STAT 665 Midterm

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February 21, 2019

Problem 1

Suppose that $\mathbf{X} = (X_1, X_2, X_3)'$ is a trivariate normal random vector with the moment generating function

$$m(\mathbf{t}) = \exp\left(t_1 - t_2 + 2t_3 + t_1^2 + \frac{1}{2}t_2^2 + 2t_3^2 - \frac{1}{2}t_1t_2 - t_1t_3\right),\ \mathbf{t} \in \mathcal{R}^3$$

Let $\bar{X} = (X_1 + X_2 + X_3)/3$. Find the distribution of X_1, conditional on $\bar{X} = 6$.

From the mgf we find μ and Σ by recalling that the mgf of a multivariate normal random variable takes the form

$$M_X(\mathbf{t}) = \exp\{\mathbf{t}'\mu + \frac{1}{2}\mathbf{t}'\Sigma\mathbf{t}\}$$

Clearly,

$$\mu = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

and

$$\Sigma = \begin{bmatrix} 2 & -\frac{1}{2} & -1 \\ -\frac{1}{2} & 1 & 0 \\ -1 & 0 & 4 \end{bmatrix}$$

We need to find the conditional probability of X_1 given $\bar{X} = 6$. From Result 5.2.6 we know that $[X_1, \bar{X}]' \sim N(\mathbf{B}\mu, \mathbf{B\Sigma B'})$. We can rewrite $\bar{X}$ as $\frac{1}{3}X_1 + \frac{1}{3}X_2 + \frac{1}{3}X_3$ such that

$$\begin{bmatrix} X_1 \\ \bar{X} \end{bmatrix} = \begin{bmatrix} X_1 \\ \frac{1}{3}X_1 + \frac{1}{3}X_2 + \frac{1}{3}X_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

Solving for $\mathbf{B}\mu$ we get

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix}$$

and

$$\mathbf{B\Sigma B'} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 & -\frac{1}{2} & -1 \\ -\frac{1}{2} & 1 & 0 \\ -1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & \frac{1}{3} \\ 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 2 & \frac{1}{6} \\ \frac{1}{6} & \frac{4}{9} \end{bmatrix}$$

From Result 5.2.10 we know that if $\mathbf{x} \sim N_k(\mu, \Sigma)$ with rank of Σ k, then the conditional distribution of x_1 given that $x_2 = c_2$ is multivariate normal with mean vector

$$\mu_{1.2} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{c_2} - \mu_2)$$

and

$$\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

We can partition $\mathbf{x}$, $\mathbf{B}\mu$, and $\mathbf{B}'\Sigma\mathbf{B}$ as

$$\left[\begin{array}{c} X_1 \\ \hline \bar{X} \end{array}\right],\ \left[\begin{array}{c} 1 \\ \hline \frac{2}{3} \end{array}\right],\ \left[\begin{array}{c|c} 2 & \frac{1}{6} \\ \hline \frac{1}{6} & \frac{4}{9} \end{array}\right].$$

Plugging into Result 5.2.10 we have that the distribution of X_1 given $\bar{X} = 6$ is normal with mean

$$\mu_{1.2} = 1 + (\frac{1}{6})(\frac{4}{9})^{-1}(6 - \frac{2}{3}) = 1 + \frac{9}{24}\frac{16}{3} = 1 + 2 = 3$$

and variance

$$\Sigma_{11.2} = 2 - (\frac{1}{6})(\frac{4}{9})^{-1}(\frac{1}{6}) = 2 - \frac{9}{24}\frac{1}{6} = 2 - \frac{1}{16} = \frac{31}{16}$$

Problem 2

Suppose that $\mathbf{Y} = X\beta + \epsilon$, where ϵ are $N(\mathbf{0}, \sigma^2\mathbf{I})$. The design matrix is given as follows:

$$\mathbf{X} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(a) Compute the rank of $P = X(X'X)^{-1}X'$.

