Boundary Effects in Stochastic Cyclic Competition Models on a Two-Dimensional Lattice

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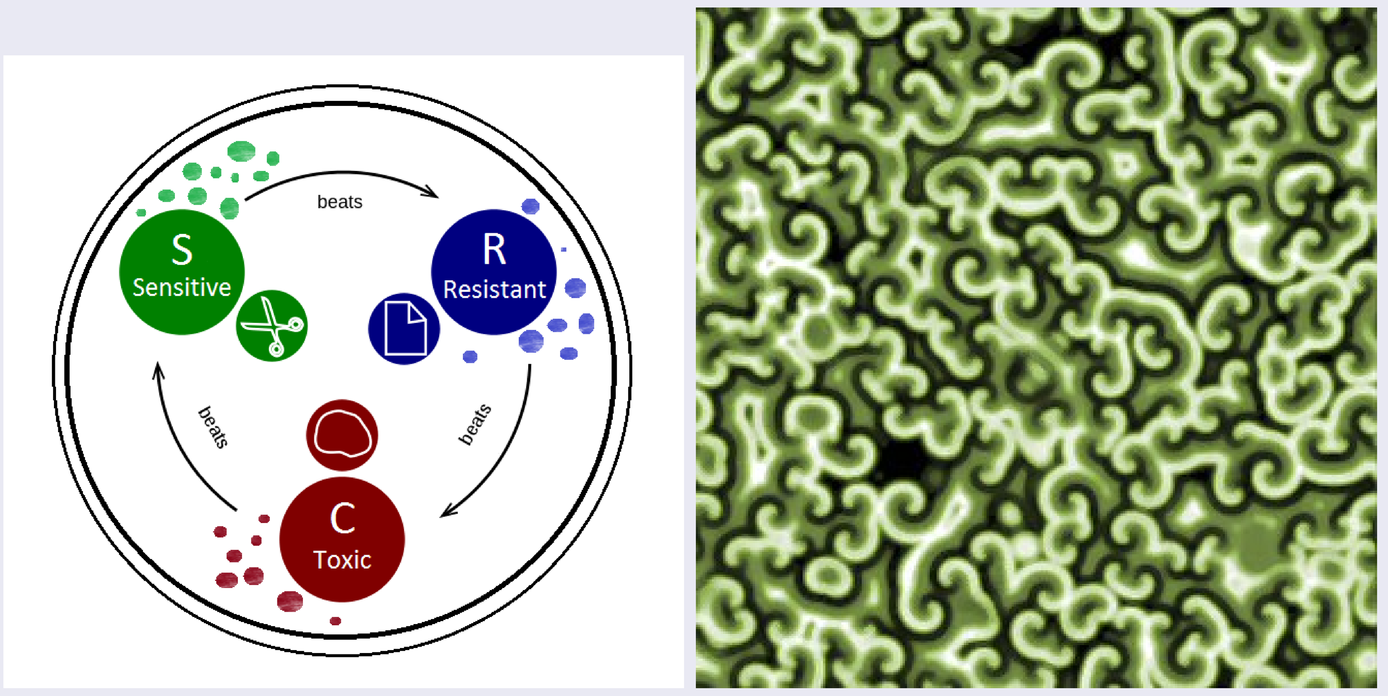
We study noise-induced and -stabilized spatial patterns in two distinct stochastic population model variants for cyclic competition of three species, namely the Rock-Paper-Scissors (RPS) and the May-Leonard (ML) models. In two dimensions, it is well established that the ML model can display (quasi-)stable spiral structures, in contrast to simple species clustering in the RPS system. Our ultimate goal is to impose control over such competing structures in systems where both RPS and ML reactions are implemented. To this end, we have employed Monte Carlo computer simulations to investigate how changing the microscopic rules in a subsection of a two-dimensional lattice influences the macroscopic behavior in the rest of the lattice. Specifically, we implement the ML reaction scheme on a torus, except on a ring-shaped patch, which is set to follow the cyclic Lotka-Volterra predation rules of the RPS model. There, we observe a marked disruption of the usual spiral patterns in the form of plane waves emanating from the RPS region. Furthermore, the overall population density drops considerably in the vicinity of the interface between both regions.

Introduction



(a) (b)

Figure 1: (a) Cyclic dominance in *E. coli* (b) Spiral pattern formation in the Belousov-Zhabotinsky reaction.

