STAT 666 Homework 2

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Problem 7.12

In the less-than-full-rank linear model, suppose we wish to test $H : \mathbf{C}'\beta = \mathbf{d}$. Let $\mathbf{G} = (\mathbf{X}'\mathbf{X})^-$. Show that $\mathbf{C}'\mathbf{G}\mathbf{C}$ is nonsingular whenever H is a testable hypothesis.

Let H be a testable hypothesis. That is, for $H : \mathbf{C}'\beta = \mathbf{d}$, we know that $\mathbf{C}'_{s\times p}\mathbf{H}_{N\times p} = \mathbf{C}'_{s\times p}$, where $H = (X'X)^-X'X$. We need to show that $\mathbf{C}'\mathbf{G}\mathbf{C}$ is nonsingular, which happens when $rank(\mathbf{C}'\mathbf{G}\mathbf{C}) = \dim(\mathbf{C}'\mathbf{G}\mathbf{C})$.

Consider that $C'H = C'(X'X)^-X'X = C'$ has dimensions $s \times p$ and rank s. By Result 1.2.12, we know that

$$rank(C'(X'X)^-X'X) \leq rank(C'(X'X)^-X') \leq rank(C')$$

but since we know $rank(C'(X'X)^-X'X) = rank(C')$, we have that $rank(C'(X'X)^-X') = s$ also. Now, recalling that $rank(A) = rank(AA')$, we show that

$$\begin{aligned}
rank(C') &= rank(C'(X'X)^-X') \\
&= rank(C'(X'X)^-X'(C'(X'X)^-X')') \\
&= rank(C'(X'X)^-X'X(X'X)^-C) \\
&= rank(C'(X'X)^-C) \\
&= rank(C'GC)
\end{aligned}$$

Since the dimensions of $\mathbf{C}'\mathbf{G}\mathbf{C}$ are $s \times s$ and its rank is also s, $\mathbf{C}'\mathbf{G}\mathbf{C}$ is nonsingular when the hypothesis H is testable.

Problem 7.14

[Wu, Hosking and Ravishanker, 1993]. For $i = 1, \ldots, N$, let

$$\begin{aligned}
Y_i &= \epsilon_i, & i \neq k, k+1, k+2 \\
Y_k &= \lambda_1 + \epsilon_k \\
Y_{k+1} &= -c\lambda_1 + \lambda_2 + \epsilon_{k+1} \\
Y_{k+2} &= -c\lambda_2 + \epsilon_{k+2},
\end{aligned}$$

where k is a fixed integer, $1 \leq k \leq N-2$, $|c| < 1$ is a known constant, and ϵ_i's are iid $N(0, \sigma^2)$ variables. Let $\beta = (\lambda_1, \lambda_2)'$, and suppose σ^2 is known.

(a) Derive the least squares estimate of β, and the variance of the estimate.

We have the following model:

$$\begin{bmatrix} Y_i \\ Y_k \\ Y_{k+1} \\ Y_{k+2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -c & 1 \\ 0 & -c \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} \epsilon_i \\ \epsilon_k \\ \epsilon_{k+1} \\ \epsilon_{k+2} \end{bmatrix}$$

such that we can calculate $\hat{\beta} = \mathbf{G}\mathbf{X}'\mathbf{y}$ as

$$
\begin{aligned}
\mathbf{G} = (\mathbf{X}'\mathbf{X})^{-1} &= \left(\begin{bmatrix} 0 & 1 & -c & 0 \\ 0 & 0 & 1 & -c \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -c & 1 \\ 0 & -c \end{bmatrix} \right)^{-1} \\
&= \left(\begin{bmatrix} c^2+1 & -c \\ -c & c^2+1 \end{bmatrix} \right)^{-1} \\
&= \frac{1}{c^4+c^2+1} \begin{bmatrix} c^2+1 & c \\ c & c^2+1 \end{bmatrix} \\
\mathbf{G}\mathbf{X}'\mathbf{y} &= \frac{1}{c^4+c^2+1} \begin{bmatrix} c^2+1 & c \\ c & c^2+1 \end{bmatrix} \begin{bmatrix} 0 & 1 & -c & 0 \\ 0 & 0 & 1 & -c \end{bmatrix} \begin{bmatrix} Y_i \\ Y_k \\ Y_{k+1} \\ Y_{k+2} \end{bmatrix} \\
&= \frac{1}{c^4+c^2+1} \begin{bmatrix} 0 & c^2+1 & -c^3 & -c^2 \\ 0 & c & 1 & -c(c^2+1) \end{bmatrix} \begin{bmatrix} Y_i \\ Y_k \\ Y_{k+1} \\ Y_{k+2} \end{bmatrix}
\end{aligned}
$$

