On rank and orbits in tensor products over finite fields

Michel Lavrauw

Antalya Algebra Days 2014

Tensor products have many applications

- Computational complexity theory
- Tensors describe quantum mechanical systems (entanglement)
- Data analysis (chemistry, biology, physics, ...)
- Signal processing, source separation
- ...

More about these applications can be found in [Landsberg 2012]: "The geometry of tensors", American Mathematical Society

Our original motivation:

- Theory of finite semifields: finite non-associative division algebras.

Tensor product $\bigotimes_{i \in I} V_i$

Consider m vectorspaces V_i over the field $\mathbb{F}$, $I = \{1, \ldots, m\}$.

- fundamental or pure tensors: $v_1 \otimes \ldots \otimes v_m$, $v_i \in V_i$.
- general element $\tau \in \bigotimes_{i \in I} V_i$

$$\tau = \sum_i v_{1i} \otimes \ldots \otimes v_{mi}$$

- the map $(v_1, \ldots, v_m) \mapsto v_1 \otimes \ldots \otimes v_m$ is multilinear
- choosing bases for each V_i we obtain a hypercube $(a_{i_1 i_2 \ldots i_m})$

$$\tau = \sum_{i_1, \ldots, i_m} a_{i_1 i_2 \ldots i_m} e_{1 i_1} \otimes \ldots \otimes e_{m i_m}$$

Tensor products generalise the concept of a matrix

- For $m = 2$, $\tau \in U \otimes V \longleftrightarrow A_\tau = (a_{ij})$

  $$\tau = \sum_{i,\ j} a_{ij} u_i \otimes v_j$$

  where $\{u_i\}$ and $\{v_i\}$ are bases for U and V.

  Example:

  $$\begin{pmatrix} 0 & 0 & 3 \\ 0 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix} \longleftrightarrow 3u_1 \otimes v_3 + 2u_2 \otimes v_2 + u_3 \otimes v_1 - u_3 \otimes v_3$$

- $(\lambda u) \otimes v = u \otimes (\lambda v) = \lambda(u \otimes v)$ (bilinear)
- The fundamental tensors are the rank one matrices:

  $$(1, 0, 2) \otimes (1, 2, 3) \longleftrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 2 & 4 & 6 \end{pmatrix}$$

Matrices and linear maps

- With $\tau = \sum_{i,\ j} a_{ij} u_i \otimes v_j \longleftrightarrow A_\tau = (a_{ij})$, there corresponds a linear map $L_\tau\ :\ U \to V$

$$L_\tau(u) = L_\tau\left(\sum_i x_i u_i\right) = \sum_i x_i L_\tau(u_i) = \sum_i x_i \sum_j a_{ij} v_j$$

Example:

$$\begin{pmatrix} 0 & 0 & 3 \\ 0 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix} \longleftrightarrow 3u_1 \otimes v_3 + 2u_2 \otimes v_2 + u_3 \otimes v_1 - u_3 \otimes v_3$$

$$L_\tau(x, y, z) = (z, 2y, 3x - z)\ (L_\tau\ :\ (x\ y\ z) \mapsto (x\ y\ z)A_\tau)$$

Dual vector space

Alternatively, one defines the linear map L_τ associated to $\tau \in U \otimes V$ using the dual vector space $U^\vee$: the vector space of all linear functionals $u^\vee : U \to \mathbb{F}$.

- basis $\{u_i^\vee\}$ of $U^\vee$
- $u_i^\vee(u_j) = 0$ for $i \neq j$ and $u_i^\vee(u_i) = 1$
- $L_\tau \ : \ U^\vee \to V$:

$$L_\tau(u_k^\vee) = \sum_j a_{kj} v_j = \sum_{i,j} a_{ij} u_k^\vee(u_i) v_j$$

$U \otimes V \cong \mathrm{Hom}(U^\vee, V)$

- linear maps $\leftrightarrow$ 2-fold tensor products
- The isomorphism $U \otimes V \cong \mathrm{Hom}(U^\vee, V)$ is given by:

$$\Psi \ : \ u \otimes v \mapsto \left[w^\vee \mapsto w^\vee(u)v\right]$$

  extended linearly to all vectors of $U \otimes V$.
- Example: $\tau = 3u_1 \otimes v_3 + 2u_2 \otimes v_2 + u_3 \otimes v_1 - u_3 \otimes v_3$
  $\tau^\Psi \in \mathrm{Hom}(U^\vee, V)$ with

