On linear sets on a projective line

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Abstract

Linear sets generalise the concept of subgeometries in a projective space. They have many applications in finite geometry. In this paper we address two problems for linear sets: the equivalence problem and the intersection problem. We consider linear sets as quotient geometries and determine the exact conditions for two linear sets to be equivalent. This is then used to determine in which cases all linear sets of rank 3 of the same size on a projective line are (projectively) equivalent. In [4], the intersection problem for subgeometries of $\mathrm{PG}(n,q)$ is solved. The intersection of linear sets is much more difficult. We determine the intersection of a subline $\mathrm{PG}(1,q)$ with a linear set in $\mathrm{PG}(1,q^h)$ and investigate the existence of *irregular* sublines, contained in a linear set. We also derive an upper bound, which is sharp for odd q, on the size of the intersection of two different linear sets of rank 3 in $\mathrm{PG}(1,q^h)$.

1 Introduction and preliminaries

Linear sets have been intensively used in recent years in order to classify, construct or characterise various structures related to finite incidence geometry. Besides their relation to blocking sets (see below), linear sets also appear in the study of translation ovoids (see e.g. [16]), and are of high relevance in the theory of semifields (e.g. [13]).

In contrast to the number of papers that make use of linear sets, few papers have been published in which linear sets are the main object. To our knowledge the only general treatment of linear sets is by Polverino [18]. Apart from the lack of knowledge about linear sets, in particular about the possible intersections of linear sets, the main motivation for this research arose from the study of blocking sets in finite projective spaces.

If V is a vector space, then we denote by $\mathrm{PG}(V)$ the corresponding projective space. If V has dimension n over the finite field $\mathbb{F}_q$ with q elements, then we also write $\mathrm{PG}(n-1,q)$.

A small minimal k-blocking set is a set B of points in the projective space $\mathrm{PG}(n,q)$, meeting every $(n-k)$-space, where $|B| < 3(q^k+1)/2$ and such that no proper subset of B is a k-blocking set. The term "linear" was used by Lunardon to describe a particular construction of a small minimal blocking set [15], which led to the first construction of small blocking sets that are not of Rédei type

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[17]. This construction opened up a new perspective on blocking sets and soon people conjectured that all small minimal k-blocking sets in $\mathrm{PG}(n, q)$ must be linear. However, it took until 2008 for this so-called “Linearity conjecture” to be formally stated in the literature, see Sziklai [19]. The Linearity conjecture is still open, apart from a few specific instances. Recently, a proof of the Linearity conjecture in $\mathrm{PG}(n, p^3)$, p prime[1], was given in [14], where the authors use that an $\mathbb{F}_q$-linear set S of rank 3 and an $\mathbb{F}_q$-subline of $\mathrm{PG}(1, q^3)$, not contained in S, have at most 3 points in common. This served as an extra motivation and it is our hope that the results obtained here will bring us closer to solving the Linearity conjecture for blocking sets. One of the theorems in this paper, of relevance here, determines the intersection of an $\mathbb{F}_q$-linear set of rank k with an $\mathbb{F}_q$-subline in $\mathrm{PG}(1, q^h)$.

We also study the possible intersections of two linear sets of rank 3 (Section 5). In the case where the linear sets are subgeometries this problem was recently solved in [4] by Durante and Donati (completing a study originated by Bose et al. in 1980) showing that the intersection of two subgeometries is necessarily the union of subgeometries in independent subspaces. The intersection of linear sets is considerably more difficult, and it is not necessarily the union of linear sets in independent subspaces.

Let V be an r-dimensional vector space over a finite field $\mathbb{F}$. A set S of points of $\mathrm{PG}(V)$ is called a *linear set* (*of rank* t) if there exists a subset U of V that forms a (t-dimensional) $\mathbb{F}_q$-vector space for some $\mathbb{F}_q \subset \mathbb{F}$, such that $S = \mathcal{B}(U)$, where

$$\mathcal{B}(U) := \{\langle u \rangle_{\mathbb{F}} \ : \ u \in U \setminus \{0\}\}.$$

If we want to specify the subfield we call S an $\mathbb{F}_q$-*linear set*. In other words, if $\mathbb{F} = \mathbb{F}_{q^n}$, we have the following diagram

$$\begin{array}{ccccccc}
 & & \mathbb{F}_{q^n}^r & \longleftrightarrow & \mathbb{F}_q^{rn} & \supseteq & U \\
 & & \updownarrow & & \updownarrow & & \updownarrow \\
\mathcal{B}(U) & \subseteq & \mathrm{PG}(r-1, q^n) & \longleftrightarrow & \mathrm{PG}(rn-1, q) & \supseteq & \mathrm{PG}(U)
\end{array}$$

It should be clear that any subset T of $\mathbb{F}_q^{rn}$ or $\mathrm{PG}(rn-1, q)$ induces a subset in $\mathrm{PG}(r-1, q^n)$ in this way. In what follows we will use the same notation $\mathcal{B}(\pi)$ for the set of points of $\mathrm{PG}(r-1, q^n)$ induced by π, regardless of π being a subspace or a subset of $\mathbb{F}_q^{rn}$ or $\mathrm{PG}(rn-1, q)$. Since the points of $\mathrm{PG}(r-1, q^n)$ correspond to 1-dimensional subspaces of $\mathbb{F}_{q^n}^r$, and by field reduction to n-dimensional subspaces of $\mathbb{F}_q^{rn}$, they correspond to a set $\mathcal{D}$ of $(n-1)$-dimensional subspaces of $\mathrm{PG}(rn-1, q)$, which partitions the point set of $\mathrm{PG}(rn-1, q)$. The set $\mathcal{D}$ is called a *Desarguesian spread*, and we have a one-to-one correspondence between the points of $\mathrm{PG}(r-1, q^n)$ and the elements of $\mathcal{D}$. This gives us a more geometric perspective on the notion of a linear set; namely, an $\mathbb{F}_q$-linear set is a set S of points of $\mathrm{PG}(r-1, q^n)$ for which there exists a subspace π in $\mathrm{PG}(rn-1, q)$ such that the points of S correspond to the elements of $\mathcal{D}$ that have a non-empty intersection with π. Also in what follows, we will often identify the elements of $\mathcal{D}$ with the points of $\mathrm{PG}(r-1, q^n)$, which allows us to view $\mathcal{B}(\pi)$ as a subset of $\mathcal{D}$. This is illustrated by the following diagram, where

[1] The same result was proven in [10], [11].

$\mathcal{P}$ denotes the set of points of $\mathrm{PG}(r-1,q^n)$.

$$\begin{array}{ccccccc} & & \mathrm{PG}(r-1,q^n) & \longleftrightarrow & \mathrm{PG}(rn-1,q) & \supseteq & \pi \\ & & \downarrow & & \downarrow & & \Downarrow \\ \mathcal{B}(\pi) & \subseteq & \mathcal{P} & \longleftrightarrow & \mathcal{D} & \supseteq & \mathcal{B}(\pi) \end{array}$$

If P is a point of $\mathcal{B}(\pi)$ in $\mathrm{PG}(r-1,q^n)$, where π is a subspace of $\mathrm{PG}(rn-1,q)$, then we define the *weight of* P as $wt(P) := \dim(P\cap\pi)+1$, where P is identified with an element of $\mathcal{D}$. This makes a point to have weight 1 if its corresponding spread element intersects π in a point. Note that if $\pi = \mathrm{PG}(U)$, then $wt(P) = \dim\langle\{v \in U \mid \mathrm{PG}(v) = P\}\rangle$. If α is a collineation of $\mathrm{PG}(r-1,q^n)$, then α is induced by a non-singular semi-linear map φ of $V(r,q^n)$. Let $\langle v_1,\ldots,v_{wt(P)}\rangle$ be a basis for $\langle\{v \in U \mid \mathrm{PG}(v) = P\}\rangle$. The vectors v_i^φ are linearly independent and clearly belong to $\langle\{v \in U^\varphi \mid \mathrm{PG}(v) = P^\alpha\}\rangle$, hence, $wt(P) \leq wt(P^\alpha)$. Repeating this argument for the collineation α^{-1} shows that $wt(P) = wt(P^\alpha)$, hence, a collineation preserves the weight of a point.

2 Equivalent linear sets

Two subsets S and T of $\mathrm{PG}(n,q)$ are called *projectively equivalent* if there is an element $\phi \in \mathrm{PGL}(n+1,q)$ such that $\phi(S) = T$, and *equivalent* if there is an element $\phi \in \mathrm{P\Gamma L}(n+1,q)$ such that $\phi(S) = T$.

The proofs in this section make use of yet another perspective on linear sets, namely as the quotient geometry of a canonical subgeometry. If $\mathcal{G}$ is a frame in $\mathrm{PG}(m,q^n)$, then the set of points (and subspaces) obtained by restricting the coordinates with respect to $\mathcal{G}$ to $\mathbb{F}_q$ is called a *canonical subgeometry* $\mathrm{PG}(m,q)$ of $\mathrm{PG}(m,q^n)$. It should be clear that a canonical subgeometry is a linear set, but not conversely. However, the following theorem of Lunardon and Polverino shows that every linear set is a projection of a canonical subgeometry. Let $\Sigma = \mathrm{PG}(m,q)$ be a canonical subgeometry of $\Sigma^* = \mathrm{PG}(m,q^n)$ and suppose there exists an $(m-r)$-dimensional subspace Ω^* of Σ^* disjoint from Σ. Let $\Omega = \mathrm{PG}(r-1,q^n)$ be an $(r-1)$-dimensional subspace of Σ^* disjoint from Ω^*, and let Γ be the projection of Σ from Ω^* to Ω. Let $p_{\Omega^*,\Omega}$ denote the map defined by $x \mapsto \langle\Omega^*, x\rangle \cap \Omega$ for each point $x \in \Sigma^* \setminus \Omega^*$.