A number of systems in the fields of ecology, epidemiology, and chemistry follow a paradigm of cyclic dominance or have been shown to exhibit noise-induced pattern formation (e.g. certain subspecies of Lizards in California [1], experiments on cyclically competing *E. coli* bacteria [2], and the Belousov-Zhabotinsky reaction [3]). The formation of noise-induced and -stabilized spiral patterns in this class of system is captured by the spatially-extended May-Leonard (ML) model. The formation of these spirals is in stark contrast to the species clustering seen in the Rock-Paper-Scissors (RPS) model.

The RPS Model

The RPS Model is defined by the following binary reactions:

- Replacement reaction:

$$AB \to AA$$
$$BC \to BB$$
$$CA \to CC$$

with rate ζ

- Pair swap / diffusion reaction:

$$XY \to YX$$

where $X, Y \in \{A, B, C, \varnothing\}$ with rate ϵ_r

The May-Leonard Model

The ML Model is defined by the following binary reactions:

- Predation reaction:

$$AB \to A\varnothing$$
$$BC \to B\varnothing$$
$$CA \to C\varnothing$$

with rate σ

- Reproduction reaction:

$$A\varnothing \to AA$$
$$B\varnothing \to BB$$
$$C\varnothing \to CC$$

with rate μ

- Pair swap / diffusion reaction:

