STAT 666 HW #3

Maggie Buffum

May 21, 2019

Problem 7.22

In the regression model

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2(3X_i^2 - 2) + \epsilon_i,\ i = 1, 2, 3$$

with $X_1 = -1$, $X_2 = 0$, and $X_3 = 1$, what happens to the least squares estimates of β_0 and β_1 when $\beta_2 = 0$? Why?

We are given the model

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} + \epsilon_i$$

The sum of squared errors is $S(\beta) = (\mathbf{y} - \mathbf{X}\beta)'(\mathbf{y} - \mathbf{X}\beta)$:

$$\begin{aligned}
\mathbf{y} - \mathbf{X}\beta &= \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} \\
&= \begin{bmatrix} Y_1 - \beta_0 - \beta_1 + \beta_2 \\ Y_2 - \beta_0 - 2\beta_2 \\ Y_3 - \beta_0 + \beta_1 + \beta_2 \end{bmatrix} \\
S(\beta) &= \begin{bmatrix} Y_1 - \beta_0 - \beta_1 + \beta_2 & Y_2 - \beta_0 - 2\beta_2 & Y_3 - \beta_0 + \beta_1 + \beta_2 \end{bmatrix} \begin{bmatrix} Y_1 - \beta_0 - \beta_1 + \beta_2 \\ Y_2 - \beta_0 - 2\beta_2 \\ Y_3 - \beta_0 + \beta_1 + \beta_2 \end{bmatrix} \\
&= (Y_1 - \beta_0 - \beta_1 + \beta_2)^2 + (Y_2 - \beta_0 - 2\beta_2)^2 + (Y_3 - \beta_0 + \beta_1 + \beta_2)^2
\end{aligned}$$

The least squares estimate of β_1 is

$$\begin{aligned}
\frac{\partial S(\beta)}{\partial \beta_1} &= 2(Y_1 - \beta_0 - \beta_1 + \beta_2)(-1) + 2(Y_3 - \beta_0 + \beta_1 + \beta_2) = \text{ (set) } 0 \\
&\implies -Y_1 + \beta_0 + \beta_1 - \beta_2 + Y_3 - \beta_0 + \beta_1 + \beta_2 = 0 \\
&\implies \hat{\beta}_1 = \frac{1}{2}(Y_1 - Y_3)
\end{aligned}$$

and $\hat{\beta}_2$:

$$\begin{aligned}
\frac{\partial S(\beta)}{\partial \beta_2} &= 2(Y_1 - \beta_0 - \beta_1 + \beta_2) + 2(Y_2 - \beta_0 - 2\beta_2)(-2) + 2(Y_3 - \beta_0 + \beta_1 + \beta_2) = \text{ (set) } 0 \\
&\implies Y_1 - \beta_0 - \beta_1 + \beta_2 - 2Y_2 + 2\beta_0 + 4\beta_2 + Y_3 - \beta_0 + \beta_1 + \beta_2 = 0 \\
&\implies Y_1 + 6\beta_2 - 2Y_2 + Y_3 = 0 \\
&\implies \hat{\beta}_2 = \frac{1}{6}(-Y_1 + 2Y_2 - Y_3)
\end{aligned}$$

Now β_0:

$$\begin{aligned}\frac{\partial S(\beta)}{\partial \beta_0} &= 2(Y_1-\beta_0-\beta_1+\beta_2)(-1)+2(Y_2-\beta_0-2\beta_2)(-1)+2(Y_3-\beta_0+\beta_1+\beta_2)(-1) = \text{ (set) } 0\\ &\implies Y_1-3\beta_0+Y_2+Y_3=0\\ &\implies \hat\beta_0=\frac{1}{3}(Y_1+Y_2+Y_3)\end{aligned}$$

