STAT 665 - HW 1

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Problem 5.1

Let $x = (X_1, \dots, X_k) \sim N_k(\mu, \Sigma)$, with $r(\Sigma) = k$.

(a) Show that

$$
\begin{aligned}
I &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} exp\Big\{ -\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\Big\} dx_1 \dots dx_k \\
&= (2\pi)^{k/2}|\Sigma|^{1/2}.
\end{aligned}
$$

What we have is part of a multivariate normal distribution integrated over all possible values of $\mathbf{x}$. Let's first note that by definition of positive-definite, Σ^{-1} must be a positive definite matrix, otherwise its inverse would not exist. Now, we know that

$$
f(\mathbf{x};\mu,\Sigma) = \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}} exp\Big\{ -\frac{1}{2}(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)\Big\} \text{ for all } \mathbf{x} \in \mathcal{R}^k
$$

and the integration of $f(\mathbf{x};\mu,\Sigma)$ over all possible values of $\mathbf{x}$ is 1. Therefore, we have that

$$
\begin{aligned}
1 &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}} exp\Big\{ -\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\Big\} dx_1 \dots dx_k \\
\implies (2\pi)^{k/2}|\Sigma|^{1/2} &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} exp\Big\{ -\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\Big\} dx_1 \dots dx_k
\end{aligned}
$$

(b) Evaluate $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} exp\{-(x_1^2 + 2x_1x_2 + 4x_2^2)\} dx_1 dx_2$.

Let's rewrite the expression in the following way:

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} exp\{-(x_1^2 + 2x_1x_2 + 4x_2^2)\} dx_1 dx_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} exp\{-\frac{1}{2}(2x_1^2 + 4x_1x_2 + 8x_2^2)\} dx_1 dx_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} exp\{-\frac{1}{2}(\mathbf{x}'\mathbf{A}\mathbf{x})\} dx_1 dx_2
\end{aligned}
$$

where $\mathbf{A} = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}$, a 2×2 positive-definite matrix. Applying Aitken's integral, we have the solution

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} exp\{-\frac{1}{2}(\mathbf{x}'\mathbf{A}\mathbf{x})\} dx_1 dx_2 \\
&= (2\pi)^{2/2}|\mathbf{A}|^{-1/2} \\
&= (2\pi)\Bigg| \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} \Bigg|^{-1/2} \\
&= (2\pi)(4)^{-1/2} \\
&= \pi
\end{aligned}
$$

Problem 5.2

[Graybill, 1961]. Let $x = (X_1, X_2)$ have a bivariate normal distribution with pdf

$$f(x; \mu, \Sigma) = \frac{1}{k} exp[-Q/2]$$

where $Q = 2x_1^2 - x_1x_2 + 4x_2^2 - 11x_1 - 5x_2 + 19$, and k is a constant. Find a constant a such that $P(3X_1 - X_2 < a) = 0.01$.

We need to find the pdf of a new random variable $Y_1 = 3X_1 - X_2$. Let $Y_2 = X_1$. Solving for X_1 and X_2 in terms of Y_1 and Y_2, we have

$$\begin{aligned} X_1 &= Y_2 \\ X_2 &= 3Y_2 - Y_1 \end{aligned}$$

Next we solve for the Jacobian:

$$\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} = (0)(3) - (1)(-1) = 1$$

Now we can solve for the joint distribution of Y_1 and Y_2

$$\begin{aligned} f(\mathbf{y}) &= f(\mathbf{x})J(\mathbf{y}) \\ &= \frac{1}{k} exp\{-Q/2\}(1) \\ &= \frac{1}{k} exp\{-\frac{1}{2}(2x_1^2 - x_1x_2 + 4x_2^2 - 11x_1 - 5x_2 + 19)\} \\ &= \frac{1}{k} exp\{-\frac{1}{2}(\mathbf{x}'\mathbf{B}\mathbf{x} + \mathbf{x}'\mathbf{b} + b_0)\} \end{aligned}$$

Let's solve for $\mathbf{B}$ first:

$$\begin{aligned} \mathbf{x}'\mathbf{B}\mathbf{x} &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} b_{11}x_1 + b_{21}x_2 & b_{12}x_1 + b_{22}x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= b_{11}x_1^2 + b_{21}x_1x_2 + b_{12}x_1x_2 + b_{22}x_2^2 \\ &= (2)x_1^2 + (-1/2)x_1x_2 + (-1/2)x_1x_2 + (4)x_2^2 \\ &\implies \mathbf{B} = \begin{bmatrix} 2 & -1/2 \\ -1/2 & 4 \end{bmatrix} \end{aligned}$$

and now $\mathbf{b}$:

$$\begin{aligned} \mathbf{x}'\mathbf{b} &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \\ &= b_1x_1 + b_2x_2 \\ &\implies \mathbf{b} = \begin{bmatrix} -11 \\ -5 \end{bmatrix} \end{aligned}$$

and clearly $b_0 = 19$.

