mathbb{F}_q^2 \otimes \mathbb{F}_q^2$ for $q \in \{2, 3\}$ [Bremner-Stavrou 2013]
2. $S^4\mathbb{F}_q^2$ for $q \in \{2, 3, 5, 7\}$ [Stavrou 2014]
3. $S^d\mathbb{F}_q^3$ for $d \in \{3, 4\}$ and $q \in \{2, 3\}$, [Stavrou 2015]
4. $\mathbb{F}_2^2 \otimes \mathbb{F}_2^2 \otimes \mathbb{F}_2^2 \otimes \mathbb{F}_2^2 \otimes \mathbb{F}_2^2$ [Stavrou-Low-Hernandez 2016] [Stavrou-Low-Hernandez 2018] [Betten-ML 202*]

(i) $\mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^r$

Complete classification of the G-orbits [ML-Sheekey 2015], [ML-Sheekey 2017].

The following table gives a representative for each of the 18 G-orbits of tensors in $\mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$. The last column gives the rank distribution $r_1(A)$ of the first contraction space $C_1(A)$ of the representative A.

Orbit	**Canonical form**	**Condition**	$r_1(A)$
o_0	0		$[0, 0, 0]$
o_1	$e_1 \otimes e_1 \otimes e_1$		$[1, 0, 0]$
o_2	$e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2)$		$[0, 1, 0]$
o_3	$e_1 \otimes e$		$[0, 0, 1]$
o_4	$e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_1 \otimes e_2$		$[q+1, 0, 0]$
o_5	$e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2$		$[2, q-1, 0]$
o_6	$e_1 \otimes e_1 \otimes e_1 + e_2 \otimes (e_1 \otimes e_2 + e_2 \otimes e_1)$		$[1, q, 0]$
o_7	$e_1 \otimes e_1 \otimes e_3 + e_2 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2)$		$[1, q, 0]$
o_8	$e_1 \otimes e_1 \otimes e_1 + e_2 \otimes (e_2 \otimes e_2 + e_3 \otimes e_3)$		$[1, 1, q-1]$
o_9	$e_1 \otimes e_3 \otimes e_1 + e_2 \otimes e$		$[1, 0, q]$
o_{10}	$e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2 + ue_1 \otimes e_2) + e_2 \otimes (e_1 \otimes e_2 + ve_2 \otimes e_1)$	$(*)$	$[0, q+1, 0]$
o_{11}	$e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2) + e_2 \otimes (e_1 \otimes e_2 + e_2 \otimes e_3)$		$[0, q+1, 0]$
o_{12}	$e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2) + e_2 \otimes (e_1 \otimes e_3 + e_3 \otimes e_2)$		$[0, q+1, 0]$
o_{13}	$e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2) + e_2 \otimes (e_1 \otimes e_2 + e_3 \otimes e_3)$		$[0, 2, q-1]$
o_{14}	$e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2) + e_2 \otimes (e_2 \otimes e_2 + e_3 \otimes e_3)$		$[0, 3, q-2]$
o_{15}	$e_1 \otimes (e + ue_1 \otimes e_2) + e_2 \otimes (e_1 \otimes e_2 + ve_2 \otimes e_1)$	$(*)$	$[0, 1, q]$
o_{16}	$e_1 \otimes e + e_2 \otimes (e_1 \otimes e_2 + e_2 \otimes e_3)$		$[0, 1, q]$
o_{17}	$e_1 \otimes e + e_2 \otimes (e_1 \otimes e_2 + e_2 \otimes e_3 + e_3 \otimes (\alpha e_1 + \beta e_2 + \gamma e_3))$	$(**)$	$[0, 0, q+1]$

Condition $(*)$ **is:** $v\lambda^2 + uv\lambda - 1 \neq 0$ **for all** $\lambda \in \mathbb{F}_q$.
Condition $(**)$ **is:** $\lambda^3 + \gamma\lambda^2 - \beta\lambda + \alpha \neq 0$ **for all** $\lambda \in \mathbb{F}_q$.

Orbit identification

This classification of the G-orbits includes various other useful data of the G-orbits, such as tensor rank, geometric description, and rank distribution of the contraction spaces, see
`http://people.sabanciuniv.edu/~mlavrauw/T233/table1.html`.

[Alnajjarine-ML2020]

In some cases one needs to understand more of the geometry to tell orbits apart. For example, for o_{15} and o_{16}.

