STAT 666 - Midterm

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Problem 1

Suppose we want to fit the following model:

$$Y_{ij} = \mu + \tau_i + \epsilon_{ij},\ i = 1, 2,\ j = 1, 2$$

where $\epsilon_{ij} \sim\ (iid)\ N(0, \sigma^2)$ and σ^2 is unknown. Derive a suitable test for $H : \mu + \tau_1 = \mu + \tau_2$.

We can rewrite the hypothesis test as $H : \mu + \tau_1 - \mu - \tau_2 = \tau_1 - \tau_2 = 0$. Consider formatting the hypothesis test as $H : \mathbf{C}'\beta = \mathbf{d}$, where

$$\mathbf{C}' = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix},\ \beta = \begin{bmatrix} \mu \\ \tau_1 \\ \tau_2 \end{bmatrix},\ \text{and } d = 0$$

Following Example 7.2.7, we know that we can derive a test statistic for the one-way fixed-effects ANOVA, testing for the equality of treatment means τ_1 and τ_2, as

$$F(H) = \frac{SSTr/(a-1)}{SSE/(N-a)} \sim F(a-1, N-a)$$

where $SSTr = \sum_{i=1}^{a} n_i(\bar{Y}_i - \bar{Y}_{..})^2$ is the treatment sum of squares and $SSE = \sum_{i=1}^{a}\sum_{j=1}^{n_i}(Y_{ij} - \bar{Y}_i)^2$ is the error sum of squares. In this case we are given $a = 2$ and $n_i = 2\ \forall i$, such that $N = n_1 + n_2 = 2 + 2 = 4$.

$$\begin{aligned}
&\frac{SSTr/(a-1)}{SSE/(N-a)} \\
&= \frac{\sum_{i=1}^{2}(2)(\bar{Y}_i - \bar{Y}_{..})^2/(2-1)}{\sum_{i=1}^{2}\sum_{j=1}^{2}(Y_{ij} - \bar{Y}_i)^2/(4-2)} \\
&= \frac{4\sum_{i=1}^{2}(\bar{Y}_i - \bar{Y}_{..})^2}{\sum_{i=1}^{2}\sum_{j=1}^{2}(Y_{ij} - \bar{Y}_i)^2} \\
&\sim F(1, 2)
\end{aligned}$$

Problem 2

Consider the model $\mathbf{Y} = \beta + \epsilon$, where $\epsilon \sim N_4(\mathbf{0}, \sigma^2\mathbf{I})$ with σ^2 being unknown and $\sum_{i=1}^{4}\beta_i = 0$. Derive an appropriate test statistic for testing $H : \beta_1 = \frac{\beta_2+\beta_3}{2}$.

Rewriting the null hypothesis, we have $H : 2\beta_2 - \beta_2 - \beta_3 = 0$, which we can format as $H : \mathbf{C}'\beta = \mathbf{d}$, where

$$\mathbf{C}' = \begin{bmatrix} 2 & -1 & -1 & 0 \end{bmatrix},\ \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix},\ \text{and } \mathbf{d} = 0.$$

From Result 7.2.1 we know that for $\mathbf{Q} = (\mathbf{C}'\beta^0 - \mathbf{d})'[\mathbf{C}'\mathbf{G}\mathbf{C}]^{-1}(\mathbf{C}'\beta^0 - \mathbf{d})$,

$$F(H) = \frac{\mathbf{Q}/s}{SSE/(N-r)} \sim F(s, N-r, \lambda)$$

Let's find $\mathbf{Q}$. It's clear that $\mathbf{C}'\beta - \mathbf{d} = 2\beta_1 - \beta_2 - \beta_3$. However, we also have the constraint that $\sum_{i=1}^4 \beta_i = 0 \implies \mathbf{1}'\beta = 0$. Under this restriction, $\hat{\beta}_r = \mathbf{y} - \bar{y}\mathbf{1}$, where $\bar{y} = \frac{1}{4}\sum_{i=1}^4 y_i$. Now we can calculate $\mathbf{C}'\beta - \mathbf{d}$ as

$$\begin{aligned}
\mathbf{C}'\beta - \mathbf{d} &= 2\beta_1 - \beta_2 - \beta_3 \\
&= 2(Y_1 - \bar{y}) - (Y_2 - \bar{y}) - (Y_3 - \bar{y}) \\
&= 2Y_1 - 2\bar{y} - Y_2 + \bar{y} - Y_3 + \bar{y} \\
&= 2Y_1 - Y_2 - Y_3
\end{aligned}$$

