University of Primorska
Prof. Dr. M. Lavrauw

Final exam **Algebra IV** Koper, 20 June 2024

Name: Student number:

Time: 3 hours.

The questions are to be answered with adequate explanation.

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1. (4 points) Give the definition of the following notions:
   (a) an ideal of a commutative ring R;
   (b) a principal ideal domain (PID);
   (c) the Galois group $G(K/F)$ of an extension field K of F;
   (d) a separable extension K of a field F.

2. (8 points) State and prove primitive element theorem.

3. (8 points) Let F be a field. Prove the following statement.

   Every polynomial in $F[X]$ has a root in some extension field of F.

4. (8 points) Prove the following statement.

   Let α, β be algebraic over a field F with $n = \deg(\alpha, F) = \deg(\beta, F)$. The map
   $$\psi_{\alpha,\beta} \ : \ F(\alpha) \to F(\beta) \ : \ \sum a_i \alpha^i \mapsto \sum a_i \beta^i$$
   is an isomorphism if and only if α and β are conjugate over F.

5. (2 points) Find all prime and maximal ideals in $\mathbb{Z}_2 \times \mathbb{Z}_4$.

6. (10 points)
   (a) Give a construction of a field F with 27 elements.
   (b) Determine a primitive element α in F (a generator of the multiplicative group).
   (c) Give a basis B for the field F as a vector space over $\mathbb{Z}_3$.
   (d) Write α^{11} as a linear combination of the elements of B.
   (e) Determine the order of the element $\alpha^4 + \alpha^7$ in the multiplicative group of F.

7. (10 points) Let K denote the splitting field of $f(X) = X^4 + 1$ over $\mathbb{Q}$.
   (a) Determine whether $f(X)$ is irreducible over $\mathbb{Q}$.
   (b) Determine $\alpha \in K$ such that $K = \mathbb{Q}(\alpha)$.
   (c) Describe the elements of the Galois group $G(K/\mathbb{Q})$, in terms of the element α determined in (b).
   (d) For each subgroup H of $G(K/\mathbb{Q})$ of order two, and determine its fixed field (as a subfield of K).
   (e) Write each of the subfields of K determined in (d) as a simple extension of $\mathbb{Q}$.