STAT 665 - HW 1

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Problem 5.1

Let $x = (X_1, \ldots, X_k) \sim N_k(\mu, \Sigma)$, with $r(\Sigma) = k$.

(a) Show that

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} exp\Big\{ - \frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\Big\} dx_1 \ldots dx_k \\ &= (2\pi)^{k/2}|\Sigma|^{1/2}. \end{aligned}$$

What we have is part of a multivariate normal distribution integrated over all possible values of **x**. Let's first note that by definition of positive-definite, Σ^{-1} must be a positive definite matrix, otherwise its inverse would not exist. Now, we know that

$$f(\mathbf{x}; \mu, \Sigma) = \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}} exp\Big\{ - \frac{1}{2}(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)\Big\} \text{ for all } \mathbf{x} \in \mathcal{R}^k$$

and the integration of $f(\mathbf{x}; \mu, \Sigma)$ over all possible values of **x** is 1. Therefore, we have that

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}} exp\Big\{ - \frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\Big\} dx_1 \ldots dx_k \\ \implies (2\pi)^{k/2}|\Sigma|^{1/2} &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} exp\Big\{ - \frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\Big\} dx_1 \ldots dx_k \end{aligned}$$

(b) Evaluate $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} exp\{-(x_1^2 + 2x_1x_2 + 4x_2^2)\} dx_1 dx_2$.

Let's rewrite the expression in the following way:

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} exp\{-(x_1^2 + 2x_1x_2 + 4x_2^2)\} dx_1 dx_2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} exp\{-\frac{1}{2}(2x_1^2 + 4x_1x_2 + 8x_2^2)\} dx_1 dx_2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} exp\{-\frac{1}{2}(\mathbf{x}'\mathbf{A}\mathbf{x})\} dx_1 dx_2 \end{aligned}$$

where $\mathbf{A} = \begin{bmatrix} 2 & 2 \\ 2 & 8 \end{bmatrix}$, a 2×2 positive-definite matrix. Applying Aitken's integral, we have the solution

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} exp\{-\frac{1}{2}(\mathbf{x}'\mathbf{A}\mathbf{x})\} dx_1 dx_2 \\ &= (2\pi)^{2/2}|\mathbf{A}|^{-1/2} \\ &= (2\pi)\left| \begin{bmatrix} 2 & 2 \\ 2 & 8 \end{bmatrix} \right|^{-1/2} \\ &= (2\pi)((2)(8) - (2)(2))^{-1/2} \\ &= (2\pi)(12)^{-1/2} \\ &= (2\pi)\frac{2}{\sqrt{3}} \\ &= \frac{\pi}{\sqrt{3}} \end{aligned}$$

Problem 5.2

[Graybill, 1961]. Let $x = (X_1, X_2)$ have a bivariate normal distribution with pdf

$$f(x; \mu, \Sigma) = \frac{1}{k} exp[-Q/2]$$

where $Q = 2x_1^2 - x_1x_2 + 4x_2^2 - 11x_1 - 5x_2 + 19$, and k is a constant. Find a constant a such that $P(3X_1 - X_2 < a) = 0.01$.

The above bivariate normal distribution can be rewritten as

$$f(\mathbf{x}) = \frac{1}{k} \exp\left\{ -\frac{1}{2}(\mathbf{x}'\mathbf{\Sigma}^{-1}\mathbf{x} + \mathbf{x}'\mu + b_0)\right\}$$

where

$$\begin{aligned}
\mathbf{x}'\mathbf{\Sigma}^{-1}\mathbf{x} &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \sigma_{11}^2 & \sigma_{12}^2 \\ \sigma_{21}^2 & \sigma_{22}^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= \begin{bmatrix} \sigma_1^2 x_1 + \sigma_{212}^2 x_2 & \sigma_{12}^2 x_1 + \sigma_2^2 x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= \sigma_1^2 x_1^2 + \sigma_{12}^2 x_1 x_2 + \sigma_{12}^2 x_1 x_2 + \sigma_2^2 x_2^2 \\
&= (2)x_1^2 + (-1/2)x_1x_2 + (-1/2)x_1x_2 + (4)x_2^2 \\
&\implies \mathbf{\Sigma}^{-1} = \begin{bmatrix} 2 & -1/2 \\ -1/2 & 4 \end{bmatrix} \\
&\implies \mathbf{\Sigma} = \begin{bmatrix} 1/2 & -2 \\ -2 & 1/4 \end{bmatrix}
\end{aligned}$$

and we know the variances of the two normal random variables are $\sigma_1^2 = 1/2$ and $\sigma_2^2 = 1/4$. The means of the random variabes are $\mu_1 = -11$ and $\mu_2 = -5$.