$$\tau^\Psi(u_3^\vee) = (3u_1 \otimes v_3 + 2u_2 \otimes v_2 + u_3 \otimes v_1 - u_3 \otimes v_3)^\Psi \ (u_3^\vee)$$

$$= u_3^\vee(3u_1)v_3 + u_3^\vee(2u_2)v_2 + u_3^\vee(u_3)v_1 + u_3^\vee(-u_3)v_3$$

$$= v_1 - v_3$$

Multilinear maps

The isomorphism $U \otimes V \cong \mathrm{Hom}(U^\vee, V)$ can be extended to tensor products with more factors, for instance

- $U \otimes V \otimes W \cong \mathrm{Bil}(U^\vee \otimes V^\vee, W)$

  $$(u \otimes v \otimes w)^\Psi(u^\vee, v^\vee) = u^\vee(u)v^\vee(v)w$$

  extended linearly to $U^\vee \otimes V^\vee$.
- bilinear maps $\leftrightarrow$ 3-fold tensor product
- multilinear maps $\leftrightarrow$ n-fold tensor product

The tensor associated to an algebra

For $U = V = W$, we have $U^\vee \otimes U^\vee \otimes U \cong \mathrm{Bil}(U \otimes U, U)$

- an algebra A is a vectorspace U and a multiplication $f : U \times U \to U$
- f is bilinear
- so with an algebra A there corresponds a three-fold tensor τ_A in $U^\vee \otimes U^\vee \otimes U$

$$(\tau_A)^\Psi = f$$

If for example $\tau_A = \sum_{i=1}^r u_i^\vee \otimes v_i^\vee \otimes w_i$ then

$$f(u, v) = \sum_{i=1}^r u_i^\vee(u) v_i^\vee(v) w_i$$

Main issue for applications: "decomposition"

An expression

$$\tau = \sum_{i=1}^{r} v_{1i} \otimes \ldots \otimes v_{mi} \qquad (1)$$

is called a decomposition of $\tau \in V_1 \otimes \ldots \otimes V_m$.

Four important problems:

- Algorithm
- Uniqueness
- Existence: given τ and r, does (1) exist? $\rightarrow$ rank
- Orbits: how many "different" tensors are there?

Algorithm

Given a tensor $\tau \in \bigotimes_{i=1}^{m} V_i$, is there an algorithm to obtain a decomposition of τ into

$$\tau = \sum_{i=1}^{r} v_{1i} \otimes \ldots \otimes v_{mi} \ ?$$

- Generalizes the "Singular value decomposition": given a complex matrix A, find unitary X, Y and (rectangular) diagonal matrix B such that $A = XBY^*$.
- "PARAFAC", "CANDECOMP", "CP decomposition", ...
- In general no such algorithms exist.

Uniqueness

Given a tensor $\tau \in \bigotimes_{i=1}^m V_i$, is the decomposition of τ into

$$\tau = \sum_{i=1}^{r} v_{1i} \otimes \ldots \otimes v_{mi} \text{ "essentially" unique?}$$

or is there a finite number of expressions?

- [Kruskal 1977]: a condition in which uniqueness holds ($m = 3$)
- Are there similar results known?

This talk

1. "Existence"

2. "Orbits"

1. Existence → rank of a tensor

Consider

$$\tau = \sum_{i=1}^{r} v_{1i} \otimes \ldots \otimes v_{mi} \qquad (1)$$

- the rank of τ is the minimum r such that (1) exists
- notation $\mathrm{rk}(\tau)$
- examples (for suitable u, v, w, x, u_i, v_i)
  - $\mathrm{rk}(u \otimes v \otimes w \otimes x) = 1$;
  - $\mathrm{rk}(u_1 \otimes v + u_2 \otimes v) = 1$, since $u_1 \otimes v + u_2 \otimes v = (u_1 + u_2) \otimes v$;
  - $\mathrm{rk}(u_1 \otimes v_1 + u_2 \otimes v_2) = 2$.

Introduced in [F.L. Hitchcock, The expression of a tensor or a polyadic as a sum of products, J. of Mathematics and Physics 6 (1927), 164–189.]

Tensor rank

Frank Lauren Hitchcock

From Wikipedia, the free encyclopedia

Frank Lauren Hitchcock (1875–1957) was an American mathematician and physicist notable for vector analysis. He formulated the transportation problem in 1941. He was also an expert in mathematical chemistry and quaternions.