Lemma

Let x_2 be the unique rank 2 point on $C_1(A)$ and x_1 be a point among the q points of rank 3 on $C_1(A)$. Then, there exists a unique solid V containing x_2 which intersects $S_{3,3}(\mathbb{F}_q)$ in a subvariety $Q(x_2)$ equivalent to a Segre variety $S_{2,2}(\mathbb{F}_q)$. There is no rank one point in $U \setminus Q(x_2)$ for o_{16} where $U := \langle V, x_1 \rangle$, and there is one for o_{15}.

Implementation in GAP [Alnajjarine - ML 2020]

(joint work with Nour Alnajjarine)

The geometric and combinatorial data from the classification of tensors in $\mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$ has been used to implement algorithms in GAP (relying on the FinInG package) which allow orbit identification and rank computation in this tensor space.

[Alnajjarine - ML 2020] *Determining the rank of tensors in* $\mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$ (2020).

The GAP code is available from
`http://people.sabanciuniv.edu/mlavrauw/T233/T233_paper.html`.

This research is part of Nour's PhD.

The code is illustrated on the next slide with an example in the 17-dimensional projective space over the field of size 397.

```
gap> q:=397; sv:=SegreVariety([PG(1,q),PG(2,q),PG(2,q)]);
397
Segre Variety in ProjectiveSpace(17, 397)
gap> Size(Points(sv));
9936552395502
gap> Size(Points(PG(17,397)));
15154232143809814799565590114693875696752607 8
gap> A:=VectorSpaceToElement(pg,[Z(397)^0,Z(397)^336,Z(397)^339,
Z(397)^37,Z(397)^233,Z(397)^56,Z(397)^268,Z(397)^363,Z(397)^342,
Z(397)^297,Z(397)^146,Z(397)^71,Z(397)^57,Z(397)^84,Z(397)^33,
Z(397)^203,Z(397)^229,Z(397)^191]);
gap> OrbitOfTensor(A)[1]; time;
14
94
gap> RankOfTensor(A); time;
3
141
gap> NrCombinations([1..Size(Points(sv))], 3);
163514371865202881474954561407873423500
```

(ii) $S^2\mathbb{F}_q^3 \otimes \mathbb{F}_q^2$

Complete classification of the G-orbits.

[ML-Popiel 2019]

Symmetric tensors, subspaces of dimension two in $S^2\mathbb{F}_q^3$

Algebraic variety: quadric Veronesean in $\mathbb{P}^5$

Linear systems: webs of conics, and for q odd, also pencils of conics
[Dickson 1908]

Stabilisers

Correspondence with the orbits in $\mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$ (which orbits split?)

15 orbits for each q

(iii) $S^2\mathbb{F}_q^3 \otimes \mathbb{F}_q^3$

Not complete.

Symmetric tensors, subspaces of dimension three in $S^2\mathbb{F}_q^3$

Algebraic variety: quadric Veronesean in $\mathbb{P}^5$

Linear systems: nets of conics

[ML-Popiel-Sheekey 2020]: classification of nets of rank one for q odd.

(This completes/corrects [Wilson 1914])

[Alnajjarine-ML 202*]: classification of nets of rank one for q even.

(iii) $S^2\mathbb{F}_q^3 \otimes \mathbb{F}_q^3$ continued

Nets of conics of rank one

Stabilisers

Combinatorial invariants for q odd

point-orbit distributions

line-orbit distributions

[ML-Popiel-Sheekey 2021]