$$XY \to YX$$

$X, Y \in \{A, B, C, \varnothing\}$ with rate ϵ_m

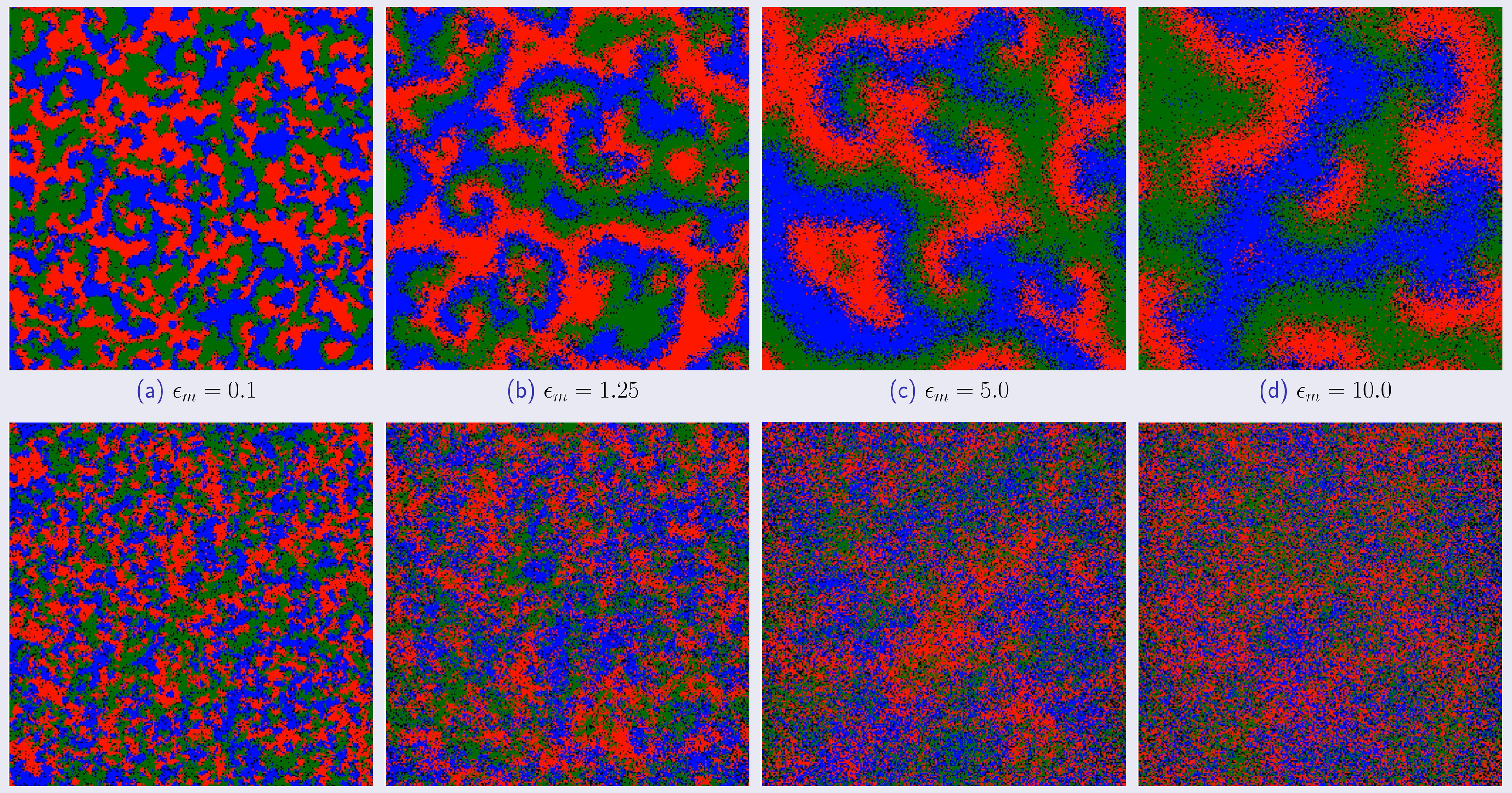
Simulation

We define x to be the vertical (short) axis and y to be the horizontal (long) axis. A lattice of size $L_x \times L_y$ is then initialized with each cell being assigned a random species with probability $p(A) = p(B) = p(C) = \rho_0/3$ (where ρ_0 is the initial net particle density). We limit each lattice point to contain at most one particle. The lattice is given a toroidal topology (i.e. $x = 0$ is equivalent to $x = L_x$ and likewise for $y = L_y$). The simulation then proceeds according to the following algorithm:

1. A random coordinate (x, y) is selected from a uniform distribution and time is advanced by $\delta t = N^{-1}$ (where $N = L_x \times L_y$).
2. If that lattice point is not empty, then a nearest neighbor is chosen at random. If the cell is empty, the simulation returns to step 1.
3. One of the possible reactions (as determined by the model) is selected at random (with probability determined by the reaction rates), and excecuted if possible. The simulation returns to step 1.

In cases where there are both RPS and ML lattice points, all lattice points in the range $0 \le y < d_i$ are governed by the RPS model and all remaining lattice points are governed by the ML model.

Normal Pattern Formation



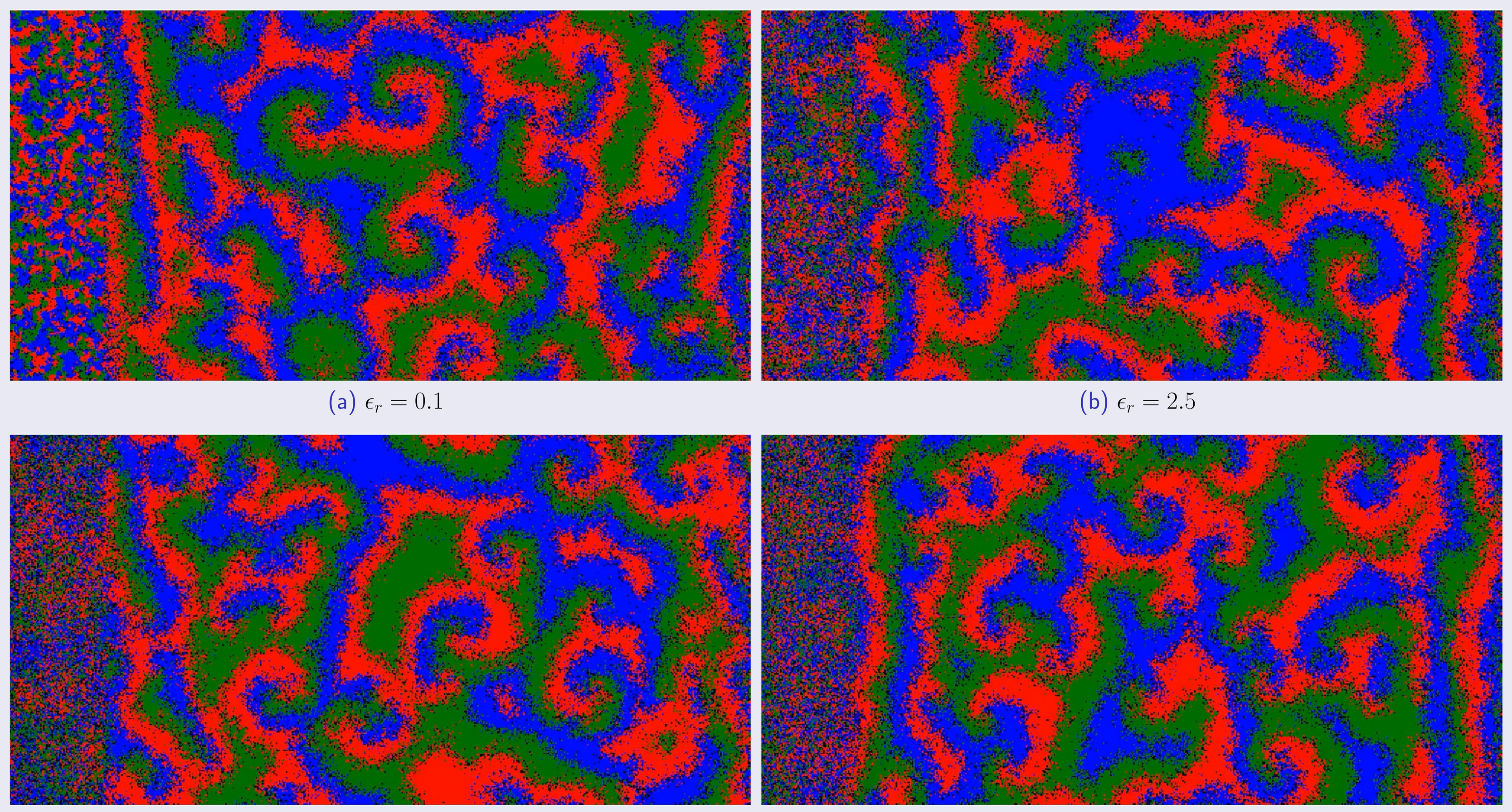
(a) $\epsilon_m = 0.1$ (b) $\epsilon_m = 1.25$ (c) $\epsilon_m = 5.0$ (d) $\epsilon_m = 10.0$

