STAT 562 HW #2

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Problem 5.7

In Example 5.2.10, a partial fraction decomposition is needed to derive the distribution of the sum of two independent Cauchy random variables. This exercise provides the details that are skipped in that example.

(a) Find the constants A, B, C, and D that satisfy

$$\frac{1}{1+(w/\sigma)^2}\frac{1}{1+((z-w)/\tau)^2}$$
$$=\frac{Aw}{1+(w/\sigma)^2}+\frac{B}{1+(x/\sigma)^2}-\frac{Cw}{1+((z-w)/\tau)^2}-\frac{D}{1+(z+((z-w)/\tau)^2)}$$

where A, B, C, and D may depend on z but not on w.

We are given $\pi\sigma f_X(w)$ and $\pi\tau f_Y(z-w)$, where X and Y are independent Cauchy random variables with parameters $(0,\sigma)$ and $(0,\tau)$, respectively. Consider than integrating both sides of the equation with respect to w will maintain the equality while also removing any dependence on w from the constants A, B, C, and D. However, per the note, we already know that integrals involving A and C on their own do not exist, but if we group these two terms under one integral, their difference does exist. Now we can solve the following integrals:

$$\int_{-\infty}^{\infty}\frac{Aw}{1+(w/\sigma)^2}+\frac{B}{1+(x/\sigma)^2}-\frac{Cw}{1+((z-w)/\tau)^2}-\frac{D}{1+(z+((z-w)/\tau)^2)}dw$$
$$=\int_{-\infty}^{\infty}\frac{Aw}{1+(w/\sigma)^2}-\frac{Cw}{1+((z-w)/\tau)^2}dw+\int_{-\infty}^{\infty}\frac{B}{1+(x/\sigma)^2}dw-\int_{-\infty}^{\infty}\frac{D}{1+(z+((z-w)/\tau)^2)}dw$$

Let's start by solving the first integral, involving A and C:

$$\int_{-\infty}^{\infty}\frac{Aw}{1+(w/\sigma)^2}-\frac{Cw}{1+((z-w)/\tau)^2}dw$$

If we take a sneak peek at part (b) we see that the solution to the first integral is almost given to us. We can rewrite to conform:

$$\int_{-\infty}^{\infty}\frac{Aw}{1+(w/\sigma)^2}-\frac{Cw}{1+((z-w)/\tau)^2}dw$$
$$=\int_{-\infty}^{\infty}A\sigma^2\frac{w}{\sigma^2+w^2}-C\tau^2\frac{(z-w)}{\tau^2+(z-w)^2}+Cz\frac{1}{\tau^2+(z-w)^2}dw$$
$$=A\sigma^2\int_{-\infty}^{\infty}\frac{w}{\sigma^2+w^2}-C\tau^2\int_{-\infty}^{\infty}\frac{(z-w)}{\tau^2+(z-w)^2}+Cz\int_{-\infty}^{\infty}\frac{1}{\tau^2+(z-w)^2}dw$$
$$=A\sigma^2\left[\frac{1}{2}log\big(w^2+\sigma^2\big)\right]_{-\infty}^{\infty}-C\tau^2\left[\frac{1}{2}log\big((z-w)^2+\tau^2\big)\right]_{-\infty}^{\infty}+Cz\left[\frac{1}{\tau}arctan\big(\frac{z-w}{\tau}\big)\right]_{-\infty}^{\infty}$$
$$=\pi\tau Cz$$

Now we can solve integrals involving B and D:

$$\begin{aligned}\int_{-\infty}^{\infty} &\frac{B}{1+(w/\sigma)^2}dw \\ &= B\left[\frac{1}{2}log(1+(w/\sigma)^2\right]_{-\infty}^{\infty} \\ &= \pi\sigma B\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{\infty} &\frac{D}{1+((z-w)/\tau)^2}dw \\ &= D\left[\frac{1}{2}log(1+((z-w)/\tau)^2\right]_{-\infty}^{\infty} \\ &= \pi\tau D\end{aligned}$$

We are given that the integral of $f_X(w)f_((z-w)$ with respect to w is

$$f_Z(z) = \frac{1}{\pi(\sigma+\tau)}\frac{1}{1+(z/(\sigma+\tau))^2}$$

so now we need to solve

$$\pi^2\sigma\tau\frac{1}{\pi(\sigma+\tau)}\frac{1}{1+(z/(\sigma+\tau))^2} = \pi\tau Cz + \pi\sigma B - \pi\tau D$$

I'm not sure how to find a solution from here because there aren't enough equations. . . ?

