On linear systems of conics over finite fields[1]

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Linear systems

A form on $\mathbb{P}^n$ is a homogeneous polynomial $f \in \mathbb{F}[X_0, \ldots, X_n]$, and the forms of degree d on $\mathbb{P}^n$ form a vector space W of dimension

$$\dim W = \binom{n+d}{d}.$$

A hypersurface of degree d in $\mathbb{P}^n$ is the zero locus

$$\mathcal{Z}(f) = \{p \in \mathbb{P}^n \; : \; f(p) = 0\}$$

of a form f on $\mathbb{P}^n$ of degree d.

A (k-dimensional) linear system of hypersurfaces of degree d is a (k-dimensional) subspace of $\mathbb{P}W$. Linear systems of dimension 1, 2, and 3 are called pencils, nets, and webs, respectively.

Classification

A classification of the k-dimensional linear systems of hypersurfaces of degree d in $\mathbb{P}^n$: a list of the (representatives of the) orbits of the k-dimensional subspaces of $\mathbb{P}W$ under the "induced" action of $\mathrm{PGL}(n+1, \mathbb{F})$ (the projectivity group of $\mathbb{P}^n$) on $\mathbb{P}W$.

If possible: a complete invariant for each orbit, the stabiliser group of each representative, the size of each orbit, an algorithm to determine the orbit of a given linear system, Schreier elements, ...

Classification: finite number of orbits

Is such a classification possible? Are there a finite number of orbits?

For hypersurfaces ($k = 0$) we have the following theorem.

Theorem

Let W denote the vector space of forms of degree d on $\mathbb{P}^n$. If K is infinite, then $\mathrm{PGL}(n+1, K)$ has a finite number of orbits on $\mathbb{P}W$ if and only if $d \leq 2$ or $(n, d) = (1, 3)$.

Proof.

(Sketch) Consider $\mathrm{PGL}(n+1, K)$ as an algebraic group $\mathcal{G}$. For a hypersurface $\mathcal{H}$ of $\mathbb{P}^n$ of degree d we have

$$\dim(\mathcal{G}) = \dim(\mathcal{H}^{\mathcal{G}}) + \dim(\mathrm{stab}_{\mathcal{G}}(\mathcal{H})).$$

This implies that $\dim(\mathcal{H}^{\mathcal{G}}) \leq n^2 + 2n$. In order to have a finite number of orbits, it is therefore required that $\dim \mathbb{P}W \leq n^2 + 2n$. $\qquad \square$

Finite number of orbits

If $d = 1$ then there is one orbit of hyperplanes.

If $d = 2$ then Sylvester's law of inertia gives a classification for $K = \mathbb{R}$. For algebraically closed fields there is one orbit for each rank. For finite fields there are one or two orbits for each rank.

The case $(n, d) = (1, 3)$ corresponds to cubics on $\mathbb{P}^1$.

Results on other cases

For $(n, d) = (2, 3)$, $k = 0$ [Schoof 1986] determined the number of projectively equivalent nonsingular cubics over $\mathbb{F}_q$ with a fixed number of $\mathbb{F}_q$-rational points.
([Bedocchi 1981], [Cicchese 1965, 1971], [De Groote - Hirschfeld 1980], [Duering 1941], [Waterhouse 1969])

For $d = 2$, $k = 1$, and $K = \mathbb{C}$: two "non-singular" pencils $\mathcal{P}_1$ and $\mathcal{P}_2$ of quadrics, are projectively equivalent if and only if they have the same Segre symbol and there is a projectivity on $\mathbb{P}^1$ taking the points $(x_0^{(i)}, x_1^{(i)})$ parametrizing the singular elements of $\mathcal{P}_1$ to those of $\mathcal{P}_2$. [Kronecker, Weierstrass, C. Segre]

For $(n, d) = (3, 2)$, $k = 1$, and $K = \mathbb{F}_q$: [Bruen - Hirschfeld 1986] [Bruen - Hirschfeld 1988] non-singular pencils of quadrics with reducible base.

For $(n, d) = (1, 3)$, $k = 1$, and $K = \mathbb{F}_q$, q odd, and $q \not\equiv 0(3)$: partial results [ML - Günay]

Linear systems of conics

From now on $(n, d) = (2, 2)$, $\dim \mathbb{P}W = 5$.

Linear systems of conics: old results

[Jordan 1906] pencils of conics over $\mathbb{R}$ (13 orbits) and $\mathbb{C}$ (8 orbits).