Line orbit distributions

Orbit	q (3)	o_5	o_6	$o_{8,1}$	$o_{8,2}$	o_9	o_{10}	o_{12}	$o_{13,1}$	$o_{13,2}$	$o_{14,1}$	$o_{14,2}$	$o_{15,1}$	$o_{15,2}$	o_{16}	o_{17}
Σ_1		$\frac{q(q+1)}{2}$	$q+1$				$\frac{q(q-1)}{2}$									
Σ_2		3		$\frac{3(q-1)}{2}$	$\frac{3(q-1)}{2}$						$\frac{(q-1)^2}{4}$	$\frac{3(q-1)^2}{4}$				
Σ_3		1	1	q		$q-1$			$\frac{q(q-1)}{2}$	$\frac{q(q-1)}{2}$						
Σ_4		1		$2q$				1	$\frac{q(q-3)}{2}$	$\frac{q(q-1)}{2}$					$q-1$	
Σ_5		1		$q-1$	$q-1$	2			$\frac{q-1}{2}$	$\frac{q-1}{2}$	$\frac{(q-1)(q-3)}{8}$	$\frac{(q-1)(3q-5)}{8}$	$\frac{(q-1)^2}{4}$	$\frac{(q+1)(q-1)}{4}$		
Σ_6				$\frac{q+1}{2}$	$\frac{q+1}{2}$		1						$\frac{(q+1)(q-1)}{2}$	$\frac{(q+1)(q-1)}{2}$		
Σ_7			$q+1$					q^2								
Σ_8			1	q				1	$q(q-1)$						$q-1$	
Σ_9			1	q					q		$\frac{q(q-1)}{2}$		$\frac{q(q-1)}{2}$			
Σ_{10}			1		q					q		$\frac{q(q-1)}{2}$	$\frac{q(q-1)}{2}$			
Σ_{11}	0			q		1			q		$\frac{q(q-3)}{6}$		$\frac{q(q-1)}{2}$			$\frac{q^2}{3}$
	$\not\equiv 0$			q		1			$q-1$		$\frac{(q-1)(q-2)}{6}$		$\frac{q(q-1)}{2}$		1	$\frac{(q+1)(q-1)}{3}$
Σ_{12}	1			$\frac{q-1}{2}$	$\frac{q-1}{2}$	2			$\frac{q-7}{2}$	$\frac{q-1}{2}$	$\frac{(q-1)(q-7)}{24}+1$	$\frac{(q-1)(q-3)}{8}$	$\frac{(q-1)^2}{4}$	$\frac{(q+1)(q-1)}{4}$	3	$\frac{(q-1)(q+2)}{3}$
	$\not\equiv 1$			$\frac{q-1}{2}$	$\frac{q-1}{2}$	2			$\frac{q-3}{2}$	$\frac{q-1}{2}$	$\frac{(q-3)(q-5)}{24}$	$\frac{(q-1)(q-3)}{8}$	$\frac{(q-1)^2}{4}$	$\frac{(q+1)(q-1)}{4}$	1	$\frac{q(q+1)}{3}$
Σ_{13}	-1			$\frac{q+1}{2}$	$\frac{q+1}{2}$				$\frac{q-5}{2}$	$\frac{q+1}{2}$	$\frac{(q+1)(q-5)}{24}+1$	$\frac{(q+1)(q-1)}{8}$	$\frac{(q+1)(q-3)}{4}$	$\frac{(q+1)(q-1)}{4}$	3	$\frac{(q+1)(q-2)}{3}$
	$\not\equiv -1$			$\frac{q+1}{2}$	$\frac{q+1}{2}$				$\frac{q-1}{2}$	$\frac{q+1}{2}$	$\frac{(q-1)(q-3)}{24}$	$\frac{(q+1)(q-1)}{8}$	$\frac{(q+1)(q-3)}{4}$	$\frac{(q+1)(q-1)}{4}$	1	$\frac{q(q-1)}{3}$
Σ_{14}	1			$\frac{q-1}{2}$	$\frac{q-1}{2}$	2			$\frac{q-1}{2}$	$\frac{q-1}{2}$	$\frac{(q-1)(q-7)}{24}$	$\frac{(q-1)(q-3)}{8}$	$\frac{(q-1)^2}{4}$	$\frac{(q+1)(q-1)}{4}$		$\frac{(q-1)^2}{3}+q$
	-1			$\frac{q+1}{2}$	$\frac{q+1}{2}$				$\frac{q+1}{2}$	$\frac{q+1}{2}$	$\frac{(q+1)(q-5)}{24}$	$\frac{(q+1)(q-1)}{8}$	$\frac{(q-1)(q-3)}{4}$	$\frac{(q+1)(q-1)}{4}$		$\frac{(q-1)^2}{3}-q$
Σ'_{14}	0			q		1					$\frac{q(q-1)}{6}$		$\frac{q(q-1)}{2}$		q	$\frac{q(q-1)}{3}$
Σ_{15}			1			q									q^2	

TABLE 1. Line-orbit distributions of planes in $\mathrm{PG}(5,q)$, q odd, that intersect the quadric Veronesean. Where applicable, the second column indicates the congruence class(es) of q modulo 3.

This table is taken from [ML-Popiel-Sheekey 2021]

(iv) $S^2\mathbb{F}_q^3 \otimes \mathbb{F}_q^4$

Complete classification of the G-orbits.

Symmetric tensors, subspaces of dimension four in $S^2\mathbb{F}_q^3$

Algebraic variety: quadric Veronesean in $\mathbb{P}^5$

Linear systems: pencils of conics

[ML-Popiel 2019], [Alnajjarine-ML-Popiel 2022]

(This completes/corrects [Campbell 1927])

Theorem

There are exactly 15 orbits of solids in $\mathrm{PG}(5,q)$ *under the induced action of* $\mathrm{PGL}(3,q) \leqslant \mathrm{PGL}(6,q)$.