Theorem 1. [16] *If* Γ *is a projection of* $\mathrm{PG}(m,q)$ *into* $\Omega = \mathrm{PG}(r-1,q^n)$, *then* Γ *is an* $\mathbb{F}_q$*-linear set of rank* $m+1$ *and* $\langle\Gamma\rangle = \Omega$. *Conversely, if* L *is an* $\mathbb{F}_q$*-linear set of* Ω *of rank* $m+1$ *and* $\langle L\rangle = \Omega = \mathrm{PG}(r-1,q^n)$, *then either* L *is a canonical subgeometry of* Ω *or there are an* $(m-r)$*-dimensional subspace* Ω^* *of* $\Sigma^* = \mathrm{PG}(m,q^n)$ *disjoint from* Ω *and a canonical subgeometry* Σ *of* Σ^* *disjoint from* Ω^* *such that* $L = p_{\Omega^*,\Omega}(\Sigma)$.

If we consider the quotient space Σ^*/Ω^* instead of the projection we obtain the following.

Theorem 2. *If* Γ *is the quotient of* $\mathrm{PG}(m,q)$ *in* $\Sigma^*/\Omega^* \cong \mathrm{PG}(r-1,q^n)$, *then* Γ *is an* $\mathbb{F}_q$*-linear set of rank* $m+1$ *and* $\langle\Gamma\rangle = \Sigma^*/\Omega^*$. *Conversely, if* L *is an* $\mathbb{F}_q$*-linear set of rank* $m+1$ *and* $\langle L\rangle = \mathrm{PG}(r-1,q^n)$, *then there are an* $(m-r)$*-dimensional subspace* Ω^* *of* $\Sigma^* = \mathrm{PG}(m,q^n)$ *and a canonical subgeometry* Σ *of* Σ^* *disjoint from* Ω^* *such that* L *is the quotient of* Σ *in* $\Sigma^*/\Omega^* \cong \mathrm{PG}(r-1,q^n)$.

Using this perspective on linear sets we derive the following equivalences which will be used in this section. For the particular case of $\mathbb{F}_q$-linear sets of rank $n+1$ in $\mathrm{PG}(2, q^n)$, this was proven in [3].

Theorem 3. *Let S_i be the $\mathbb{F}_q$-linear set of rank $m+1$ in $\mathrm{PG}(r-1, q^n)$, defined as the quotient of $\Sigma_i \cong \mathrm{PG}(m, q)$ in Σ^*/Ω_1^*, where $\langle \Sigma_i \rangle = \Sigma^* \cong \mathrm{PG}(m+1, q^n)$, $i = 1, 2$ and suppose that S_i is not a linear set of rank n with $n < m+1$. The following statements are equivalent.*

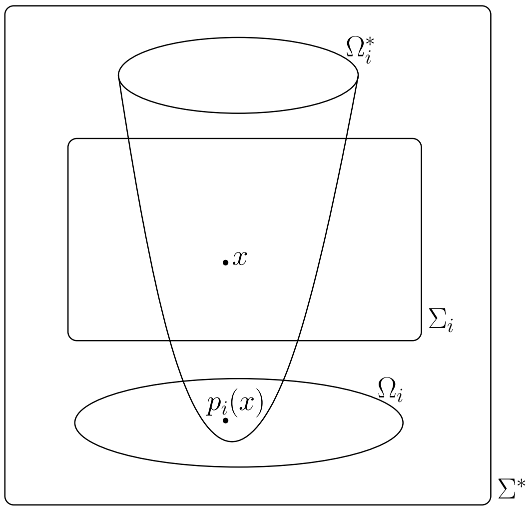
(i) There exists an element $\alpha \in \mathrm{P\Gamma L}(r, q^n)$ such that $S_1^\alpha = S_2$.

(ii) There exists an element $\beta \in \mathrm{Aut}(\Sigma^*)$ such that $\Sigma_1^\beta = \Sigma_2$ and $(\Omega_1^*)^\beta = \Omega_2^*$.

(iii) For all canonical subgeometries $\Sigma \cong \mathrm{PG}(m, q)$ in Σ^*, there exist elements $\delta, \phi, \psi \in \mathrm{Aut}(\Sigma^*)$, such that $\Sigma^\delta = \Sigma$ and $(\Omega_1^*)^{\phi\delta} = (\Omega_2^*)^\psi$, $\Sigma_1^\phi = \Sigma$ and $\Sigma_2^\psi = \Sigma$.

Proof. (ii)⇒(i) Define $\alpha : S_1 \to S_2$: $\langle \Omega_1^*, x \rangle / \Omega_1^* \mapsto \langle \Omega_2^*, x^\beta \rangle / \Omega_2^*$ with $x \in \Sigma_1$. The map α is well defined: every element of S_1 can be written as $\langle \Omega_1^*, z \rangle / \Omega_1^*$ for some $z \in \Sigma_1$. Suppose that $\langle \Omega_1^*, z \rangle / \Omega_1^* = \langle \Omega_1^*, z' \rangle / \Omega_1^*$ for some z, z' in Σ_1, then, since $\dim \langle z, z', \Omega_1^* \rangle = \dim \langle z^\beta, z'^\beta, (\Omega_1^*)^\beta \rangle$, $(\langle \Omega_1^*, z \rangle / \Omega_1^*)^\alpha = (\langle \Omega_1^*, z' \rangle / \Omega_1^*)^\alpha$. The map α is a collineation: if the points $\langle \Omega_1^*, z \rangle / \Omega_1^*$, $\langle \Omega_1^*, z' \rangle / \Omega_1^*$, and $\langle \Omega_1^*, z'' \rangle / \Omega_1^*$ are collinear, then $\dim \langle z, z', z'', \Omega_1^* \rangle = m - r + 2 = \dim \langle z^\beta, z'^\beta, z''^\beta, \Omega_2^* \rangle$, hence, the points $(\langle \Omega_1^*, z \rangle / \Omega_1^*)^\alpha$, $(\langle \Omega_1^*, z' \rangle / \Omega_1^*)^\alpha$, and $(\langle \Omega_1^*, z'' \rangle / \Omega_1^*)^\alpha$ are collinear. Moreover, $S_1^\alpha = (\langle \Omega_1^*, \Sigma \rangle / \Omega_1^*)^\alpha = \langle \Omega_2^*, \Sigma \rangle / \Omega_2^* = S_2$.

(i)⇒ (ii) Let Ω_i be an $(r-1)$-dimensional space in Σ^*, skew to Ω_i^*, and denote the projection of the point $x \notin \Omega_i^*$ from Ω_i^* on Ω_i, i.e. $\langle \Omega_i^*, x \rangle \cap \Omega_i$, by $p_i(x)$, $i = 1, 2$.



The collineation α induces a collineation χ of Σ^*, with $\Omega_1^{*\chi} = \Omega_2^*$. Part 1 of this proof, applied to χ, implies that $\Sigma_1/\Omega_1^* \cong \Sigma_1^\chi/\Omega_1^{*\chi} = \Sigma_1^\chi/\Omega_2^*$. Hence, $\Sigma_1^\chi/\Omega_2^* \cong \Sigma_2/\Omega_2^*$.

We claim that there exist m linearly independent points $a_0, \ldots, a_m$ in Σ_1^χ and m linearly independent points $b_0, \ldots, b_m$ in Σ_2 such that $p_2(a_i) = p_2(b_i)$. Note that, since $\Sigma_1^\chi/\Omega_2^* \cong \Sigma_2/\Omega_2^*$, for every x in Σ_1^χ, there is at least one point x' in Σ_2 with $p_2(x) = p_2(x')$. Suppose that for every choice for a basis $\{x_0, \ldots, x_m\}$, the set $\{x'_0, \ldots, x'_m\}$ (of possibly coinciding points), does not span the space Σ_2. Let $\{a_0, \ldots, a_m\}$ be a basis for Σ_1^χ such that $\dim\langle a'_0, \ldots, a'_m\rangle = \mu < m$ is maximal. W.l.o.g. the set $\{a'_0, \ldots, a'_\mu\}$ is a basis for $\langle a'_0, \ldots, a'_m\rangle$. Denote the space $\langle a_0, \ldots, a_\mu\rangle$ by π_μ, and $\langle a'_0, \ldots, a'_\mu\rangle$ by π'_μ. Let $\{a'_0, \ldots, a'_\mu, b_{\mu+1}, \ldots, b_m\}$ be a basis of Σ_2. Define the projectivity γ from Σ_1^χ to Σ_2, with $a_i^\gamma = a'_i$ for all $i = 0, \ldots, \mu$, and $a_i^\gamma = b_i$ for all $i = \mu+1, \ldots, m$. For all $x \in \pi_\mu$, $x = \sum_{i=0}^{\mu} \lambda_i a_i$, so

$$p_2(x^\gamma) = p_2(\sum_{i=0}^{\mu} \lambda_i a_i^\gamma) = \sum_{i=0}^{\mu} \lambda_i p_2(a_i^\gamma) = \sum_{i=0}^{\mu} \lambda_i p_2(a_i) = p_2(\sum_{i=0}^{\mu} \lambda_i a_i) = p_2(x).$$

Suppose that there is a point $z \in \Sigma_1^\chi \setminus \langle a_0, \ldots, a_\mu\rangle$, such that there is a point z' in $\Sigma_2 \setminus \langle a'_0, \ldots, a'_\mu\rangle$ with $p_2(z) = p_2(z')$. Let $\{a_0, \ldots, a_\mu, z, c_{\mu+1}, \ldots, c_m\}$ be a basis of Σ_1^χ, then $p_2(a_i) = p_2(a'_i)$ for all $i = 0, \ldots, \mu$, $p_2(z) = p_2(z')$, and $\dim\langle a'_0, \ldots, a'_\mu, z'\rangle > \mu$, a contradiction. Hence, for all points $z \in \Sigma_1^\chi \setminus \langle a_0, \ldots, a_\mu\rangle$, there is a point z' in $\langle a'_0, \ldots, a'_\mu\rangle$ with $p_2(z) = p_2(z')$. As shown before, for all points $t \in \langle a_0, \ldots, a_\mu\rangle$, there is a point $t' \in \langle a'_0, \ldots, a'_\mu\rangle$ with $p_2(t) = p_2(t')$. But this implies that $\langle a'_0, \ldots, a'_\mu\rangle/\Omega_2^* = \Sigma_1^\chi/\Omega_2^*$, a contradiction since S_2 is a linear set of rank $m+1$ with the property that it is not a linear set of lower rank. This proves our claim.