F. L. Hitchcock

Frank Lauren Hitchcock (1875–1957)

Born	March 6, 1875 New York, USA
Died	May 31, 1957 Los Angeles, USA
Residence	USA
Nationality	American
Fields	Physicist and mathematician
Institutions	Massachusetts Institute of Technology North Dakota State University
Alma mater	Harvard University Phillips Andover Academy
Doctoral students	Gleason Kenrick Claude Shannon
Known for	Transportation problem

Contents [hide]

Education [edit]

He first attended the Phillips Andover Academy. He received his AB from Harvard in 1896. Before his PhD he taught at Paris and at Kenyon College in Gambier, Ohio. In 1910 he completed his PhD at Harvard with a thesis entitled, *Vector Functions of a Point*.

Career [edit]

In 1904–1906 he was a professor of chemistry at North Dakota State University, Fargo, and then he moved to become a professor of mathematics at Massachusetts Institute of Technology.

Personal life [edit]

His mother was Susan Ida Porter (b. 1 January 1848, Middlebury, Vermont) and

The rank of a tensor is not so easy ...

- the rank of $\tau \in V_1 \otimes V_2$, corresponds to usual rank of $\tau^{\Psi} \in \mathrm{Hom}(V_1^{\vee}, V_2)$.
- Gaussian elimination: computationally the rank of a matrix (naively) takes about $n \cdot n^3$ multiplications
- in general, computing the rank in $\bigotimes_{i=1}^{m} V_i$ is very difficult.
- in most cases no algorithm available!

Illustration of a rank problem

The rank of matrix multiplication: computational complexity

- $M_{n,n,n} \in \mathrm{Bil}(K^{n^2} \times K^{n^2}, K^{n^2})$ matrix multiplication.
- $\mathrm{rk}(M_{n,n,n})$ is the rank of the associated tensor in $(K^{n^2})^\vee \otimes (K^{n^2})^\vee \otimes K^{n^2}$, using the isomorphism $\mathrm{Bil}(A \times B, C) \cong A^\vee \otimes B^\vee \otimes C$:

$$\alpha \otimes \beta \otimes c \mapsto [(a,b) \mapsto \alpha(a)\beta(b)c]$$

- $\mathrm{rk}(M_{n,n,n})$ measures the number of multiplications needed
- [Strassen1969] $\mathrm{rk}(M_{2,2,2}) \leq 7$.
- [Winograd1971] $\mathrm{rk}(M_{2,2,2}) = 7$.
- for 3×3 matrices,

$$19 \leq \mathrm{rk}(M_{3,3,3}) \leq 23$$

The rank of a subspace and contraction

Definition
(i) The rank of a subspace U ($\mathrm{rk}(U)$) of $\bigotimes_{i=1}^{m} V_i$ is the minimum number of fundamental tensors that is needed to span a subspace containing U.
(ii) For every $u_i^\vee \in V_i^\vee$, define the contraction $u_i^\vee(\tau)$ of $\tau = v_1 \otimes v_2 \otimes \ldots \otimes v_m$ by

$$u_i^\vee(\tau) = u_i^\vee(v_i) v_1 \otimes \ldots \otimes v_{i-1} \otimes v_{i+1} \otimes \ldots \otimes v_m,$$

and extend linearly to define the contraction of any tensor.
Proposition
If $\tau \in \bigotimes_{i=1}^{m} V_i$, then for each $j \in \{1, \ldots, m\}$: $\mathrm{rk}(\tau) = \mathrm{rk}(T_j)$, where $T_j = \langle u_j^\vee(\tau) \ : \ u_j^\vee \in V_j^\vee \rangle$.