Problem 5.5

(a) Show that (X_1, X_2) has a bivariate normal distribution with means μ_1, μ_2, variances σ_1^2, σ_2^2, and a correlation coefficient ρ if and only if every linear combination $c_1X_1 + c_2X_2$ has a univariate normal distribution with mean $c_1\mu_1 + c_2\mu_2$, and variance $c_1^2\sigma_1^2 + c_2^2\sigma_2^2 + 2c_1c_2\rho\sigma_{1,2}$, where c_1 and c_2 are real constants, not both equal to zero.

Let's first show that if (X_1, X_2) has a bivariate normal distribution with means μ_1, μ_2, variances σ_1^2, σ_2^2, and a correlation coefficient ρ then every linear combination $c_1X_1 + c_2X_2$ has a univariate normal distribution with mean $c_1\mu_1 + c_2\mu_2$ and variance $c_1^2\sigma_1^2 + c_2\sigma_2^2 + 2c_1c_2\rho\sigma_{1,2}$.

Let $(X_1, X_2) \sim N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$. Consider the transformation $Y_1 = c_1X_1 + c_2X_2$. Let $Y_2 = X_2$. We can solve for X_1 and X_2:

$$X_2 = Y_2$$
$$X_1 = \frac{1}{c_1}(Y_1 - c_2Y_2)$$

The Jacobian is

$$\mathbf{J} = \begin{bmatrix} \frac{1}{c_1} & -\frac{1}{c_2} \\ 0 & 1 \end{bmatrix} = \frac{1}{c_1}$$

(b) Let $Y_i = X_i/\sigma_i, i = 1, 2$. Show that $Var(Y_1 - Y_2) = 2(1 - \rho)$.

Problem 5.6

(a) Let $(X_1, X_2) \sim N_2(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ where $\mu_1 = \mu_2 = 0$ and $\rho \neq 1$. The polar coordinate transformation is defined by $X_1 = Rcos\Theta$, $X_2 = Rsin\Theta$. Show that the joint pdf of R and Θ is given by

$$r(2\pi)^{-1}(1-\rho^2)^{-1/2}exp\left[-\frac{1}{2(1-\rho^2)}r^2(1-\rho sin2\theta)\right],$$

$0 \leq r < \infty$, and $0 \leq \theta \leq 2\pi$, and that the marginal pdf of Θ is

$$(2\pi)^{-1}(1-\rho^2)^{1/2}(1-\rho sin2\theta)^{-1},\ 0 \leq \theta \leq 2\pi.$$

(b) Suppose (X_1, X_2) has a bivariate normal distribution $N_2(0, 0, \sigma_1^2, \sigma_2^2, \rho,\ |\rho| \neq 1$. Show that

$$P(X_1 > 0, X_2 > 0) = \frac{1}{4} + \frac{1}{2\pi}sin^{-1}(\rho).$$

Problem 5.7

The random vector $x = (X_1, X_2, \ldots, X_k)'$ is said to have a symmetric multivariate normal distribution if $x \sim N_k(\mu, \Sigma)$ where $\mu = \mu 1_k$, i.e., the mean of each X_j is equal to the same constant μ, and Σ is the equicorrelation dispersion matrix, i.e.,

$$\Sigma = \sigma^2 = \begin{bmatrix} 1 & \rho & \ldots & \rho \\ \rho & 1 & \ldots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \ldots & 1 \end{bmatrix}$$

When $k = 3$, $\mu = 0$, $\sigma^2 = 2$, and $\rho = 1/2$, find the probability that $X_3 = min(X_1, X_2.X_3)$.

(*Hint:* Recall that if $x = (X_1, \ldots, X_k)'$ has a continuous symmetric distribution, then all possible permutations of $X_1, \ldots, X_k$ are equally likely, each having probability $P(X_{i1} < \cdots < X_{ik}) = 1/k!$ for any permutation $(i1, \ldots, i_k)$ for the first k positive integers.

Problem 5.8

Let $\mathbf{x} \sim N_k(0, \Sigma)$ with pdf $f(x)$ where $\Sigma = \{\Sigma_{ij}\}$. The entropy $h(x)$ is defined as

$$h(x) = -\int f(x) ln f(x)$$

(a) Show that $h(x) = \frac{1}{2} ln\big[(2\pi e)^k |\Sigma|\big]$.

