

STAT 562 Homework 4

Maggie Buffum

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Problem 5.35

Stirling's Formula (derived in Exercise 1.28), which gives an approximation for factorials, can be easily derived using the CLT.

- (a) Argue that, if $X_i \sim \text{exponential}(1), i = 1, 2, \dots$, all independent, then for every x ,

$$P\left(\frac{\bar{X}_n - 1}{1/\sqrt{n}} \leq x\right) \rightarrow P(Z \leq x),$$

where Z is a standard normal random variable.

Suppose we have $X_i \sim \text{exponential}(1), i = 1, 2, \dots$, all independent. We know that $\mu = 1$ and $\sigma^2 = 1$.

The *Central Limit Theorem* tells that

$$\frac{(\bar{X}_n - \mu)}{\sigma/\sqrt{n}} = \frac{(\bar{X}_n - 1)}{1/\sqrt{n}}$$

has (approximately) a standard normal distribution. Therefore, it directly follows that

$$P\left(\frac{\bar{X}_n - 1}{1/\sqrt{n}} \leq x\right) \rightarrow P(Z \leq x)$$

where $Z \sim n(0, 1)$.

- (b) Show that differentiating both sides of the approximation in part (a) suggests

$$\frac{\sqrt{n}}{\Gamma(n)} (x\sqrt{n} + n)^{n-1} e^{-(x\sqrt{n}+n)} \approx \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

and that $x = 0$ gives Stirling's Formula.

Let's start with the right side of the equation:

$$\frac{d}{dx} P(Z \leq x) = \frac{d}{dx} F_Z(x) = f_Z(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Now the left side of the equation,

$$\begin{aligned}
\frac{d}{dx} P\left(\frac{\bar{X}_n - 1}{1/\sqrt{n}} \leq x\right) &= \frac{d}{dx} \left(\frac{\left(\frac{1}{n} \sum_{i=1}^n X_i\right) - 1}{1/\sqrt{n}} \leq x \right) \\
&= \frac{d}{dx} \left(\frac{\sqrt{n}}{n} \sum_{i=1}^n X_i \leq x + \sqrt{n} \right) \\
&= \frac{d}{dx} \left(\sum_{i=1}^n X_i \leq (x + \sqrt{n}) \frac{n}{\sqrt{n}} \right) \\
&= \frac{d}{dx} \left(\sum_{i=1}^n X_i \leq x\sqrt{n} + n \right)
\end{aligned}$$

Recall that $\sum_{i=1}^n X_i \sim \text{gamma}(n, 1)$:

$$\begin{aligned}
&= \frac{d}{dx} F_{\sum X_i}(x\sqrt{n} + n) \\
&= f_{\sum X_i}(x\sqrt{n} + n) \sqrt{n} \\
&= \frac{1}{\Gamma(n)} (x\sqrt{n} + n)^{n-1} e^{-(x\sqrt{n} + n)} \sqrt{n} \\
&\approx \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \text{ as } n \rightarrow \infty,
\end{aligned}$$

because of the approximation determined in part (a) and substituting the derivative of the right hand side. Now, for $x = 0$ we see that

$$n! \approx \sqrt{2\pi n} n^{n+(1/2)} e^{-n}$$

Problem 5.39

This exercise, and the two following, will look at some of the mathematical details of convergence.

- (a) Prove Theorem 5.5.4. (*Hint*: since h is continuous, given $\epsilon > 0$ we can find a δ such that $|h(x_n) - h(x)| < \epsilon$ whenever $|x_n - x| < \delta$. Translate this into probability statements.)

Theorem 5.5.4 states that if X_1, X_2, \dots converges in probability to a random variable X , then if h is a continuous function, $h(X_1), h(X_2), \dots$ converges in probability to $h(X)$.

Let's first suppose that X_1, X_2, \dots converge in probability to a random variable X . This means that

$$\lim_{n \rightarrow \infty} P(|X_n - X| < \delta) = 1.$$

Consider a continuous function h . We want to show that

$$\lim_{n \rightarrow \infty} P(|h(x_n) - h(x)| < \epsilon) = 1.$$

Since h is continuous, we know by theorem that given $\epsilon > 0$ we can find δ such that $|h(x_n) - h(x)| < \epsilon$ whenever $|x_n - x| < \delta$. That is, $P(|h(x_n) - h(x)| < \epsilon) = 1$ given we have that $|x_n - x| < \delta$, so $P(|h(x_n) - h(x)| < \epsilon)$ comes down knowing under what conditions $|x_n - x| < \delta$. But we've already established that $|x_n - x| < \delta$ is always true as $n \rightarrow \infty$, which implies that $\lim_{n \rightarrow \infty} P(|h(x_n) - h(x)| < \epsilon) = 1$ under the starting assumptions.