But consider what happens when $\beta_2=0$. The model is

$$\begin{bmatrix}Y_1\\Y_2\\Y_3\end{bmatrix}=\begin{bmatrix}1 & -1\\1 & 0\\1 & 1\end{bmatrix}\begin{bmatrix}\beta_0\\\beta_1\end{bmatrix}+\epsilon_i$$

and we have sum of squares

$$S(\beta)=(Y_1-\beta_0-\beta_1)^2+(Y_2-\beta_0)^2+(Y_3-\beta_0+\beta_1)^2$$

Let's re-estimate β_1:

$$\begin{aligned}\frac{\partial S(\beta)}{\partial \beta_1} &= 2(Y_1-\beta_0-\beta_1)(-1)+2(Y_3-\beta_0+\beta_1) = \text{ (set) } 0\\ &\implies -Y_1+Y_3+2\beta_1=0\\ &\implies \hat\beta_1=\frac{1}{2}(Y_1-Y_3)\end{aligned}$$

that is, the least squares estimate of $\hat\beta_1$ doesn't change when $\beta_2=0$.

$$\begin{aligned}\frac{\partial S(\beta)}{\partial \beta_0} &= 2(Y_1-\beta_0-\beta_1)(-1)+2(Y_2-\beta_0)(-1)+2(Y_3-\beta_0+\beta_1)(-1) = \text{ (set) } 0\\ &\implies Y_1-3\beta_0+Y_2+Y_3=0\\ &\implies \hat\beta_0=\frac{1}{3}(Y_1+Y_2+Y_3)\end{aligned}$$

Again the least squares estimate doesn't change. This happens because the least squares estimates of β_0 and β_1 were independent of β_2.

Problem 7.28

Let $Y_{ij}=\mu+\tau_i+\epsilon_{ij}$, $j=1,\ldots,n$, $i=1,\ldots,3$, and $\epsilon_{ij}\sim N(0,\sigma^2)$. Derive a test for $H:\tau_2=(\tau_1+\tau_3)/2$.

We can rewrite the hypothesis test as $H: 2\tau_2-\tau_1-\tau_3=0$ such that we are testing $\mathbf{C}'\beta=\mathbf{d}$, where

$$\mathbf{C}'=\begin{bmatrix}0 & -1 & 2 & -1\end{bmatrix},\ \beta=\begin{bmatrix}\mu\\\tau_1\\\tau_2\\\tau_3\end{bmatrix},\ \text{and } \mathbf{d}=\mathbf{0}$$

Our design matrix looks like

$$\mathbf{X} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 1 \end{bmatrix}_{(3n\times 4)}$$

and $\mathbf{X}'\mathbf{X}$ is

$$\begin{aligned} \mathbf{X}'\mathbf{X} &= \begin{bmatrix} 1 & \dots & 1 & 1 & \dots & 1 & 1 & \dots & 1 \\ 1 & \dots & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3n & n & n & n \\ n & n & 0 & 0 \\ n & 0 & n & 0 \\ n & 0 & 0 & n \end{bmatrix} \end{aligned}$$

This is a singular matrix, so we need to find the generalized inverse instead. Let's use Rao's method and take the inverse of the lower right 3×3 matrix, giving us

$$(X'X)^{-} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{n} & 0 & 0 \\ 0 & 0 & \frac{1}{n} & 0 \\ 0 & 0 & 0 & \frac{1}{n} \end{bmatrix}$$

Now we can find Q. Let's start by evaluating $\mathbf{C}'\beta^0 - \mathbf{d}$:

$$\mathbf{C}'\beta^0 - \mathbf{d} = \begin{bmatrix} 0 & -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} \mu^0 \\ \tau_1^0 \\ \tau_2^0 \\ \tau_3^0 \end{bmatrix} = -\tau_1^0 + 2\tau_2^0 - \tau_3^0$$