Let δ be the projectivity from Σ_1^χ onto Σ_2 with $a_i^\delta = b_i$, where $\{a_0, \ldots, a_m\}$ is a basis for Σ_1^χ and $\{b_0, \ldots, b_m\}$, for which $p_2(a_i) = p_2(b_i)$, $i = 0, \ldots, m$.

Since $\langle \Sigma_1^\chi \rangle = \Sigma^*$, a point x of Σ^* can be written as a linear combination $\sum_{i=0}^m \lambda_i a_i$ of points a_i of Σ_1^χ, with $\lambda_i \in \mathbb{F}_{q^n}$. If $x \notin \Omega_2^*$, $p_2(x)$ is well defined, and

$$p_2(x) = \sum_{i=0}^{m} \lambda_i p_2(a_i) = \sum_{i=0}^{m} \lambda_i p_2(a_i^\delta).$$

If $x \notin \Omega_2^*$ and $x^\delta \notin \Omega_2^*$, then

$$p_2(x^\delta) = \sum_{i=0}^{m} \lambda_i p_2(a_i^\delta) = p_2(x). \tag{1}$$

Let P be a point of Ω_2^*. We will show that $P^\delta \in \Omega_2^*$. Let P_1 and P_2 be different points of Ω_2. Let $x \neq P, P_1$ be a point of P_1P and let $y \neq P, P_2$ be a point of PP_2. Either $x^\delta \in \Omega_2^*$, or, by (1), $p_2(x^\delta) = p_2(x) = P_1$, and either $P_1^\delta \in \Omega_2^*$, or, by (1), $p_2(P_1^\delta) = p_2(P_1) = P_1$. In any case, $x^\delta P_1^\delta \in \langle \Omega_2^*, P_1\rangle$ and similarly $y^\delta P_2^\delta \in \langle \Omega_2^*, P_2\rangle$. Now $P^\delta = x^\delta P_1^\delta \cap y^\delta P_2^\delta$, hence, $P^\delta \in \langle \Omega_2^*, P_1\rangle \cap \langle \Omega_2^*, P_2\rangle$. Since $P_1 \neq P_2$, the spaces $\langle \Omega_2^*, P_1\rangle$ and $\langle \Omega_2^*, P_2\rangle$ are distinct, so $P^\delta \in \Omega_2^*$. This implies that $P^\delta \in \Omega_2^*, \forall P \in \Omega_2^*$, and since the collineation δ maps an $(m-r)$-space to an $(m-r)$-space, we get that ${\Omega_2^*}^\delta = \Omega_2^*$.

If $\beta := \delta \circ \chi$, then $\Omega_1^{*\beta} = (\Omega_1^{*\chi})^\delta = \Omega_2^{*\delta} = \Omega_2^*$ and $\Sigma_1^\beta = (\Sigma_1^\chi)^\delta = \Sigma_2$.

(ii)⇒(iii) Let Σ be a canonical subgeometry in Σ^*. Let ϕ, resp. ψ, be the collineation of Σ^* mapping Σ_1, resp. Σ_2 onto Σ. The previous part shows that the linear sets $\Sigma/\Omega_1^{*\phi}$ and Σ_1/Ω_1^* are isomorphic, and the linear sets $\Sigma/\Omega_2^{*\psi}$ and Σ_2/Ω_2^* are isomorphic. Let $\delta := \psi \circ \beta \circ \phi^{-1}$, then δ is a collineation of Σ^* with $\Sigma^\delta = \Sigma$ and $(\Omega_1^{*\phi})^\delta = \Omega_2^{*\psi}$.

(iii)⇒(ii) The collineation $\beta := \psi^{-1} \circ \delta \circ \phi$ maps Σ_1 onto Σ_2 and Ω_1^* onto Ω_2^*. □

Now we turn our attention to the $\mathbb{F}_q$-linear sets of rank 3 in $\mathrm{PG}(1, q^h)$. Let $\mathcal{D}$ be the Desarguesian $(h-1)$-spread in $\mathrm{PG}(2h-1, q)$. Consider the linear set $\mathcal{B}(\pi)$, with π a plane of $\mathrm{PG}(2h-1, q)$, not contained in an element of $\mathcal{D}$. If there is a spread element H intersecting π in a line then, using the terminology introduced by Fancsali and Sziklai in [6], $\mathcal{B}(\pi)$ is called a *club* and H is called the *head* of $\mathcal{B}(\pi)$. If all elements of $\mathcal{D}$ intersecting π, intersect π in a point, then $\mathcal{B}(\pi)$ is a *scattered linear set of rank 3.*

In general, if all elements of $\mathcal{D}$ intersect a k-space μ in at most a point, then μ is called *scattered with respect to* $\mathcal{D}$ and $\mathcal{B}(\mu)$ is a *scattered linear set of rank* $k+1$ (see [2] for more on scattered spaces).

Remark 1.6 of [6] states without proof that a club in $\mathrm{PG}(1, q^3)$ is projectively equivalent to the set of points $\{x \in \mathbb{F}_{q^3} || \mathrm{Tr}(x) = x + x^q + x^{q^2} = 0\} \cup \{\infty\}$. In Theorem 5(i), we show that indeed in the case $h = 3$, all clubs of $\mathrm{PG}(1, q^h)$ are projectively equivalent, and that all scattered linear sets of rank 3 are projectively equivalent too. If $h > 3$ however, the situation is different and linear sets of the same size are not necessarily (projectively) equivalent (see Theorem 5).

Lemma 4. *Let $H \leq \mathrm{P\Gamma L}(3, q^3)$ denote the setwise stabiliser of a subplane $\pi \cong \mathrm{PG}(2, q)$ of $\mathrm{PG}(2, q^3)$, and put $H' := H \cap \mathrm{PGL}(3, q)$, let T denote the set of points that do not lie on a secant line of π, and let S denote the set of points of $\mathrm{PG}(2, q^3) \setminus \pi$ that lie on a secant line of π.*

(i) For each point X of π, the stabiliser H'_X of X in H' acts sharply transitively on the set T, and for each point R of T, the stabiliser H'_R of R in H' acts transitively on the points of π.

(ii) The group H' acts transitively on the set S. The stabiliser H'_Z of a point $Z \in S$ acts transitively on the points of π not lying on the secant line through Z.

Proof. (i) The set T has size

$$t := q^6 + q^3 + 1 - (q^2 + q + 1) - (q^2 + q + 1)(q^3 - q) = q^6 - q^5 - q^4 + q^3.$$

Let $X \in \pi$. Since H' acts transitively on the points of π and has size $(q^2+q+1)t$, we have $|H'_X| = t$. We show that H'_X acts sharply transitively on the points of T, by proving that H'_{XY} is trivial, for each $Y \in T$. An element of H'_{XY} corresponds to a matrix A with entries in $\mathbb{F}_q$ with an eigenvalue in $\mathbb{F}_q$, with eigenvector X, and an eigenvalue in $\mathbb{F}_{q^3} \setminus \mathbb{F}_q$, corresponding to Y. But Y^q and Y^{q^2} are also fixed by A. Since $Y \in T$, Y, Y^q and Y^{q^2} are linearly independent. Since a matrix A can have at most three eigenvalues which correspond to linearly independent points, and there are already three linearly independent points with eigenvalue in $\mathbb{F}_{q^3} \setminus \mathbb{F}_q$, there cannot be an eigenvector with eigenvalue in $\mathbb{F}_q$. This implies that H'_{XY} is trivial.

For each point $R \in T$, there exists an element $\alpha \in \mathrm{PGL}(3, q)$ of order q^2+q+1 (generating a Singer cycle) that fixes R. This implies that the size of H'_R is at least q^2+q+1. Since the stabiliser H'_{RX} of a point $X \in \pi$ is trivial, and the orbit of a point of π can have length at most $q^2 + q + 1$, from $|H'_R| = |H'_{RX}||X^{H'_R}|$, we derive that $|X^{H'_R}| = q^2 + q + 1$. So H'_R acts transitively on the points of π.