Next we need to find $(\mathbf{C}'\mathbf{G}\mathbf{C})^{-1}$, where $\mathbf{G} = (\mathbf{X}'\mathbf{X})^{-}$. From the model we know that $\mathbf{X} = \mathbf{I}$, and therefore $\mathbf{G} = \mathbf{I}$.

$$\begin{aligned}
(\mathbf{C}'\mathbf{C})^{-1} &= \left(\begin{bmatrix} 2 & -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix} \right)^{-1} \\
&= \left[2^2 + (-1)^2 + (-1)^2 + (0)^2 \right]^{-1} \\
&= [6]^{-1}
\end{aligned}$$

and

$$Q = \frac{(2Y_1 - Y_2 - Y_3)^2}{6}$$

We also need to find SSE under the restriction:

$$\begin{aligned}
SSE &= (\mathbf{y} - \mathbf{X}\beta^0)'(\mathbf{y} - \mathbf{X}\beta^0) \\
&= (\mathbf{y} - (\mathbf{y} - \bar{y}\mathbf{1}))'(\mathbf{y} - (\mathbf{y} - \bar{y}\mathbf{1})) \\
&= (\mathbf{y} - \mathbf{y} + \bar{y}\mathbf{1})'(\mathbf{y} - \mathbf{y} + \bar{y}\mathbf{1}) \\
&= (\bar{y}\mathbf{1})'(\bar{y}\mathbf{1}) \\
&= 4\left(\frac{1}{4}\sum_{i=1}^4 Y_i \right)^2 \\
&= \frac{1}{4}\left(\sum_{i=1}^4 Y_i \right)^2
\end{aligned}$$

Now we can calculate the F-statistics:

$$
\begin{aligned}
F(H) &= \frac{\frac{(2Y_1-Y_2-Y_3)^2}{6}/s}{\frac{1}{4}\left(\sum_{i=1}^{4} Y_i\right)^2/(N-r)} \\
&= \frac{(N-r)(4)(2Y_1-Y_2-Y_3)^2}{6\left(\sum_{i=1}^{4} Y_i\right)^2} \\
&= \frac{(4-3)(4)(2Y_1-Y_2-Y_3)^2}{6\left(\sum_{i=1}^{4} Y_i\right)^2} \\
&= \frac{4(2Y_1-Y_2-Y_3)^2}{6\left(\sum_{i=1}^{4} Y_i\right)^2} \\
&= \left(\frac{2}{3}\right)\frac{(2Y_1-Y_2-Y_3)^2}{\left(\sum_{i=1}^{4} Y_i\right)^2} \sim F(1,1)
\end{aligned}
$$

Problem 3

Consider the Gauss-Markov model, and suppose that **d** has a multivariate normal distribution. There are n observations in total, and it is possible that some of these are repeated observations, i.e., that some of the rows in the X-matrix are identical. In accordance with this, the rows of X can be partitioned into m, say, distinct sets consisting of identical rows; we shall call each of these sets an experimental combination. Assume that $m > p$. Let y_{ij} represent the jth response at the ith experimental combination ($j = 1, \ldots, n_i;\ i = 1, \ldots, m$); here, $\sum_{i=1}^{m} n_i = n$. Thus, there are n_i repeated obserations at the ith experimental combination. In this context the residual sum of squares, $\mathbf{y}'(\mathbf{I}-\mathbf{P})\mathbf{y}$, can be expressed as

$$\sum_{i=1}^{m}\sum_{j=1}^{n_i}(y_{ij}-\hat{y}_i)^2,$$

where $\hat{y}_i$ is the fitted value of the response at the ith experimental combination. Consider the following decomposition of this residual sum of squares:

$$\sum_{i=1}^{m}\sum_{j=1}^{n_i}(y_{ij}-\hat{y}_i)^2 = \sum_{i=1}^{m}\sum_{j=1}^{n_i}(y_{ij}-\bar{y}_i)^2 + \sum_{i=1}^{m} n_i(\bar{y}_i-\hat{y}_i)^2,$$

where $\bar{y}_i$ is the average response at the ith experimental combination. This decomposition can be written as $RSS = SS(\text{Pure Error}) + SS(\text{Lack of Fit})$.