We know the estimates of the coefficients are

$$\mu^0 = \bar{Y}_{...}, \text{ and } \quad \tau_i^0 = \bar{Y}_{i..} - \bar{Y}_{...}$$

and so we can substitute these into the evaluated $\mathbf{C}'\beta^0 - \mathbf{d}$:

$$\begin{aligned} \mathbf{C}'\beta^0 - \mathbf{d} &= -\tau_1^0 + 2\tau_2^0 - \tau_3^0 \\ &= -(\bar{Y}_{1..} - \bar{Y}_{...}) + 2(\bar{Y}_{2..} - \bar{Y}_{...}) - (\bar{Y}_{3..} - \bar{Y}_{...}) \\ &= -\bar{Y}_{1..} + \bar{Y}_{...} + 2\bar{Y}_{2..} - 2\bar{Y}_{...} - \bar{Y}_{3..} + \bar{Y}_{...} \\ &= -\bar{Y}_{1..} + 2\bar{Y}_{2..} - \bar{Y}_{3..} \end{aligned}$$

Next we need to evaluate $\mathbf{C}'(\mathbf{X}'\mathbf{X})^{-}\mathbf{C}$:

$$
\begin{aligned}
\mathbf{C}'(\mathbf{X}'\mathbf{X})^{-}\mathbf{C} &= \begin{bmatrix} 0 & -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{n} & 0 & 0 \\ 0 & 0 & \frac{1}{n} & 0 \\ 0 & 0 & 0 & \frac{1}{n} \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 2 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -\frac{1}{n} & \frac{2}{n} & -\frac{1}{n} \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 2 \\ -1 \end{bmatrix} \\
&= \frac{1}{n} + \frac{4}{n} + \frac{1}{n} \\
&= \frac{6}{n}
\end{aligned}
$$

Therefore,

$$
Q = \frac{n}{6}(-\bar{Y}_{1..} + 2\bar{Y}_{2..} - \bar{Y}_{3..})^2
$$

The SSE is

$$
\begin{aligned}
SSE &= \sum_{i=1}^{3}\sum_{j=1}^{n}(Y_{ij} - \mu^0 - \tau_i^0)^2 \\
&= \sum_{i=1}^{3}\sum_{j=1}^{n}(Y_{ij} - \bar{Y}_{..} - (\bar{Y}_{i.} - \bar{Y}_{..}))^2 \\
&= \sum_{i=1}^{3}\sum_{j=1}^{n}(Y_{ij} - \bar{Y}_{i.})^2
\end{aligned}
$$

and now we can find the test statistic:

$$
\begin{aligned}
F(H) &= \frac{Q/s}{SSE/(N-r)} \\
&= \frac{\frac{n}{6}(-\bar{Y}_{1..} + 2\bar{Y}_{2..} - \bar{Y}_{3..})^2/(1)}{\sum_{i=1}^{3}\sum_{j=1}^{n}(Y_{ij} - \bar{Y}_{i.})^2/(3n-3)} \\
&= \frac{3(n-1)n(-\bar{Y}_{1..} + 2\bar{Y}_{2..} - \bar{Y}_{3..})^2}{6\sum_{i=1}^{3}\sum_{j=1}^{n}(Y_{ij} - \bar{Y}_{i.})^2} \\
&= \frac{(n-1)n(-\bar{Y}_{1..} + 2\bar{Y}_{2..} - \bar{Y}_{3..})^2}{2\sum_{i=1}^{3}\sum_{j=1}^{n}(Y_{ij} - \bar{Y}_{i.})^2}
\end{aligned}
$$

Problem 7.29

Consider the three-factor model with normal errors, viz., $Y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \gamma_k + (\beta\gamma)_{jk} + \epsilon_{ijk}$, $i = 1, \ldots, a$, $j = 1, \ldots, b$, $k = 1, \ldots, c$, with the constraints $\sum_i \tau_i = 0$, $\sum_j \beta_j = 0$, $\sum_k \gamma_k = 0$, $\sum_i (\tau\beta)_{ij} = 0$, $\sum_j (\tau\beta)_{ij} = 0$, $\sum_j (\beta\gamma)_{jk} = 0$, and $\sum_k (\beta\gamma)_{jk} = 0$. Derive a suitable sequence of nested hypotheses, giving the relevant sum of squares. Complete the ANOVA table.