Groups and geometry

- Segre embedding:

$$\sigma \ : \ \mathrm{PG}(V_1) \times \mathrm{PG}(V_2) \times \ldots \times \mathrm{PG}(V_m) \ \rightarrow \ \mathrm{PG}(\bigotimes_i V_i)$$

$$(\langle v_1 \rangle, \langle v_2 \rangle, \ldots, \langle v_m \rangle) \mapsto \langle v_1 \otimes v_2 \otimes \ldots \otimes v_m \rangle$$

- $S_{n_1, n_2, \ldots, n_m}(F) = Im(\sigma)$ is the Segre variety

Group action

- An element $(g_1, g_2, \ldots g_m)$ of $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \times \ldots \times \mathrm{GL}(V_m)$ acts on points of the Segre variety as follows:

$$\langle v_1 \otimes v_2 \otimes \ldots \otimes v_m \rangle \mapsto \langle v_1^{g_1} \otimes v_2^{g_2} \otimes \ldots \otimes v_m^{g_m} \rangle.$$

- If $V_i = V = V(n, F)$ for all i, then we also have an action of S_m as follows:

$$\pi : \langle v_1 \otimes v_2 \otimes \ldots \otimes v_m \rangle \mapsto \langle v_{\pi(1)} \otimes v_{\pi(2)} \otimes \ldots \otimes v_{\pi(m)} \rangle.$$

$V_i = V = V(n, F)$

- The wreath product $\mathrm{GL}(V) \wr S_m$ induces a subgroup G_m of $\mathrm{PGL}(n^m - 1, F)$.
- G_m stabilizes $X := S_{n,\dots,n}$ and the set of maximal subspaces of X,

  for example $\sigma(PG(V_1) \times v_2 \times \ldots \times v_m)$.

  NOTE: the rank is invariant under the action of G_m

The 3-fold tensor product

- $m = 1$: $\mathrm{rk}(u) \leq 1$, $\forall u \in V_1$
- $m = 2$: $V_1 \otimes V_2 \cong \mathcal{M}(n_1, n_2, K)$: $\mathrm{rk}(u) = \mathrm{rk}(M_u)$
- $m = 3$: What is the maximum rank in $V_1 \otimes V_2 \otimes V_3$?

$\rightarrow$ Maximum rank depends on the dimension of the factors and on the ground field.
$\rightarrow$ Known results are over $\mathbb{C}$.
$\rightarrow$ We will focus on the case $n_1 = n_2 = n_3$.

The rank in $K^m \otimes K^m \otimes K^m$

Put $U = V = W = K^m$.
The **Proposition** implies: if $\tau \in K^m \otimes K^m \otimes K^m$, $\mathrm{rk}(\tau) = \mathrm{rk}(T_1)$, where $T_1 = \langle u^\vee(\tau) \ : \ u^\vee \in U^\vee \rangle$ is a subspace of $K^m \otimes K^m$.

- $\Rightarrow$ trivial upper bound is m^2

Theorem (Atkinson-Stephens 1979)

If $K = \mathbb{C}$, then the maximum rank $\leq \frac{1}{2}m^2 + O(m)$

- As far as we know, this is still the best result of its kind.

The 3-fold tensor product for general fields K

- the proof of Atkinson-Stephens depends on the fact that $\mathbb{C}$ is algebraically closed and separable.

$m = 1$ The rank in $K \otimes K \otimes K$: trivial

$m = 2$ The rank in $K^2 \otimes K^2 \otimes K^2$

Theorem

The rank of a $2 \times 2 \times 2$ tensor is at most 3 over any field.

Proof Each line in $\mathrm{PG}(3, K)$ lies in a plane spanned by three points on $S_{2,2}(K)$. $\square$

$m = 3$ The rank in $K^3 \otimes K^3 \otimes K^3$

The rank in $K^3 \otimes K^3 \otimes K^3$

Theorem (ML - A. Pavan - C. Zanella 2013)

The rank of a $3 \times 3 \times 3$ tensor is at most six over any field.

Proof (sketch)

- We need to prove: each point of $\langle S_{3,3,3}(K) \rangle$ is contained in a subspace spanned by six points of $S_{3,3,3}(K)$.
- $\tau \in U \otimes V \otimes W \to N = \langle u^\vee(\tau) \ : \ u^\vee \in U^\vee \rangle \subset \mathrm{PG}(V \otimes W)$
- N is contained in a plane of $\langle S_{3,3}(K) \rangle \Rightarrow N = \langle \sigma, L \rangle$, $\sigma \in V \otimes W$, for some line L (w.l.o.g.)
- choose bases v_1, v_2, v_3 and w_1, w_2, w_3 s.t.

$$\sigma \in \langle v_1 \otimes w_1, v_2 \otimes w_2, v_3 \otimes w_3 \rangle =: D$$

- show that L is contained in the span of D and $\leq$ three other points of $S_{3,3}(K)$. $\square$

The rank in $K^3 \otimes K^3 \otimes K^3$

Theorem (ML - A. Pavan - C. Zanella 2013)

The rank of a $3 \times 3 \times 3$ tensor is at most six over any field.