Solving for P_X, we get

```
X <- matrix(c(
  rep(1,6),rep(1,3),rep(0,3),rep(c(1,0),3)
),nrow = 6,byrow = F)

P_X <- X %*% solve(t(X) %*% X) %*% t(X)
print(P_X)
```

```
##             [,1]       [,2]        [,3]        [,4]       [,5]        [,6]
## [1,]  0.41666667  0.1666667  0.41666667 -0.08333333  0.1666667 -0.08333333
## [2,]  0.16666667  0.6666667  0.16666667  0.16666667 -0.3333333  0.16666667
## [3,]  0.41666667  0.1666667  0.41666667 -0.08333333  0.1666667 -0.08333333
## [4,] -0.08333333  0.1666667 -0.08333333  0.41666667  0.1666667  0.41666667
## [5,]  0.16666667 -0.3333333  0.16666667  0.16666667  0.6666667  0.16666667
## [6,] -0.08333333  0.1666667 -0.08333333  0.41666667  0.1666667  0.41666667
```

After row reduction, we have the following matrix, which has 3 linearly independent columns, and thus a rank of 3.

```
rref(P_X)
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6]
## [1,]    1    0    1    0    1    0
## [2,]    0    1    0    0   -1    0
## [3,]    0    0    0    1    1    1
## [4,]    0    0    0    0    0    0
## [5,]    0    0    0    0    0    0
## [6,]    0    0    0    0    0    0
```

(b) Compute the rank of $Q = I - P$.

We know that P_X is idempotent, so $I_n - P_X$ is also idempotent with rank equal to $n - tr(P_X) = n - rank(P_X) = 6 - 3 = 3$ (by Result 2.3.9). So the rank of Q is 3.

(c) If $\rho(Q)$ is the rank of Q, prove that $\mathbf{Y}'Q\mathbf{Y}/\rho(Q)$ is an unbiased estimator of σ^2, where $\mathbf{Y} = (Y_1, \ldots, Y_6)'$.

As a result of Exercise 5.23, we have that $E(\mathbf{x}'\mathbf{A}\mathbf{x}) = tr(\mathbf{A}\Sigma) + \mu'\mathbf{A}\mu$. Therefore,

$$E[\mathbf{Y}'Q\mathbf{Y}/\rho(Q)] = \frac{1}{\rho(Q)}E[\mathbf{Y}'Q\mathbf{Y}] = \frac{1}{\rho(Q)}\left[tr(Q\Sigma) + \mu'Q\mu\right]$$

Note that $Var(\mathbf{Y}) = Var(\mathbf{X}\beta + \epsilon) = Var(\epsilon) = \mathbf{I}\sigma^2$ and $\mu = E[\mathbf{Y}] = E[\mathbf{X}\beta + \epsilon] + E[\mathbf{X}\beta] + E[\epsilon] = \mathbf{X}\beta$.

Now we have

$$\begin{aligned}
E[\mathbf{Y}'Q\mathbf{Y}/\rho(Q)] &= \frac{1}{\rho(Q)}\left[tr(Q\Sigma) + \mu'Q\mu\right] \\
&= \frac{1}{3}\left[tr(QI\sigma^2) + (X\beta)'(I-P)X\beta\right] \\
&= \frac{1}{3}\left[tr(Q\sigma^2) + \beta'X'X\beta - \beta'X'PX\beta\right] \\
&\text{Recalling that } PX = X, \\
&= \frac{1}{3}\left[\sigma^2 tr(Q) + \beta'X'X\beta - \beta'X'X\beta\right] \\
&= \frac{1}{3}\left[3\sigma^2\right] \\
&= \sigma^2
\end{aligned}$$

Since $E\left[\frac{\mathbf{Y}'Q\mathbf{Y}}{\rho(Q)}\right] = \sigma^2$, $\frac{\mathbf{Y}'Q\mathbf{Y}}{\rho(Q)}$ is an unbiased estimator for σ^2.

