STAT 563 - Midterm

Maggie Buffum

May 14, 2019

Assume that $X_1, X_2, \ldots, X_{10}$ is a random sample from a distribution having pdf of the form

$$f(x) = \begin{cases} \lambda x^{\lambda-1}, & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$$

---

1. Find the best critical region of level 0.05 for testing $H_0 : \lambda = \frac{1}{2}$ against $H_1 : \lambda = 1$.

Since both the null and alternative hypotheses are simple hypotheses, we can use the Neyman-Pearson Lemma to determine the best critcal region of size $\alpha = 0.05$. The likelihood function for λ is

$$L(\lambda) = \prod_{i=1}^{10} \lambda x_i^{\lambda-1} = \lambda^n \left(\prod_{i=1}^{10} x_i \right)^{\lambda-1}$$

and the ratio of likelihoods under the null and alternative is

$$\frac{L(\lambda_0)}{L(\lambda_1)} = \frac{\left(\frac{1}{2}\right)^n \left(\prod_{i=1}^{10} x_i\right)^{\frac{1}{2}-1}}{(1)^n \left(\prod_{i=1}^{10} x_i\right)^{1-1}} = \left(\frac{1}{2}\right)^n \left(\prod_{i=1}^{10} x_i \right)^{-\frac{1}{2}} < k$$

But this doesn't have a known distribution. Instead, consider that the inequality is equivalent to

$$-\sum_{i=1}^{n} \ln(x_i) < k'$$

Note that the X_i's follow a Beta$(\lambda, 1)$ distribution:

$$\frac{\Gamma(\lambda+1)}{\Gamma(\lambda)\Gamma(1)} x^{\lambda-1}(1-x)^{1-1} = \frac{\lambda\Gamma(\lambda)}{\Gamma(\lambda)} x^{\lambda-1} = \lambda x^{\lambda-1}$$

for $0 < x < 1$, assuming $\lambda > 0$. We already know that if $X \sim \text{Beta}(\lambda, 1)$, then $-\ln(X) = Y \sim \text{Exponential}(\lambda)$. Furthermore, $\sum_{i=1}^{n} Y_i \sim \text{Gamma}(n, \frac{1}{\lambda})$. Under the null hypothesis, $\lambda = \frac{1}{2}$ such that the test statistic is the special case of the gamma random variable where $\alpha = p/2 \implies n = p/2 \implies p = 2n$ and $\beta = 2$, that is, a chi-squared random variable with degrees of freedom p. Now, substituting $n = 10$ as given, we have that

$$-\sum_{i=1}^{10} \ln(x_i) \sim \chi^2_{20} < k'$$

If the test statistic is less than k' then there is sufficient evidence to reject the null hypothesis. At the 5% significance level, k'=10.8508114.

---

2. Find the power of the test in (1)

The power of a test is $1 - P(\text{Fail to Reject } H_0|H_1 \text{ is true}) = P(\text{Reject } H_0|H_1 \text{ is true})$. From part (1) we will reject H_0 when $-\sum_{i=1}^{10} \ln x_i$<10.8508114. Let $t = -\sum_{i=1}^{10} \ln(x_i)$. We are looking for

$$P(\text{Reject } H_0|H_1 \text{ is true}) = P(t < 10.85081|\lambda = 1)$$

Under the alternative hypothesis, $-\sum_{i=1}^{10} \ln(x_i) \sim \text{Gamma}(10, 1)$. Therefore,

$$\text{Power} = P(t < 10.85081|\lambda = 1) = \int_0^{10.85081} \frac{1}{\Gamma(10)1^{10}} t^{10-1} e^{-t/1} dt = \frac{1}{(10-1)!} \int_0^{10.85081} t^9 e^{-t} dt$$

```
integrand <- function(t) 1/(factorial(9)) * t^9 * exp(-t)
power <- integrate(integrand,lower = 0,upper = qchisq(0.05,20))
```

which integrates numerically as 0.6430816.

---

3. Is your answer from (1) uniformly most powerful for testing $H_0 : \lambda = \frac{1}{2}$ against $H_1 : \lambda > \frac{1}{2}$?

Note that the ratio of likelihoods under the null and under the alternative is a monotone-increasing function of x under the range of x given $(0 < x < 1)$. Therefore, by the Karlin-Rubin theorem, $T = -\sum_{i=1}^{10} \ln(x_i)$ is a UMP level α test.