Orbit	Point OD	Hyperplane OD	Stabiliser	Orbit size
Ω_1	$[1, q+1, 2q^2-1, q^3-q^2]$	$[1, q/2, q/2, 0]$	$E_q^2 \rtimes (E_q \times C_{q-1})$	$(q^3-1)(q+1)$
Ω_2	$[q+1, q+1, 2q^2-q-1, q^3-q^2]$	$[1, q, 0, 0]$	$E_q^{1+2} \rtimes C_{q-1}^2$	$(q^2+q+1)(q+1)$
Ω_3	$[1, q^2+q+1, q^2-1, q^3-q^2]$	$[q+1, 0, 0, 0]$	$E_q^2 \rtimes \mathrm{GL}(2,q)$	q^2+q+1
Ω_4	$[q+2, 1, 2q^2-2, q^3-q^2]$	$[0, q+1, 0, 0]$	$\mathrm{GL}(2,q)$	$q^2(q^2+q+1)$
Ω_5	$[1, q+1, q^2-1, q^3]$	$[1, 0, 0, q]$	$E_q^2 \rtimes C_{q-1}$	$q(q^3-1)(q+1)$
Ω_6	$[2, q+1, q^2+q-2, q^3-q]$	$[1, 1, 0, q-1]$	$C_{q-1}^2 \rtimes C_2$	$\frac{1}{2}q^3(q^2+q+1)(q+1)$
Ω_7	$[0, q+1, q^2+q, q^3-q]$	$[1, 0, 1, q-1]$	$D_{2(q+1)} \times C_{q-1}$	$\frac{1}{2}q^3(q^3-1)$
Ω_8	$[3, 1, q^2+2q-3, q^3-q]$	$[0, 2, 0, q-1]$	$C_{q-1} \times C_2$	$\frac{1}{2}q^3(q^3-1)(q+1)$
Ω_9	$[4, 1, q^2+3q-4, q^3-2q]$	$[0, 3, 0, q-2]$	Sym_4	$\frac{1}{24}q^3(q^3-1)(q^2-1)$
Ω_{10}	$[1, 1, q^2+2q-1, q^3-q]$	$[0, 1, 1, q-1]$	$C_{q-1} \times C_2$	$\frac{1}{2}q^3(q^3-1)(q+1)$
Ω_{11}	$[2, 1, q^2+q-2, q^3]$	$[0, 1, 0, q]$	$E_q \rtimes C_{q-1}$	$q^2(q^3-1)(q+1)$
Ω_{12}	$[2, 1, q^2+q-2, q^3]$	$[0, 1, 0, q]$	C_2^2	$\frac{1}{4}q^3(q^3-1)(q^2-1)$
Ω_{13}	$[0, 1, q^2+3q, q^3-2q]$	$[0, 1, 2, q-2]$	$C_2^2 \rtimes C_2$	$\frac{1}{8}q^3(q^3-1)(q^2-1)$
Ω_{14}	$[0, 1, q^2+q, q^3]$	$[0, 0, 1, q]$	C_4	$\frac{1}{4}q^3(q^3-1)(q^2-1)$
Ω_{15}	$[1, 1, q^2-1, q^3+q]$	$[0, 0, 0, q+1]$	C_3	$\frac{1}{3}q^3(q^3-1)(q^2-1)$

This table is taken from [ML-Popiel 2019], [Alnajjarine-ML-Popiel 2022]

(v) $S^3\mathbb{F}_q^2 \otimes \mathbb{F}_q^2$

Not complete.

Symmetric tensors, subspaces of dimension two in $S^3\mathbb{F}_q^2$

Algebraic variety: twisted cubic in $\mathrm{PG}(2, q)$

Linear systems: pencils of cubics in $\mathrm{PG}(1, q)$

For q odd: not complete.

For q even: complete (*)

Related to coset leader weight enumerator of Reed-Solomon codes, multiple covering codes.