We need to find the pdf of a new random variable $Y_1 = 3X_1 - X_2$. We know that the sum of two normal random variables is normal, so let's find the expected value and variance of the bivariate distribution.

$$\begin{aligned}
E[3X_1 - X_2] &= E[3X_1] - E[X_2] \\
&= 3E[X_1] - E[X_2] \\
&= 3(-11) - (-5) \\
&= -33 + 5 \\
&= -28
\end{aligned}$$

and the variance is

$$Var(3X_1 - X_2) = Var(3X_1) + Var(X_2) + 2Cov(3X_1, -X_2)$$

Problem 5.5

(a) Show that (X_1, X_2) has a bivariate normal distribution with means μ_1, μ_2, variances σ_1^2, σ_2^2, and a correlation coefficient ρ if and only if every linear combination $c_1X_1 + c_2X_2$ has a univariate normal distribution with mean $c_1\mu_1 + c_2\mu_2$, and variance $c_1^2\sigma_1^2 + c_2^2\sigma_2^2 + 2c_1c_2\rho\sigma_{1,2}$, where c_1 and c_2 are real constants, not both equal to zero.

Let's first show that if (X_1, X_2) has a bivariate normal distribution with means μ_1, μ_2, variances σ_1^2, σ_2^2, and a correlation coefficient ρ then every linear combination $c_1X_1 + c_2X_2$ has a univariate normal distribution with mean $c_1\mu_1 + c_2\mu_2$ and variance $c_1^2\sigma_1^2 + c_2\sigma_2^2 + 2c_1c_2\rho\sigma_{1,2}$.

Let $(X_1, X_2) \sim N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$. Consider the transformation $Y_1 = c_1X_1 + c_2X_2$. Let $Y_2 = X_2$. We can solve for X_1 and X_2:

$$\begin{aligned} X_2 &= Y_2 \\ X_1 &= \frac{1}{c_1}(Y_1 - c_2Y_2) \end{aligned}$$

The Jacobian is

$$\mathbf{J} = \begin{bmatrix} \frac{1}{c_1} & -\frac{1}{c_2} \\ 0 & 1 \end{bmatrix} = \frac{1}{c_1}$$

(b) Let $Y_i = X_i/\sigma_i, i = 1, 2$. Show that $Var(Y_1 - Y_2) = 2(1 - \rho)$.

Problem 5.6

(a) Let $(X_1, X_2) \sim N_2(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ where $\mu_1 = \mu_2 = 0$ and $\rho \neq 1$. The polar coordinate transformation is defined by $X_1 = Rcos\Theta$, $X_2 = Rsin\Theta$. Show that the joint pdf of R and Θ is given by

$$r(2\pi)^{-1}(1-\rho^2)^{-1/2}exp\left[-\frac{1}{2(1-\rho^2)}r^2(1-\rho sin2\theta)\right],$$

$0 \leq r < \infty$, and $0 \leq \theta \leq 2\pi$, and that the marginal pdf of Θ is

$$(2\pi)^{-1}(1-\rho^2)^{1/2}(1-\rho sin2\theta)^{-1},\ 0 \leq \theta \leq 2\pi.$$

(b) Suppose (X_1, X_2) has a bivariate normal distribution $N_2(0, 0, \sigma_1^2, \sigma_2^2, \rho, |\rho| \neq 1$. Show that

$$P(X_1 > 0, X_2 > 0) = \frac{1}{4} + \frac{1}{2\pi}sin^{-1}(\rho).$$

Problem 5.7

The random vector $x = (X_1, X_2, \ldots, X_k)'$ is said to have a symmetric multivariate normal distribution if $x \sim N_k(\mu, \Sigma)$ where $\mu = \mu 1_k$, i.e., the mean of each X_j is equal to the same constant μ, and Σ is the equicorrelation dispersion matrix, i.e.,

$$\Sigma = \sigma^2 = \begin{bmatrix} 1 & \rho & \ldots & \rho \\ \rho & 1 & \ldots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \ldots & 1 \end{bmatrix}$$

When $k = 3$, $\mu = 0$, $\sigma^2 = 2$, and $\rho = 1/2$, find the probability that $X_3 = min(X_1, X_2.X_3)$.

(*Hint:* Recall that if $x = (X_1, \ldots, X_k)'$ has a continuous symmetric distribution, then all possible permutations of $X_1, \ldots, X_k$ are equally likely, each having probability $P(X_{i1} < \cdots < X_{ik}) = 1/k!$ for any permutation $(i1, \ldots, i_k)$ for the first k positive integers.

Problem 5.8

Let $\mathbf{x} \sim N_k(0, \Sigma)$ with pdf $f(x)$ where $\Sigma = \{\Sigma_{ij}\}$. The entropy $h(x)$ is defined as

$$h(x) = -\int f(x)lnf(x)$$

(a) Show that $h(x) = \frac{1}{2}ln\left[(2\pi e)^k|\Sigma|\right]$.