[Dickson 1906] pencils of conics over $\mathbb{F}_q$, q odd (15 orbits).

[Wilson 1914] nets of conics over $\mathbb{F}_q$, q odd (incomplete)

[Campbell 1927] pencils of conics over $\mathbb{F}_q$, q even (incomplete)

[Campbell 1928] nets of conics over $\mathbb{F}_q$, q even (incomplete)

[Wall 1977] nets of conics over $\mathbb{R}$ and $\mathbb{C}$.

Linear systems of conics: recent results

[ML- Popiel 2020] pencils of conics over finite fields $\mathbb{F}_q$, q odd (15 orbits), and over algebraically closed fields of characteristic $\neq 2$.

[ML- Popiel - Sheekey 2020] nets of conics of rank one over $\mathbb{F}_q$, q odd (15 orbits).

[ML- Popiel - Sheekey 202*] combinatorial invariants of nets.

[Alnajjarine - ML - Popiel 202*] pencils of conics over $\mathbb{F}_q$, q even (15 orbits).

Nets of conics

A quadratic form f on $\mathbb{P}^2$ defines a conic $\mathcal{C} = \mathcal{Z}(f)$ in $\mathbb{P}^2$. A net of conics $\mathcal{N}$ is defined by three conics $\mathcal{C}_i = \mathcal{Z}(f_i)$ ($i = 1, 2, 3$), not contained in a pencil:

$$\mathcal{N} = \{\mathcal{Z}(af_1 + bf_2 + cf_3) : a, b, c \in \mathbb{F}, (a, b, c) \neq (0, 0, 0)\}.$$

Then

$$xf_1 + yf_2 + zf_3 =$$

$$a_{00}(x, y, z)X_0^2 + a_{01}(x, y, z)X_0X_1 + \cdots + a_{22}(x, y, z)X_2^2$$

is a quadratic form whose coefficients are linear forms in x, y, z.

We thus obtain a symmetric matrix $A_{\mathcal{N}}$ over $\mathbb{F}[x, y, z]$.

Nets of conics of rank one

$(\operatorname{char}\mathbb{F} \neq 2)$

We define the discriminant of the net $\mathcal{N}$ as

$$\Delta_{\mathcal{N}} = \det(A_{\mathcal{N}}).$$

The discriminant $\Delta_{\mathcal{N}}$ defines a cubic curve $\mathcal{Z}(\Delta_{\mathcal{N}})$.

For each $a, b, c \in \mathbb{F}_q$, not all zero, we obtain a conic

$$\mathcal{N}(a, b, c) = \mathcal{Z}(af_1 + bf_2 + cf_3).$$

Lemma
The conic $\mathcal{N}(a, b, c)$ is singular if and only if (a, b, c) lies on the cubic $\mathcal{Z}(\Delta_{\mathcal{N}})$.

A net has rank one if it contains a repeated line.

(rank two if it contains no repeated lines but contains a conic which is not absolutely irreducible, and rank three if every conic in the net is absolutely irreducible)

The Veronese surface in $\mathbb{P}^5$

We represent points $y = (y_0, y_1, y_2, y_3, y_4, y_5, y_6)$ of $\mathbb{P}^5$ by symmetric matrices

$$M_y = \begin{bmatrix} y_0 & y_1 & y_2 \\ y_1 & y_3 & y_4 \\ y_2 & y_4 & y_5 \end{bmatrix}. \tag{1}$$

The Veronese surface $\mathcal{V}(\mathbb{F})$ in $\mathbb{P}^5$ is defined by setting the 2×2 minors of the above matrix to zero, and we have the corresponding Veronese map from $\mathbb{P}^2$ to $\mathcal{V}(\mathbb{F}) \subset \mathbb{P}^5$:

$$\nu : (x_0, x_1, x_2) \mapsto (x_0^2, x_0 x_1, x_0 x_2, x_1^2, x_1 x_2, x_2^2).$$

The Veronese surface in $\mathbb{P}^5$

The rank of a point in y in $\mathbb{P}^5$ is $\mathrm{rank}(M_y)$. The points contained in $\mathcal{V}(\mathbb{F})$ are points of rank 1, and the points of rank 2 are those contained in the secant variety of $\mathcal{V}(\mathbb{F})$.