Problem 3

Suppose that X_1, X_2, and X_3 are iid $N(0,1)$.

(a) Find the distribution of Q, where

$$Q = \frac{1}{6}\left(5x_1^2 + 2x_2^2 + 5x_3^2 + 4x_1x_2 - 2x_1x_3 + 4x_2x_3\right)$$

We can write Q as

$$\begin{aligned}
Q = \mathbf{x}'\mathbf{A}\mathbf{x} &= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
&= \begin{bmatrix} x_1a_{11} + x_2a_{21} + x_3a_{31} & x_1a_{12} + x_2a_{22} + x_3a_{32} & x_1a_{13} + x_2a_{23} + x_3a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
&= x_1^2a_{11} + x_2^2a_{22} + x_3^2a_{33} + x_1x_2(a_{21} + a_{12}) + x_1x_3(a_{31} + a_{13}) + x_2x_3(a_{32} + a_{23})
\end{aligned}$$

which implies that

$$A = \frac{1}{6}\begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}$$

If A is idempotent, we can use Result 5.4.1. Let's check:

```
A <- matrix(c(
  5/6,2/6,-1/6,
  2/6,2/6,2/6,
  -1/6,2/6,5/6
),nrow = 3)
A
```

```
##            [,1]      [,2]       [,3]
## [1,]  0.8333333 0.3333333 -0.1666667
## [2,]  0.3333333 0.3333333  0.3333333
## [3,] -0.1666667 0.3333333  0.8333333
```

```
A%*%A
```

```
##            [,1]      [,2]       [,3]
## [1,]  0.8333333 0.3333333 -0.1666667
## [2,]  0.3333333 0.3333333  0.3333333
## [3,] -0.1666667 0.3333333  0.8333333
```

Since A is idempotent, we can use Result 5.4.1 to show that $Q\ \chi^2_m$, where $m = rank(A)$ which, in this case, is 2 (see A in row-reduced Echelon form below).

```
rref(A)
```

```
##      [,1] [,2] [,3]
## [1,]    1    0   -1
## [2,]    0    1    2
## [3,]    0    0    0
```

Therefore, $Q \sim \chi^2_2$.

(b) Find the distribution of Y and $Var(Y)$, where

$$Y = 4x_1^2 + 3x_2^2 + 4x_3^2 + 2\sqrt{2}x_2x_3.$$

(Hint: Express Y as a quadratic form with a matrix being nonidempotent. Try to use a famous theorem to represent the matrix in the quadratic form as a linear combination of some matrices.)

Similar to part (a), we can express Y as

$$Y = \mathbf{x}'\mathbf{A}\mathbf{x}$$

where

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & \sqrt{2} \\ 0 & \sqrt{2} & 4 \end{bmatrix}$$

```
A <- matrix(c(
  4,0,0,
  0,3,sqrt(2),
  0,sqrt(2),4
),nrow = 3)

A%*%A
```

```
##      [,1]     [,2]     [,3]
## [1,]   16 0.000000 0.000000
```

```
## [2,]    0 11.000000  9.899495
## [3,]    0  9.899495 18.000000
```

Clearly, A is not idempotent. Instead, let's consider the spectral decomposition theorem, since A is symmetric. Using this theorem we can rewrite $\mathbf{Y}$ as

$$\begin{aligned}
Y &= x'Ax \\
&= x'(\lambda_1 p_1 p_1' + \lambda_2 p_2 p_2' + \lambda_3 p_3 p_3')x \\
&= x'(\lambda_1 p_1 p_1')x + x'(\lambda_2 p_2 p_2')x + x'(\lambda_3 p_3 p_3')x \\
&= \lambda_1 x'(p_1 p_1')x + \lambda_2 x'(p_2 p_2')x + \lambda_3 x'(p_3 p_3')x
\end{aligned}$$

where p_k are the eigenvectors for corresponding λ_k. Note that $p_i p_i'$ are 3×3 orthongal, idempotent matrices of rank 1. Using Result 5.4.1 we know that since $p_i p_i'$ are idempotent and $X \sim iid\ N(0,1)$, $x_i'(p_i p_i')x_i$ are χ^2_1 independent random variables.