So that we have

$$
\begin{aligned}
\hat{\lambda}_1 &= \frac{(c^2+1)Y_k - c^3 Y_{k+1} - c^2 Y_{k+2}}{c^4+c^2+1}, \\
\hat{\lambda}_2 &= \frac{cY_k + Y_{k+1} - c(c^2+1)Y_{k+2}}{c^4+c^2+1}
\end{aligned}
$$

To the variance around the parameter estimates, we note that rank of $\mathbf{X}$ is equal to the number of parameters p, and we can use Corollary 7.1.1 (1):

$$
Var(\hat{\beta}) = \sigma^2 \mathbf{G} = \frac{\sigma^2}{c^4+c^2+1} \begin{bmatrix} c^2+1 & c \\ c & c^2+1 \end{bmatrix}
$$

(b) Derive the least squares estimate of β subject to the restriction $\lambda_1 + \lambda_2 = 0$. What is the variance of this estimate?

For $\lambda_1 + \lambda_2 = 0$, we know that $\lambda_1 = -\lambda_2 = \lambda$. Now our model is

$$
\begin{bmatrix} Y_i \\ Y_k \\ Y_{k+1} \\ Y_{k+2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -c-1 \\ c \end{bmatrix} \begin{bmatrix} \lambda \end{bmatrix} + \begin{bmatrix} \epsilon_i \\ \epsilon_k \\ \epsilon_{k+1} \\ \epsilon_{k+2} \end{bmatrix}
$$

So that

$$
\begin{aligned}
\mathbf{X}'\mathbf{X} &= (1)^2 + (-c-1)^2 + c^2 \\
&= 1 + c^2 + 2c + 1 + c^2 \\
&= 2c^2 + 2c + 2 \\
&= 2(c^2 + c + 1) \neq 0 \text{ (since } b^2 - 4ac = -3 < 0)
\end{aligned}
$$

and the inverse is

$$
\mathbf{X}'\mathbf{X}^{-1} = \frac{1}{2(c^2+c+1)}
$$

such that

$$
\begin{aligned}
\hat{\lambda}_0 &= (X'X)^{-1}X'y \\
&= \frac{1}{2(c^2+c+1)}\begin{bmatrix}0 & 1 & -c-1 & -\end{bmatrix}\begin{bmatrix}y_i \\ y_k \\ y_{k+1} \\ y_{k+2}\end{bmatrix} \\
&= \frac{1}{2(c^2+c+1)}\begin{bmatrix}y_k - (c+1)y_{k+1} + cy_{k+2}\end{bmatrix}
\end{aligned}
$$

with variance $Var(\hat{\lambda}_0) = \sigma^2(X'X)^{-1} = \frac{\sigma^2}{2c^2+2c+2}$.

(c) Derive a statistic for testing $H : \lambda_1 + \lambda_2 = 0$ versus the alternative hypothesis that λ_1 and λ_2 are unrestricted.

We can rewrite the hypothesis test in the form

$$H : \mathbf{C}'\beta = \mathbf{d}$$

where

$$
\begin{aligned}
\mathbf{C}' &= \begin{bmatrix}1 & 1\end{bmatrix} \\
\beta &= \begin{bmatrix}\lambda_1 \\ \lambda_2\end{bmatrix} \\
\mathbf{d} &= \mathbf{0}
\end{aligned}
$$

We can define Q as

$$
\begin{aligned}
Q &= (\mathbf{C}'\beta - \mathbf{d})'(\mathbf{C}'\mathbf{G}\mathbf{C})^{-1}(\mathbf{C}'\beta - \mathbf{d}) \\
&= (\hat{\lambda}_1 + \hat{\lambda}_2)\left(\frac{1}{c^4+c^2+1}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1+c^2 & c \\ c & c^2+1\end{bmatrix}\begin{bmatrix}1 \\ 1\end{bmatrix}\right)^{-1}(\hat{\lambda}_1 + \hat{\lambda}_2) \\
&= (\hat{\lambda}_1 + \hat{\lambda}_2)\left(\frac{2(c^2+c+1)}{c^4+c^2+1}\right)^{-1}(\hat{\lambda}_1 + \hat{\lambda}_2) \\
&= (\hat{\lambda}_1 + \hat{\lambda}_2)\frac{c^4+c^2+1}{2(c^2+c+1)}(\hat{\lambda}_1 + \hat{\lambda}_2)
\end{aligned}
$$