(a) Verify the above decomposition by showing that $SS(\text{Pure Error}) = \mathbf{y}'(\mathbf{I}-\mathbf{P})\mathbf{y}$, $SS(\text{Lack of Fit}) = \mathbf{y}'(\mathbf{Q}-\mathbf{P})\mathbf{y}$, and $(\mathbf{I}-\mathbf{Q})(\mathbf{Q}-\mathbf{P}) = 0$, for some matrix $\mathbf{Q}$. (You may assume that the elements of $\mathbf{y}$ have been permuted, if necessary, so that all responses coresponding to the 1st experimental combinations appear first, etc.)

Let $\hat{y}_e = P_e y$, where $\mathcal{C}(P) \subset \mathcal{C}(P_e)$. Then we can rewrite SS(Pure Error) as

$$\begin{aligned}
SS(\text{Pure Error}) &= \sum_{i=1}^{m}\sum_{j=1}^{n_i}(y_{ij} - \bar{y}_i)^2 \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n_i}(y_{ij}^2 + \bar{y}_i^2 - 2y_{ij}\bar{y}_i) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n_i} y_{ij}^2 + \sum_{i=1}^{m}\sum_{j=1}^{n_i} \bar{y}_i^2 - 2\sum_{i=1}^{m}\sum_{j=1}^{n_i} y_{ij}\bar{y}_i \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n_i} y_{ij}^2 + \sum_{i=1}^{m} n_i\bar{y}_i^2 - 2\sum_{i=1}^{m} n_i\bar{y}_{i.}^2 \\
&= y'y - \sum_{i=1}^{m} n_i\bar{y}_{i.}^2 \\
&= y'y - y'P_e y \\
&= y'(I - P_e)y
\end{aligned}$$

and SS(Lack of Fit) as

$$\begin{aligned}
SS(\text{Lack of Fit}) &= \sum_{i=1}^{m} n_i(\bar{y}_{i.} - \hat{y}_{i.})^2 \\
&= \sum_{i=1}^{m} n_i(\bar{y}_{i.}^2 + \hat{y}_{i.}^2 - 2\bar{y}_{i.}\hat{y}_{i.}) \\
&= \sum_{i=1}^{m} n_i\bar{y}_{i.}^2 + \sum_{i=1}^{m} n_i\hat{y}_{i.}^2 - 2\sum_{i=1}^{m} n_i\bar{y}_{i.}\hat{y}_{i.} \\
&= y'P_e y + y'Py - 2y'P_e Py
\end{aligned}$$

But since $\mathcal{C}(P) \subset \mathcal{C}(P_e)$, we know that $P_e P = P$, and therefore

$$\begin{aligned}
SS(\text{Lack of Fit}) &= y'P_e y + y'Py - 2y'P_e Py \\
&= y'P_e y + y'Py - 2y'Py \\
&= y'P_e y - y'Py \\
&= y'(P_e - P)y
\end{aligned}$$

which suggests that $Q = P_e$. Now, consider $(I - Q)(Q - P)$:

$$\begin{aligned}
(I - Q)(Q - P) &= (I - P_e)(P_e - P) \\
&= IP_e - IP - P_e P_e + P_e P \\
&= P_e - P - P_e + P \\
&= 0
\end{aligned}$$

Now consider the following problem, related to a special case of the above decomposition. Data are available on a single regressor variable x and a response y. It is desired to test the hypotheses

$$H_0 : E(y_{ij}) = \beta_0 + \beta_1 x_i \text{ (for } j = 1, \ldots, n_i, \text{ and all } i)$$

versus

$$H_a : E(y_{ij}) = \beta_0 + \beta_1 x_i + \sum_{l=1}^{h} \beta_{l+1} f_l(x_i) \text{ (for } j = 1, \ldots, n_i, \text{ and all } i)$$

where the $\{f_l(.)\}$ are unspecified, possibly nonlinear functions. Assume $x_1 \neq x_2$.