Let's determine the distribution of $\lambda_1(x_i' p_i p_i' x) = \lambda(x'pp'x)$. Let $y = \lambda x$. Then $x = y/\lambda$, and $\frac{dx}{dy}\frac{y}{\lambda} = \frac{1}{\lambda}$. The distribution of y is

$$f_Y(y) = \frac{\left(\frac{y}{\lambda}\right)^{1/2-1} e^{-(1/2)(y/\lambda)} \frac{1}{\lambda}}{\Gamma\left(\frac{1}{2}\right) 2^{1/2}} = \frac{y^{1/2-1} e^{-(y/2\lambda)}}{\Gamma\left(\frac{1}{2}\right)(2\lambda)^{1/2}},$$

that is, $Y \sim Gamma(\frac{1}{2}, 2\lambda)$. Now we have the sum of three independent Gamma$(\frac{1}{2}, 2\lambda_i)$, with $\lambda_1 = 5, \lambda_2 = 4, \lambda_3 = 2$. Since they do not have the same scale parameter (since λ_is are unique), they don't form a known distribution. But they are independent, making the variance of $\mathbf{Y}$ easy to calculate:

$$\begin{aligned}
Var(\mathbf{Y}) &= Var(Y_1) + Var(Y_2) + Var(Y_3) \\
&= \frac{1}{2}(2\lambda_1)^2 + \frac{1}{2}(2\lambda_2)^2 + \frac{1}{2}(2\lambda_3)^2 \\
&= 2\sum_{i=1}^{3} \lambda_i^2
\end{aligned}$$

```
eig(A)
```

```
## [1] 5 4 2
```

The eigenvalues of $\mathbf{A}$ are 5, 4, and 2, giving us

$$Var(\mathbf{Y}) = 2\sum_{i=1}^{3} \lambda_i^2 = 2(5^2 + 4^2 + 2^2) = 90.$$

Problem 4

Consider the two-way crossed classification model without interaction with exactly one observation per cell, i.e.,

$$y_{ij} = \mu + \alpha_i + \beta_j + d_{ij},\ (i = 1, \ldots, p;\ j = 1, \ldots, q).$$

Suppose further that $\mathbf{d} = \{d_{ij}\}$ has a multivariate normal distribution with mean vector $\mathbf{0}$ and variances and covariances given by

$$\begin{aligned}
cov(d_{ij}, d_{i',j'}) &= \lambda + 2\gamma_i, \quad \text{if } i' = i \text{ and } j' = j \\
&= \gamma_i + \gamma_{i'}, \quad \text{if } i' \neq i \text{ and } j' = j \\
&= 0, \qquad\qquad \text{if } j' \neq j.
\end{aligned}$$

Here, λ and γ_i are known quantities satisfying $\lambda > 0$ and $\lambda + 2\gamma_i > 0$, for all i.

(a) Express $q\sum_{i=1}^{p}(\bar{y}_{i.} - \bar{y}_{..})^2$ as a quadratic form in the vector $\mathbf{z}$, where $\mathbf{z'} = (\bar{y}_1, \ldots, \bar{y}_p)$.

Let's expand $q\sum_{i=1}^{p}(\bar{y}_{i.}-\bar{y}_{..})^2$:

$$
\begin{aligned}
q\sum_{i=1}^{p}(\bar{y}_{i.}-\bar{y}_{..})^2 &= q\sum_{i=1}^{p}\left(\bar{y}_{i.}^2+\bar{y}_{..}^2-2\bar{y}_{i.}\bar{y}_{..}\right)\\
&= q\left[\sum_{i=1}^{p}\bar{y}_{i.}^2+\sum_{i=1}^{p}\bar{y}_{..}^2-\sum_{i=1}^{p}2\bar{y}_{i.}\bar{y}_{..}\right]\\
&= q\left[\sum_{i=1}^{p}\bar{y}_{i.}^2+p\bar{y}_{..}^2-2p\bar{y}_{..}^2\right]\\
&= q\left[\sum_{i=1}^{p}\bar{y}_{i.}^2-p\bar{y}_{..}^2\right]
\end{aligned}
$$