Given our model, consider the following sequence of subspaces

$$\begin{aligned}
S_{H_0} &= \{Y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \gamma_k + (\beta\gamma)_{jk} + \epsilon_{ijk}\} \supset \\
S_{H_1} &= \{Y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \gamma_k + \epsilon_{ijk}\} \supset \\
S_{H_2} &= \{Y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk}\} \supset \\
S_{H_3} &= \{Y_{ijk} = \mu + \tau_i + \beta_j + \epsilon_{ijk}\} \supset \\
S_{H_4} &= \{Y_{ijk} = \mu + \tau_i + \epsilon_{ijk}\} \supset \\
S_{H_5} &= \{Y_{ijk} = \mu + \epsilon_{ijk}\} \supset \\
S_{H_5} &= \{Y_{ijk} = \epsilon_{ijk}\} \supset
\end{aligned}$$

assuming the constraints given in the problem hold. We need to find the fitted values under the different hypotheses to complete the sum of squares, meaning we need to solve for the least squares estimates of our parameters. Let's start with H_5:

$$\begin{aligned}
\frac{d}{d\mu} S(H_5) &= \frac{d}{d\mu} \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{c} (Y_{ijk} - \mu)^2 \\
&= 2 \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{c} (Y_{ijk} - \mu)(-1) = \text{ (set) } 0 \\
&= \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{c} \mu = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{c} Y_{ijk} \\
&= abc\mu = Y_{...} \\
&= \mu^0 = \bar{Y}_{...}
\end{aligned}$$

We can find the estimate of τ_i using H_4:

$$\begin{aligned}
\frac{d}{d\tau_i} S(H_4) &= \frac{d}{d\tau_i} \sum_{j=1}^{b} \sum_{k=1}^{c} (Y_{ijk} - \mu - \tau_i)^2 \\
&\implies 2 \sum_{j=1}^{b} \sum_{k=1}^{c} (Y_{ijk} - \mu - \tau_i)(-1) = \text{ (set) } 0 \\
&\implies \sum_{j=1}^{b} \sum_{k=1}^{c} \tau_i = \sum_{j=1}^{b} \sum_{k=1}^{c} bY_{ijk} - \sum_{j=1}^{b} \sum_{k=1}^{c} \mu \\
&\implies bc\tau_i^0 = Y_{i..} - bc\mu^0 \\
&\implies \tau_i^0 = \bar{Y}_{i..} - \bar{Y}_{...}
\end{aligned}$$

We can find the estimate of β_j using H_3:

$$\begin{aligned}
\frac{d}{d\beta_j} S(H_3) &= \frac{d}{d\beta_j} \sum_{i=1}^{a} \sum_{k=1}^{c} (Y_{ijk} - \mu - \tau_i - \beta_j)^2 \\
&\implies 2 \sum_{i=1}^{a} \sum_{k=1}^{c} (Y_{ijk} - \mu - \tau_i - \beta_j)(-1) = \text{ (set) } 0 \\
&\implies \sum_{i=1}^{a} \sum_{k=1}^{c} \beta_j = \sum_{i=1}^{a} \sum_{k=1}^{c} Y_{ijk} - \sum_{i=1}^{a} \sum_{k=1}^{c} \mu - \sum_{i=1}^{a} \sum_{k=1}^{c} \tau_i \\
&\implies ac\beta_j^0 = Y_{.j.} - ac\mu^0 \\
&\implies \beta_j^0 = \bar{Y}_{.j.} - \bar{Y}_{...}
\end{aligned}$$