---

4. Find the Cramer-Rao lower bound for the variance of an unbiased estimator of λ.

The Cramer-Rao lower bound is defined as

$$CRLB = \frac{1}{nI(\lambda)}, \text{ where } I(\lambda) = -E\left[\frac{d^2 \ln(f(x|\lambda))}{d\lambda^2}\right]$$

First let's find the information criterion:

$$\begin{aligned}
\ln(f(x|\lambda)) &= \ln(\lambda) + (\lambda - 1)\ln(x) \\
\frac{d\ln(f(x|\lambda))}{d\lambda} &= \frac{1}{\lambda} + \ln(x) \\
\frac{d^2\ln(f(x|\lambda))}{d\lambda^2} &= \frac{-1}{\lambda^2} \\
I(\lambda) = -E\left[\frac{-1}{\lambda^2}\right] &= E[\lambda^{-2}] = \lambda^{-2}
\end{aligned}$$

Now, the Cramer-Rao Lower Bound is:

$$CRLB = \frac{1}{nI(\lambda)} = \frac{1}{n\lambda^{-2}} = \frac{\lambda^2}{n}$$

---

5. Find the MVUE of λ.

Consider that the likelihood function can be written as

$$\lambda^n e^{(\lambda-1)\sum_{i=1}^{10}\ln(x_i)} = \lambda^n e^{\lambda\sum_{i=1}^{10}\ln(x_i)} e^{-\sum_{i=1}^{10}\ln(x_i)}$$

so that clearly, $T(X) = -\sum_{i=1}^{10}\ln(X_i)$ is a complete and sufficient statistic for λ. If we can find some function of $T(X)$ that is also an unbiased estimator for λ, then we know it is MVUE. Let's take the expected value of $T(X)$. We've already shown that $T(X) \sim \text{Gamma}(10, \frac{1}{\lambda})$, so consider the following function of $T(X)$:

$$E[T(X)^{-1}] = \frac{\lambda\Gamma(n-1)}{\Gamma(n)} = \frac{\lambda\Gamma(n-1)}{(n-1)\Gamma(n-1)} = \frac{\lambda}{n-1}$$

which is biased. But $(n-1)[T(X)^{-1}] = (9)[T(X)^{-1}]$ is an unbiased estimator for λ, so it is MVUE.

---

6. Show that the MVUE of λ is asymptotically efficient.

From part (5) we know that $\hat{\lambda}_{MVUE} = (n-1)T(X)^{-1}$ is an unbiased estimator for λ:

$$E[(n-1)T(X)^{-1}] = (n-1)E[T(X)^{-1}] = (n-1)\frac{\lambda\Gamma(n-1)}{\Gamma(n-1)} = (n-1)\frac{\lambda\Gamma(n-1)}{(n-1)\Gamma(n-1)} = \lambda$$

We need to find its variance in order to test for asymptotic efficiency.

$$Var(\hat{\lambda}_{MVUE}) = Var((n-1)T(X)^{-1}) = (n-1)^2 Var(T(X)^{-1})$$

where $Var(T(X)^{-1}) = E[T(X)^{-2}] - (E[T(X)^{-1}])^2$:

$$E[T(X)^{-2}] = \frac{\lambda^2\Gamma(n-2)}{\Gamma(n)} = \frac{\lambda^2\Gamma(n-2)}{(n-1)(n-2)\Gamma(n-2)} = \frac{\lambda^2}{(n-1)(n-2)}$$

and

$$(E[T(X)^{-1}])^2 = \left(\frac{\lambda}{n-1}\right)^2 = \frac{\lambda^2}{(n-1)^2}$$

such that

$$\begin{aligned} Var(T(X)^{-1}) &= \frac{\lambda^2}{(n-1)(n-2)} - \frac{\lambda^2}{(n-1)^2} \\ &= \frac{(n-1)\lambda^2 - (n-2)\lambda^2}{(n-1)^2(n-2)} \\ &= \frac{n\lambda^2 - \lambda^2 - n\lambda^2 + 2\lambda^2}{(n-1)^2(n-2)} \\ &= \frac{\lambda^2}{(n-1)^2(n-2)} \end{aligned}$$

and finally,

$$Var(\hat{\lambda}_{MVUE}) = (n-1)^2 Var(T(X)^{-1}) = (n-1)^2\frac{\lambda^2}{(n-1)^2(n-2)} = \frac{\lambda^2}{n-2}$$

Now we can find the asymptotic efficiency of $\hat{\lambda}_{MVUE}$:

$$\lim_{n\to\infty}\frac{CRLB}{Var(\hat{\lambda}_{MVUE})} = \lim_{n\to\infty}\frac{\frac{\lambda^2}{n}}{\frac{\lambda^2}{n-2}} = \lim_{n\to\infty}\frac{n-2}{n} = 1$$

Therefore, $\hat{\lambda}_{MVUE}$ is asymptotically efficient.