(ii) The number of points in S is equal to

$$s := (q^2 + q + 1)(q^3 - q).$$

Let Z be a point of S, and let L be the secant line to $\mathrm{PG}(2,q)$ through Z. Since an element of H'_Z fixes three different points Z, Z^q, Z^{q^2} on L, it must fix L pointwise. It follows that an element of H'_{ZX}, with $X \in \pi \setminus L$, is a homology with center X and axis L, and each homology with center X and axis L clearly belongs to H'_{ZX}. It follows that $|H'_{ZX}| = q-1$. Since the group of elations of π with axis L acts transitively on the points not on L, $|X^{H'_Z}| = q^2$. Now $|H'_Z| = |H'_{ZX}||X^{H'_Z}| = (q-1)q^2$, $|H'| = |H'_Z||Z^{H'}|$ and $|H'| = (q^2+q+1)(q^6-q^5-q^4+q^3)$, hence $|Z^{H'}| = (q^2+q+1)(q^3-q) = s$. This implies that H' acts transitively on the points of S. $\square$

Theorem 5. *(i) All $\mathbb{F}_q$-clubs in $\mathrm{PG}(1,q^3)$, $q>2$, and all scattered $\mathbb{F}_q$-linear spaces in $\mathrm{PG}(1,q^3)$ are projectively equivalent.*

(ii) All $\mathbb{F}_2$-clubs and all scattered $\mathbb{F}_2$-linear sets of rank 3 in $\mathrm{PG}(1,2^5)$ are equivalent, but there exist projectively inequivalent $\mathbb{F}_2$-clubs and projectively inequivalent scattered $\mathbb{F}_2$-linear sets of rank 3 in $\mathrm{PG}(1,2^5)$.

(iii) There exist inequivalent $\mathbb{F}_q$-clubs and inequivalent scattered $\mathbb{F}_q$-linear sets of rank 3 in $\mathrm{PG}(1,q^h)$, $q>2$, with $h>3$ and $(h,q)\neq(5,2)$.

Proof. Let H be the setwise stabiliser in $\mathrm{P\Gamma L}(3,q^h)$ of a subplane $\pi \cong \mathrm{PG}(2,q)$ of $\mathrm{PG}(2,q^h)$. Let T denote the set of points that do not lie on a secant line of π, and let S denote the set of points of $\mathrm{PG}(2,q^3)\setminus\pi$ that lie on a secant line of π. By the equivalent perspective on linear sets using quotient geometries, Theorem 3, it suffices to study the transitivity of the action of H on the sets T and S. Since the group $\mathrm{PGL}(3,q)$ acts sharply transitively on the frames of $\mathrm{PG}(2,q)$ and the only element of $\mathrm{PGL}(3,q^h)$ fixing a frame is the identity element, it follows that

$$|H| = h|\mathrm{P\Gamma L}(3,q)| = hkq^3(q^3-1)(q^2-1),$$

where $q=p^k$, p prime. Calculating the size of T and S we get

$$|T| = q^{2h}-q^{h+2}-q^{h+1}+q^3 \text{ and } |S| = q^{h+2}+q^{h+1}+q^h-q^3-q^2-q.$$

Using Theorem 3, it follows that there are inequivalent scattered linear sets of rank 3 in $\mathrm{PG}(1,q^h)$, $h\geq 6$, and in $\mathrm{PG}(1,q^5)$, for $q>2$, and that there are inequivalent clubs in $\mathrm{PG}(1,q^h)$, $h>7$, and in $\mathrm{PG}(1,q^7)$ for $q>5$. If H acts transitively on S, then $|S|$ has to divide $|H|$. If $h=5$, this yields that $p^{2k}+1$ has to divide $5k(p^k-1)$. This is only possible in the cases $k=1, p=2,3$. If $h=7$, this argument yields that $p^{4k}+p^{2k}+1$ has to divide $7k(p^{3k}-p^k)$, which is not possible. If h is not a prime, then by the induced action of H on subplanes containing π, it follows that H does not act transitively on S neither on T. If $h=3$, then H acts transitively on both T and S by Lemma 4. This leaves only the cases $\mathrm{PG}(2,q^5)$ with $q=2,3$. Let Z be a point of S on a secant L of π. Since H_Z fixes $Z, Z^q, Z^{q^2}, Z^{q^3}, Z^{q^4}$ on L, L is fixed pointwise. The elements of H_Z consist of an element ϕ of $\mathrm{PGL}(3,q)$ and an element α of $\mathrm{Aut}(\mathbb{F}_{q^h})$ and since Z is fixed pointwise, α is trivial and $H_Z = H'_Z$, with $H' = \mathrm{PGL}(3,q)$. As in the proof of Lemma 4, one shows that the size of H'_Z is equal to $(q-1)q^2$. If $q^h = 2^5$, then $|H| = 840 = |H_Z|\cdot|Z^H| = 4\cdot|Z^H|$. Since $|S| = 210$, H acts transitively on the points of S. Together with Theorem 3, this shows that all clubs in $\mathrm{PG}(1,2^5)$ are equivalent. Since $|H_Z| = |H'_Z| = 4$ and $|H'| = 168$, $|Z^{H'}| = 42 < 210$, hence, not all clubs in $\mathrm{PG}(1,2^5)$ are projectively equivalent.

If $q = 3^5$, then $|H_Z| = 18$ and $|H| = 28080$, from which it follows that $|Z^H| = 1560 < |S| = 3120$, which implies that there are inequivalent clubs in $\mathrm{PG}(1,3^5)$. Let X be a point of T in $\mathrm{PG}(2,2^5)$. An element of H_X corresponds to a 3×3-matrix A with entries in $\mathbb{F}_q$, having 5 eigenvectors, 3 of which are linearly independent, each corresponding to an eigenvalue of A in $\mathbb{F}_{2^5}\setminus\mathbb{F}_2$, a contradiction unless A is the identity matrix. Hence, $|H_X| = 1$ and $|X^H| = |H| = 840 = |T|$. So H acts transitively on the points of T, and, by Theorem 3, all scattered linear sets of rank 3 in $\mathrm{PG}(1,2^5)$ are equivalent. Since $H'_X < H_X$ and H_X is trivial, $|H'| = 168 = |X^{H'}| < 840$ hence, there are scattered linear sets of rank 3 in $\mathrm{PG}(1,2^5)$ that are projectively inequivalent. $\square$

3 The intersection of a subline and an $\mathbb{F}_q$-linear set in $\mathrm{PG}(1,q^h)$

The intersection of an $\mathbb{F}_q$-subline and a club of $\mathrm{PG}(1,q^h)$ was already investigated in [6]. However, in this proof, the authors use that all clubs of $\mathrm{PG}(1,q^h)$ are projectively equivalent, which is in general not true[2] (see Theorem 5). Theorem 8 shows that their result is correct and Lemma 6 gives a description of the geometric structure of the intersection points of the subline with the linear set. Consider a subline $\ell \cong \mathrm{PG}(1,q)$ of $\mathrm{PG}(1,q^h)$. Using the notation introduced above, ℓ is an $\mathbb{F}_q$-linear set of $\mathrm{PG}(1,q^h)$ of rank two. In the spread representation $\ell = \mathcal{B}(L)$, where L is a line in $\mathrm{PG}(2h-1,q)$ not contained in an element of $\mathcal{D}$, so an $\mathbb{F}_q$-subline ℓ of $\mathrm{PG}(1,q^h)$ corresponds to an $(h-1)$-regulus $\mathcal{B}(L)$ in $\mathrm{PG}(2h-1,q)$.

Lemma 6. *Let $\mathcal{R} = \{\sigma_1,\ldots,\sigma_{q+1}\}$ be an $(h-1)$-regulus in $\mathrm{PG}(2h-1,q)$, $q > 2$, and let π be a plane in $\mathrm{PG}(2h-1,q)$ such that $\pi\cap\sigma_i$ is a point P_i, $i = 1,\ldots,4$, where no three points of $\{P_1,P_2,P_3,P_4\}$ are collinear. Then $\pi\cap\sigma_i$ is a point P_i for all $1 \le i \le q+1$ and $\{P_1,\ldots,P_{q+1}\}$ are the points of a conic in π.*

Proof. Let $\mathcal{R} = \{\sigma_1,\ldots,\sigma_{q+1}\}$ be an $(h-1)$-regulus, and let π be a plane such that $\pi\cap\sigma_i$ is a point P_i, $i = 1,\ldots,4$, where no three points of $\{P_1,P_2,P_3,P_4\}$ are collinear. Let t be the transversal line to $\mathcal{R}$ through P_1. Let $P'_1 = P_1$ and let P'_j be the points $\sigma_j\cap t$, $j = 2,\ldots,q+1$. The 3-dimensional space $\langle t,\pi\rangle$ intersects σ_j in the line $\ell_j = P_jP'_j$, $j = 2,3,4$. Hence, t is a transversal line to the 1-regulus $\mathcal{R}(\ell_2,\ell_3,\ell_4)$. The transversal line t' through P_2 to the regulus $\mathcal{R}(\ell_2,\ell_3,\ell_4)$ intersects the spread elements σ_i in points P''_i, with $i = 1,5,6,\ldots,q+1$ if $q \ge 4$ and $i = 1$ if $q = 3$. This implies that the line $P'_iP''_i$, contained in σ_i, with $i = 1,5,6,\ldots,q+1$ if $q \ge 4$ and $i = 1$ if $q = 3$, is a line of the regulus $\mathcal{R}(\ell_2,\ell_3,\ell_4)$. Hence, the elements of $\mathcal{R}$ intersect the 3-dimensional space $\langle t,\pi\rangle$ in the lines of a regulus of a hyperbolic quadric. Since π is a plane of $\langle t,\pi\rangle$, not containing a line of an element of $\mathcal{R}$, the line $P'_iP''_i$, with $i = 1,5,6,\ldots,q+1$ if $q \ge 4$ and $i = 1$ if $q = 3$, meets π in a point P_i and $\{P_1,\ldots,P_{q+1}\}$ are the points of a conic. $\square$

Lemma 7. *If a $(2k-1)$-space π intersects 3 elements of an $(h-1)$-regulus $\mathcal{R}$ in a $(k-1)$-dimensional subspace, $\mathcal{R}$ intersects π in a $(k-1)$-regulus.*

[2] The same authors provide a correct proof and a thorough description of the clubs in [7].