We are looking for some matrix $\mathbf{A}$ such that $\mathbf{z'Az}=q\left[\sum_{i=1}^{p}\bar{y}_{i.}^2-p\bar{y}_{..}^2\right]$. Consider $\mathbf{A}=\mathbf{B}-\mathbf{C}$ such that $\mathbf{z'Az}=\mathbf{z'}q(\mathbf{B}-\mathbf{C})\mathbf{z}$. Note that

$$
\mathbf{z'I_pz}=\begin{bmatrix}\bar{y}_{1.} & \bar{y}_{2.} & \dots & \bar{y}_{p.}\end{bmatrix} I_p \begin{bmatrix}\bar{y}_{1.}\\ \bar{y}_{2.}\\ \vdots\\ \bar{y}_{p.}\end{bmatrix} = \bar{y}_{1.}^2+\bar{y}_{2.}^2+\cdots+\bar{y}_{p.}^2=\sum_{i=1}^{p}\bar{y}_{i.}
$$

and

$$
\begin{aligned}
\mathbf{z'\frac{1}{p}J_pz} &= \begin{bmatrix}\bar{y}_{1.} & \bar{y}_{2.} & \dots & \bar{y}_{p.}\end{bmatrix}\frac{1}{p}J_pz\begin{bmatrix}\bar{y}_{1.}\\ \bar{y}_{2.}\\ \vdots\\ \bar{y}_{p.}\end{bmatrix}\\
&= \frac{1}{p}\begin{bmatrix}\sum_{i=1}^{p}\bar{y}_{i.} & \sum_{i=1}^{p}\bar{y}_{i.} & \dots & \sum_{i=1}^{p}\bar{y}_{i.}\end{bmatrix}\begin{bmatrix}\bar{y}_{1.}\\ \bar{y}_{2.}\\ \vdots\\ \bar{y}_{p.}\end{bmatrix}\\
&= \frac{1}{p}\sum_{i=1}^{p}\bar{y}_{i.}\begin{bmatrix}1 & 1 & \dots & 1\end{bmatrix}\begin{bmatrix}\bar{y}_{1.}\\ \bar{y}_{2.}\\ \vdots\\ \bar{y}_{p.}\end{bmatrix}\\
&= \frac{1}{p}\sum_{i=1}^{p}\bar{y}_{i.}\times\sum_{i=1}^{p}\bar{y}_{i.}\\
&= p\bar{y}_{..}^2
\end{aligned}
$$

Therefore, $q\sum_{i=1}^{p}(\bar{y}_{i.}-\bar{y}_{..})^2=\mathbf{z'}q(\mathbf{I_p}-\mathbf{\frac{1}{p}J_p})\mathbf{z}$.

(b) Obtain the covariance matrix of $\mathbf{z}$.

To begin, let's look at the full model variance-covariance matrix, based on the description of $cov(d_{ij},d_{i'j'})$ given to us. We know that within each $q\times q$ submatrix where $i=i'$, that is, the submatrices along the diagonal, the covariance is $\lambda+2\gamma_i$ whenever $j=j'$, giving us $(\lambda+2\gamma_i)I_q$ submatrices along the diagonal. The off-diagonal $q\times q$ partitions include all cases where $i\neq i'$. Within these off-diagonal submatrices, the covariance is $(\gamma_i + \gamma_{i'})I_q$. For all submatrices, the covariance is 0 whenever $j \neq j'$, which is why each submatrix is a scalar multiplied by the identity matrix.