We can find the estimate of $(\tau\beta)_{ij}$ using H_2:

$$
\begin{aligned}
\frac{d}{d(\tau\beta)_{ij}} S(H_2) &= \frac{d}{d(\tau\beta)_{ij}} \sum_{k=1}^{c} (Y_{ijk} - \mu - \tau_i - \beta_j - (\tau\beta)_{ij})^2 \\
&\implies 2 \sum_{k=1}^{c} (Y_{ijk} - \mu - \tau_i - \beta_j - (\tau\beta)_{ij})(-1) = \text{ (set) } 0 \\
&\implies \sum_{k=1}^{c} (\tau\beta)_{ij} = \sum_{k=1}^{c} Y_{ijk} - \sum_{k=1}^{c} \mu - \sum_{k=1}^{c} \tau_i - \sum_{k=1}^{c} \beta_j \\
&\implies c(\tau\beta)^0_{ij} = Y_{ij.} - c\mu^0 - c\tau^0_i - c\beta^0_j \\
&\implies (\tau\beta)^0_{ij} = \bar{Y}_{ij.} - \bar{Y}_{...} - (\bar{Y}_{i..} - \bar{Y}_{...}) - (\bar{Y}_{.j.} - \bar{Y}_{...}) \\
&\implies (\tau\beta)^0_{ij} = \bar{Y}_{ij.} - \bar{Y}_{...} - \bar{Y}_{i..} + \bar{Y}_{...} - \bar{Y}_{.j.} + \bar{Y}_{...} \\
&\implies (\tau\beta)^0_{ij} = \bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...}
\end{aligned}
$$

We can find the estimate of γ_k using H_1:

$$
\begin{aligned}
\frac{d}{d\beta_j} S(H_1) &= \frac{d}{d\beta_j} \sum_{i=1}^{a} \sum_{j=1}^{b} (Y_{ijk} - \mu - \tau_i - \beta_j - (\tau\beta)_{ij} - \gamma_k)^2 \\
&\implies 2 \sum_{i=1}^{a} \sum_{j=1}^{b} (Y_{ijk} - \mu - \tau_i - \beta_j - (\tau\beta)_{ij} - \gamma_k)(-1) = \text{ (set) } 0 \\
&\implies \sum_{i=1}^{a} \sum_{j=1}^{b} \gamma_k = \sum_{i=1}^{a} \sum_{j=1}^{b} Y_{ijk} - \sum_{i=1}^{a} \sum_{j=1}^{b} \mu - \sum_{i=1}^{a} \sum_{j=1}^{b} \tau_i - \sum_{i=1}^{a} \sum_{j=1}^{b} \beta_j - \sum_{i=1}^{a} \sum_{j=1}^{b} (\tau\beta)_{ij} \\
&\implies ab\gamma^0_k = Y_{..k} - ab\mu^0 \\
&\implies \gamma^0_k = \bar{Y}_{..k} - \bar{Y}_{...}
\end{aligned}
$$

Finally, we can find the estimate of $(\beta\gamma)_{jk}$ using H_0:

$$
\begin{aligned}
\frac{d}{d((\beta\gamma)_{jk}} S(H_0) &= \frac{d}{d(\beta\gamma)_{jk}} \sum_{i=1}^{a} (Y_{ijk} - \mu - \tau_i - \beta_j - (\tau\beta)_{ij} - \gamma_k - (\beta\gamma)_{jk})^2 \\
&\implies 2 \sum_{i=1}^{a} (Y_{ijk} - \mu - \tau_i - \beta_j - (\tau\beta)_{ij} - \gamma_k - (\beta\gamma)_{jk})(-1) = \text{ (set) } 0 \\
&\implies \sum_{i=1}^{a} (\beta\gamma)_{jk} = \sum_{i=1}^{a} Y_{ijk} - \sum_{i=1}^{a} \mu - \sum_{i=1}^{a} \tau_i - \sum_{i=1}^{a} \beta_j - \sum_{i=1}^{a} (\tau\beta)_{ij} - \sum_{i=1}^{a} \gamma_k \\
&\implies a(\beta\gamma)^0_{jk} = Y_{.jk} - a\mu^0 - a\beta^0_j - a\gamma^0_k \\
&\implies (\beta\gamma)^0_{jk} = \bar{Y}_{.jk} - \bar{Y}_{...} - (\bar{Y}_{.j.} - \bar{Y}_{...}) - (\bar{Y}_{..k} - \bar{Y}_{...}) \\
&\implies (\beta\gamma)^0_{jk} = \bar{Y}_{.jk} - \bar{Y}_{...} - \bar{Y}_{.j.} + \bar{Y}_{...} - \bar{Y}_{..k} + \bar{Y}_{...} \\
&\implies (\beta\gamma)^0_{jk} = \bar{Y}_{.jk} - \bar{Y}_{.j.} - \bar{Y}_{..k} + \bar{Y}_{...}
\end{aligned}
$$