(e) $\epsilon_r = 0.1$ (f) $\epsilon_r = 1.25$ (g) $\epsilon_r = 5.0$ (h) $\epsilon_r = 10.0$

Figure 2: Steady state snapshots. All lattices have $L_x = L_y = 256$: (a)-(d) Typical ML pattern formation. $\sigma = \sigma = \mu = 1.0$ (e)-(h) Typical RPS pattern formation. $\zeta = 1.0$.

Our simulations produce (quasi-)stable patterns similar to those seen by Qian He *et al.* [4] and Peltomäki and Alava [5].

Plane Wave Formation



(a) $\epsilon_r = 0.1$ (b) $\epsilon_r = 2.5$

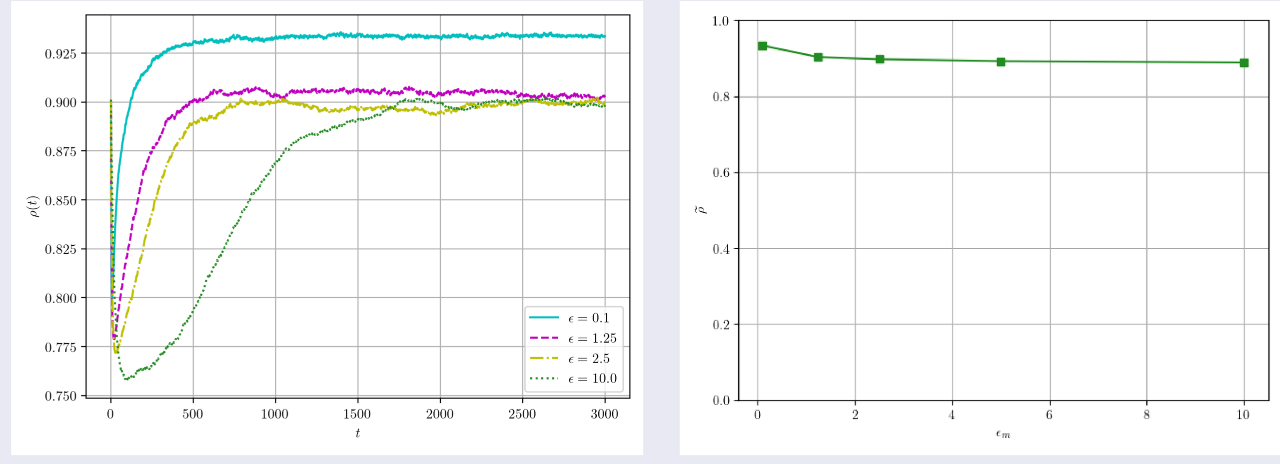
(c) $\epsilon_r = 5.0$ (d) $\epsilon_r = 10.0$

Figure 3: Steady state snapshots of plane wave formation in 256×512 lattices. In all cases $\epsilon_m = 2.5$, $\sigma = \mu = 1.0$, $\zeta = 1.0$. The interface is placed at $y = 64$.

