STAT 562 Final

Maggie Buffum

March 18, 2019

Problem 1

$X_1, X_2, \ldots, X_n$ is a random sample from a distribution having a pdf of the form

$$f(x) = \begin{cases} \lambda x^{\lambda-1}, & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$$

Find a complete and sufficient statistic for λ. Justify your answer.

If we can show that a minimally sufficient statistic exists for λ, then by Theorem 6.2.28 any complete statistic is also a minimally sufficient statistic. To find a minmally sufficient statistic, consider

$$\frac{f(\mathbf{x}|\lambda)}{f(\mathbf{y}|\lambda)} = \frac{\prod_{i=1}^n \lambda x_i^{\lambda-1}}{\prod_{i=1}^n \lambda y_i^{\lambda-1}} = \frac{\left(\prod_{i=1}^n x_i\right)^{\lambda-1}}{\left(\prod_{i=1}^n y_i\right)^{\lambda-1}}$$

Clearly this ratio is only constant as a function of λ when $\prod_{i=1}^n x_i = \prod_{i=1}^n y_i$, so $\prod_{i=1}^n X_i$ is a minimally sufficient statistic for λ.

We need to final a complete statistic for λ. First, rewrite the equation as

$$f(x|\lambda) = \lambda^n \left(\prod_{i=1}^n x_i\right)^{\lambda-1} = \lambda^n \exp\left[(\lambda-1)\sum_{i=1}^n \ln(x_i)\right]$$

so that it's clear $X_1, \ldots, X_n$ are observations from an exponential family, where $h(x) = 1$, $c(\theta) = \lambda^n$, $w(\lambda) = (\lambda - 1)$, and $t(x) = \sum_{i=1}^n x_i$. Because $\lambda > 0$, the set is open, and by Theorem 6.2.25, $T(X) = \sum_{i=1}^n \ln(X_i)$ is a complete statistic for λ. Note that by Theorem 6.2.28, $T(X) = \sum_{i=1}^n \ln(X_i)$ is also minimally sufficient.

Problem 2

Let Y_n be the nth order statistic of a random sample of size n from the normal distribution $N(\theta, \sigma^2)$. Prove that $Y_n - \bar{Y}$ and $\bar{Y}$ are independent.

We want to prove that $Y_n - \bar{Y}$ and $\bar{Y}$ are independent. First, let's note that $Y_n - \bar{Y}$ is the deviation between the maximum sample drawn and the sample mean. Also, recall that $S^2 = \sum_{i=1}^n (Y_i - \bar{Y})^2$ is independent from $\bar{Y}$ by Theorem 5.3.1. But by Lemma 5.3.3, for $\bar{Y}$ and $\sum_{i=1}^n (Y_i - \bar{Y})^2$ to be independent, all pairs of $Y_i - \bar{Y}$ and $\bar{Y}$ must be independent, including when $i = n$, the nth order statistic. Therefore, $Y_n - \bar{Y}$ and $\bar{Y}$ are independent.

Problem 3

Suppose that $X_1, \ldots, X_n \sim$ iid Exponential(θ), i.e. $f(x|\theta) = \theta e^{-\theta x}$, $x > 0$. Assume that the prior distribution of θ is $\pi(\theta) = \lambda e^{-\lambda\theta}$, $\theta > 0$.

(a) Find the posterior distribution $\pi(\theta|\vec{x})$.

The posterior distribution of $\theta|\vec{x}$ is $g(\vec{x},\theta)/m(\vec{x})$, where the joint distribution of $(X_1, \ldots, X_n, \theta)$ is $g(\vec{x},\theta) = \pi(\theta)\prod_{i=1}^n f(x_i|\theta)$ and the marginal distribtion of $(X_1, \ldots, X_n)$ is $m(\vec{x}) = \int_{-\infty}^{\infty} g(\vec{x},\theta)d\theta$. Given $f(x|\theta) = \theta e^{-\theta x}$, $x > 0$ and $\pi(\theta) = \lambda e^{-\lambda\theta}$, $\theta > 0$, we have

$$\begin{aligned} g(\vec{x},\theta) &= \pi(\theta)\prod_{i=1}^n f(x_i|\theta) \\ &= \lambda e^{-\lambda\theta}\prod_{i=1}^n \theta e^{-\theta x_i} \\ &= \lambda\theta^n e^{-\theta\left(\sum_{i=1}^n x_i+\lambda\right)} \end{aligned}$$

and

$$\begin{aligned} m(\vec{x}) &= \int_{-\infty}^{\infty} g(\vec{x},\theta)d\theta \\ &= \int_0^{\infty} \lambda\theta^n e^{-\theta\left(\sum_{i=1}^n x_i+\lambda\right)}d\theta \\ &= \lambda\frac{\Gamma(n+1)}{\left(\sum_{i=1}^n x_i+\lambda\right)^{n+1}}\int_0^{\infty}\frac{\left(\sum_{i=1}^n x_i+\lambda\right)^{n+1}}{\Gamma(n+1)}\theta^n e^{-\theta\left(\sum_{i=1}^n x_i+\lambda\right)}d\theta \end{aligned}$$