Now we have the following fitted values under each hypothesis:

$$\begin{aligned}
\hat{\mathbf{y}}_{H_0} &= \{\mu^0 + \tau_i^0 + \beta_j^0 + (\tau\beta)_{ij}^0 + \gamma_k + (\beta\gamma)_{jk}\} = \{\bar{Y_{ij.}} + \bar{Y_{.jk}} - \bar{Y_{.j.}}\} \\
\hat{\mathbf{y}}_{H_1} &= \{\mu^0 + \tau_i^0 + \beta_j^0 + (\tau\beta)_{ij}^0 + \gamma_k\} = \{\bar{Y_{ij.}} + \bar{Y_{..k}} - \bar{Y_{...}}\} \\
\hat{\mathbf{y}}_{H_2} &= \{\mu^0 + \tau_i^0 + \beta_j^0 + (\tau\beta)_{ij}^0\} = \{\bar{Y_{ij.}}\} \\
\hat{\mathbf{y}}_{H_3} &= \{\mu^0 + \tau_i^0 + \beta_j^0\} = \{\bar{Y_{i..}} + \bar{Y_{.j.}} - \bar{Y_{...}}\} \\
\hat{\mathbf{y}}_{H_4} &= \{\mu^0 + \tau_i^0\} = \{\bar{Y_{i..}}\} \\
\hat{\mathbf{y}}_{H_5} &= \{\mu^0\} = \{\bar{Y_{...}}\}
\end{aligned}$$

Now we can estimate the sum of squares attributed to each parameter. $R(H_5|H_6)$ will give us the sum of squares for μ:

$$\begin{aligned}
R(H_5|H_6) &= (\hat{\mathbf{y}}_{H_5} - \hat{\mathbf{y}}_{H_6})'(\hat{\mathbf{y}}_{H_5} - \hat{\mathbf{y}}_{H_6}) \\
&= (\bar{Y_{...}})'(\bar{Y_{...}}) = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\bar{Y_{...}}^2 \\
&= abc\bar{Y_{...}}^2
\end{aligned}$$

The sum of squares for τ_i is:

$$\begin{aligned}
R(H_4|H_5) &= (\hat{\mathbf{y}}_{H_4} - \hat{\mathbf{y}}_{H_5})'(\hat{\mathbf{y}}_{H_4} - \hat{\mathbf{y}}_{H_5}) \\
&= (\bar{Y_{i..}} - \bar{Y_{...}})'(\bar{Y_{i..}} - \bar{Y_{...}}) \\
&= \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}(\bar{Y_{i..}} - \bar{Y_{...}})^2 \\
&= bc(\sum_{i=1}^{a}\bar{Y_{i..}}^2 + \sum_{i=1}^{a}\bar{Y_{...}}^2 - 2\sum_{i=1}^{a}\bar{Y_{i..}}\bar{Y_{...}}) \\
&= bc\sum_{i=1}^{a}\bar{Y_{i..}}^2 + abc\bar{Y_{...}}^2 - 2abc\bar{Y_{...}}^2 \\
&= bc\sum_{i=1}^{a}\bar{Y_{i..}}^2 - abc\bar{Y_{...}}^2
\end{aligned}$$