Plane Wave Behavior

In Fig. 3 we see the formation of plane waves at the interface between the ML and RPS regions. The plane waves appear to be stable features of the interface. Furthermore, initial visual inspection seems to indicate that the coherence of these plane-waves seems to be dictated by both ϵ_m and ϵ_r. Further numerical analysis is required to confirm this, however.

Well-Mixing Effects



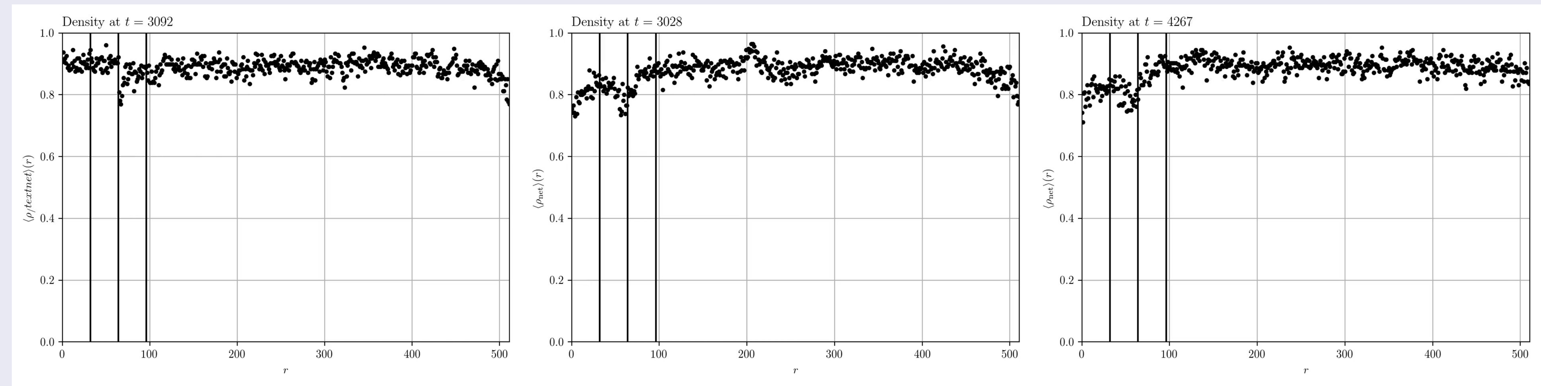
(a) Net density vs. time for a variety of ϵ_m values. Here we can clearly see the transient minimum caused by the homogeneous initial conditions.

(b) Net population density at $t = 3000\text{MCS}$ as a function of ϵ_m

Figure 4: In the above simulations we set $\sigma = \mu = 1.0$

We simulate four $N = 256 \times 256$ May-Leonard lattices, each with a different mobility rate, for 3000 Monte-Carlo steps and track the total population density $\rho = \frac{n_a + n_b + n_c}{N}$. We begin each simulation from a homogeneous initial condition and an initial density of $\rho_0 = 0.9$. In each simulation the total density quickly reaches a transient minimum $\rho_{\min}$ (Fig. 4 (a)) before relaxing to a steady state density $\tilde{\rho}$ determined, in part, by the mobility rate of the system (Fig. 4 (b)). Note that in each simulation $\rho_{\min} \sim \rho^* = 3\left(\frac{\mu}{3\mu+\sigma}\right) = 0.75$, where ρ^* is the mean field steady-state density [4].

Interface Density Effects



(a) $\epsilon_m = 0.1$ (b) $\epsilon_m = 5.0$ (c) $\epsilon_m = 10.0$

Figure 5: Typical interface density effects in 256×512 lattices for various mobility rates. In all cases $\sigma = \mu = 1.0$, $\zeta = 1.0$. The interface is placed at $y = 64$.

We observe a marked drop in density near the interface between the RPS and ML regions (the center black line in Fig. 5). This is likely caused by a local increase in spatial homogeneity induced by the RPS region. This appears to be similar to the transient minima that we observe above.

Ongoing and Future Work

- We are currently working to quantify their spatial coherence and permeation distance of the plane waves.
- We will also measure local homogeneity and effective reaction rates near the interface to confirm that the local mixing is the cause of the drop in density.
- As we advance our goal of enhancing and/or disrupting the formation of noise induced patterns we will use the understanding that we have developed of these boundary effects to inform potential new control schemes.

References

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