Proof. Let σ_i, with $i = 1, 2, 3$, be the 3 elements of $\mathcal{R}$, with $\mathcal{R}$ an $(h-1)$-regulus, intersecting some $(2k-1)$-space π in a $(k-1)$-space S_i. The spaces S_1, S_2, S_3 determine a unique $(k-1)$-regulus $\mathcal{R}'$. Let S_l be an element of $\mathcal{R}'$. A transversal line t through a point P of S_l to $\mathcal{R}'$ intersects the elements $\sigma_i, i = 1, 2, 3$, in a point of S_i. Hence, t is the unique transversal line through P to $\mathcal{R}$ and it follows that every element of the regulus through S_1, S_2, S_3 is contained in an element of the regulus $\mathcal{R}$, and conversely every element of $\mathcal{R}$ contains an element of the $(k-1)$-regulus $\mathcal{R}'$ through S_1, S_2, S_3. □

Theorem 8. *An $\mathbb{F}_q$-subline intersects an $\mathbb{F}_q$-linear set of rank k of* $\mathrm{PG}(1, q^h)$ *in* $0, 1, \ldots, \min\{q + 1, k\}$ *or* $q + 1$ *points.*

Proof. We proceed by induction on the rank k. For $k = 2$, the theorem follows from the observation that 3 points determine a unique $\mathbb{F}_q$-subline. So now suppose $k > 2$ and assume that the statement holds for $k' < k$. Let π be a $(k-1)$-dimensional space. Let $\mathcal{B}(L_1)$ be an $\mathbb{F}_q$-subline of $\mathrm{PG}(1, q^h)$, intersecting $\mathcal{B}(\pi)$ in at least $k+1$ points. Let $\sigma_1, \ldots, \sigma_{k+1}$ be elements of $\mathcal{B}(L_1)$, intersecting π. We may choose L_1 to go through a point R_1 of $\sigma_1 \cap \pi$. Let R_i be a point in $\sigma_i \cap \pi$. If one of the intersections $\sigma_i \cap \pi$, say $\sigma_2 \cap \pi$, contains a line M, then the space $\mu = \langle R_1, R_3, \ldots, R_k\rangle$ intersects M in a point, and we have that $\mathcal{B}(L_1)$ intersects $\mathcal{B}(\mu)$, which has rank $k-1$, in k points. By induction, $\mathcal{B}(L_1)$ is contained in $\mathcal{B}(\mu) \subset \mathcal{B}(\pi)$.

So from now on, we assume that all intersections $\sigma_i \cap \pi$, $i = 1, \ldots, k+1$, are points R_i. Suppose that $r+1$ points of $\{R_1, \ldots, R_{k+1}\}$ are contained in an $(r-1)$-dimensional subspace ν of π, $r < k$, then again by induction on k, $\mathcal{B}(L_1)$ is contained in $\mathcal{B}(\nu) \subset \mathcal{B}(\pi)$.

Hence, from now on, we also assume that no $r+1$ points of $\{R_1, \ldots, R_{k+1}\}$ are contained in an $(r-1)$-dimensional space, $r < k$.

Let L_i be the transversal line through R_i, $i = 2, \ldots, k-2$, to the regulus $\mathcal{B}(L_1)$.

Let ϕ_i be the space $\langle L_1 \cap \sigma_i, \ldots, L_{k-2} \cap \sigma_i\rangle$, with $\dim \phi_i = k - 3 - x$, for all $1 \leq i \leq q+1$. The $(k-3)$-space $\psi = \langle R_1, \ldots, R_{k-2}\rangle$ is contained in the $(2k-2x-5)$-space $\langle \phi_1, \phi_2\rangle$. Hence, if $x > 0$, $\phi_i \subset \sigma_i$ meets $\psi \subset \pi$ for all $i \leq q+1$, so, $\mathcal{B}(L_1) \subset \mathcal{B}(\pi)$.

Assume that $\dim \phi_i = k-3$. If one of the points $R_j \in \phi_j$ for some $j \in \{k-1, k, k+1\}$, then $\dim\langle \pi, L_1 \ldots, L_{k-2}\rangle \leq 2k-4$, and hence π intersects each ϕ_i, i.e., $\mathcal{B}(L_1) \subset \mathcal{B}(\pi)$.

If $R_j \notin \phi_j$ for all $j \in \{k-1, k, k+1\}$, the $(k-2)$-spaces $\langle R_i, \phi_i\rangle$, $i = k-1, k, k+1$, contained in the $(2k-3)$-space $\nu = \langle \pi, L_1, \ldots, L_{k-2}\rangle$, determine a $(k-2)$-regulus $\{\tau_1, \ldots, \tau_{q+1}\}$, by Lemma 7. Since for all $1 \leq i \leq q+1$, the $(k-2)$-space $\tau_i \subset \sigma_i$ and the $(k-1)$-space π, contained in the $(2k-3)$-space ν, intersect in a point, $\mathcal{B}(L_1) \subset \mathcal{B}(\pi)$. □

Theorem 9. *For every subline $L \cong \mathrm{PG}(1, q)$ of $\mathrm{PG}(1, q^h)$, there is a linear set S of rank k, $k \leq h$ and $k \leq q+1$, intersecting L in exactly j points, for all $0 \leq j \leq k$.*

Proof. Let $\mathcal{B}(L) = \{\sigma_1, \ldots, \sigma_{q+1}\}$ be a subline of $\mathrm{PG}(1, q^h)$. Let t_P denote the transversal line to $\mathcal{B}(L)$ through a point P that is contained in one of the elements of the regulus $\mathcal{B}(L)$. Throughout the proof we will use the notation

$$\mu_i := \langle P_1, \ldots, P_i\rangle, \text{ and } \tau_i := \langle t_{P_1}, \ldots, t_{P_i}\rangle.$$

(i) First we show that whenever we choose j points $\{P_i, \ldots, P_j\}$, $j \leq h$, such that P_i is a point in $\sigma_i \setminus \tau_{i-1}$, it holds that

$$\mathcal{B}(\mu_j) \cap \mathcal{B}(L) = \{\sigma_1, \ldots, \sigma_j\}.$$

This statement is trivial for $j \in \{1, 2\}$. We proceed by induction on j, i.e., suppose that the statement is true for less than j points. By way of contradiction, suppose μ_j intersects more than j elements of $\mathcal{B}(L)$. Then, by Theorem 8, $\mathcal{B}(L)$ is contained in $\mathcal{B}(\mu_j)$.

Suppose $\mu_j \cap \sigma_i$ is contained in τ_{j-2}, for all $i \in \{j+1, \ldots, q+1\}$. If $\mu_j \cap \sigma_i$ is contained in μ_{j-2}, for all $i \in \{j+1, \ldots, q+1\}$, then

$$|\mathcal{B}(\mu_{j-2}) \cap \mathcal{B}(L)| > j - 2,$$

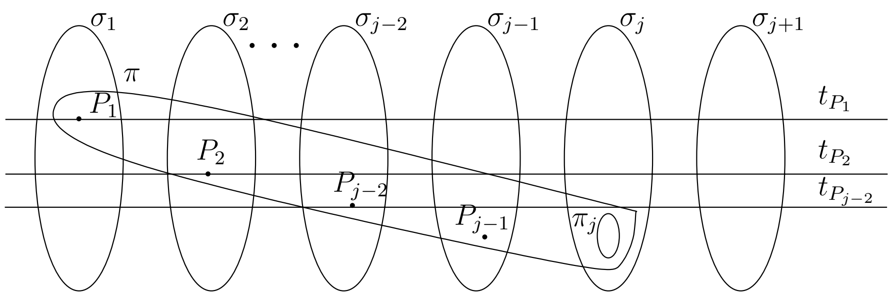
contradicting the induction hypothesis. Hence there is at least one point

$$P_l \in (\sigma_l \cap \mu_j) \setminus \mu_{j-2}, \ l \in \{j+1, \ldots, q+1\}.$$

If $P_{j-1} \in \langle \mu_{j-2}, P_l \rangle$, then $P_{j-1} \in \tau_{j-2}$, a contradiction. Hence $P_{j-1} \notin \langle \mu_{j-2}, P_l \rangle$, and it follows that $\mu_j = \langle \mu_{j-1}, P_l \rangle$. But then $P_j \in \mu_j \subset \tau_{j-1}$, again a contradiction.

We have shown that there exists an $l \in \{j+1, \ldots, q+1\}$, for which $P_l \in \mu_j \cap \sigma_l$ is not contained in τ_{j-2}. It follows from Lemma 7 that the $(2j-3)$-space $\langle \tau_{j-2}, P_{j-1}, P_j \rangle = \langle \tau_{j-2}, \mu_j \rangle$, which intersects each of the spread elements σ_{j-1}, σ_j, and σ_l in a $(j-2)$-space, intersects the $(h-1)$-regulus $\mathcal{B}(L)$ in a $(j-2)$-regulus. But this implies that $P_j \in \tau_{j-1}$, a contradiction.