[Davydov-Marcugini-Pambianco 2021,2021,202*,202*,202*], [Günay-ML 2022], [Blokhuis-Pellikaan-Szőnyi 2022], [Ceria-Pavese 202*]

The 10 line orbits in $\mathrm{PG}(3,q)$ with combinatorial invariants in the line classes $O_{i<6}$ (table from [Günay-ML 2022]).

orbit	OD_2	OD_0	base
$\mathcal{L}_1 = \mathcal{O}_1^{\perp}$	$[2, 0, 0, (q-1), 0]$	$[0, 2, (q-1), 0, 0]$	$\emptyset$
$-3 \in \square$	$[2, 0, \frac{(q-1)}{3}, 0, \frac{2(q-1)}{3}]$	$[0, 2, (q-1), 0, 0]$	$\emptyset$
$\mathcal{L}_2 = \mathcal{O}_2$	$[1, q, 0, 0, 0]$	$[1, q, 0, 0, 0]$	$\mathcal{Z}(X_1^2)$
$\mathcal{L}_3 = \mathcal{O}_4$	$[1, 1, \frac{(q-1)}{2}, \frac{(q-1)}{2}, 0]$	$[1, 1, \frac{(q-1)}{2}, \frac{(q-1)}{2}, 0]$	$\mathcal{Z}(X_1)$
$\mathcal{L}_4 \subset \mathcal{O}_5^{\perp}$	$[1, 2, \frac{(q-5)}{6}, \frac{(q-3)}{2}, \frac{(q+1)}{3}]$,	$[0, 3, \frac{(q-3)}{2}, \frac{(q-1)}{2}, 0]$	$\emptyset$
$-3 \in \square$	$[1, 2, \frac{(q-7)}{6}, \frac{(q-1)}{2}, \frac{(q-1)}{3}]$,	$[0, 3, \frac{(q-3)}{2}, \frac{(q-1)}{2}, 0]$	$\emptyset$
$\mathcal{L}_5 \subset \mathcal{O}_5^{\perp}$	$[1, 0, \frac{(q+1)}{6}, \frac{(q-1)}{2}, \frac{(q+1)}{3}]$	$[0, 1, \frac{q-1}{2}, \frac{q+1}{2}, 0]$,	$\emptyset$
$-3 \in \square$	$[1, 0, \frac{(q-1)}{6}, \frac{(q+1)}{2}, \frac{(q-1)}{3}]$	$[0, 1, \frac{q-1}{2}, \frac{q+1}{2}, 0]$	$\emptyset$
$\mathcal{L}_6 = \mathcal{O}_1$	$[0, 2, (q-1), 0, 0]$	$[2, 0, (q-1), 0, 0]$	$\mathcal{Z}(X_0, X_1)$
$-3 \in \square$	$[0, 2, (q-1), 0, 0]$	$[2, 0, \frac{(q-1)}{3}, 0, \frac{2(q-1)}{3}]$	$\mathcal{Z}(X_0, X_1)$
$\mathcal{L}_7 \subset \mathcal{O}_5$	$[0, 3, \frac{(q-3)}{2}, \frac{(q-1)}{2}, 0]$	$[1, 2, \frac{(q-5)}{6}, \frac{(q-3)}{2}, \frac{(q+1)}{3}]$	$\mathcal{Z}(X_1)$
$-3 \in \square$	$[0, 3, \frac{(q-3)}{2}, \frac{(q-1)}{2}, 0]$	$[1, 2, \frac{(q-7)}{6}, \frac{(q-1)}{2}, \frac{(q-1)}{3}]$	$\mathcal{Z}(X_1)$
$\mathcal{L}_8 \subset \mathcal{O}_5$	$[0, 1, \frac{(q-1)}{2}, \frac{(q+1)}{2}, 0]$	$[1, 0, \frac{(q+1)}{6}, \frac{(q-1)}{2}, \frac{(q+1)}{3}]$	$\mathcal{Z}(X_1)$
$-3 \in \square$	$[0, 1, \frac{q-1}{2}, \frac{q+1}{2}, 0]$	$[1, 0, \frac{(q-1)}{6}, \frac{(q+1)}{2}, \frac{(q-1)}{3}]$	$\mathcal{Z}(X_1)$
$\mathcal{L}_9 = \mathcal{O}_3$	$[0, 0, 0, (q+1), 0]$	$[0, 0, \frac{(q+1)}{3}, 0, \frac{2(q+1)}{3}]$	$\mathcal{Z}(X_0^2 + bX_1^2)$
$-3 \in \square$	$[0, 0, 0, (q+1), 0]$	$[0, 0, 0, (q+1), 0]$	$\mathcal{Z}(X_0^2 + bX_1^2)$
$\mathcal{L}_{10} = \mathcal{O}_3^{\perp}$	$[0, 0, \frac{(q+1)}{3}, 0, \frac{2(q+1)}{3}]$	$[0, 0, 0, (q+1), 0]$	$\emptyset$
$-3 \in \square$	$[0, 0, 0, (q+1), 0]$	$[0, 0, 0, (q+1), 0]$	$\emptyset$

Table: Plane orbit distributions (OD_2) and point orbit distributions (OD_0) of lines in $\mathrm{PG}(3,q)$, q even