Hyperplanes of $\mathbb{P}^5$:

- $\mathcal{H}_1$ is the set of hyperplanes intersecting $\mathcal{V}(\mathbb{F})$ in a conic,
- $\mathcal{H}_3$ is the set of hyperplanes intersecting $\mathcal{V}(\mathbb{F})$ in a normal rational curve,
- $\mathcal{H}_2$ is the set of hyperplanes not contained in $\mathcal{H}_1 \cup \mathcal{H}_3$.

Let δ denote the map from the set of conics in $\mathbb{P}^2$ to the set of hyperplanes in $\mathbb{P}^5$ which takes the conic $\mathcal{C} = \mathcal{Z}(f)$ with $f(X_0, X_1, X_2) = \sum_{i \leq j} a_{ij} X_i X_j$ to

$$\delta(\mathcal{C}) = H[a_{00}, a_{01}, a_{02}, a_{11}, a_{12}, a_{22}],$$

where $H[a_0, \ldots, a_5]$ denotes the hyperplane $\mathcal{Z}(a_0 Y_0 + \ldots + a_5 Y_5)$.

We also extend the definition of δ from the set of conics to the sets of pencils and nets of conics as follows. Given any set $\mathcal{S}$ of conics in $\mathbb{P}^2$, define

$$\delta(\mathcal{S}) = \cap_{\mathcal{C} \in \mathcal{S}} \delta(\mathcal{C}).$$

In this way we obtain the following one-to-one correspondences.

Lemma
The equivalence classes of nets of conics in $\mathbb{P}^2$ are in one-to-one correspondence with the K-orbits of planes in $\mathbb{P}^5$, where $K = \alpha(\mathrm{PGL}(3, \mathbb{F})) \leq \mathrm{PGL}(6, \mathbb{F})$ with

$$\alpha(\varphi_A) \in \mathrm{PGL}(6, \mathbb{F}) \ : \ y \mapsto z \text{ where } M_z = AM_yA^T.$$

K-invariants

From now on assume that $\mathbb{F} = \mathbb{F}_q$ with q odd.

There are four K-orbits of points in $\mathbb{P}^5$, namely: $\mathcal{P}_1$, the points of rank 1; $\mathcal{P}_{2,e}$, the exterior rank-2 points; $\mathcal{P}_{2,i}$, the interior rank-2 points; and $\mathcal{P}_3$, the points of rank 3.

The point-orbit distribution of a subspace W is the list $[n_1, n_2, n_3, n_4]$, where n_i is the number of points in W which belong to the i-th point-orbit.

The rank distribution of W is $[n_1, n_2 + n_3, n_4]$.

The line-orbits were determined in [ML-Popiel 2020] using [ML-Sheekey]'s classification of tensors in $\mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$.

Their point-orbit distributions and stabilisers were determined in [ML-Popiel 2020].

Representatives of the K-orbits on lines in $\mathbb{P}^5$

Orbit	Representative	Conditions	Orbit	Representative	Conditions
o_5	$\begin{bmatrix} \alpha & \cdot & \cdot \\ \cdot & \beta & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$		$o_{13,1}$	$\begin{bmatrix} \cdot & \alpha & \cdot \\ \alpha & \beta & \cdot \\ \cdot & \cdot & -\beta \end{bmatrix}$	
o_6	$\begin{bmatrix} \alpha & \beta & \cdot \\ \beta & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$		$o_{13,2}$	$\begin{bmatrix} \cdot & \alpha & \cdot \\ \alpha & \beta & \cdot \\ \cdot & \cdot & -\varepsilon\beta \end{bmatrix}$	$\varepsilon \in \boxtimes$
$o_{8,1}$	$\begin{bmatrix} \alpha & \cdot & \cdot \\ \cdot & \beta & \cdot \\ \cdot & \cdot & -\beta \end{bmatrix}$		$o_{14,1}$	$\begin{bmatrix} \alpha & \cdot & \cdot \\ \cdot & -(\alpha+\beta) & \cdot \\ \cdot & \cdot & \beta \end{bmatrix}$	
$o_{8,2}$	$\begin{bmatrix} \alpha & \cdot & \cdot \\ \cdot & \beta & \cdot \\ \cdot & \cdot & -\varepsilon\beta \end{bmatrix}$	$\varepsilon \in \boxtimes$	$o_{14,2}$	$\begin{bmatrix} \alpha & \cdot & \cdot \\ \cdot & -\varepsilon(\alpha+\beta) & \cdot \\ \cdot & \cdot & \beta \end{bmatrix}$	$\varepsilon \in \boxtimes$
o_9	$\begin{bmatrix} \alpha & \cdot & \beta \\ \cdot & \beta & \cdot \\ \beta & \cdot & \cdot \end{bmatrix}$		$o_{15,1}$	$\begin{bmatrix} v\beta & \alpha & \cdot \\ \alpha & u\alpha+\beta & \cdot \\ \cdot & \cdot & \alpha \end{bmatrix}$	$-v \in \square,\ (*)$
o_{10}	$\begin{bmatrix} v\alpha & \beta & \cdot \\ \beta & \alpha+u\beta & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$	$(*)$	$o_{15,2}$	$\begin{bmatrix} v\beta & \alpha & \cdot \\ \alpha & u\alpha+\beta & \cdot \\ \cdot & \cdot & \alpha \end{bmatrix}$	$-v \in \boxtimes,\ (*)$
o_{12}	$\begin{bmatrix} \cdot & \alpha & \cdot \\ \alpha & \cdot & \beta \\ \cdot & \beta & \cdot \end{bmatrix}$		o_{16}	$\begin{bmatrix} \cdot & \cdot & \alpha \\ \cdot & \alpha & \beta \\ \alpha & \beta & \cdot \end{bmatrix}$	
			o_{17}	$\begin{bmatrix} v^{-1}\alpha & \beta & \cdot \\ \beta & u\beta-w\alpha & \alpha \\ \cdot & \alpha & \beta \end{bmatrix}$	$(**)$