From here we recognize the gamma distribution, where $\alpha = n+1$ and $1/\beta = \sum_{i=1}^n x_i + \lambda$, and we find that $m(\vec{x}) = \frac{\lambda\Gamma(n+1)}{\left(\sum_{i=1}^n x_i+\lambda\right)^{n+1}}$. Now we can calculate the posterior distribution of $\theta|\vec{x}$:

$$\begin{aligned} \pi(\theta|\vec{x}) = \frac{g(\vec{x},\theta)}{m(\vec{x})} &= \frac{\lambda\theta^n e^{-\theta\left(\sum_{i=1}^n x_i+\lambda\right)}}{\frac{\lambda\Gamma(n+1)}{\left(\sum_{i=1}^n x_i+\lambda\right)^{n+1}}} \\ &= \frac{\theta^n e^{-\theta\left(\sum_{i=1}^n x_i+\lambda\right)}\left(\sum_{i=1}^n x_i+\lambda\right)^{n+1}}{\Gamma(n+1)} \\ &= \frac{\left(\sum_{i=1}^n x_i+\lambda\right)^{n+1}}{\Gamma(n+1)}\theta^n e^{-\theta\left(\sum_{i=1}^n x_i+\lambda\right)} \end{aligned}$$

which is again a gamma distribution with $\alpha = n+1$ and $1/\beta = \sum_{i=1}^n x_i + \lambda$.

(b) Find the Bayes estimator of θ, assuming squared-error loss.

We are given a squared-error loss function, implying that $\mathcal{L}_\theta(\hat{\theta}) = (\hat{\theta}-\theta)^2$. We want to find the Bayes estimator of θ, that is, the value of θ, $\hat{\theta}$, that minimizes $E\left[\mathcal{L}_\theta(\hat{\theta})|\vec{x}\right]$, which we already know is $\hat{\theta} = E[\theta|\vec{x}]$, i.e., the posterior mean. Since the posterior distribution is Gamma$(n+1, \left(\sum_{i=1}^n x_i+\lambda\right)^{-1})$, the Bayes estimator for θ is

$$\hat{\theta} = \frac{n+1}{\sum_{i=1}^n x_i + \lambda}$$

(c) Write this estimator as a weighted (arithmetic, geometric, or harmonic) average of the MLE and some prior constant.

The prior distribution was exponential, so the mean of the proir is simply $1/\lambda$, and the MLE of an exponential random variable is just the mean across the x_is, $(1/n)\sum_{i=1}^n x_i$. Looking at the harmonic mean, we see that

$$\frac{1}{E[\hat{\theta}|\vec{x}]} = \frac{\sum_{i=1}^n x_i + \lambda}{n+1} = \frac{\frac{n}{n}\sum_{i=1}^n x_i + \lambda}{n+1} = \frac{n\bar{x}}{n+1} + \frac{\lambda}{n+1} = \frac{n}{n+1}\bar{x} + \frac{\lambda^2}{n+1}\frac{1}{\lambda}$$

(d) Find the Bayes estimator of θ, assuming absolute loss.

We know that assuming aboslute loss, the Bayes estimator of θ, $\hat{\theta}$, is equal to the posterior median. But the posterior distribution is Gamma$(n+1, \sum_{i=1}^n +\lambda)$, which doesn't have a median with a closed form.

(e) Find the Bayes estimator of θ, assuming binary loss.

Using a binary loss function, we know that the Bayes estimator $\hat{\theta}$ is equal to the mode of the posterior distribution. The mode of a Gamma(α, β) random variable is $(\alpha - 1)\beta$, and so the mode of the Bayes estimator is

$$\hat{\theta} = \frac{n+1-1}{\sum_{i=1}^n x_i + \lambda} = \frac{n}{\sum_{i=1}^n x_i + \lambda}$$

Problem 4

Redo all of Problem 3, using the non-informative prior $\pi(\theta) = 1$, $\theta > 0$. Note that this is not a valid density function since its integral is infinite, but proceed with it anyways.