(b) Obtain the distribution of $SS(\text{Pure Error})/\sigma^2$ under H_0.

Under the null hypothesis, we have that $y \sim N(X\beta, \sigma^2 I)$. The sum of squares due to pure error has a quadratic form, specifically $SS(\text{Pure Error}) = y'(I - P_e)y$. We can use Result 5.4.5 to determine the distribution of $SS(\text{Pure Error})/\sigma^2$, so long as $\frac{(I-P_e)}{\sigma^2} I\sigma^2$ is idempotent:

$$\begin{aligned}
A\Sigma A\Sigma &= \frac{(I - P_e)}{\sigma^2} I\sigma^2 \frac{(I - P_e)}{\sigma^2} I\sigma^2 \\
&= (I - P_e)(I - P_e) \\
&= I - P_e - P_e + P_e P_e \\
&= I - 2P_e + P_e \\
&= I - P_e
\end{aligned}$$

so we can use Result 5.4.5, and $\frac{y'(I-P_e)y}{\sigma^2} \sim \chi^2_{n-m}$.

(c) Obtain the expected value of $SS(\text{Lack of Fit})/(m - 2)$ under H_a.

We can use Result 5.4.2 to find the expected value of the sum of squares from lack of fit under the alternative hypothesis. Let P_a be the projection matrix P under the alternative hypothesis, β_a be the vector of β's under the alternative hypothesis, and X_a be the design matrix under the alternative hypothesis. We can find the expected value of $SS(\text{Lack of Fit})/(m - 2)$ under H_a as

$$\begin{aligned}
E\left[\frac{y'(P_e - P_a)y}{(m-2)}\right] &= \frac{1}{(m-2)} E[y'(P_e - P_a)y] \\
&= \frac{1}{(m-2)} \left[tr[(P_e - P_a)I\sigma^2] + \mu'(P_e - P)\mu\right] \\
&= \frac{1}{(m-2)} \left[\sigma^2 tr[(P_e - P_a)] + (X_a\beta_a)'(P_e - P_a)(X_a\beta_a)\right] \\
&= \frac{1}{(m-2)} \left[\sigma^2 tr[(P_e - P_a)] + \beta_a' X_a'(P_e - P_a)X_a\beta_a\right] \\
&= \frac{1}{(m-2)} \left[\sigma^2 tr[(P_e - P_a)] + \beta_a' X_a' P_e X_a\beta_a - \beta_a' X_a' P_a X_a \beta_a\right] \\
&= \frac{1}{(m-2)} \left[\sigma^2 tr[(P_e - P_a)] + \beta_a' X_a' X_a\beta_a - \beta_a' X_a' X_a \beta_a\right] \\
&= \frac{\sigma^2 tr(P_e - P_a)}{(m-2)}
\end{aligned}$$

(d) Obtain the distribution of $SS(\text{Lack of Fit})/\sigma^2$ under H_0.

Under the null hypothesis, we have that $y \sim N(X\beta, \sigma^2 I)$. The sum of squares due to lack of it has a quadratic form, specifically $SS(\text{Lack of Fit}) = y'(P_e - P)y$. We can use Result 5.4.5 to determine the distribution of $SS(\text{Lack of Fit})/\sigma^2$, so long as $\frac{(P_e-P)}{\sigma^2} I\sigma^2$ is idempotent:

$$\begin{aligned}
A\Sigma A\Sigma &= \frac{(P_e - P)}{\sigma^2} I\sigma^2 \frac{(P_e - P)}{\sigma^2} I\sigma^2 \\
&= (P_e - P)(P_e - P) \\
&= P_e P_e - P_e P - P P_e + PP \\
&= P_e - P - P + P \\
&= P_e - P
\end{aligned}$$

so we can use Result 5.4.5, and $\frac{y'(P_e-P)y}{\sigma^2} \sim \chi^2_{m-2}$.