Conditions $(*)$ and $(**)$: $v\lambda^2 + uv\lambda - 1 \neq 0$, and $\lambda^3 + w\lambda^2 - u\lambda + v \neq 0$ for all $\lambda \in \mathbb{F}_q$.

Point-orbit distributions of the K-orbits on lines in $\mathbb{P}^5$

Orbit	Point-orbit distribution
o_5	$[2, \frac{q-1}{2}, \frac{q-1}{2}, 0]$
o_6	$[1, q, 0, 0]$
$o_{8,1}$	$[1, 1, 0, q-1]$
$o_{8,2}$	$[1, 0, 1, q-1]$
o_9	$[1, 0, 0, q]$
o_{10}	$[0, \frac{q+1}{2}, \frac{q+1}{2}, 0]$
o_{12}	$[0, q+1, 0, 0]$
$o_{13,1}$	$[0, 2, 0, q-1]$
$o_{13,2}$	$[0, 1, 1, q-1]$
$o_{14,1}$	$[0, 3, 0, q-2]$
$o_{14,2}$	$[0, 1, 2, q-2]$
$o_{15,1}$	$[0, 1, 0, q]$
$o_{15,2}$	$[0, 0, 1, q]$
o_{16}	$[0, 1, 0, q]$
o_{17}	$[0, 0, 0, q+1]$

K-orbits of planes meeting $\mathcal{V}(\mathbb{F}_q)$ in $\mathbb{P}^5$, q odd

Steps in the classification

1. Planes spanned by points of $\mathcal{V}(\mathbb{F}_q)$: Σ_1 and Σ_2.
2. Planes meeting the quadric Veronesean in exactly two points: Σ_3, Σ_4 and Σ_5.
3. Planes that are spanned by points of the secant variety of the quadric Veronesean and meet the quadric Veronesean in exactly one point: eight further orbits $\Sigma_6, \ldots, \Sigma_{13}$ (independent of q), one orbit Σ_{14} which only appears in characteristic $\neq 3$, and one orbit Σ'_{14} which only appears in characteristic 3.
4. Exactly one remaining orbit, Σ_{15}, consisting of planes that meet the Veronesean but are not spanned by points in the secant variety of the Veronesean.

Classification

Theorem (ML - Popiel - Sheekey 2020)

Let q be a power of an odd prime. There are 15 orbits of planes in $\mathbb{P}^5$ that meet the quadric Veronesean in at least one point, under the action of $\mathrm{PGL}(3, q) \leq \mathrm{PGL}(6, q)$.

As a corollary, we complete Wilson's classification of nets of rank one, rectifying some of the statements made in his paper.

Corollary

There are 15 orbits of nets of conics of rank one in $\mathbb{P}^2_q$, q odd.

Comparison with Wilson's paper

Wilson was aware of the fact that he had not completely classified the orbits, pointing out that

"All questions of inter-relations between these types have been considered and answered, except ..."

Although Wilson's work was in general very thorough, there are also some other issues with his classification besides the open cases he mentioned; there are some inaccuracies, and some missing orbits.

Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	Σ_7	Σ_8	Σ_9	Σ_{10}	Σ_{11}	Σ_{12}	Σ_{13}	Σ_{14}	Σ'_{14}	Σ_{15}
	III	IV	VI	XI	VIII	I	V	VII	VII	IX	XI	X, XI	X, XI		II

TABLE 1. Correspondence between K-orbits of planes in $\mathbb{P}^5$ and canonical types I, ..., XI of nets of conics of rank one in $\mathbb{P}^2$ obtained by [Wilson 1941].

Representatives of the K-orbits on planes in $\mathbb{P}^5$

Orbit	Representative	Conditions	Orbit	Representative	Conditions
Σ_1	$\begin{bmatrix} \alpha & \gamma & \cdot \\ \gamma & \beta & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$		Σ_9	$\begin{bmatrix} \alpha & \beta & \cdot \\ \beta & \gamma & \cdot \\ \cdot & \cdot & -\gamma \end{bmatrix}$	
Σ_2	$\begin{bmatrix} \alpha & \cdot & \cdot \\ \cdot & \beta & \cdot \\ \cdot & \cdot & \gamma \end{bmatrix}$		Σ_{10}	$\begin{bmatrix} \alpha & \beta & \cdot \\ \beta & \gamma & \cdot \\ \cdot & \cdot & -\varepsilon\gamma \end{bmatrix}$	$\varepsilon \in \boxtimes$
Σ_3	$\begin{bmatrix} \alpha & \cdot & \gamma \\ \cdot & \beta & \cdot \\ \gamma & \cdot & \cdot \end{bmatrix}$		Σ_{11}	$\begin{bmatrix} \cdot & \beta & \gamma \\ \beta & \alpha & \alpha \\ \gamma & \alpha & \alpha+\gamma \end{bmatrix}$	
Σ_4	$\begin{bmatrix} \alpha & \cdot & \gamma \\ \cdot & \beta & \gamma \\ \gamma & \gamma & \cdot \end{bmatrix}$		Σ_{12}	$\begin{bmatrix} \alpha & \beta & \cdot \\ \beta & \gamma & \beta \\ \cdot & \beta & \gamma \end{bmatrix}$	
Σ_5	$\begin{bmatrix} \alpha & \cdot & \gamma \\ \cdot & \beta & \gamma \\ \gamma & \gamma & \gamma \end{bmatrix}$		Σ_{13}	$\begin{bmatrix} \alpha & \beta & \cdot \\ \beta & \gamma & \beta \\ \cdot & \beta & \varepsilon\gamma \end{bmatrix}$	$\varepsilon \in \boxtimes$
Σ_6	$\begin{bmatrix} \alpha & \beta & \cdot \\ \beta & \varepsilon\alpha & \cdot \\ \cdot & \cdot & \gamma \end{bmatrix}$	$\varepsilon \in \boxtimes$	Σ_{14}	$\begin{bmatrix} \alpha & \beta & \cdot \\ \beta & c\gamma & \beta-\gamma \\ \cdot & \beta-\gamma & \gamma \end{bmatrix}$	$q \not\equiv 0 \pmod 3$, (†)
Σ_7	$\begin{bmatrix} \alpha & \beta & \gamma \\ \beta & \cdot & \cdot \\ \gamma & \cdot & \cdot \end{bmatrix}$		Σ_{14}'	$\begin{bmatrix} \alpha+\gamma & \gamma & \gamma \\ \gamma & \beta+\gamma & \gamma \\ \gamma & \gamma & -\beta \end{bmatrix}$	$q \equiv 0 \pmod 3$
Σ_8	$\begin{bmatrix} \alpha & \beta & \cdot \\ \beta & \cdot & \gamma \\ \cdot & \gamma & \cdot \end{bmatrix}$		Σ_{15}	$\begin{bmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \cdot \\ \gamma & \cdot & \cdot \end{bmatrix}$	