$$\begin{bmatrix} (\lambda + 2\gamma_1)I_q & (\gamma_1 + \gamma_2)I_q & \dots & (\gamma_1 + \gamma_p)I_q \\ (\gamma_2 + \gamma_1)I_q & (\lambda + 2\gamma_2)I_q & \dots & (\gamma_2 + \gamma_p)I_q \\ \vdots & \vdots & \ddots & \vdots \\ (\gamma_p + \gamma_1)I_q & (\gamma_p + \gamma_2)I_q & \dots & (\lambda + 2\gamma_p)I_q \end{bmatrix}_{pq \times pq}$$

But we want to find the covariance of $\mathbf{z'} = [\bar{y}_{1.}, \bar{y}_{2.}, \dots, \bar{y}_{p.}]'$. Because $\bar{y}_{i.}$ is the sum across all q within group i, and $\mathbf{y}$ is normally distributed (since its variation comes from the d_{ij}s), $\bar{y}_{i.}$s will be normally distributed (the reproductive property) with the following vaiance-covariance structure:

$$Cov(\bar{Y}_{i.}, \bar{Y}_{i'.}) = \frac{1}{q} \begin{bmatrix} \lambda + 2\gamma_1 & \gamma_1 + \gamma_2 & \dots & \gamma_1 + \gamma_p \\ \gamma_2 + \gamma_1 & \lambda + 2\gamma_2 & \dots & \gamma_2 + \gamma_p \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_p + \gamma_1 & \gamma_p + \gamma_2 & \dots & \lambda + 2\gamma_p \end{bmatrix}_{p \times p}$$

(c) Show that $q \sum_{i=1}^{p} (\bar{y}_{i.} - \bar{y}_{..})^2$ is distributed as a scalar of a noncentral chi-square random variable, and determine the noncentrality parameter. You may assume that the covariance matrix of $\mathbf{z}$ is positive definite.

By Result 5.4.5, if we can show that $q(\mathbf{I_p} - \frac{\mathbf{1}}{\mathbf{p}}\mathbf{J_p})\mathbf{\Sigma_z}$ is idempotent, then $q \sum_{i=1}^{p} (\bar{y}_{i.} - \bar{y}_{..})^2$ has a noncentral chi-square distribution. Note that Σ_z can be rewritten as $\lambda \mathbf{I} + \mathbf{1}\gamma' + \gamma \mathbf{1'}$.

Let's check for idempotency by first simplifying $A\Sigma_z$:

$$\begin{aligned} \mathbf{A\Sigma_z} &= q(\mathbf{I_p} - \frac{1}{p}\mathbf{J_p})\frac{1}{q}(\lambda \mathbf{I_p} + \mathbf{1}\gamma' + \gamma \mathbf{1'}) \\ &= \lambda \mathbf{I_p} + \mathbf{1}\gamma' + \gamma \mathbf{1'} - \frac{1}{p}\mathbf{J_p}\lambda \mathbf{I_p} - \frac{1}{p}\mathbf{J_p}\mathbf{1}\gamma' - \frac{1}{p}\mathbf{J_p}\gamma \mathbf{1'} \\ &= \lambda \mathbf{I_p} + \mathbf{1}\gamma' + \gamma \mathbf{1'} - \frac{1}{p}\mathbf{J_p}\lambda \mathbf{I_p} - \frac{1}{p}\mathbf{1}\mathbf{1'}\mathbf{1}\gamma' - \frac{1}{p}\mathbf{J_p}\gamma \mathbf{1'} \\ &= \lambda \mathbf{I_p} + \mathbf{1}\gamma' + \gamma \mathbf{1'} - \frac{1}{p}\mathbf{J_p}\lambda \mathbf{I_p} - \frac{1}{p}\mathbf{1}p\gamma' - \frac{1}{p}\mathbf{J_p}\gamma \mathbf{1'} \\ &= \lambda \mathbf{I_p} + \gamma \mathbf{1'} - \frac{1}{p}\mathbf{J_p}\lambda \mathbf{I_p} - \frac{1}{p}\mathbf{J_p}\gamma \mathbf{1'} \\ &= \lambda(\mathbf{I_p} - \frac{1}{p}\mathbf{J_p}) + \gamma \mathbf{1'}(\mathbf{I_p} - \frac{1}{p}\mathbf{J_p}) \\ &= (\lambda + \gamma \mathbf{1'})(\mathbf{I_p} - \frac{1}{p}\mathbf{J_p}) \end{aligned}$$