(ii) Now, we show that there exists a linear set of rank k, $k \leq h$ and $k \leq q+1$, intersecting L in exactly j points, for all $0 \leq j \leq k$. The statement is trivial for $j \in \{0, 1, 2\}$, so fix $3 \leq j \leq k$.



Let P_1 be the point $\sigma_1 \cap L$, let P_i be a point in $\sigma_i \setminus \tau_{i-1}$, for $2 \leq i \leq j-1$, and let π_j be a $(k-j)$-space in σ_j, skew to τ_{j-1}. Let π be the subspace

$$\pi := \langle \mu_{j-1}, \pi_j \rangle,$$

hence $\mathcal{B}(\pi)$ is a linear set of rank k. We show that $\mathcal{B}(\pi) = \{\sigma_1, \ldots, \sigma_j\}$.

Suppose that one extra element of $\mathcal{B}(L)$, say σ_l, is contained in $\mathcal{B}(\pi)$ and let P_l be a point in the intersection of σ_l with π. By the first part of the proof, it follows that the dimension of $\mu := \langle P_1, \ldots, P_{j-1}, P_l \rangle$ is $j-1$, and since μ and π_j are contained in π, there is a point $P_j \in \mu \cap \pi_j$. This implies that $P_l \in \langle P_1, \ldots, P_j \rangle$. This contradicts the first part of the proof. $\square$

4 Sublines contained in a linear set

Throughout this section, we let $\mathcal{D} = \{\sigma_1, \ldots, \sigma_{q^h+1}\}$ denote the Desarguesian $(h-1)$-spread of $\mathrm{PG}(2h-1, q)$ and by a subline, we mean an $\mathbb{F}_q$-subline $\mathrm{PG}(1, q)$. If π is a subspace of $\mathrm{PG}(2h-1, q)$ and s and r are points of π, then the subline $\mathcal{B}(rs)$ is clearly contained in $\mathcal{B}(\pi)$. In this section, we investigate the possibility of other sublines through $\mathcal{B}(r)$ and $\mathcal{B}(s)$, contained in $\mathcal{B}(\pi)$. A subline $\mathcal{B}(L)$ of $\mathcal{B}(\pi)$ is called *irregular* if there is no line M of π such that $\mathcal{B}(M) = \mathcal{B}(L)$.

Lemma 10. *Let π be a 3-dimensional space in $\mathrm{PG}(2h-1, q)$. The intersection of the elements of $\mathcal{D}$ with π is one of the following:*

1. *π is contained in an element of $\mathcal{D}$,*
2. *π is scattered with respect to $\mathcal{D}$,*
3. *one element of $\mathcal{D}$ intersects π in a plane, and q^3 elements of $\mathcal{D}$ intersect π in a point,*
4. *one or two elements of $\mathcal{D}$ intersect π in a line and the other elements of $\mathcal{D}$ that intersect π, intersect π in a point,*
5. *$q+1$ elements of $\mathcal{D}$ intersect π in a line, and q^3-q elements of $\mathcal{D}$ intersect π in a point. In this case, the $q+1$ lines that are the intersection of an element of $\mathcal{D}$ with π form a (line-)regulus in π, or*
6. *all elements of $\mathcal{D}$, intersecting π, intersect π in a line and in this case, the elements intersecting π define a subline $\mathrm{PG}(1, q^2)$.*

Moreover, if h is odd, possibility 6 cannot occur and if $h = 3$, only the possibilities 3 and 5 occur.

Proof. If there is an element of $\mathcal{D}$ that intersects π in a plane, it is clear that all other intersections of an element of $\mathcal{D}$ with π are points. Suppose now that only lines and points occur as intersection of an element of $\mathcal{D}$ with π. Let L_1, L_2, L_3 be three lines in π that occur as the intersection of $\sigma_1, \sigma_2, \sigma_3 \in \mathcal{D}$ with π and let t be a transversal line to L_1, L_2, L_3, which exists since the lines L_i are contained in a 3-dimensional space. The line t is a transversal line to $\sigma_1, \sigma_2, \sigma_3$, hence, intersects $q-2$ other spread elements of $\mathcal{D}$, say $\sigma_4, \ldots, \sigma_{q+1}$. A transversal line $t' \neq t$ to $\mathcal{B}(t)$ intersects $\sigma_1, \sigma_2, \sigma_3$, hence, also $\sigma_4, \ldots, \sigma_{q+1}$, which implies that the intersection of σ_i, $i = 1, \ldots, q+1$, with π is a line L_i, and that the lines L_i form a regulus. Suppose now that there is a line $M \neq L_i$ contained in π, with $\mathcal{B}(M) \cap \pi = M$. Since the regulus through three of the lines $\{L_1, \ldots, L_{q+1}, M\}$ is contained in π, we easily see that in that case, every element of $\mathcal{D}$ that intersects π, intersects π in a line. The q^2+1 elements of $\mathcal{D}$ intersecting π in the Desarguesian line spread form a $\mathrm{PG}(1, q^2)$, embedded in $\mathrm{PG}(1, q^h)$, hence, h is even.

Let A be the number of elements of $\mathcal{D}$ intersecting π in a line, and suppose that no element of $\mathcal{D}$ intersects π in a plane. If $h = 3$, then $A(q+1) + (q^3 + 1 - A) = q^3 + q^2 + q + 1$ since all $q^3 + 1$ elements of the plane spread $\mathcal{D}$ in $\mathrm{PG}(5, q)$ intersect the 3-dimensional space π. If follows that $A = q + 1$. $\square$

4.1 Sublines contained in a club

In this subsection, we show that there are no irregular sublines contained in a club $S \ncong \mathrm{PG}(1, q^2)$.

Lemma 11. *If $S \cong \mathrm{PG}(1, q^2)$, then there are exactly $q+1$ different $\mathbb{F}_q$-sublines through two points of S, contained in S.*

Proof. If $S = \mathcal{B}(\pi) \cong \mathrm{PG}(1, q^2)$, then there is a 3-dimensional space μ through π such that $\mathcal{D}$ intersects μ in $q^2 + 1$ lines. Let $\mathcal{B}(r)$ and $\mathcal{B}(s)$ be two different points of S, with $r, s \in \pi$, and let $\mathcal{B}(s) \cap \mu$ be the line L. Any of the $q+1$ lines t_i through r and a point of L intersects $\mathcal{D}$ in $q + 1$ elements of $\mathcal{B}(\pi)$, hence, $\mathcal{B}(t_i)$ is contained in S. If $\mathcal{B}(t_i) = \mathcal{B}(t_j)$ for some $i \neq j$, then $t_i = t_j$ since t_i and t_j are transversal lines to the regulus $\mathcal{B}(t_i)$ through the point r. $\square$

Lemma 12. *Let π be a plane in $\mathrm{PG}(2h - 1, q)$, such that $\mathcal{B}(\pi)$ is a club $\ncong$ $\mathrm{PG}(1, q^2)$. If there is a line L for which $\mathcal{B}(L) \subset \mathcal{B}(\pi)$, then there is a line $M \subseteq \pi$ for which $\mathcal{B}(M) = \mathcal{B}(L)$.*

Proof. Suppose that π is a plane such that $\mathcal{B}(\pi) \ncong \mathrm{PG}(1, q^2)$ and that L is a line, intersecting π in exactly one point, such that $\mathcal{B}(L) \subset \mathcal{B}(\pi)$. This implies that at least $q + 1$ elements of $\mathcal{D}$ intersect $\langle \pi, L\rangle$ in a line. But then there are exactly $q + 1$ elements of $\mathcal{D}$ intersecting $\langle \pi, L\rangle$ in a line (see Lemma 10), where the $q + 1$ intersection lines form a regulus $\mathcal{R} = \{L_1, \ldots, L_{q+1}\}$. One of these lines L_i of $\mathcal{R}$, say L_1, is contained in π. The plane π contains one line L_1 of the regulus $\mathcal{R}$, hence, it contains a transversal line M to this regulus. $\square$

If $\mathcal{B}(\pi)$ is a club $\ncong \mathrm{PG}(1, q^2)$ with head H and $\mathcal{B}(r)$ and $\mathcal{B}(s)$, $r, s \in \pi$, are two non-head points, then the line rs meets H in a point p. Hence, the subline $\mathcal{B}(rs)$ through $\mathcal{B}(r)$ and $\mathcal{B}(s)$ always contains the head H. As a corollary to the previous lemma, we have shown that every subline contained in $\mathcal{B}(\pi)$ is regular and contains the head.

Corollary 13. *If S is a club of $\mathrm{PG}(1, q^h)$, where $S \ncong \mathrm{PG}(1, q^2)$, then there are no irregular sublines contained in S. Hence, through 2 non-head points of a club $S \ncong \mathrm{PG}(1, q^2)$ of $\mathrm{PG}(1, q^h)$, there is exactly one subline contained in S, which contains the head of the club.*

4.2 Sublines contained in a scattered linear set of rank 3

In this subsection, we show that there are irregular sublines contained in a linear set of rank 3.