Point-orbit distributions of the K-orbits on planes in $\mathbb{P}^5$

Orbit	Point-orbit distribution	Condition
Σ_1	$[q+1, q(q+1)/2, q(q-1)/2, 0]$	
Σ_2	$[3, 3(q-1)/2, 3(q-1)/2, q^2-2q+1]$	
Σ_3	$[2, (3q-1)/2, (q-1)/2, q^2-q]$	
Σ_4	$[2, (3q-1)/2, (q-1)/2, q^2-q]$	
Σ_5	$[2, q-1, q-1, q^2-q+1]$	
Σ_6	$[1, (q+1)/2, (q+1)/2, q^2-1]$	
Σ_7	$[1, q^2+q, 0, 0]$	
Σ_8	$[1, 2q, 0, q^2-q]$	
Σ_9	$[1, 2q, 0, q^2-q]$	
Σ_{10}	$[1, q, q, q^2-q]$	
Σ_{11}	$[1, q, 0, q^2]$	
Σ_{12}	$[1, (q-1)/2, (q-1)/2, q^2+1]$	
Σ_{13}	$[1, (q+1)/2, (q+1)/2, q^2-1]$	
Σ_{14}	$[1, (q\mp 1)/2, (q\mp 1)/2, q^2\pm 1]$	$q \equiv \pm 1 \pmod 3$
Σ'_{14}	$[1, q, 0, q^2]$	$q \equiv 0 \pmod 3$
Σ_{15}	$[1, q, 0, q^2]$	

The hard basket (whiteboard)

3. Planes spanned by $\mathcal{V}(\mathbb{F}_q)^{(2)}$ and meeting $\mathcal{V}(\mathbb{F}_q)$ in one point.

($\rightarrow$ orbits $\Sigma_6, \ldots, \Sigma_{13}$, and Σ_{14} or Σ'_{14} depending on $q \mod 3$)

Let $\pi = \langle x, y, z \rangle$, with $x \in \mathcal{V}(\mathbb{F}_q)$ and $y, z \in \mathcal{V}(\mathbb{F}_q)^{(2)} \setminus \mathcal{V}(\mathbb{F}_q)$.

Consider the conics $\mathcal{C}_y$ and $\mathcal{C}_z$ determined by y and z, and let p_x and ℓ_y, ℓ_z be such that $x = \nu(p_x)$, $\mathcal{C}_y = \nu(\ell_y)$ and $\mathcal{C}_z = \nu(\ell_z)$.

If $x \in \mathcal{C}_y = \mathcal{C}_z$, then π is a conic plane, and so lies in the orbit Σ_1.

The remaining possibilities (up to symmetry) are as follows:

(a) $x \notin \mathcal{C}_y = \mathcal{C}_z$, ($\rightarrow \Sigma_6$)

(b) $x = \mathcal{C}_y \cap \mathcal{C}_z$, ($\rightarrow \Sigma_7$)

(c) $x \in \mathcal{C}_y \setminus \mathcal{C}_z$, ($\rightarrow \Sigma_8, \Sigma_9, \Sigma_{10}$)

(d) $\mathcal{C}_y \neq \mathcal{C}_z$ and $x \notin \mathcal{C}_y \cup \mathcal{C}_z$.

The hard basket continued (whiteboard)

(d) Suppose that $x \notin \mathcal{C}_y \cup \mathcal{C}_z$, and write $w = \mathcal{C}_y \cap \mathcal{C}_z$.

Without loss of generality, we may fix p_w, p_x, ℓ_y and ℓ_z.

Then the stabiliser of this configuration in $\mathbb{P}^2$ is contained in the group of perspectivities with centre p_w.

On the plane $\langle \mathcal{C}_y \rangle$ in $\mathbb{P}^5$, the induced group acts as the stabiliser of the conic $\mathcal{C}_y$ and the point w. There are two possibilities:

(i) y lies on the tangent line $t_w(\mathcal{C}_y)$,

(ii) y does not lie on $t_w(\mathcal{C}_y)$.

(d-i) Suppose that y lies on the tangent line $t_w(\mathcal{C}_y)$. Let $u \in \mathbb{P}^5$ be such that $y = t_u(\mathcal{C}_y) \cap t_w(\mathcal{C}_y)$. Then the stabiliser of x, y, w and $\mathcal{C}_z$ is induced by the group of perspectivities with centre p_w and axis $\langle p_x, p_u \rangle$, and acts on $\langle \mathcal{C}_z \rangle$ as the stabiliser of $\mathcal{C}_z$ and the two points w and $v := \nu(\langle p_x, p_u \rangle \cap \ell_z)$ of $\mathcal{C}_z$. Consider $t_v(\mathcal{C}_z)$ to $\mathcal{C}_z$. There are five possibilities for z:

(A) $z = t_w(\mathcal{C}_y) \cap t_v(\mathcal{C}_z)$,

(B) $z \in t_w(\mathcal{C}_y) \setminus t_v(\mathcal{C}_z)$,

(C) $z \in t_v(\mathcal{C}_z) \setminus t_w(\mathcal{C}_y)$,

(D) $z \in \langle w, v \rangle$, and

(E) $z \notin t_v(\mathcal{C}_z) \cup t_w(\mathcal{C}_y) \cup \langle w, v \rangle$.