(e) Obtain the distribution of $\frac{SS(\text{Lack of Fit})/(m-2)}{SS(\text{Pure Error})/(n-m)}$ under H_0.

We showed in part (d) that $\frac{y'(P_e-P)y}{\sigma^2} = \frac{SS(\text{Lack of Fit})}{\sigma^2} =\sim \chi^2_{m-2}$, and in part (b) that $\frac{y'(I-P_e)y}{\sigma^2} = \frac{SS(\text{Pure Error})}{\sigma^2} \sim \chi^2_{n-m}$. Consider now

$$\frac{\frac{SS(\text{Lack of Fit})}{\sigma^2}/(m-2)}{\frac{SS(\text{Pure Error})}{\sigma^2}/(n-m)} = \frac{SS(\text{Lack of Fit})/(m-2)}{SS(\text{Pure Error})/(n-m)}$$

is the ratio of two chi-squared random variables, each divided by their degrees of freedom, which we know to have an F distribution with degrees of freedom $m-2$ and $n-m$.

Problem 4

Figure out LSE for Exercise 7.10, assuming the intersection of the two lines in two ways, one using the Result 7.1, and the other using the origin-shifted model.

In Exercise 7.10, we are given two lines (x_0, y_0):

$$\begin{aligned}
Y_{1,i} &= \beta_{1,0} + \beta_{1,1} X_{1,i} + \epsilon_{1,i} \\
Y_{2,i} &= \beta_{2,0} + \beta_{2,1} X_{2,i} + \epsilon_{2,i}
\end{aligned}$$

where $\epsilon_i \sim N(0, \sigma^2)$. We can estimate $\beta_{l,0}$ and $\beta_{l,2}$ by minimizing the function $S(\beta)$, From Example 7.2.4 we have

$$\hat{\beta}_{l,1} = \frac{\sum_{i=1}^{n_l} (Y_{l,i} - \bar{Y}_{l.})(X_{l,i} - \bar{X}_{l.})}{\sum_{i=1}^{n_l} (X_{l,i} - \bar{X}_{l.})^2}$$

and

$$\hat{\beta}_{l,0} = \bar{Y}_{l.} - \hat{\beta}_{l,1} \bar{X}_{l.}$$

providing the sum of squared errors as

$$SSE = \sum_{i=1}^{n_1} (Y_{1,i} - \bar{Y}_{1.})^2 - \hat{\beta}_{1,1}^2 \sum_{i=1}^{n_1} (X_{1,i} - \bar{X}_{1.})^2 + \sum_{i=1}^{n_2} (Y_{2,i} - \bar{Y}_{2.})^2 - \hat{\beta}_{2,1}^2 \sum_{i=1}^{n_2} (X_{2,i} - \bar{X}_{2.})^2$$

We want to test the hypothesis that the two lines intersect at a point (x_0, y_0).

First, let's use the shifted model approach to derive the F-test

Consider the case where we shift each line by x_0 and y_0 such that if they intersect, they intersect at the origin. Our model (under no restrictions) becomes

$$\begin{aligned}
Y_{1,i} - y_0 &= \beta_{1,0} + \beta_{1,1}(X_{1,i} - x_0) + \epsilon_{1,i} \\
Y_{2,i} - y_0 &= \beta_{2,0} + \beta_{2,1}(X_{2,i} - x_0) + \epsilon_{2,i}
\end{aligned}$$

We want to test the hypothesis $H : (\beta_{1,0} + y_0) = (\beta_{2,0} + y_0) = 0$. The sum of squared errors under the null hypothesis is

$$\begin{aligned}
SSE_H &= \sum_{i=1}^{n_1}(Y_{1,i} - y_0 - \hat{\beta}_{1,0} - \hat{\beta}_{1,1}(X_{1,i} - x_0))^2 + \sum_{i=1}^{n_2}(Y_{2,i} - y_0 - \hat{\beta}_{2,0} - \hat{\beta}_{2,1}(X_{2,i} - x_0))^2 \\
&= \sum_{i=1}^{n_1}(Y_{1,i} - (\hat{\beta}_{1,0} + y_0) + \hat{\beta}_{1,1}(X_{1,i} - x_0))^2 + \sum_{i=1}^{n_2}(Y_{2,i} - (\hat{\beta}_{2,0} + y_0) + \hat{\beta}_{2,1}(X_{2,i} - x_0))^2 \\
&= \sum_{i=1}^{n_1}(Y_{1,i} + \hat{\beta}_{1,1}(X_{1,i} - x_0))^2 + \hat{\sum}_{i=1}^{n_2}(Y_{2,i} + \hat{\beta}_{2,1}(X_{2,i} - x_0))^2
\end{aligned}$$