$\rightarrow$ This bound is sharp

- the tensor rank of in an algebra $\mathbb{S}$: $\mathrm{Trk}(\mathbb{S})$
- we have $\mathrm{Trk}(\mathbb{F}_{q^n}/\mathbb{F}_q) \geq 2n-1$, with equality iff $q \geq 2n-2$ [Winograd 1979], [de Groote 1983]
- for $n = 3$ it follows that $\mathrm{Trk}(\mathbb{F}_{2^3}/\mathbb{F}_2) = \mathrm{Trk}(\mathbb{F}_{3^3}/\mathbb{F}_3) = 6$

2. Orbits

- aim: determine the orbits of points of $\mathrm{PG}(\bigotimes_{i=1}^{m} V_i)$ under the action of G_m
- since the rank is invariant, the number of orbits is at least the maximum rank in $\mathrm{PG}(\bigotimes_{i=1}^{m} V_i)$
- a tensor T is nonsingular if applying any $m-1$ consecutive nonzero contractions never returns the zero vector
- nonsingularity is invariant under G_m
- observe that $v \in V_1 \otimes V_2$ is nonsingular if and only if the corresponding homomorphism in $\mathrm{Hom}(V_1^\vee, V_2)$ is nonsingular

Nonsingular tensors and semifields (planes)

[Liebler1981]

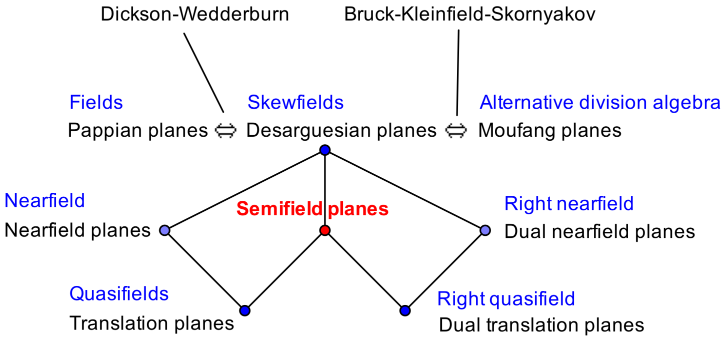
semifield $\mathbb{S} \mapsto T_{\mathbb{S}} \in V_1 \otimes V_2 \otimes V_3, \; V_i \cong K^n$

Theorem

(i) *The tensor* $T_{\mathbb{S}} \in V_1 \otimes V_2 \otimes V_3$ *is nonsingular.*
(ii) *To every nonsingular tensor* $T \in V_1 \otimes V_2 \otimes V_3$ *there corresponds a presemifield* $\mathbb{S}$ *for which* $T = T_{\mathbb{S}}$.
(iii) *The map* $\mathbb{S} \mapsto T_{\mathbb{S}}$ *is injective.*

- semifields $\leftrightarrow$ projective planes (with a lot of symmetry)

Types of finite translation planes



[Hughes - Piper, Projective Planes, Springer, 1973]

Orbits and isotopism (isomorphism)

- isomorphism classes (planes) $\leftrightarrow$ isotopism classes (semifields) $\leftrightarrow$ orbits on tensors
- the Knuth orbit of a semifield $\mathbb{S}$ is represented in the projective space $\mathrm{PG}(n^3 - 1, q)$ as the orbit of $P_{\mathbb{S}}$ under the group G_n.
- the tensor rank of a semifield is an invariant for the Knuth orbit of a semifield [ML2012]

Geometric characterisation of singular tensors

Theorem (ML 2012)

A tensor $\tau \in K^n \otimes K^n \otimes K^n$ is singular if and only if

$$\langle \tau \rangle \subset \langle x_1, \ldots, x_j, S_{k_1,k_2,k_3} \rangle$$

for some $j < n$ points and a S_{k_1,k_2,k_3} properly contained in $S_{n,n,n}$.

We will use this result for $n = 2$.

The orbits in $K^2 \otimes K^2 \otimes K^2$

Theorem (ML-J. Sheekey 2014)

There exist precisely four G_3-orbits of singular tensors in $K^2 \otimes K^2 \otimes K^2$.

- Glynn et al (2006) showed this computationally for $\mathbb{F}_2$.
- Havlicek-Odehnal-Saniga (2011) proved this geometrically for $\mathbb{F}_2$.

