STAT 565 HW 3

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Problem 5.22

Let X and Y be iid $N(0,1)$ random variables, and define $Z = \min(X, Y)$. Prove that $Z^2 \sim \chi^2_1$.

We know that the square of a standard normal random variable has a chi-squared distribution with one degree of freedom. Therefore, we just need to prove that $Z \sim N(0,1)$.

Let's look at the cdf of Z^2:

$$\begin{aligned}
F_Z^2(z) &= P(\min(X,Y) \leq z) \\
&= P(-\sqrt{z} \leq \min(X,Y) \leq \sqrt{z}) \\
&= P(-\sqrt{z} \leq X \leq \sqrt{z}, X \leq Y) + P(-\sqrt{z} \leq Y \leq \sqrt{z}, Y \leq X) \\
&= P(-\sqrt{z} \leq X \leq \sqrt{z} | X \leq Y)P(X \leq Y) + P(-\sqrt{z} \leq Y \leq \sqrt{z}, Y \leq X)P(Y \leq X)
\end{aligned}$$

Because X and Y are independent, when know that the probability that X or Y is less than $|\sqrt{z}|$ is the same with or without the conditional. Also, X and Y are identically distributed, so the chances of one being bigger than the other are identifical, that is, 50/50. Therefore, we have

$$F_Z^2(z) = \frac{1}{2}P(-\sqrt{z} \leq X \leq \sqrt{z}) + \frac{1}{2}P(-\sqrt{z} \leq Y \leq \sqrt{z})$$

But additionally, because the distributions are identical, we know that

$$P(Z^2 < z) = P(-\sqrt{z} < X < \sqrt{z})$$

We know that the cdf of a normal distribution does not have a closed form. Let's immediately take the derivative of the cdf to find the pdf:

$$\begin{aligned}
f_{Z^2}(z) &= \frac{d}{dz}P(-\sqrt{z} < X < \sqrt{z}) \\
&= \frac{1}{\sqrt{2\pi}}\left(e^{-z/2}\frac{1}{2}z^{-1/2} + e^{-z/2}\frac{1}{2}z^{-1/2}\right) \\
&= \frac{1}{\sqrt{2\pi}}z^{-1/2}e^{-z/2}
\end{aligned}$$

which is the pdf of a chi-square random variable with one degree of freedom.

Problem 5.24

Let $X_1, \ldots, X_n$ be a random sample from a population with pdf

$$f_X(x) = \begin{cases} 1/\theta & \text{if } 0 < x < \theta \\ 0 & \text{otherwise.} \end{cases}$$

Let $X_{(1)} < \cdots < X_{(n)}$ be the order statistics. Show that $X_{(1)}/X_{(n)}$ and $X_{(n)}$ are independent random variables.

The random variables $X_1, \dots, X_n$ each have a uniform distribution on the bound 0 to θ. From Example 5.4.7 we know that their joint distribution is

$$f_{X_{(1)},X_{(n)}}(x_1, x_n) = \frac{n(n-1)(x_n - x_1)^{n-2}}{\theta^n},\ 0 < x_1 < x_n < \theta$$

Now we need to find the joint distribution of $X_{(1)}/X_{(n)}$. Let $W = X_{(1)}/X_{(n)}$ and $Z = X_{(n)}$. Then, $x_n = z$ and $x_1 = wz$. The Jacobian is

$$J = \begin{bmatrix} z & w \\ 0 & 1 \end{bmatrix} = z$$

The joint distribution of W and Z is now

$$\begin{aligned} f_{W,Z}(w,z) &= f_{X_{(1)},X_{(n)}}(x_1, x_n)|J| \\ &= \frac{n(n-1)(z - wz)^{n-2}}{\theta^n} z \\ &= \frac{n(n-1)z^{n-2}(1-w)^{n-2}}{\theta^n} z \\ &= \frac{n(n-1)z^{n-1}(1-w)^{n-2}}{\theta^n},\ 0 < w < 1, 0 < z < \theta \end{aligned}$$

By factorization, W and Z, and thus $X_{(1)}/X_{(n)}$ and $X_{(n)}$ are independent.

Problem 5.25

As a generalization of the previous exercise, let $X_1, \dots, X_n$ be iid with pdf

$$f_X(x) = \begin{cases} \frac{a}{\theta^a} x^{a-1} & \text{if } 0 < x < \theta \\ 0 & \text{otherwise.} \end{cases}$$

Let $X_{(1)}, \dots, X_{(n)}$ be the order statistics. Show that $X_{(1)}/X_{(2)}, X_{(2)}/X_{(3)}, \dots, X_{(n-1)}/X_{(n)}$, and $X_{(n)}$ are mutually independent random variables. Find the distribution of each of them.

We are looking to show that $X_{(1)}/X_{(2)}, X_{(2)}/X_{(3)}, \dots, X_{(n-1)}/X_{(n)}$, and $X_{(n)}$ are all mutually independent from each. Similarly to the last problem, we can prove this through the factorization theorem.

Let's find the joint distribution of $X_{(1)}, \dots, X_{(n)}$:

$$\begin{aligned} f_{X_{(1)},\dots,X_{(n)}}(x_1, \dots, x_n) &= n! f_X(x_1) \times \cdots \times f_X(x_n) \\ &= n! \frac{a}{\theta^a} x_1^{a-1} \times \cdots \times \frac{a}{\theta^a} x_n^{a-1} \\ &= n! \frac{a^n}{\theta^{na}} x_1^{a-1} \times \cdots \times x_n^{a-1},\quad 0 < x_1 < \cdots < x_n < \theta \end{aligned}$$

Now we need to make the transformation $Y_1 = X_{(1)}/X_{(2)}, \dots, Y_{n-1} = X_{(n-1)}/X_{(n)}, Y_n = X_{(n)}$. We know from the last problem that the Jacobian was z when $Z = X_{(n)}$, and now the Jacobian is $y_2 y_3^2 \times \cdots \times y_n^{n-1}$ and the pdf of $Y_1, \dots, Y_n$ is

$$\begin{aligned} f_{Y_1,\dots,Y_n}(y_1, \dots, y_n) &= \frac{n! a^n}{\theta^{na}} (y_1 \times \cdots \times y_n)^{a-1} (y_2 \times \cdots \times y_n)^{a-1} \dots (y_n)^{a-1} (y_2 y_3^2 \times \cdots \times y_n^{n-1}) \\ &= \frac{n! a^n}{\theta^{na}} y_1^{a-1} y_2^{2a-1} \times \cdots \times y_n^{na-1},\quad 0 < y_i < 1, 0 < y_n < \theta \end{aligned}$$

By factorization, we know that $Y_1, \dots, Y_n = X_{(1)}/X_{(2)}, \dots, X_{(n-1)}/X_{(n)}, X_{(n)}$ are mutually independent.