(b) Using the facts that

$$\int \frac{t}{1+t^2}dt = \frac{1}{2}log(1+t^2) + c \text{ and } \int \frac{1}{1+t^2}dt = arctan(t) + c$$

evaluate (5.2.4) and hence verify (5.2.5).

$$\begin{aligned}\frac{1}{\pi^2\tau\sigma}\int_{-\infty}^{\infty} &\frac{Aw}{1+(w/\sigma)^2} + \frac{B}{1+(x/\sigma)^2} - \frac{Cw}{1+((z-w)/\tau)^2} - \frac{D}{1+(z+((z-w)/\tau)^2)} \\ &= \frac{1}{\pi^2\tau\sigma}(\pi\tau Cz + \pi\sigma B - \pi\tau D) \\ &= \frac{1}{\pi(\sigma+\tau)}\frac{1}{1+(z/(\sigma+\tau))^2}\end{aligned}$$

(Note that the integration in part (b) is quite delicate. Since the mean of a Cauchy does not exist, the integrals $\int_{-\infty}^{\infty}\frac{\mathbf{Aw}}{\mathbf{1+(w/\sigma)^2}}\mathbf{dw}$ and $\int_{-\infty}^{\infty}\frac{\mathbf{Cw}}{\mathbf{1+((z-w)/\tau)^2}}\mathbf{dw}$ do not exist. However, the integral of the difference *does exist*, which is all that is needed.)

Problem 5.10

Let $\mathbf{X_1, \ldots, X_n}$ be a random sample from a $\mathbf{n(\mu, \sigma^2)}$ population.

(c) Calculate Var$(\mathbf{S^2})$ a completely different (and easier) way: use the fact that $\mathbf{(n-1)S^2/\sigma^2 \sim \chi^2_{n-1}}$.

Knowing the distribution of $\frac{(n-1)S^2}{\sigma^2}$ has a chi-square distribution with $n-1$ degrees of freedom does make it easier to calculate the variance of S^2. Consider

$$Var(S^2) = Var\left(\frac{(n-1)S^2}{\sigma^2}\frac{\sigma^2}{(n-1)}\right) = \frac{\sigma^4}{(n-1)^2}Var\left(\frac{(n-1)S^2}{\sigma^2}\right)$$

We know the variance of a chi-square random variable, so we have:

$$Var(S^2) = \frac{\sigma^4}{(n-1)^2} Var\left(\frac{(n-1)S^2}{\sigma^2}\right) = \frac{\sigma^4}{(n-1)^2} 2(n-1) = \frac{2\sigma^4}{n-1}$$

Problem 5.13

Let $\mathbf{X_1, \ldots, X_n}$ be iid $\mathbf{n(\mu, \sigma^2)}$. Find a function of $\mathbf{S^2}$, the sample variance, say $\mathbf{g(S^2)}$, that satisfies $\mathbf{Eg(S^2) = \sigma}$. (*Hint*: try $\mathbf{g(S^2) = c\sqrt{S^2}}$, where c is a constant.)