- (b) In Example 5.5.8, find a subsequence of the X_i s that converges almost surely, that is, converges pointwise.

A sequence of random variables X_1, X_2, \dots converges almost surely to a random variable X is, for every $\epsilon > 0$,

$$P(\lim_{n \rightarrow \infty} |X_n - X| < \epsilon) = 1.$$

In Example 5.5.8, the sample space S is the closed interval $[0,1]$, and X_1, \dots, X_6 are defined as

$$\begin{aligned} X_1(s) &= s + I_{[0,1]}(s) & X_2(s) &= s + I_{[0, \frac{1}{2}]}(s) & X_3(s) &= s + I_{[\frac{1}{2}, 1]}(s) \\ X_4(s) &= s + I_{[0, \frac{1}{3}]}(s) & X_5(s) &= s + I_{[\frac{1}{3}, \frac{2}{3}]}(s) & X_6(s) &= s + I_{[\frac{2}{3}, 1]}(s) \end{aligned}$$

As the example shows, the sequence of all X_i s converges in probability but not pointwise. This will always be the case, as the example shows, depending on the value of s . So let's define our subsequence using s :

$$X_i(s) \rightarrow \begin{cases} s & \text{if } s > 0 \\ s + 1 & \text{if } s = 0 \end{cases}$$

Problem 5.41

Prove Theorem 5.5.13; that is, show that

$$P(|X_n - \mu| > \epsilon) \rightarrow 0 \text{ for every } \epsilon \iff P(X_n \leq x) \rightarrow \begin{cases} 0 & \text{if } x < \mu \\ 1 & \text{if } x \geq \mu. \end{cases}$$

- (a) Set $\epsilon = |x - \mu|$ and show that if $x > \mu$, then $P(X_n \leq x) \geq P(|X_n - \mu| \leq \epsilon)$, while if $x < \mu$, then $P(X_n \leq x) \leq P(|X_n - \mu| > \epsilon)$. Deduce the \implies implication.

We will start by showing that if $P(|X_n - \mu| \leq \epsilon) \rightarrow 0$, then $P(X_n \leq x) \rightarrow 1$ for $x \geq \mu$ and $P(X_n \leq x) \rightarrow 0$ for $x < \mu$. Let $\epsilon = |x - \mu|$.

Consider the case where $x > \mu$.

$$\begin{aligned} P(|X_n - \mu| \leq \epsilon) &= P(|X_n - \mu| \leq |x - \mu|) \\ &= P(|X_n - \mu| \leq x - \mu) \text{ since } x > \mu \\ &= P(-(x - \mu) \leq X_n - \mu \leq x - \mu) \\ &= P(X_n - \mu \leq x - \mu) - P(X_n - \mu \leq -(x - \mu)) \\ &\leq P(X_n - \mu \leq x - \mu) \\ &= P(X_n \leq x) \end{aligned}$$

which gives us that

$$P(X_n \leq x) \geq P(|X_n - \mu| \leq \epsilon)$$

We know that $P(|X_n - \mu| > \epsilon) \rightarrow 0$ as $n \rightarrow \infty$, implying that $P(|X_n - \mu| \leq \epsilon) \rightarrow 1$ as $n \rightarrow \infty$. Therefore, $P(X_n \leq x) \rightarrow 1$ for $x > \mu$.

Consider the case where $x < \mu$.

$$\begin{aligned} P(|X_n - \mu| > \epsilon) &= P(|X_n - \mu| > |x - \mu|) \\ &= P(|X_n - \mu| > -(x - \mu)) \\ &= P(X_n - \mu > -(x - \mu)) + P(X_n - \mu < x - \mu) \\ &\geq P(X_n - \mu < x - \mu) \\ &= P(X_n < x) \end{aligned}$$

such that

$$P(X_n \leq x) \leq P(|X_n - \mu| > \epsilon)$$

We began with the assumption that $P(|X_n - \mu| > \epsilon) \rightarrow 0$ when $n \rightarrow \infty$, and since $P(X_n \leq x)$ cannot be less than 0, $P(X_n \leq x) \rightarrow 0$ as $n \rightarrow \infty$.

(b) Use the fact that $\{x : |x - \mu| > \epsilon\} = \{x : x - \mu < -\epsilon\} \cup \{x : x - \mu > \epsilon\}$ to deduce the \Leftarrow implication.
(See Billingsley 1995, Section 25, for a detailed treatment of the above results.)

Now we need to show that if $P(X_n \leq x) \rightarrow 0$ when $x < \mu$ and $P(X_n \leq x) \rightarrow 1$ when $x > \mu$ then $P(|X_n - \mu| > \epsilon) \rightarrow 0$.

Consider $\{x : |x - \mu| > \epsilon\}$