To derive the F-statistic, we need to evaluate $SEE_H - SSE$:

$$\begin{aligned}
SSE_H - SSE =& \sum_{i=1}^{n_1}(Y_{1,i} + \hat{\beta}_{1,1}(X_{1,i} - x_0))^2 - \sum_{i=1}^{n_1}(Y_{1,i} - \bar{Y}_{1.})^2 - \hat{\beta}_{1,1}^2 \sum_{i=1}^{n_1}(X_{1,i} - \bar{X}_{1.})^2 \\
=& \sum_{i=1}^{n_1} Y_{1,i}^2 + \hat{\beta}_{1,1}^2 \sum_{i=1}^{n_1} X_{1,i}^2 + \hat{\beta}_{1,1}^2 n_1 x_0^2 - 2\hat{\beta}_{1,1}^2 x_0 n_1 \bar{X}_{1.} - 2\hat{\beta}_{1,1} \sum_{i=1}^{n_1} Y_{1,i} X_{1,i} - 2\hat{\beta}_{1,1} x_0 n_1 \bar{Y}_{1.} \\
& - \sum_{i=1}^{n_1} Y_{1,i}^2 + n_1 \bar{Y}_{1.}^2 - \hat{\beta}_{1,1}^2 \sum_{i=1}^{n_1} X_{1,i}^2 + \hat{\beta}_{1,1}^2 n_1 \bar{X}_{1.}^2 \\
=& \hat{\beta}_{1,1}^2 n_1 x_0^2 - 2\hat{\beta}_{1,1}^2 x_0 n_1 \bar{X}_{1.} - 2\hat{\beta}_{1,1} \sum_{i=1}^{n_1} Y_{1,i} X_{1,i} - 2\hat{\beta}_{1,1} x_0 n_1 \bar{Y}_{1.} + n_1 \bar{Y}_{1.}^2 + \hat{\beta}_{1,1}^2 n_1 \bar{X}_{1.}^2 \\
=& \hat{\beta}_{1,1}^2 n_1 (\bar{X}_{1.} - x_0)^2 - 2\hat{\beta}_{1,1}(\sum_{i=1}^{n_1} Y_{1,i} X_{1,i} + x_0 n_1 \bar{Y}_{1.}) + n_1 \bar{Y}_{1.}^2
\end{aligned}$$

Next let's derive the F-statistic from the original model.

We know that shifting the model should not change our estimates of the slope coefficients from the first part. Also, because of how I shifted the model in the first part, imposing the restriction that the original intercept shifted by y_0 must equal 0 (instead of hypothesising that $\beta_{l,0} = 0$), the estimates of the model intercept should not change. However, using a non-shifted model does change our hypothesis. Now the restriction imposed by H is

$$\begin{aligned}
& \beta_{1,0} + \beta_{1,1} x_0 = \beta_{2,0} + \beta_{2,1} x_0 \\
\implies & \beta_{1,0} - \beta_{2,0} + \beta_{1,1} x_0 - \beta_{2,1} x_0 = 0
\end{aligned}$$

such that we are testing the hypothesis $H : C'\beta = d$, where

$$\begin{aligned}
C' &= \begin{bmatrix} 1 & -1 & x_0 & -x_0 \end{bmatrix} \\
\beta' &= \begin{bmatrix} \beta_{1,0} & \beta_{2,0} & \beta_{1,1} x_0 & \beta_{2,1} x_0 \end{bmatrix} \\
d &= 0
\end{aligned}$$