Let's first simplify $E[g(S^2)]$, substituing $c\sqrt{S^2}$ for $g(S^2)$

$$E[g(S^2)] = E\big[c\sqrt{S^2}\big] = cE\big[\sqrt{S^2}\big]$$

We know that a function of the sample variance, $(n-1)S^2/\sigma^2$, has a chi-squared distribution with $n-1$ degrees of freedom. If we plug in this value for S^2 is our expected value, and then divide out $(n-1)/\sigma^2$, we get

$$\begin{aligned}
cE\big[\sqrt{S^2}\big] &= cE\left[\sqrt{\frac{(n-1)S^2}{\sigma^2}\frac{\sigma^2}{(n-1)}}\right] \\
&= c\frac{\sigma}{\sqrt{n-1}}E\left[\sqrt{\frac{(n-1)S^2}{\sigma^2}}\right] \\
&= c\frac{\sigma}{\sqrt{n-1}}E[\sqrt{W}]
\end{aligned}$$

Now we just need to find the expected value of the square root of a chi-squared random variable with $n-1$ degrees of freedom:

$$\begin{aligned}
E[\sqrt{W}] &= \int_0^\infty \sqrt{w}\frac{1}{\Gamma(\frac{n-1}{2})2^{(n-1)/2}}w^{(n-1)/2-1}e^{-w/2}dw \\
&= \int_0^\infty w^{1/2}\frac{1}{\Gamma(\frac{n-1}{2})2^{(n-1)/2}}w^{(n-1)/2-1}e^{-w/2}dw \\
&= \int_0^\infty \frac{1}{\Gamma(\frac{n-1}{2})2^{(n-1)/2}}w^{(n-1)/2-1+1/2}e^{-w/2}dw \\
&= \int_0^\infty \frac{1}{\Gamma(\frac{n-1}{2})2^{(n-1)/2}}w^{(n-1+1)/2-1}e^{-w/2}dw \\
&= \int_0^\infty \frac{1}{\Gamma(\frac{n-1}{2})2^{-1/2}2^{n/2}}w^{n/2-1}e^{-w/2}dw \\
&= \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})2^{-1/2}}\int_0^\infty \frac{1}{\Gamma(\frac{n}{2})2^{n/2}}w^{n/2-1}e^{-w/2}dw \\
&= \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})2^{-1/2}} \text{ (Gamma distribution with } \alpha = n/2)
\end{aligned}$$

Now we can solve for c by setting $E[g(S^2)] = \sigma$:

$$\begin{aligned}
c\frac{\sigma}{\sqrt{n-1}}E[\sqrt{W}] = c\frac{\sigma}{\sqrt{n-1}}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})2^{-1/2}} &= \sigma \\
\implies c = \sigma\frac{\sqrt{n-1}\Gamma(\frac{n-1}{2})2^{-1/2}}{\Gamma(\frac{n}{2})}
\end{aligned}$$

Problem 5.15

Establish the following recursion relations for means and variances. Let $\bar{\mathbf{X}}_{\mathbf{n}}$ and $\mathbf{S}_{\mathbf{n}}^{\mathbf{2}}$ be the mean and variance, respectively,of $\mathbf{X_1}, \dots, \mathbf{X_n}$. Then suppose another observation, $\mathbf{X_{n+1}}$, becomes available. Show that

(a) $\bar{\mathbf{X}}_{\mathbf{n+1}} = \frac{\mathbf{X_{n+1}} + \mathbf{n}\bar{\mathbf{X}}_{\mathbf{n}}}{\mathbf{n+1}}$**.**

The mean of the sample including the new observation is

$$\frac{1}{n+1}\sum_{i=1}^{n+1} X_i = \frac{1}{n+1}\left[\sum_{i=1}^{n} X_i + X_{n+1}\right] = \frac{1}{n+1}\left[n\bar{X}_n + X_{n+1}\right]$$

(b) $\mathbf{nS_{n+1}^2} = (\mathbf{n}-\mathbf{1})\mathbf{S_n^2} + \left(\frac{\mathbf{n}}{\mathbf{n+1}}\right)(\mathbf{X_{n+1}} - \bar{\mathbf{X}}_{\mathbf{n}})^{\mathbf{2}}$.