Now we can check for idempotency of $A\Sigma_z$.

$$\begin{aligned}
\mathbf{A\Sigma_zA\Sigma_z} &= (\lambda + \gamma\mathbf{1}')(\mathbf{I_p} - \frac{1}{p}\mathbf{J_p})(\lambda + \gamma\mathbf{1}')(\mathbf{I_p} - \frac{1}{p}\mathbf{J_p}) \\
&= (\lambda\mathbf{I_p} - \frac{1}{p}\lambda\mathbf{J_p} + \gamma\mathbf{1}' - \frac{1}{p}\gamma\mathbf{1}'\mathbf{J_p})(\lambda\mathbf{I_p} - \frac{1}{p}\lambda\mathbf{J_p} + \gamma\mathbf{1}' - \frac{1}{p}\gamma\mathbf{1}'\mathbf{J_p}) \\
&= (\lambda\mathbf{I_p} - \frac{1}{p}\lambda\mathbf{J_p} + \gamma\mathbf{1}' - \frac{1}{p}\gamma\mathbf{1}'\mathbf{1}\mathbf{1}')(\lambda\mathbf{I_p} - \frac{1}{p}\lambda\mathbf{J_p} + \gamma\mathbf{1}' - \frac{1}{p}\gamma\mathbf{1}'\mathbf{1}\mathbf{1}') \\
&= (\lambda\mathbf{I_p} - \frac{1}{p}\lambda\mathbf{J_p} + \gamma\mathbf{1}' - \frac{1}{p}\gamma p\mathbf{1}')(\lambda\mathbf{I_p} - \frac{1}{p}\lambda\mathbf{J_p} + \gamma\mathbf{1}' - \frac{1}{p}\gamma p\mathbf{1}') \\
&= (\lambda\mathbf{I_p} - \frac{1}{p}\lambda\mathbf{J_p} + \gamma\mathbf{1}' - \gamma\mathbf{1}')(\lambda\mathbf{I_p} - \frac{1}{p}\lambda\mathbf{J_p} + \gamma\mathbf{1}' - \gamma\mathbf{1}') \\
&= (\lambda\mathbf{I_p} - \frac{1}{p}\lambda\mathbf{J_p})(\lambda\mathbf{I_p} - \frac{1}{p}\lambda\mathbf{J_p}) \\
&= \lambda^2(\mathbf{I_p} - \frac{1}{p}\mathbf{J_p})
\end{aligned}$$

λ is obviously a scalar. If we define $\mathbf{A}^* = \frac{1}{\lambda}\mathbf{A}$, we quickly see that $\mathbf{z}'\mathbf{A}^*\mathbf{z}$ has a noncentral chi-square distribution with $rank(A^*) = rank(\frac{q}{\lambda}(I_p - \frac{1}{p}J_p)) = rank(I_p - \frac{1}{p}J_p) = p - rank(J_p) = p - 1$ and noncentrality parameter