The joint distribution of $(X_1, \ldots, X_n, \theta)$ is just the likelihood function of θ since the prior distribution is simply 1. Therefore, the marginal of $\vec{x}$ is

$$\begin{aligned}
m(\vec{x}) &= \int_0^\infty \theta^n e^{-\theta\sum_{i=1}^n x_i} d\theta \\
&= \frac{\Gamma(n+1)}{(\sum_{i=1}^n x_i)^{n+1}} \int_0^\infty \frac{(\sum_{i=1}^n x_i)^{n+1}}{\Gamma(n+1)} \theta^n e^{-\theta\sum_{i=1}^n x_i} d\theta \\
&= \frac{\Gamma(n+1)}{(\sum_{i=1}^n x_i)^{n+1}}
\end{aligned}$$

and the posterior distribution, $\pi(\theta|\vec{x})$ is

$$\begin{aligned}
\pi(\theta|\vec{x}) &= \frac{g(\theta, \vec{x})}{m(\vec{x})} - \frac{\theta^n e^{-\theta\sum_{i=1}^n x_i}}{\frac{\Gamma(n+1)}{(\sum_{i=1}^n x_i)^{n+1}}} \\
&= \frac{(\sum_{i=1}^n x_i)^{n+1}\theta^n e^{-\theta\sum_{i=1}^n x_i}}{\Gamma(n+1)} \\
&= \frac{(\sum_{i=1}^n x_i)^{n+1}}{\Gamma(n+1)}\theta^n e^{-\theta\sum_{i=1}^n x_i}
\end{aligned}$$

which is a Gamma$(n+1, \sum_{i=1}^n x_i)$ random variable.

Given a squared-error loss function, we know that posterior mean is the Bayes estimator. Therefore,

$$\hat{\theta} = \alpha\beta = \frac{n+1}{\sum_{i=1}^n x_i}$$

We can rewrite $\hat{\theta}$ in terms of the prior mean (1) and MLE of the posterior distribution ($\bar{x}$) by looking at the harmonic mean:

$$\frac{1}{E[\hat{\theta}|\vec{x}]} = \frac{\sum_{i=1}^{n} x_i}{n+1} = \frac{n}{n+1}\bar{x}_i + \frac{0}{n+1}1$$

Now, let's assume an absolute loss funtion. That is, we're looking for the median of the posterior distribution. However, the median of a gamma random variable has no closed form.

Assuming a binary loss function, the Bayes estimator is the mode of the posterior distribution, which for a gamma random variable is equal to $(\alpha - 1)\beta$. Using the posterior α and β, we have

$$\hat{\theta} = \frac{n+1-1}{\sum_{i=1}^{n} x_i} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{x}}$$

Problem 5

Let $X_1, X_2, \ldots, X_n \sim$ iid $f(x|\theta) = \theta x^{-\theta-1}$, $x_i > 1$, $\theta > 2$.

(a) Find $\hat{\theta}_{MLE}$, the maximum likelihood estimator of θ.

The likelihood function θ is

$$L(\theta) = \prod_{i=1}^{n} \theta x_i^{-\theta-1} = \theta^n \left(\prod_{i=1}^{n} x_i\right)^{-\theta-1}$$

and the log-likelihood function is

$$l(\theta) = \ln\left(\theta^n \left(\prod_{i=1}^{n} x_i\right)^{-\theta-1}\right) = n\ln(\theta) - (\theta+1)\sum_{i=1}^{n}\ln(x_i)$$

We want to find θ such that $l(\theta)$ is maximized. Let's take the derivative of $l(\theta)$ with respect to θ, set equal to 0, and solve for $\hat{\theta}$:

$$\begin{aligned}
\frac{d}{d\theta}l(\theta) = \frac{n}{\theta} - \sum_{i=1}^{n}\ln(x_i) = 0 \\
\implies \frac{n}{\theta} = \sum_{i=1}^{n}\ln(x_i) \\
\implies \hat{\theta} = \frac{n}{\sum_{i=1}^{n}\ln(x_i)}
\end{aligned}$$

To ensure $\hat{\theta} = n/(\sum_{i=1}^{n}\ln(x_1))$ is the MLE, the second derivative of $l(\theta)$ must be negative, which it is ($l''(\theta) = -n/\theta^2$). Therefore,

$$\hat{\theta}_{MLE} = \frac{n}{\sum_{i=1}^{n}\ln(x_i)}$$

(b) Find the expected value of $\hat{\theta}_{MLE}$.