The sum of squares for β_j is:

$$\begin{aligned}
R(H_3|H_4) &= (\hat{\mathbf{y}}_{H_3} - \hat{\mathbf{y}}_{H_4})'(\hat{\mathbf{y}}_{H_3} - \hat{\mathbf{y}}_{H_4}) \\
&= (\bar{Y_{.j.}} - \bar{Y_{...}})'(\bar{Y_{.j.}} - \bar{Y_{...}}) \\
&= \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}(\bar{Y_{.j.}} - \bar{Y_{...}})^2 \\
&= ac(\sum_{j=1}^{b}\bar{Y_{.j.}}^2 + \sum_{j=1}^{b}\bar{Y_{...}}^2 - 2\sum_{j=1}^{b}\bar{Y_{.j.}}\bar{Y_{...}}) \\
&= ac\sum_{j=1}^{b}\bar{Y_{.j.}}^2 + abc\bar{Y_{...}}^2 - 2abc\bar{Y_{...}}^2 \\
&= ac\sum_{j=1}^{b}\bar{Y_{.j.}}^2 - abc\bar{Y_{...}}^2
\end{aligned}$$

The sum of squares for $(\tau\beta)_{ij}$ is:

$$
\begin{aligned}
R(H_2|H_3) &= (\hat{\mathbf{y}}_{H_2} - \hat{\mathbf{y}}_{H_3})'(\hat{\mathbf{y}}_{H_2} - \hat{\mathbf{y}}_{H_3}) \\
&= (\bar{Y_{ij.}} - \bar{Y_{i..}} - \bar{Y_{.j.}} + \bar{Y_{...}})'(\bar{Y_{ij.}} - \bar{Y_{i..}} - \bar{Y_{.j.}} + \bar{Y_{...}}) \\
&= \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}((\bar{Y_{ij.}} - \bar{Y_{i..}}) + (\bar{Y_{...}} - \bar{Y_{.j.}}))^2 \\
&= \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}((\bar{Y_{ij.}} - \bar{Y_{i..}})^2 + (\bar{Y_{...}} - \bar{Y_{.j.}})^2 + 2(\bar{Y_{ij.}} - \bar{Y_{i..}})(\bar{Y_{...}} - \bar{Y_{.j.}})) \\
&= \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}(\bar{Y_{ij.}}^2 + \bar{Y_{i..}}^2 - 2\bar{Y_{ij.}}\bar{Y_{i..}} + \bar{Y_{...}}^2 + \bar{Y_{.j.}}^2 - 2\bar{Y_{.j.}}\bar{Y_{...}} + 2\bar{Y_{ij.}}\bar{Y_{...}} - 2\bar{Y_{ij.}}\bar{Y_{.j.}} - 2\bar{Y_{i..}}\bar{Y_{...}} + 2\bar{Y_{i..}}\bar{Y_{.j.}}) \\
&= c(\sum_{i=1}^{a}\sum_{j=1}^{b}\bar{Y_{ij.}}^2 - b\sum_{i=1}^{a}\bar{Y_{i..}}^2 + ab\bar{Y_{...}}^2 + a\sum_{j=1}^{b}\bar{Y_{.j.}}^2 - 2ab\bar{Y_{...}}^2 + 2ab\bar{Y_{...}}^2 - 2a\sum_{j=1}^{b}\bar{Y_{.j.}}^2 - 2ab\bar{Y_{...}}^2 + 2ab\bar{Y_{...}}^2) \\
&= c\sum_{i=1}^{a}\sum_{j=1}^{b}\bar{Y_{ij.}}^2 - bc\sum_{i=1}^{a}\bar{Y_{i..}}^2 - ac\sum_{j=1}^{b}\bar{Y_{.j.}}^2 + abc\bar{Y_{...}}^2
\end{aligned}
$$