Lemma 14. *Let π be a plane in $\mathrm{PG}(2h-1, q)$, let $\mathcal{B}(r)$ and $\mathcal{B}(s)$ be two different points of $\mathcal{B}(\pi)$, with $r, s \in \pi$. Then the following statements hold.*

1. *There is exactly one 3-dimensional space μ through π such that μ intersects $\mathcal{B}(r)$ and $\mathcal{B}(s)$ in a line.*
2. *If there is a line L through r, $L \not\subset \pi$, such that $\mathcal{B}(s) \in \mathcal{B}(L)$ and $\mathcal{B}(L)$ is contained in $\mathcal{B}(\pi)$, then $\langle \pi, L\rangle$ intersects $\mathcal{B}(s)$ and $\mathcal{B}(r)$ in a line.*

Proof. (1) Let π be a plane in $\mathrm{PG}(2h-1, q)$, let $\mathcal{B}(r)$ and $\mathcal{B}(s)$ be two different points of $\mathcal{B}(\pi)$, with $r, s \in \pi$. Since $\langle \mathcal{B}(s), \pi \rangle$ is a $(h+1)$-space, it intersects the $(h-1)$-space $\mathcal{B}(r)$ in a subspace L_r of dimension at least 1. It is not possible that $\langle \mathcal{B}(s), \pi \rangle \cap \mathcal{B}(r)$ has dimension more than one, because then the spread elements $\mathcal{B}(r)$ and $\mathcal{B}(s)$ would intersect, so L_r is a line.

Now $\langle L_r, \pi \rangle$ meets $\mathcal{B}(s)$ in a line L_s since the 3-dimensional space $\langle L_r, \pi \rangle$ is contained in the $(h+1)$-space $\langle \pi, \mathcal{B}(s) \rangle$ and $\mathcal{B}(s)$ is $(h-1)$-dimensional. Using the same reasoning as above, we get that $\langle L_r, \pi \rangle \cap \mathcal{B}(s)$ cannot have dimension larger than one. Hence, $\langle \pi, L_r \rangle$ intersects both $\mathcal{B}(s)$ and $\mathcal{B}(r)$ in a line. Suppose that there is a 3-dimensional space μ' through π, intersecting $\mathcal{B}(r)$ in the line L'_r and $\mathcal{B}(s)$ in the line L'_s, then L'_r is the intersection L_r of $\langle \mathcal{B}(s), \pi \rangle$ with $\mathcal{B}(r)$ and L'_s is the intersection L_s of $\langle \mathcal{B}(r), \pi \rangle$ with $\mathcal{B}(s)$. Hence, μ is uniquely determined.

(2) Suppose that there is a line L through r, $L \not\subset \pi$, such that $\mathcal{B}(s) \in \mathcal{B}(L)$ and $\mathcal{B}(L)$ is contained in $\mathcal{B}(\pi)$. An element $\mathcal{B}(x) \in \mathcal{B}(L)$, $x \in L \setminus \{r\}$, intersects $\langle \pi, L \rangle$ in the line $L_x = \langle L \cap \mathcal{B}(x), \pi \cap \mathcal{B}(x) \rangle$. The q lines L_x, $x \in L \setminus \{r\}$, belong to a 1-regulus with transversal line L, so $\mathcal{B}(r) \in \mathcal{B}(L)$ intersects $\langle \pi, L \rangle$ in a line too (see Lemma 10). □

Corollary 15. *Through two points of a scattered linear set $\mathcal{B}(\pi)$ of rank 3 in $\mathrm{PG}(1, q^h)$, $q > 2$, there are at most two sublines contained in $\mathcal{B}(\pi)$. If $h = 3$, through two points of $\mathcal{B}(\pi)$, there are exactly two sublines contained in $\mathcal{B}(\pi)$.*

Proof. Let $\mathcal{B}(\pi)$ be a scattered linear set of rank 3 and let $r, s \in \pi$. The subline $\mathcal{B}(rs)$ is contained in $\mathcal{B}(\pi)$. By Lemma 14 (1), there is a unique 3-dimensional space $\langle L_r, L_s \rangle$, $L_r \in \mathcal{B}(r), L_s \in \mathcal{B}(s)$, through π. If there are exactly two elements of $\mathcal{D}$ that intersect the space $\langle L_r, L_s \rangle$ in a line, there are no irregular sublines, and if there are $q+1$ elements of $\mathcal{D}$ that intersect the space $\langle L_r, L_s \rangle$ in a line, there is an irregular subline through $\mathcal{B}(r)$ and $\mathcal{B}(s)$. Lemma 14 (2) shows that if there is an irregular subline, this irregular subline is unique.

Lemma 10 shows that if $h = 3$, there are always $q+1$ elements of $\mathcal{D}$ intersecting $\langle L_r, L_s \rangle$ in a line. □

Remark 16. *Through two points of a scattered linear set $\mathcal{B}(\pi)$ of rank 3 in $\mathrm{PG}(1, 2^h)$, there are exactly 5 sublines contained in S. Let P, R be two points of $\mathcal{B}(\pi)$. Through every of the 5 points Q_i, different from P and R, contained in $\mathcal{B}(\pi)$, there is exactly one subline containing P, R and Q_i. Since $q = 2$, this subline only contains the points P, R, Q_i, hence, is completely contained in S.*

4.3 Irregular sublines as the projection of a subconic in $\mathrm{PG}(2, q^3)$

Using Theorem 1, we see that a linear set S of rank 3 in $\mathrm{PG}(2, q^3)$ is the projection of a subplane $\mathrm{PG}(2, q)$ from a point in $\mathrm{PG}(2, q^3) \setminus \mathrm{PG}(2, q)$. The projection of a line of $\mathrm{PG}(2, q)$ is a subline of S. The irregular sublines are sublines that are not the projection of a line of $\mathrm{PG}(2, q)$. In this subsection, we show that an irregular subline is the projection of a conic, and we investigate when the projection of a conic is a subline.

Theorem 17. *[12, Chapter 6] The points $(0,0,1), (0,1,x_1), (0,1,x_2), (0,1,x_3)$ of $\mathrm{PG}(2, q^t)$, $q = p^h$, are contained in an $\mathbb{F}_q$-subline iff $\frac{x_2 - x_1}{x_3 - x_1} \in \mathbb{F}_q$.*

Lemma 18. *The quotient space C/P of an irreducible conic in* $\mathrm{PG}(2,q)$ *over a point P, where P lies on C^*, and not on an extended line of* $\mathrm{PG}(2,q)$*, where C^* denotes the extension of C to* $\mathrm{PG}(2,q^3)$ *in* $\mathrm{PG}(2,q^3)\setminus\mathrm{PG}(2,q)$*, is an $\mathbb{F}_q$-subline.*

Proof. Let C be an irreducible conic in $\mathrm{PG}(2,q)$. There is an element ϕ of $\mathrm{P\Gamma L}(3,q)$ that maps C onto the conic $C' = \{(1,a,a^2)|a \in \mathbb{F}_q\} \cup \{(0,0,1)\}$. We will project C' on the line $X_0 = 0$ from a point $P = (1,\alpha,\alpha^2)$ on C^*, where $\alpha \in \mathbb{F}_{q^3} \setminus \mathbb{F}_q$.

The projection of C' from P on the line $X_0 = 0$ consists of the set of points $\{(0,\alpha - x, \alpha^2 - x^2)|x \in \mathbb{F}_q\} \cup \{(0,0,1)\}$, which equals the set of points $\{(0,1,\alpha+x)|x \in \mathbb{F}_q\}\cup\{(0,0,1)\}$, since $\alpha \neq x$. Result 17 shows that the 4 points $(0,0,1)$, $(0,1,\alpha+x_1)$, $(0,1,\alpha+x_2)$, and $(0,1,\alpha+x_3)$ are on an $\mathbb{F}_q$-subline iff

$$\frac{\alpha + x_2 - \alpha - x_1}{\alpha + x_3 - \alpha - x_1} \in \mathbb{F}_q.$$

Since this equality holds for all $x_i \in \mathbb{F}_q$, the lemma follows. □

Corollary 19. *(i) The quotient space C/P of an irreducible conic C in* $\mathrm{PG}(2,q)$ *over a point P, not on an extended line of* $\mathrm{PG}(2,q)$ *in* $\mathrm{PG}(2,q^3) \setminus \mathrm{PG}(2,q)$*, is an $\mathbb{F}_q$-subline if and only if P lies on C^*, where C^* denotes the extension of C to* $\mathrm{PG}(2,q^3)$.

(ii) The quotient space C/P of an irreducible conic C in $\mathrm{PG}(2,q)$ *over a point P in* $\mathrm{PG}(2,q^3)\setminus\mathrm{PG}(2,q)$ *on an extended line of* $\mathrm{PG}(2,q)$*, is not a subline.*

Proof. (i) Corollary 15, with $h = 3$, shows that there is exactly one irregular subline through 2 points Q and R of a scattered linear set S of rank 3. It is clear that the subline through the pre-images Q' and R' in $\mathrm{PG}(2,q)$ of Q and R by the projection from P, is projected onto a subline L through Q and R, contained in S.

Since Lemma 18 shows that the unique conic through the point P, P^q and P^{q^2} and the two points Q' and R' is projected onto a subline, different from L, the statement follows.

(ii) It is shown in Corollary 13 that there is no irregular subline contained in a club. Hence, the projection of a conic C from a point on an extended line cannot be a subline. □

Remark 20. *In Corollary 15, we have shown that through two points of a scattered $\mathbb{F}_q$-linear set $\mathcal{B}(\pi)$ of rank 3 in* $\mathrm{PG}(1,q^3)$*, there is exactly one irregular subline. The q^2+q+1 points of $\mathcal{B}(\pi)$ and the q^2+q+1 irregular sublines obtained in this way, form a projective plane. In the setting of Corollary 19, we see that the unique irregular subline through the points Q and R of $\mathcal{B}(\pi)$, where $\mathcal{B}(\pi)$ is obtained by projecting the subplane* $\mathrm{PG}(2,q)$ *from the point $P \in$* $\mathrm{PG}(2,q^3)$*, is obtained by projecting the unique subconic of $\mathbb{F}_q$ through the points P, P^q, P^{q^2} and the pre-images of Q and R. In this way, we obtain a set of q^2+q+1 conics, which is called a* packing [9] *or* bundle [5] of conics. *The bundle appearing here is a so-called* circumscribed *bundle.*

4.4 Sublines contained in a linear set

In the following theorem, we show that every subline is an irregular subline of some linear set.