We need to show that

$$\frac{1}{2}\ln(2\pi)^k|\Sigma| = -\int f(x)\ln(f(x))$$

But we know $\mathbf{x}$ has a multivariate normal distribution $N_k(0,\Sigma)$:

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}\exp\Bigg\{-\frac{1}{2}(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)\Bigg\},\ \mathbf{x}\in R^k$$

Inserting into the definition of entropy, we have

$$\begin{aligned}
&-\int \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}\exp\Bigg\{-\frac{1}{2}(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)\Bigg\}\ln\Bigg(\frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}\exp\Bigg\{-\frac{1}{2}(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)\Bigg\}\Bigg)\\
&\quad=-\int \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}\exp\Bigg\{-\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x}+\mathbf{x}'\mu)\Bigg\}\ln\Bigg(\frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}\exp\Bigg\{-\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x}+\mathbf{x}'\mu)\Bigg\}\Bigg)\\
&\quad=-\int \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}\exp\Bigg\{-\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x})\Bigg\}\ln\Bigg(\frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}\exp\Bigg\{-\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x})\Bigg\}\Bigg)
\end{aligned}$$

since we were given that $\mu=\mathbf{0}$. Applying log rules, we can expand the last term and continue:

$$\begin{aligned}
&=-\int \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}\exp\Bigg\{-\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x})\Bigg\}\Bigg(-\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x})-\frac{k}{2}\ln(2\pi)-\frac{1}{2}\ln(|\Sigma|)\Bigg)\\
&=\int \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}\exp\Bigg\{-\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x})\Bigg\}\Bigg(\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x})+\frac{k}{2}\ln(2\pi)+\frac{1}{2}\ln(|\Sigma|)\Bigg)\\
&=\frac{1}{2}\int \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}\exp\Bigg\{-\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x})\Bigg\}\Bigg((\mathbf{x}'\Sigma^{-1}\mathbf{x})+k\ln(2\pi)+\ln(|\Sigma|)\Bigg)\\
&=\frac{1}{2}\Bigg[\frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}\int(\mathbf{x}'\Sigma^{-1}\mathbf{x})\exp\Bigg\{-\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x})\Bigg\}\\
&\quad+k\ln(2\pi)\int \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}\exp\Bigg\{-\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x})\Bigg\}\\
&\quad+\ln(|\Sigma|)\int \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}\exp\Bigg\{-\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x})\Bigg\}\Bigg)\Bigg]
\end{aligned}$$

The integrals in the last two terms simply equal the constants pulled out front as we are integrating the multinomial normal distribution across all values of $\mathbf{x}$. The first term can be evaluated per Result 5.1.3:

$$\begin{aligned}
&\frac{1}{2}\Bigg[\frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}(2\pi)^{k/2}|\Sigma^{-1}|^{-1/2}tr(\Sigma^{-1}\Sigma)+k\ln(2\pi)+\ln(|\Sigma|)\Bigg]\\
&=\frac{1}{2}\Bigg[\frac{1}{|\Sigma|^{1/2}}|\Sigma|^{1/2}tr(I_k)+k\ln(2\pi)+\ln(|\Sigma|)\Bigg]\\
&=\frac{1}{2}\Bigg[k+k\ln(2\pi)+\ln(|\Sigma|)\Bigg]\\
&=\frac{1}{2}\Bigg[k+\ln\big[(2\pi)^k|\Sigma|\big]\Bigg]\\
&=\frac{1}{2}\Bigg[\ln(e^k)+\ln\big[(2\pi)^k|\Sigma|\big]\Bigg]\\
&=\frac{1}{2}\ln\big[(2\pi e)^k|\Sigma|\big]
\end{aligned}$$

(b) Hence, or otherwise, show that $|\Sigma|\le\prod_{i=1}^k\Sigma_{ii}$, with equality holding if and only if $\Sigma_{ij}=0$, for $i\neq j$ [Hadamard's inequality].

From part (a) we know that $-\int f(x)\ln(f(x)) = \frac{1}{2}\ln\left[(2\pi e)^k|\Sigma|\right]$.

Let's look at the determinant of Σ. Consider that we hold i fixed at one; then

$$\begin{aligned}
&-\int f(x)\ln(f(x)) = \frac{1}{2}\ln((2\pi e)^k|\Sigma|) \\
&\implies -2\int f(x)\ln(f(x)) = \ln((2\pi e)^k|\Sigma|) \\
&\implies -2\int f(x)\ln(f(x)) = \ln((2\pi e)^k) + \ln(|\Sigma|) \\
&\implies -2\int f(x)\ln(f(x)) - \ln((2\pi e)^k) = \ln(|\Sigma|) \\
&\implies \exp\left\{-2\int f(x)\ln(f(x)) - \ln((2\pi e)^k)\right\} = |\Sigma|
\end{aligned}$$