We are trying to show that

$$nS_{n+1}^2 = (n-1)S_n^2 + \left(\frac{n}{n+1}\right)\left(X_{n+1} - \bar{X}_n\right)^2$$

Note that

$$(n-1)S_n^2 = (n-1)\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 = \sum_{i=1}^{n}(X_i^2 + \bar{X}_n^2 - 2X_i\bar{X}_n) = \sum_{i=1}^{n} X_i^2 + n\bar{X}_n^2 - 2n\bar{X}_n^2$$

and

$$\left(\frac{n}{n+1}\right)\left(X_{n+1} - \bar{X}_n\right)^2 = \left(\frac{n}{n+1}\right)\left(X_{n+1}^2 + \bar{X}_n^2 - 2X_{n+1}\bar{X}_n\right)$$

Now we can solve for nS_{n+1}^2:

$$
\begin{aligned}
nS_{n+1}^2 &= \frac{n}{(n+1)-1}\sum_{i=1}^{n+1}\left(X_i-\bar{X}_{n+1}\right)^2 \\
&= \sum_{i=1}^{n}\left(X_i-\bar{X}_{n+1}\right)^2+(X_{n+1}-\bar{X}_{n+1})^2 \\
&= \sum_{i=1}^{n}\left(X_i^2+\bar{X}_{n+1}^2-2X_i\bar{X}_{n+1}\right)+(X_{n+1}^2+\bar{X}_{n+1}^2-2X_{n+1}\bar{X}_{n+1}) \\
&= \sum_{i=1}^{n}X_i^2+n\bar{X}_{n+1}^2-2\bar{X}_{n+1}n\bar{X}_n+X_{n+1}^2+\bar{X}_{n+1}^2-2X_{n+1}\bar{X}_{n+1} \\
&= \sum_{i=1}^{n}X_i^2-2\bar{X}_{n+1}(n\bar{X}_n+X_{n+1})+X_{n+1}^2+(n+1)\bar{X}_{n+1}^2 \\
&= \sum_{i=1}^{n}X_i^2-2(n+1)\bar{X}_{n+1}^2+X_{n+1}^2+(n+1)\bar{X}_{n+1}^2 \\
&= \sum_{i=1}^{n}X_i^2-(n+1)\bar{X}_{n+1}^2+X_{n+1}^2 \\
&= \sum_{i=1}^{n}X_i^2-(n+1)\left(\frac{1}{n+1}(n\bar{X}_n+X_{n+1})\right)^2+X_n^2 \\
&= \sum_{i=1}^{n}X_i^2-\frac{1}{n+1}\left(n^2\bar{X}_n^2+X_{n+1}^2+2n\bar{X}_nX_{n+1}\right)+X_{n+1}^2 \\
&= \sum_{i=1}^{n}X_i^2-\frac{1}{n+1}n^2\bar{X}_n^2-\frac{1}{n+1}X_{n+1}^2-\frac{1}{n+1}2n\bar{X}_nX_{n+1}+X_{n+1}^2 \\
&= \sum_{i=1}^{n}X_i^2-\frac{1}{n+1}n^2\bar{X}_n^2-\frac{1}{n+1}2n\bar{X}_nX_{n+1}+X_{n+1}^2\left(1-\frac{1}{n+1}\right) \\
&= \sum_{i=1}^{n}X_i^2-\left(\frac{n}{n+1}\right)n\bar{X}_n^2-\left(\frac{n}{n+1}\right)2\bar{X}_nX_{n+1}+\left(\frac{n}{n+1}\right)X_{n+1}^2 \\
&= \sum_{i=1}^{n}X_i^2+n\bar{X}_n-2n\bar{X}_n^2-\left(\frac{n}{n+1}\right)n\bar{X}_n^2-n\bar{X}_n+2n\bar{X}_n^2-\left(\frac{n}{n+1}\right)2\bar{X}_nX_{n+1}+\left(\frac{n}{n+1}\right)X_{n+1}^2 \\
&= (n-1)S_n^2-n\bar{X}_n+\frac{n^2\bar{X}_n^2+2n\bar{X}_n^2}{n+1}-\left(\frac{n}{n+1}\right)2\bar{X}_nX_{n+1}+\left(\frac{n}{n+1}\right)X_{n+1}^2
\end{aligned}
$$