$$
\begin{aligned}
\mu'\mathbf{A}^*\mu &= \frac{q}{\lambda}\mu'(\mathbf{I_p} - \frac{1}{p}\mathbf{J_p})\mu \\
&= \frac{q}{\lambda}\begin{bmatrix}\mu_{1.} & \mu_{2.} & \dots & \mu_{p.}\end{bmatrix}(\mathbf{I_p} - \frac{1}{p}\mathbf{J_p})\begin{bmatrix}\mu_{1.}\\ \mu_{2.}\\ \dots\\ \mu_{p.}\end{bmatrix} \\
&= \frac{q}{\lambda}\begin{bmatrix}\mu+\alpha_1 & \mu+\alpha_2 & \dots & \mu+\alpha_p\end{bmatrix}\begin{bmatrix}1-\frac{1}{p} & -\frac{1}{p} & \dots & -\frac{1}{p}\\ -\frac{1}{p} & 1-\frac{1}{p} & \dots & -\frac{1}{p}\\ \vdots & \vdots & \ddots & \vdots\\ -\frac{1}{p} & -\frac{1}{p} & \dots & 1-\frac{1}{p}\end{bmatrix}\begin{bmatrix}\mu+\alpha_1\\ \mu+\alpha_2\\ \dots\\ \mu+\alpha_p\end{bmatrix} \\
&= \frac{q}{\lambda}\begin{bmatrix}\mu+\alpha_1 - \frac{1}{p}(\mu+\alpha_1) - \frac{1}{p}(\mu+\alpha_2) - \frac{1}{p}(\mu+\alpha_s) & \dots & \dots & \dots\end{bmatrix}\begin{bmatrix}\mu+\alpha_1\\ \mu+\alpha_2\\ \dots\\ \mu+\alpha_p\end{bmatrix} \\
&= \frac{q}{\lambda}\begin{bmatrix}\mu+\alpha_1-\frac{1}{p}(p\mu+\sum_{i=1}^p\alpha_i) & \mu+\alpha_2-\frac{1}{p}(p\mu+\sum_{i=1}^p\alpha_i) & \dots & \mu+\alpha_p-\frac{1}{p}(p\mu+\sum_{i=1}^p\alpha_i)\end{bmatrix}\begin{bmatrix}\mu+\alpha_1\\ \mu+\alpha_2\\ \dots\\ \mu+\alpha_p\end{bmatrix} \\
&= \frac{q}{\lambda}(\mu+\alpha_1-\frac{1}{p}(p\mu+\sum_{i=1}^p\alpha_i))(\mu+\alpha_1) \\
&\quad + (\mu+\alpha_2-\frac{1}{p}(p\mu+\sum_{i=1}^p\alpha_i))(\mu+\alpha_2)+\cdots+(\mu+\alpha_p-\frac{1}{p}(p\mu+\sum_{i=1}^p\alpha_i))(\mu+\alpha_p) \\
&= \frac{q}{\lambda}(\alpha_1+\mu)(\alpha_1-\bar{\alpha}_.)+(\alpha_2+\mu)(\alpha_2-\bar{\alpha}_.)+\cdots+(\alpha_p+\mu)(\alpha_p-\bar{\alpha}_.) \\
&= \frac{q}{\lambda}\sum_{i=1}^p(\alpha_i+\mu)(\alpha_i-\bar{\alpha}_.) \\
&= \frac{q}{\lambda}\sum_{i=1}^p(\alpha_i^2-\alpha_i\bar{\alpha}_.+\mu\alpha_i-\mu\bar{\alpha}_.) \\
&= \frac{q}{\lambda}\left(\sum_{i=1}^p\alpha_i^2-p\bar{\alpha}_.^{\,2}+p\mu\bar{\alpha}_.-p\mu\bar{\alpha}_.\right) \\
&= \frac{q}{\lambda}\left(\sum_{i=1}^p\alpha_i^2-p\bar{\alpha}_.^{\,2}\right)
\end{aligned}
$$

As a result, we see that $q\sum_{i=1}^p(\bar{y}_{i.}-\bar{y}_{..})^2 = \mathbf{z'Az} = \mathbf{z'}\lambda\mathbf{A}^*\mathbf{z}$ is distributed as a scalar of a noncentral chi-square distribution with parameters $p-1$ and $\frac{q}{\lambda}\left(\sum_{i=1}^p\alpha_i^2-p\bar{\alpha}_.^{\,2}\right)$.