To derive the F-statistic, we need to evaluate Q. First we'll need to find G. We know that

$$
\begin{aligned}
X'X &= \left[\begin{array}{cc|cc}
n_1 & 0 & \sum_{i=1}^{n_1} X_{1,i} & 0 \\
0 & n_2 & 0 & \sum_{i=1}^{n_2} X_{2,i} \\
\hline
\sum_{i=1}^{n_1} X_{1,i} & 0 & \sum_{i=1}^{n_1} X_{1,i}^2 & 0 \\
0 & \sum_{i=1}^{n_2} X_{2,i} & 0 & \sum_{i=1}^{n_2} X_{2,i}^2
\end{array}\right] \\
&= \left[\begin{array}{cc|cc}
n_1 & 0 & \bar{n}_1 X_{1,.} & 0 \\
0 & n_2 & 0 & \bar{n}_2 X_{2,.} \\
\hline
n_1 \bar{X}_{1,.} & 0 & \sum_{i=1}^{n_1} X_{1,i}^2 & 0 \\
0 & n_2 \bar{X}_{2,.} & 0 & \sum_{i=1}^{n_2} X_{2,i}^2
\end{array}\right]
\end{aligned}
$$

Let

$$
\begin{aligned}
n &= n_1, \quad a &= \bar{X}_{1.}, \quad c = \sum_{i=1}^{n_1} X_{1,i}^2 \\
m &= n_2, \quad b &= \bar{X}_{2.}, \quad d = \sum_{i=1}^{n_2} X_{2,i}^2
\end{aligned}
$$

Because X is full rank, we can let $G = (X'X)^{-1}$, which is

$$
\begin{bmatrix}
-\frac{c}{n(a^2n-c)} & 0 & \frac{a}{a^2n-c} & 0 \\
0 & -\frac{d}{m(mb^2-d)} & 0 & \frac{b}{mb^2-d} \\
\frac{a}{a^2n-c} & 0 & -\frac{1}{a^2n-c} & 0 \\
0 & \frac{b}{mb^2-d} & 0 & -\frac{1}{mb^2-d}
\end{bmatrix}
$$

Now,

$$
GC = \begin{bmatrix}
\frac{x_0na-c}{n(a^2n-c)} \\
\frac{d-x_0mb}{m(mb^2-d)} \\
\frac{a-x_0}{na^2-c} \\
\frac{x_0-b}{mb^2-d}
\end{bmatrix}
$$

and

$$
\begin{aligned}
C'GC &= \frac{x_0na-c}{n(a^2n-c)} - \frac{d-x_0mb}{m(mb^2-d)} + x_0\frac{a-x_0}{na^2-c} - x_0\frac{x_0-b}{mb^2-d} \\
&= \frac{2x_0na-c-nx_0^2}{n(a^2n-c)} + \frac{2x_0mb-d-mx_0^2}{m(mb^2-d)} \\
&= \frac{2x_0a-nc-x_0^2}{a^2n-c} + \frac{2x_0b-md-x_0^2}{mb^2-d} \\
&= \frac{(mb^2-d)(2x_0a-nc-x_0^2)}{(a^2n-c)(mb^2-d)} + \frac{(a^2n-c)(2x_0b-md-x_0^2)}{(a^2n-c)(mb^2-d)} \\
&= \frac{(mb^2-d)(2x_0a-nc-x_0^2)+(a^2n-c)(2x_0b-md-x_0^2)}{(a^2n-c)(mb^2-d)}
\end{aligned}
$$

We also have that

$$
C'\beta^0 - d = \begin{bmatrix} 1 & -1 & x_0 & -x_0 \end{bmatrix} \begin{bmatrix} \beta_{1,0} \\ \beta_{2,0} \\ \beta_{1,1} \\ \beta_{2,1} \end{bmatrix} = \beta_{1,0} - \beta_{2,0} + \beta_{1,1}x_0 - \beta_{2,1}x_0
$$

such that

$$
Q = \frac{(\hat{\beta}_{1,0} - \hat{\beta}_{2,0} + \hat{\beta}_{1,1}x_0 - \hat{\beta}_{2,1}x_0)^2(a^2n-c)(mb^2-d)}{(mb^2-d)(2x_0a-nc-x_0^2)+(a^2n-c)(2x_0b-md-x_0^2)}
$$