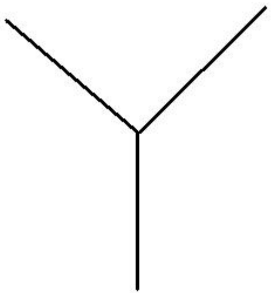
Corollary

For $n = 2$, the number of orbits of tensors is

- *five if F is finite;*
- *five if $F = \mathbb{R}$;*
- *four if F is algebraically closed;*
- *infinite if $F = \mathbb{Q}$.*

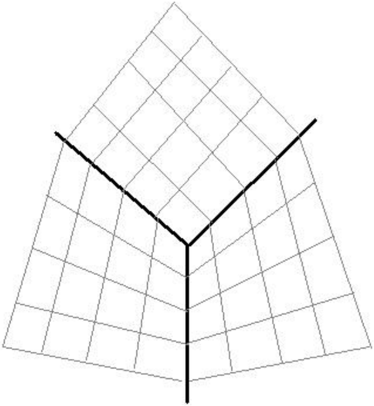
Sketch of proof

It is well known that every point $y = y_1 \otimes y_2 \otimes y_3$ lies on precisely three lines of the Segre variety X:



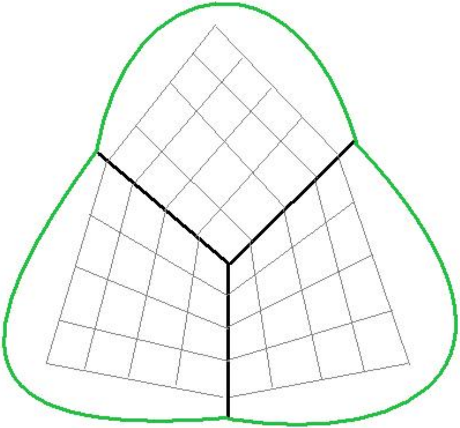
$$l_3(y) := \sigma(y_1 \times y_2 \times \mathrm{PG}(V))$$

Each pair of lines lie on a sub-Segre variety which is a hyperbolic quadric:



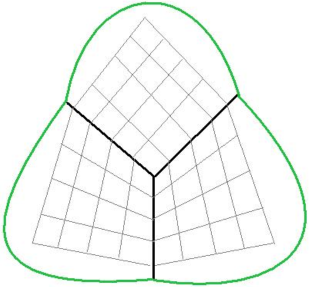
$$Q_1(y) := \sigma(y_1 \times \mathrm{PG}(V) \times \mathrm{PG}(V))$$

Each quadric spans a 3-space:



$$\mathcal{L}_i(y) := \langle Q_i(y) \rangle$$

The shamrock of a point y, denoted by $Sh(y)$, is the union of the three 3-spaces $\mathcal{L}_i(y)$, and we call $\mathcal{L}_i(y)$ a leaf.



Clearly G_3 sends a shamrock to a shamrock, a leaf to a leaf etc.

- The enumeration of the orbits goes by the rank of the points.
- We know from before that the maximum rank in $K^2 \otimes K^2 \otimes K^2$ is three.

Rank one and two

The rank one points (i.e. the points of X) form an orbit $\mathcal{O}_1$.

Any rank 2 point is contained in a line spanned by two points of X, say $\langle y, z \rangle$

Lemma

There exist precisely two orbits of rank two tensors.

Denote these by $\mathcal{O}_2$ and $\mathcal{O}_3$.

Rank three...

Next we consider planes $\pi = \langle y, z, w \rangle$, $y, z, w \in X$, with
$y = y_1 \otimes y_2 \otimes y_3$, $z = z_1 \otimes z_2 \otimes z_3$, $w = w_1 \otimes w_2 \otimes w_3$.

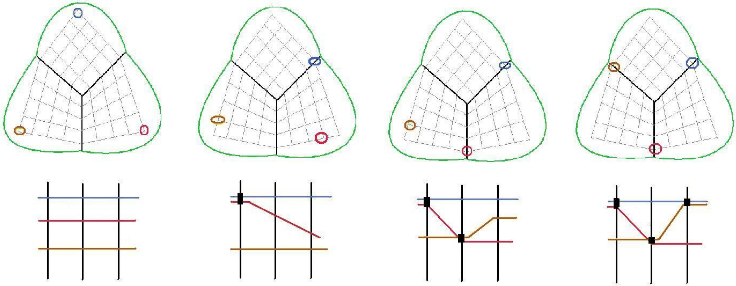
- We may assume π contains no lines of X;
- We may assume π is not contained in any leaf;

  (as then everything on π would have rank at most two).
- We will consider the shamrock of the point $u = y_1 \otimes z_2 \otimes w_3$. Then $y \in \mathcal{L}_1(u)$, $z \in \mathcal{L}_2(u)$ and $w \in \mathcal{L}_3(u)$.