To find the expected value of $\hat{\theta}_{MLE}$, let $Y_i = \ln(X_i)$. Then, $e^{y_i} = x_i = g^{-1}(y_i)$ and $\frac{d}{dy_i}e^{y_i} = e^{y_i}$ such that

$$f(y_i) = \theta(e^{y_i})^{-\theta-1}|e^{y_i}| = \theta e^{y_i(-\theta-1)}e^{y_i} = \theta e^{-y_i\theta - y_i}e^{y_i} = \theta e^{-y_i\theta}$$

which has an Exponential(θ) distribution. We are looking for the distribution of the sum of Y_is, call it Y, and since the eponential distribution is a special case of the gamma distribution, we are really looking for the sum of independent gamma random variables with $\alpha = 1$ and $\beta = \theta$. So $Y \sim Gamma(n, \theta)$. Now we can more easily find $E[\hat{\theta}_{MLE}]$:

$$E[\hat{\theta}_{MLE}] = nE[Y^{-1}] = n\left(\frac{\Gamma(n-1)}{\Gamma(n)\theta^{-1}}\right) = n\left(\frac{\Gamma(n-1)}{(n-1)\Gamma(n-1)}\theta\right) = \frac{n\theta}{n-1}$$

(c) Find the variance of $\hat{\theta}_{MLE}$.

The variance of $\hat{\theta}_{MLE}$ is $E[\hat{\theta}^2_{MLE}] - (E[\hat{\theta}_{MLE}])^2$. Using the same logic as in part (b),

$$E[\hat{\theta}^2_{MLE}] = n^2E[Y^{-2}] = n^2\frac{\Gamma(n-2)\theta^2}{\Gamma(n)} = n^2\theta^2\frac{\Gamma(n-2)}{(n-1)(n-2)\Gamma(n-2)} = \frac{n^2\theta^2}{(n-1)(n-2)}$$

and the variance is

$$\begin{aligned} V[\hat{\theta}_{MLE}] &= E[\hat{\theta}^2_{MLE}] - (E[\hat{\theta}_{MLE}])^2 \\ &= \frac{n^2\theta^2}{(n-1)(n-2)} - \left(\frac{n\theta}{n-1}\right) \\ &= \frac{n^2\theta^2}{(n-1)(n-2)} - \frac{n^2\theta^2}{(n-1)^2} \\ &= \frac{n^2\theta^2(n-1) - n^2\theta^2(n-2)}{(n-1)^2(n-2)} \\ &= \frac{n^2\theta^2}{(n-1)^2(n-2)} \end{aligned}$$

(d) Using $\hat{\theta}_{MLE}$, create an unbiased estimator $\hat{\theta}_U$.

We can scale $\hat{\theta}_{MLE}$ by $\frac{n-1}{n}$ to obtain $\hat{\theta}_U$, giving us

$$E[\hat{\theta}_U] = E\left[\frac{n-1}{n}\hat{\theta}_{MLE}\right] = \frac{n-1}{n}E[\hat{\theta}_{MLE}] = \frac{n-1}{n}\frac{n}{n-1}\theta = \theta$$

and showing that $\hat{\theta}_U = \frac{n-1}{n}\hat{\theta}_{MLE}$ is an unbiased estimator for θ.

(e) Find the variance of $\hat{\theta}_U$.

$$V[\hat{\theta}_U] = V\left[\frac{n-1}{n}\hat{\theta}_{MLE}\right] = \frac{(n-1)^2}{n^2}V[\hat{\theta}_{MLE}] = \frac{(n-1)^2}{n^2}\left(\frac{n^2\theta^2}{(n-1)^2(n-2)}\right) = \frac{\theta^2}{n-2}$$

Problem 6

Refer to problem 5.

(a) Find $\hat{\theta}_{MOM}$, the method of moments estimator of θ.

First we need to find the first moment, $E[X]$:

$$E[X] = \int_1^\infty x\theta x^{-\theta-1}dx = \int_1^\infty \theta x^{-\theta}dx = \left[\frac{\theta x^{1-\theta}}{1-\theta}\right]_1^\infty = \frac{\theta}{\theta-1}$$

Now set $\bar{x}$ equal to the first moment and solve for $\hat{\theta}_{MOM}$:

$$\bar{x} = \frac{\theta}{\theta - 1} \implies \bar{x}\theta - \bar{x} = \theta \implies \hat{\theta}_{MOM} = \frac{\bar{x}}{\bar{x} - 1}$$

(b) Use the delta method to approximate the expected value of $\hat{\theta}_{MOM}$.