The hard basket continued (whiteboard)

(d-i-D) For convenience, fix p_w, p_x, p_u and p_v as $\langle e_2 \rangle$, $\langle e_1 \rangle$, $\langle e_1 + e_3 \rangle$ and $\langle e_3 \rangle$, respectively. This implies that $\ell_y = \langle e_1 + e_3, e_2 \rangle$ and $\ell_z = \langle e_2, e_3 \rangle$. Note that y is now the point with coordinates $(0, 1, 0, 0, 1, 0)$. Since we are now assuming that z is on the line $\langle w, v \rangle$, we may take $z = (0, 0, 0, 1, 0, b)$ for some non-zero $b \in \mathbb{F}_q$. Then we obtain the orbits represented by

$$\Sigma_{12} : \begin{bmatrix} \alpha & \beta & \cdot \\ \beta & \gamma & \beta \\ \cdot & \beta & \gamma \end{bmatrix} \text{ and } \Sigma_{13} : \begin{bmatrix} \alpha & \beta & \cdot \\ \beta & \gamma & \beta \\ \cdot & \beta & \varepsilon\gamma \end{bmatrix}, \qquad (2)$$

where ε is a non-square in $\mathbb{F}_q$ ($b = 1$, $b = \epsilon$).

The point-orbit distributions of Σ_{12} and Σ_{13}

Lemma (implies Σ_{12} and Σ_{13} are $\neq$ K-orbits)

A plane belonging to the K-orbit Σ_{12} has point-orbit distribution

$$[1, (q-1)/2, (q-1)/2, q^2 + 1],$$

and a plane belonging to the K-orbit Σ_{13} has point-orbit distribution

$$[1, (q+1)/2, (q+1)/2, q^2 - 1].$$

Let π_{12}, π_{13} denote these representatives, and let C_{12}, C_{13} denote the respective cubics. Given a fixed basis for π_i, $i \in \{12, 13\}$, define a map from π_i to $\mathbb{P}^2$ in the natural way, and denote the image of C_i in $\mathbb{P}^2$ by $\overline{C}_i$.

Inflexion points of the cubics $\overline{C}_{12}$ and $\overline{C}_{13}$

Lemma (is used for future K-orbits)

Suppose that q is not a power of 3. Then the cubic curves $\overline{C}_{12}$ and $\overline{C}_{13}$ each have a double point at $P = (1, 0, 0)$. The tangents at the double point, and the points and lines of inflexion, are as given in the following table:

Plane	π_{12}	π_{13}
Tangents at P	$\beta = \pm\gamma$	$\beta = \pm\sqrt{\varepsilon}\gamma$
Inflexion points	$\{(0,1,0), (-1, \pm 4\sqrt{-1/3}, 4)\}$	$\{(0,1,0), (-\varepsilon, \pm 4\sqrt{-\varepsilon/3}, 4)\}$
Line of inflexion	$4\alpha + \gamma = 0$	$4\alpha + \varepsilon\gamma = 0$

In particular, when -3 is a square in $\mathbb{F}_q$, $\overline{C}_{12}$ has three $\mathbb{F}_q$-rational inflexion points, and $\overline{C}_{13}$ has one. When -3 is a non-square, $\overline{C}_{12}$ has one $\mathbb{F}_q$-rational inflexion point, and $\overline{C}_{13}$ has three.

The representative π_c

(d-i-E) If the point z is not on the line $\langle w, v \rangle$ and not on the tangents $t_w(\mathcal{C}_y)$ and $t_v(\mathcal{C}_z)$, then we may assume the line $\langle z, w \rangle$ passes through a point $r \in \mathcal{C}_z \setminus \{w, v\}$. We retain the same representatives as in case (d-i-D), and without loss of generality we further choose $r = \nu(p_r)$ with $p_r = \langle e_2 - e_3 \rangle$. Then we may take $z = z_c := (0, 0, 0, c, -1, 1)$ where $c \notin \{0, 1\}$ (because z_0 lies on $t_v(\mathcal{C}_z)$ and $z_1 = r$). The plane $\pi_c = \langle x, y, z_c \rangle$ is then represented by the matrix

$$\begin{bmatrix} \alpha & \beta & \cdot \\ \beta & c\gamma & \beta - \gamma \\ \cdot & \beta - \gamma & \gamma \end{bmatrix} .$$

Does π_c give a new orbit (new orbits)?