  We need to consider four possibilities...

Possibilities for rank three...

We need to consider four possibilities...



$(3,3,3)$ $(2,3,3)$ $(2,2,3)$ $(2,2,2)$

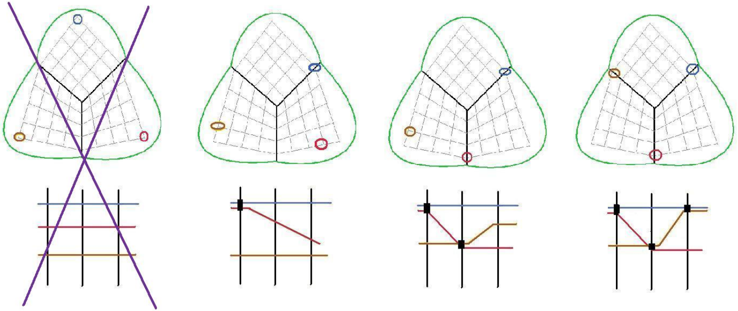
The geometric characterisation from before implies that every singular point is contained in the span of a point and a quadric $\langle y, Q_i(z) \rangle$, and hence:

Corollary

A tensor of rank three is singular if and only if it lies on a plane of type (a_1, a_2, a_3), with some $a_i = 2$.

Possibilities for rank three...

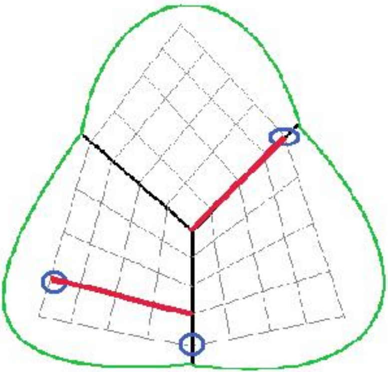
Hence we need to consider ~~four~~ three possibilities...



$(3,3,3)$ $(2,3,3)$ $(2,2,3)$ $(2,2,2)$

$(2, 2, 3)$ planes...

Every point on a $(2, 2, 3)$-plane has rank at most 2:



$(2,3,3)$ planes...

- Every point on a $(2,3,3)$-plane has rank at most 2 OR lies on a plane of type $(2,2,2)$.
- The rank three points on $(2,2,2)$-planes form a single orbit $\mathcal{O}_4$.

Hence...

Theorem (ML - J.Sheekey 2014)

For $n = 2$, there exist precisely four G_3-orbits of singular tensors over any field.

Corollary

For $n = 2$, the number of orbits of tensors is

- *five if F is finite;*
- *five if $F = \mathbb{R}$;*
- *four if F is algebraically closed;*
- *infinite if $F = \mathbb{Q}$.*

Thank you for your attention!

Tensor rank depends on the field

Example

$$\tau = e_0 \otimes e_0 \otimes e_0 + e_0 \otimes e_1 \otimes e_1 - e_1 \otimes e_0 \otimes e_1 + e_1 \otimes e_1 \otimes e_0$$

- τ has rank 3 over $\mathbb{R}$:

$$\tau = (e_0 - e_1) \otimes e_0 \otimes e_0 + (e_0 + e_1) \otimes e_1 \otimes e_1 + e_1 \otimes (e_0 + e_1) \otimes (e_0 - e_1)$$

- τ has rank 2 over $\mathbb{C}$:

$$\tau = \left(\frac{1}{2} e_0 + \frac{1}{2i} e_1\right) \otimes (e_0 + i e_1) \otimes (e_0 + i e_1)$$

$$+ \left(\frac{1}{2} e_0 - \frac{1}{2i} e_1\right) \otimes (e_0 - i e_1) \otimes (e_0 - i e_1)$$

- τ has rank 2 over $\mathbb{F}_q$ iff $q \equiv 1 \bmod 4$