We need to show that

$$\frac{1}{2}\ln(2\pi)^k|\Sigma| = -\int f(x)\ln(f(x))$$

But we know $\mathbf{x}$ has a multivariate normal distribution $N_k(0, \Sigma)$:

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}} \exp\Big\{-\frac{1}{2}(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)\Big\},\ \mathbf{x} \in R^k$$

Inserting into the definition of entropy, we have

$$\begin{aligned}
&-\int \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}} \exp\Big\{-\frac{1}{2}(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)\Big\} \ln\left(\frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}} \exp\Big\{-\frac{1}{2}(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)\Big\}\right) \\
&\quad= -\int \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}} \exp\Big\{-\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x}+\mathbf{x}'\mu)\Big\} \ln\left(\frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}} \exp\Big\{-\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x}+\mathbf{x}'\mu)\Big\}\right) \\
&\quad= -\int \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}} \exp\Big\{-\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x})\Big\} \ln\left(\frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}} \exp\Big\{-\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x})\Big\}\right)
\end{aligned}$$

since we were given that $\mu = \mathbf{0}$. Applying log rules, we can expand the last term and continue:

$$\begin{aligned}
&= -\int \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}} \exp\Big\{-\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x})\Big\}\left(-\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x}) - \frac{k}{2}\ln(2\pi) - \frac{1}{2}\ln(|\Sigma|)\right) \\
&= \int \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}} \exp\Big\{-\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x})\Big\}\left(\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x}) + \frac{k}{2}\ln(2\pi) + \frac{1}{2}\ln(|\Sigma|)\right) \\
&= \frac{1}{2}\int \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}} \exp\Big\{-\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x})\Big\}\left((\mathbf{x}'\Sigma^{-1}\mathbf{x}) + k\ln(2\pi) + \ln(|\Sigma|)\right) \\
&= \frac{1}{2}\Bigg[\frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}\int (\mathbf{x}'\Sigma^{-1}\mathbf{x}) \exp\Big\{-\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x})\Big\} \\
&\quad + k\ln(2\pi)\int \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}} \exp\Big\{-\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x})\Big\} \\
&\quad + \ln(|\Sigma|)\int \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}} \exp\Big\{-\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x})\Big\}\Big)\Bigg]
\end{aligned}$$

The integrals in the last two terms simply equal the constants pulled out front as we are integrating the multinomial normal distribution across all values of $\mathbf{x}$. The first term can be evaluated per Result 5.1.3:

$$\begin{aligned}
&\frac{1}{2}\left[\frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}(2\pi)^{k/2}|\Sigma^{-1}|^{-1/2}tr(\Sigma^{-1}\Sigma) + k\ln(2\pi) + \ln(|\Sigma|)\right] \\
&= \frac{1}{2}\left[\frac{1}{|\Sigma|^{1/2}}|\Sigma|^{1/2}tr(I_k) + k\ln(2\pi) + \ln(|\Sigma|)\right] \\
&= \frac{1}{2}\left[k + k\ln(2\pi) + \ln(|\Sigma|)\right] \\
&= \frac{1}{2}\left[k + \ln\left[(2\pi)^k|\Sigma|\right]\right] \\
&= \frac{1}{2}\left[\ln(e^k) + \ln\left[(2\pi)^k|\Sigma|\right]\right] \\
&= \frac{1}{2}\ln\left[(2\pi e)^k|\Sigma|\right]
\end{aligned}$$

(b) Hence, or otherwise, show that $|\Sigma| \leq \prod_{i=1}^{k}\Sigma_{ii}$, with equality holding if and only if $\Sigma_{ij} = 0$, for $i \neq j$ [Hadamard's inequality].

From part (a) we know that $-\int f(x)\ln(f(x)) = \frac{1}{2}\ln\left[(2\pi e)^k|\Sigma|\right]$.

Let's look at the determinant of Σ. Consider that we hold i fixed at one; then

$$\begin{aligned}
&-\int f(x)\ln(f(x)) = \frac{1}{2}\ln((2\pi e)^k|\Sigma|) \\
&\implies -2\int f(x)\ln(f(x)) = \ln((2\pi e)^k|\Sigma|) \\
&\implies -2\int f(x)\ln(f(x)) = \ln((2\pi e)^k) + \ln(|\Sigma|) \\
&\implies -2\int f(x)\ln(f(x)) - \ln((2\pi e)^k) = \ln(|\Sigma|) \\
&\implies \exp\left\{-2\int f(x)\ln(f(x)) - \ln((2\pi e)^k)\right\} = |\Sigma|
\end{aligned}$$