$(c \in \mathbb{F}_q \setminus \{0, 1\})$

Lemma
If $-3c \neq 0$ is a square in $\mathbb{F}_q$ and $\frac{\sqrt{c}+1}{\sqrt{c}-1}$ is not a cube in $\mathbb{F}_q(\sqrt{-3})$, then the plane π_c is not in any of the K-orbits $\Sigma_1, \ldots, \Sigma_{13}$.

Lemma
If π_c is contained in one of the K-orbits $\Sigma_1, \ldots, \Sigma_{13}$, then $\pi_c \in \Sigma_{12}$ if c is a square in $\mathbb{F}_q$, and $\pi_c \in \Sigma_{13}$ otherwise.

Lemma
If q is a power of 3 then $\pi_c \in \Sigma_{12} \cup \Sigma_{13}$.

How many K-orbits do $\pi_c \notin \Sigma_i$, $i \leq 13$ give?

Lemma (at least one new K-orbit)

If q is not a power of 3 then there exists $c \in \mathbb{F}_q$ such that the plane π_c is not in any of the K-orbits $\Sigma_1, \ldots, \Sigma_{13}$.

Lemma (when is $\pi_c \sim_K \pi_d$)

The planes π_c and π_d belong to the same K-orbit if one of the following conditions holds:

  (i) $cd = 1$,
 (ii) $\left(\frac{\sqrt{c}-1}{\sqrt{c}+1}\right)\left(\frac{\sqrt{d}-1}{\sqrt{d}+1}\right)$ *is a cube in* $\mathbb{F}_q(\sqrt{-3})$,
(iii) $\left(\frac{\sqrt{c}+1}{\sqrt{c}-1}\right)\left(\frac{\sqrt{d}-1}{\sqrt{d}+1}\right)$ *is a cube in* $\mathbb{F}_q(\sqrt{-3})$.

Lemma (exactly one new K-orbit)

All "new" planes π_c belong to the same K-orbit.

Conclusion. One more orbit Σ_{14} for $q \not\equiv 0(3)$.

The orbit Σ'_{14} for $q \equiv 0(3)$

(d-ii) The only planes $\pi = \langle x, y, z \rangle$ with $\mathrm{rank}(x) = 1$ and $\mathrm{rank}(y) = \mathrm{rank}(z) = 2$ that we have not yet considered are those such that y is not on the tangent $t_w(\mathcal{C}_y)$ to the conic $\mathcal{C}_y$ through the point $w = \mathcal{C}_y \cap \mathcal{C}_z$ in $\langle \mathcal{C}_y \rangle$ (and likewise z is not on $t_w(\mathcal{C}_z)$).

Lemma (no new K-orbit except for $q \equiv 0(3)$)

Such a plane π belongs to a previously considered K-orbits except when $q \equiv 0 \pmod 3$, in which case all those planes form a single K-orbit Σ'_{14}.

$$\Sigma'_{14} : \begin{bmatrix} \alpha + \gamma & \gamma & \gamma \\ \gamma & \beta + \gamma & \gamma \\ \gamma & \gamma & -\beta \end{bmatrix}, \quad \text{for } q \equiv 0 \pmod 3.$$

The final orbit Σ_{15}

Finally, planes meeting $\mathcal{V}(\mathbb{F}_q)$ in 1 pt but not spanned by $\mathcal{V}(\mathbb{F}_q)^{(2)}$.

At most one of the lines through the point $x \in \pi \cap \mathcal{V}(\mathbb{F}_q)$ meets $\mathcal{V}(\mathbb{F}_q)^{(2)} \setminus \{x\}$.

[ML-Popiel 2020] $\Rightarrow$ each such line has type o_9.

After applying some transformations and applying transitivity properties we can obtain

$$\Sigma_{15} : \begin{bmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \cdot \\ \gamma & \cdot & \cdot \end{bmatrix}.$$

Lemma
The K-orbit Σ_{15} is distinct from $\Sigma_1 \cup \cdots \cup \Sigma_{14}$, and from Σ'_{14} when $q \equiv 0(3)$.

Final comments

- The study of linear systems over finite fields seems to have been almost abandoned.
- In fact, until recently, the only complete classification was the one obtained by Dickson in 1906!
- There are definitely some more cases which can be studied.
- The geometric/combinatorial approach presented here seems novel and might help with further cases.
- Complete classifications remain very hard to obtain.