Now,

$$
\begin{aligned}
\hat{\lambda}_1 + \hat{\lambda}_2 &= \frac{(c^2+1)Y_k - c^3Y_{k+1} - c^2Y_{k+2} + cY_k + Y_{k+1} - c(c^2+1)Y_{k+2}}{c^4+c^2+1} \\
&= \frac{(c^2+1+c)Y_k + (c1-^3)Y_{k+1} - c(c^2+c+1)Y_{k+2}}{c^4+c^2+1}
\end{aligned}
$$

such that

$$
\begin{aligned}
Q &= \left(\frac{(c^2+1+c)Y_k + (c1-^3)Y_{k+1} - c(c^2+c+1)Y_{k+2}}{c^4+c^2+1}\right)^2 \frac{c^4+c^2+1}{2(c^2+c+1)} \\
&= \left((c^2+1+c)Y_k + (c1-^3)Y_{k+1} - c(c^2+c+1)Y_{k+2}\right)^2 \frac{1}{2(c^2+c+1)(c^4+c^2+1)}
\end{aligned}
$$

where we know that $Q/\sigma^2 \sim \chi^2_{(1)}$

Problem 7.16

Prove property 2 in Result 7.4.3.

Result 7.4.3 states that if the true model is $\mathbf{y} = \mathbf{X}_1\beta_1 + \epsilon$, but we fit the model $\mathbf{y} = \mathbf{X}_1\beta_1 + \mathbf{X}_2\beta_2 + \epsilon$ to the data, then

1. $E(\beta_1^0) = \mathbf{H}_1\beta_1$
2. MSE is an unbiased estimate of σ^2.

We have that $MSE_H = \mathbf{y}'(I-P)\mathbf{y}/(N-r)$. Let's find the expected value of the MSE using Result 4.2.3, where $\mu_Y = X_1\beta_1$ (based on the true model):

$$\begin{aligned}
E(MSE) &= \frac{1}{N-r}\left(\sigma^2 tr(I-P) + (X_1\beta_1)'(I-P)(X_1\beta_1)\right) \\
&= \frac{1}{N-r}\left(\sigma^2 tr(I-P) + (X_1\beta_1)'(X_1 - PX_1)\beta_1\right) \\
&= \frac{1}{N-r}\left(\sigma^2 tr(I-P)\right) \\
&= \frac{1}{N-r}\sigma^2(N-r) \\
&= \sigma^2
\end{aligned}$$

Therefore, the MSE is an unbiased estimator for σ^2.

Problem 7.19

In Example 7.5.1, test $H : X^* = 0$ versus $H_1 : X^* \neq 0$ at the 5% level of significance when $N = 40$.

From Homework 1, we know that the F-statistic under the null hypothesis $H : X^* = 0$ is

$$F(H_0) = \frac{Q/s}{SSE/(N-r)} = \frac{\hat{\beta}_1^2 \sum_{i=1}^N X_i^2/(1)}{SSE/(40-3)} = \frac{(37)\hat{\beta}_1^2 \sum_{i=1}^N X_i^2}{SSE}$$

where

$$\hat{\beta}_1 = \frac{\sum_{i=1}^N Y_i X_i}{\sum_{i=1}^N X_i^2}$$

which, when plugged into the F-statistic, gives us

$$\begin{aligned}
F(H_0) &= \frac{(37)\hat{\beta}_1^2 \sum_{i=1}^N X_i^2}{SSE} \\
&= \frac{(37)\left(\frac{\sum_{i=1}^N Y_i X_i}{\sum_{i=1}^N X_i^2}\right)^2 \sum_{i=1}^N X_i^2}{SSE} \\
&= \frac{(37)\left(\sum_{i=1}^N Y_i X_i\right)^2}{SSE \sum_{i=1}^N X_i^2}
\end{aligned}$$

This statistic follows an F distribution with 1 and 37 degrees of freedom. We will reject the null hypothesis in favor of the alternative when $F(H_0) > F_(1, 37, 0.05) = 4.1054559$.