The sum of squares for γ_k is:

$$
\begin{aligned}
R(H_1|H_2) &= (\hat{\mathbf{y}}_{H_1} - \hat{\mathbf{y}}_{H_2})'(\hat{\mathbf{y}}_{H_1} - \hat{\mathbf{y}}_{H_2}) \\
&= (\bar{Y_{ij.}} + \bar{Y_{..k}} - \bar{Y_{...}} - \bar{Y_{ij.}})'(\bar{Y_{ij.}} + \bar{Y_{..k}} - \bar{Y_{...}} - \bar{Y_{ij.}}) \\
&= \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}(\bar{Y_{..k}} - \bar{Y_{...}})^2 \\
&= \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}(\bar{Y_{..k}}^2 + \bar{Y_{...}}^2 - 2\bar{Y_{..k}}\bar{Y_{...}}) \\
&= ab(\sum_{k=1}^{c}\bar{Y_{..k}}^2 + \sum_{k=1}^{c}\bar{Y_{...}}^2 - 2\sum_{k=1}^{c}\bar{Y_{..k}}\bar{Y_{...}}) \\
&= ab\sum_{k=1}^{c}\bar{Y_{..k}}^2 - abc\bar{Y_{...}}^2
\end{aligned}
$$

and finally, the sum of squares for $(\beta\gamma)_{jk}$ is:

$$
\begin{aligned}
R(H_0|H_1) &= (\bar{Y_{.jk}} - \bar{Y_{.j.}} - \bar{Y_{..k}} + \bar{Y_{...}})'() \\
&= (\bar{Y_{.jk}} - \bar{Y_{.j.}} - \bar{Y_{..k}} + \bar{Y_{...}})'(\bar{Y_{.jk}} - \bar{Y_{.j.}} - \bar{Y_{..k}} + \bar{Y_{...}}) \\
&= \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}(\bar{Y_{.jk}} - \bar{Y_{.j.}} - \bar{Y_{..k}} + \bar{Y_{...}})^2 \\
&= a\sum_{j=1}^{b}\sum_{k=1}^{c}\bar{Y_{.jk}}^2 - ac\sum_{j=1}^{b}\bar{Y_{.j.}}^2 - ab\sum_{k=1}^{c}\bar{Y_{..k}}^2 + abc\bar{Y_{...}}^2
\end{aligned}
$$

Now we can fill out the ANOVA table. Note that the error terms are the totals minus the sums of squares from all the parameters.

Source	SS	Df
τ_i	$bc\sum_{i=1}^{a}\bar{Y_{i..}}^2 - abc\bar{Y_{...}}^2$	$(a-1)$
β_j	$ac\sum_{j=1}^{b}\bar{Y_{.j.}}^2 - abc\bar{Y_{...}}^2$	$(b-1)$
$(\tau\beta)_{ij}$	$c\sum_{i=1}^{a}\sum_{j=1}^{b}\bar{Y_{ij.}}^2 - bc\sum_{i=1}^{a}\bar{Y_{i..}}^2 - ac\sum_{j=1}^{b}\bar{Y_{.j.}}^2 + abc\bar{Y_{...}}^2$	$(a-1)(b-1)$
γ_k	$ab\sum_{k=1}^{c}\bar{Y_{..k}}^2 - abc\bar{Y_{...}}^2$	$(c-1)$
$(\beta\gamma)_{jk}$	$a\sum_{j=1}^{b}\sum_{k=1}^{c}\bar{Y_{.jk}}^2 - ac\sum_{j=1}^{b}\bar{Y_{.j.}}^2 - ab\sum_{k=1}^{c}\bar{Y_{..k}}^2 + abc\bar{Y_{...}}^2$	$(b-1)(c-1)$
Error	$\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}Y_{ijk}^2 - ac\sum_{j=1}^{b}\bar{Y_{.j.}}^2 - c\sum_{i=1}^{a}\sum_{j=1}^{b}\bar{Y_{ij.}}^2 - a\sum_{j=1}^{b}\sum_{k=1}^{c}\bar{Y_{.jk}}^2 + abc\bar{Y_{...}}^2$	$b(a-1)(c-1)$
Total	$\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}Y_{ijk}^2$	$abc-1$