The second order Taylor series is

$$g(x) = g(x_0) + g'(x_0)(x - x_0) + g''(x_0)\frac{(x - x_0)^2}{2} + R$$

Consider

$$g(x) = \frac{x}{x - 1}, \quad g'(x) = -\frac{1}{(x - 1)^2}, \quad g''(x) = \frac{2}{(x - 1)^3}$$

such that

$$g(x) = \frac{x}{x - 1} \approx \frac{x_0}{x_0 - 1} - \frac{1}{(x_0 - 1)^2}(x - x_0) + \frac{2}{(x_0 - 1)^3}\frac{(x - x_0)^2}{2}$$

Choose $x_0 = E[X] = \mu$ so that

$$\hat{\theta}_{MOM} = \frac{\bar{x}}{\bar{x} - 1} \approx \frac{\mu}{\mu - 1} - \frac{1}{(\mu - 1)^2}(\bar{x} - \mu) + \frac{(\bar{x} - \mu)^2}{(\mu - 1)^3}$$

Taking the expectation of both sides, we get

$$\begin{aligned}
E[\hat{\theta}_{MOM}] &\approx E\left[\frac{\mu}{\mu - 1} - \frac{1}{(\mu - 1)^2}(\bar{x} - \mu) + \frac{(\bar{x} - \mu)^2}{(\mu - 1)^3}\right] \\
&= E\left[\frac{\mu}{\mu - 1}\right] - E\left[\frac{1}{(\mu - 1)^2}(\bar{x} - \mu)\right] + E\left[\frac{(\bar{x} - \mu)^2}{(\mu - 1)^3}\right] \\
&= \frac{\mu}{\mu - 1} + \frac{1}{(\mu - 1)^3}V[\bar{x}] \\
&= \frac{\mu}{\mu - 1} + \frac{\sigma^2/n}{(\mu - 1)^3}
\end{aligned}$$

But we know that $\mu = \frac{\theta}{\theta - 1}$, so

$$\begin{aligned}
E[\hat{\theta}_{MOM}] &= \frac{\frac{\theta}{\theta-1}}{\frac{\theta}{\theta-1} - 1} + \frac{\sigma^2/n}{\left(\frac{\theta}{\theta-1} - 1\right)^3} \\
&= \frac{\frac{\theta}{\theta-1}}{\frac{\theta-(\theta-1)}{\theta-1}} + \frac{\sigma^2/n}{\left(\frac{\theta-(\theta-1)}{\theta-1}\right)^3} \\
&= \theta + (\theta - 1)^3\frac{\sigma^2}{n}
\end{aligned}$$

To find σ^2, we need to calculate $V(X) = E[X^2] - (E[X])^2$. First,

$$E[X^2] = \int_1^\infty x^2\theta x^{-\theta-1}dx = \int_1^\infty \theta x^{-\theta+1}dx = \frac{\theta}{\theta - 2}$$

and then

$$
\begin{aligned}
V(X) &= \frac{\theta}{\theta-2} - \left(\frac{\theta}{\theta-1}\right)^2 = \frac{\theta}{\theta-2} - \frac{\theta^2}{(\theta-1)^2} \\
&= \frac{\theta(\theta-1)^2 - \theta^2(\theta-2)}{(\theta-2)(\theta-1)^2} \\
&= \frac{\theta^3 + \theta - 2\theta^2 - \theta^3 + 2\theta^2}{(\theta-2)(\theta-1)^2} \\
&= \frac{\theta}{(\theta-2)(\theta-1)^2}
\end{aligned}
$$

Now we can plug in $V(X)$ for σ^2 and finish solving for the expected value of the method of moments estimator for θ:

$$
\begin{aligned}
E[\hat{\theta}_{MOM}] &\approx \theta + (\theta-1)^3 \frac{\sigma^2}{n} \\
&= \theta + (\theta-1)^3 \frac{\frac{\theta}{(\theta-2)(\theta-1)^2}}{n} \\
&= \theta + \frac{(\theta-1)^3\theta}{n(\theta-2)(\theta-1)^2} \\
&= \theta + \frac{\theta(\theta-1)}{n(\theta-2)}
\end{aligned}
$$

(c) Use the delta method to approximate the variance of $\hat{\theta}_{MOM}$.

To approximate the variance, we just use the first order Taylor series:

$$
\begin{aligned}
V[\hat{\theta}] &\approx V\left[\frac{\mu}{\mu-1} - \frac{1}{(\mu-1)^2}(\bar{x}-\mu)\right] \\
&= \frac{1}{(\mu-1)^4} V[\bar{x}] = \frac{1}{(\mu-1)^4}\frac{\sigma^2}{n} \\
&= \frac{1}{(\frac{\theta}{\theta-1}-1)^4} \frac{\theta}{n(\theta-2)(\theta-1)^2} \\
&= \frac{\theta(\theta-1)^4}{n(\theta-2)(\theta-1)^2} \\
&= \frac{\theta(\theta-1)^2}{n(\theta-2)}
\end{aligned}
$$