Theorem 21. *For every $k \leq h+1$ and for every subline $\mathcal{B}(L)$ in $\mathrm{PG}(1, q^h)$, there is a linear set $\mathcal{B}(\pi)$ of rank k such that $\mathcal{B}(L) \subset \mathcal{B}(\pi)$, $L \not\subset \pi$, and such that $\mathcal{B}(L) \not\subset \mathcal{B}(\pi')$ for every proper subspace π' of π.*

Proof. Let $\mathcal{B}(L) = \{\sigma_1, \ldots, \sigma_{q+1}\}$ be an $\mathbb{F}_q$-subline of $\mathrm{PG}(1, q^h)$. Let t_P denote the transversal line through a point P to $\mathcal{B}(L)$. Let P_1 be a point of σ_1, let P_2 be a point in $\sigma_2 \setminus t_{P_1}$, let P_3 be a point in $\sigma_3 \setminus \langle t_{P_1}, t_{P_2} \rangle$, let P_i, $i \leq k-1$, be a point in $\sigma_i \setminus \langle t_{P_1}, \ldots, t_{P_{i-1}} \rangle$. Let P_k be a point in $\sigma_k \cap \langle t_{P_1}, \ldots, t_{P_{k-1}} \rangle$, not in $\langle P_1, \ldots, P_{k-1} \rangle$.

The $(k-2)$-dimensional space $\langle t_{P_1} \cap \sigma_i, \ldots, t_{P_{k-1}} \cap \sigma_i \rangle \subset \sigma_i$ and the $(k-1)$-space $\pi = \langle P_1, \ldots, P_k \rangle$ are both contained in the $(2k-3)$-dimensional space $\langle \langle t_{P_1} \cap \sigma_1, \ldots, t_{P_{k-1}} \cap \sigma_1 \rangle, \langle t_{P_1} \cap \sigma_k, \ldots, t_{P_{k-1}} \cap \sigma_k \rangle \rangle$. Hence, σ_i meets π for all i.

The same arguments as in the proof of Theorem 9 show that $\mathcal{B}(L)$ cannot be contained in $\mathcal{B}(\pi')$ with π' a proper subspace of π. $\square$

5 The intersection of two linear sets of rank 3

In [8], the authors show that two different linear sets of rank 3 in $\mathrm{PG}(1, q^3)$ share at most $2q+2$ points, where $q = p^h$, $p \geq 7$, using coordinates and the fact that linear sets of the same size in $\mathrm{PG}(1, q^3)$ are projectively equivalent. Now, we prove this in a geometrical way and extend this result, at least for odd q, to $\mathrm{PG}(1, q^h)$. Moreover, we show that the bound is sharp.

Lemma 22. *Let π and π' be two planes in $\mathrm{PG}(rh-1, q)$ with $\pi \cap \pi' = L$, L a line. Let $\mathcal{D}$ be a Desarguesian $(h-1)$-spread in $\mathrm{PG}(rh-1, q)$. If $\mathcal{B}(\pi) \neq \mathcal{B}(\pi')$, then $|\mathcal{B}(\pi) \cap \mathcal{B}(\pi')| \in \{1, 2, q+1, q+2, q+3, 2q, 2q+1, 2q+2\}$.*

Proof. By Lemma 10, the number of elements of $\mathcal{D}$ intersecting the 3-dimensional space $\langle \pi, \pi' \rangle$ in a line is $1, 2$ or $q+1$ since $\mathcal{B}(\pi) \neq \mathcal{B}(\pi')$. If $|\mathcal{B}(\pi \cap \pi')| = 1$, this implies that $\mathcal{B}(\pi)$ and $\mathcal{B}(\pi')$ have 1, 2 or $q+1$ elements in common. If $|\mathcal{B}(\pi \cap \pi')| = q+1$, and no element of $\mathcal{B}(\pi \cap \pi')$ meets π or π' in a line, then $|\mathcal{B}(\pi) \cap \mathcal{B}(\pi')| = q+1, q+2, q+3$ or $2q+2$. If $|\mathcal{B}(\pi \cap \pi')| = q+1$, and there is an element of $\mathcal{B}(\pi \cap \pi')$ that meets π or π' in a line, then $|\mathcal{B}(\pi) \cap \mathcal{B}(\pi')| = q+1, q+2, 2q$ or $2q+1$. $\square$

Theorem 23. *Two $\mathbb{F}_q$-linear sets of rank 3 in $\mathrm{PG}(1, q^h)$, $q > 3$, intersect in at most $2q+2$ points if q is odd, and in at most $2q+3$ points if q is even.*

Proof. Let $\mathcal{D}$ be a Desarguesian $(h-1)$-spread in $\mathrm{PG}(rh-1, q)$, $q > 3$. Let π and π' be two planes (in $\mathrm{PG}(rh-1, q)$) with $\pi \cap \pi' \neq \emptyset$ and suppose that $|\mathcal{B}(\pi) \cap \mathcal{B}(\pi')| \geq 2q+3$. Let $X = \{P \text{ a point of } \pi | \mathcal{B}(P) \subset \mathcal{B}(\pi')\}$ and $X' = \{P \text{ a point of } \pi' | \mathcal{B}(P) \subset \mathcal{B}(\pi)\}$. From Theorem 8, we get that if a line contains 4 points of X (resp. X'), then this line is contained in X (resp. X'). If $\pi \cap \pi'$ is a line, the theorem follows from Lemma 22, hence, suppose that π intersects π' in the point P.

Suppose first that there are no lines in X or X', say in X. Then every line in π contains at most 3 points of X. In that case, $|X| = 2q+3$, and every line intersects X in 0 or 3 points and $q = 3^h$. But a maximal arc in a plane of odd order does not exist (see [1]), a contradiction. Hence, from now on, we assume that there is a line L_X in X and a line $L_{X'}$ in X'.

Case 1: $|\mathcal{B}(L_X)| = 1$. If $|\mathcal{B}(L_{X'})| = 1$ and $L_X \cap L_{X'} = \emptyset$, the plane $\langle L_X, \mathcal{B}(L_X) \cap \pi' \rangle$, contained in $\mathcal{B}(L_X)$ and $\langle L_{X'}, \mathcal{B}(L_{X'}) \cap \pi \rangle$ contained in $\mathcal{B}(L_{X'})$, are contained in the 4-space $\langle \pi, \pi' \rangle$, a contradiction. If $|\mathcal{B}(L_{X'})| = 1$ and $L_X \cap L_{X'} = \pi \cap \pi'$, either $|\mathcal{B}(\pi) \cap \mathcal{B}(\pi')| = 1, 2$, or $q+1$, or there is a point P in $X \setminus L_X$. In that case, the plane π'' through P with $\mathcal{B}(\pi') = \mathcal{B}(\pi'')$ does not contain a line M of X through P with $|\mathcal{B}(M)| = 1$, so the problem reduces to one of the other cases.

Hence, we may suppose that the $q+1$ elements of $\mathcal{B}(L_{X'})$ meet the plane π in points of a conic C. Since $|X| \geq 2q+3$, there is a point P_1 contained in $X \setminus (C \cup L_X)$ lying on a secant line M to C which meets L_X in a point not on C, hence containing 4 points of X, which implies that $M \subset X$. Now if $q > 3$, every point of π lies on a secant line to C, intersecting L_X and M in distinct points. This shows that $\pi = X$.

Case 2: $|\mathcal{B}(L_X)| = q+1$. Hence, the elements of $\mathcal{B}(L_X)$ meet π' in the points of a conic C.

Let P_1 be a point of C, and let Q be the point on the line L_X such that $\mathcal{B}(Q) = \mathcal{B}(P_1)$. There is a plane π'', through P_1, such that $\mathcal{B}(\pi) = \mathcal{B}(\pi'')$, moreover, the plane π'' contains a line L through P_1 with $\mathcal{B}(L) = \mathcal{B}(L_X)$. The elements of $\mathcal{B}(L_{X'})$ meet π'' in the points of a conic C'. Since $|X| \geq 2q+3$, X contains a point Q, not on $L \cup C'$. Let π'' play the role of π.

If there is a secant line M through Q to C', not through the possible intersection of L with C', containing 4 points of X, then M is contained in X. It is easy to see that if $q > 3$, every point R of π, different from the nucleus n of C' if q is even, lies on a secant line through C', meeting M and L. But then R lies on a line with four points of X, and we conclude that $X = \pi$. If q is odd, the secant line M always exists. If q is even, it is possible that Q is the nucleus of C'. In the latter case, if $|X| \geq 2q+4$, there is a point $Q' \in X$, lying on a secant line M to C' and we can repeat the previous arguments with $Q = Q'$ to show that $X = \pi$. The statement follows. □

Remark 24. *For general q, there are two linear sets of rank 3, intersecting in $2q+2$ points. Let π be a 3-dimensional space, such that there are $q+1$ elements of $\mathcal{D}$, say σ_i intersecting π in a line L_i. Let M be a line skew to all lines L_i. Let π and π' be two different planes through M. The sets $\mathcal{B}(\pi)$ and $\mathcal{B}(\pi')$ have exactly $2q+2$ points in common.*

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