# STAT 562 HW #6

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## Problem 6.30

Let $X_1, \ldots, X_n$ be a random sample from the pdf $f(x|\mu) = e^{-(x-\mu)}$, where $-\infty < \mu < x < \infty$.

(a) Show that $X_{(1)} = \min_i X_i$ is a complete sufficient statistic.

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The likelihood function of $\mu$ is

$$L(\theta) = \prod_{i=1}^{n} e^{-(x_i-\mu)} I_{\{-\infty<\mu<x<\infty\}} = e^{-\sum_{i=1}^{n}(x-\mu)} I_{\{-\infty<\mu<x<\infty\}}$$

Note that for $\mu$ to be less than $x$ means that $\mu$ must be less than $\min(x) = x_{(1)}$. Therefore, by factorization theorem, $X_{(1)}$ is a sufficient statistic for $\theta$.

To show completeness, consider some function of $T(\mathbf{X})$, say $g(T(\mathbf{X}))$, such that $E[g(T(\mathbf{X}))] = 0$. To find the expected value, we write

$$E[g(X_{(1)})] = \int_{\mu}^{\infty} g(t) f(t|\mathbf{x}) dt$$

We know the distribution of $T(\mathbf{X}) = X_{(1)}$ is

$$\frac{n!}{(n-1)!(i-1)!} f_X(x)[F_X(x)]^{i-1}[1 - F_X(x)]^{n-1} = n[e^{-(x-\mu)}][1 - (1 - e^{-(x-\mu)})]^{n-1} = n[e^{-(t-\mu)}]^n$$

such that

$$\begin{aligned}
E[g(X_{(1)})] = \int_{\mu}^{\infty} g(t) n[e^{-(t-\mu)}]^n dt = 0 \\
\implies ne^{n\mu} \int_{\mu}^{\infty} g(t) e^{-nt} dt = 0 \\
\implies \int_{\mu}^{\infty} g(t) e^{-nt} dt = 0 \\
\implies \int_{\mu}^{\infty} h(t) dt = 0 \\
\implies H(t)_{t\to\infty} - H(\mu) = 0
\end{aligned}$$

where $H'(t) = h(t)$. Consider that taking the derivative of both sides with respect to $\mu$ doesn't change the equality, but gives us

$$h(t) = g(t)e^{-nt} = 0$$

which only holds for all $t$ if $g(t) = 0$. Therefore, $T(X) = X_{(1)}$ is a complete statistic for $\mu$.

(b) Use Basu's Theorem to show that $X_{(1)}$ and $S^2$ are independent.

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Basu's theorem states that if $T(\mathbf{X})$ is a complete and minimally sufficient statistic, then $T(\mathbf{X})$ is independent of every ancillary statistic. Let's first show that $X_{(1)}$ is also minimally sufficient:

$$\frac{e^{-\sum_{i=1}^n x_i} e^{-n\mu} I_{\{\infty<\mu<x_{(1)}<\infty\}}}{e^{-\sum_{i=1}^n y_i} e^{-n\mu} I_{\{\infty<\mu<y_{(1)}<\infty\}}} = \frac{e^{-\sum_{i=1}^n x_i} I_{\{\infty<\mu<x_{(1)}<\infty\}}}{e^{-\sum_{i=1}^n y_i} I_{\{\infty<\mu<y_{(1)}<\infty\}}}$$

Clearly this ratio is only constant as a function of $\mu$ is when $x_{(1)} = y_{(1)}$, so $X_{(1)}$ is minimally sufficient.

Now we want to show that $S^2$ is an ancillary statistic. $S^2 \sim \chi^2_{n-1}$, whose distribution doesn't depend on $X_{(1)}$, and thus is constant as a function of $X_{(1)}$. Therefore, $S^2$ is ancillary and by Basu's Theorem, independent from $X_{(1)}$.

## Problem 7.6

Let $X_1, \ldots, X_n$ be a random sample from the pdf

$$f(x|\theta) = \theta x^{-2},\ 0 < \theta \leq x < \infty$$

(a) What is a sufficient statistic for $\theta$?

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The likelihood function of $\theta$ is

$$L(\theta) = \prod_{i=1}^n \theta x^{-2} I_{\{0<\theta<x<\infty\}} = \left(\prod_{i=1}^n x\right)^{-2} \theta^n I_{\{0<\theta<x<\infty\}} = \left(\prod_{i=1}^n x\right)^{-2} \theta^n I_{\{0<\theta<x_{(1)}<\infty\}}$$

By the factorization theorem, we see that $X_{(1)}$ is a sufficient statistic for $\theta$.

(b) Find the MLE of $\theta$.

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We now need to find the value of $\theta$ that maximizes the likelihood function. We quickly see that to maximize the likelihood function, we just need to maximize $\theta$, which has an upper bound of $x_{(1)}$. Therefore, the MLE of $\theta$ is $X_{(1)}$.

(c) Find the method of moments estimator of $\theta$.

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Let's first find the first moment:

$$EX^1 = \int_\theta^\infty x\theta x^{-2} dx = \int_\theta^\infty \theta x^{-1} dx = \theta \left[-x^{-2}\right]_\theta^\infty,$$

so the first moment doesn't exist, and the method of moments estimator doesn't exist.

## Problem 7.10

The independent random variables $X_1, \ldots, X_n$ have the common distribution

$$P(X_i \leq x|\alpha, \beta) = \begin{cases} 0 & \text{if } x < 0 \\ (x/\beta)^\alpha & \text{if } 0 \leq x \leq \beta \\ 1 & \text{if } x > \beta \end{cases}$$

where the parameters $\alpha$ and $\beta$ are positive.

(a) Find a two-dimensional sufficient statistic for $(\alpha, \beta)$.

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We are given the cdf; the pdf of $X_i$ is $\alpha(1/\beta)(x/\beta)^{\alpha-1}$, and the likelihood function is

$$L(\alpha, \beta) = \prod_{i=1}^{n} \frac{\alpha}{\beta}\left(\frac{x_i}{\beta}\right)^{\alpha-1} I_{\{0\leq x\leq\beta\}} = \frac{\alpha^n}{\beta^{n\alpha}}\left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} I_{\{0\leq x_{(n)}\leq\beta\}}$$

By the factorization theorem it's clear that $(\prod_{i=1}^{n} X_i, X_{(i)})$ is a sufficient statistic for $(\alpha, \theta)$.

(b) Find the MLEs of $\alpha$ and $\beta$.

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To find MLEs, we want to find $\hat{\alpha}$ and $\hat{\beta}$ that maximize the likelihood function. Quickly we see that to maximize $L(\alpha, \beta)$ we need to minimize $\beta$, meaning that $\hat{\beta} = x_{(n)}$. It's tricker to see how defining $\alpha$ can maximize the likelihood function, so let's take the partial derivative of the log-likelihood with respect to $\alpha$ and set equal to 0:

$$l(\alpha, \beta) = n\ln(\alpha) - n\alpha\ln(\beta) + (\alpha - 1)\sum_{i=1}^{n}\ln(x_i)$$

Now we can take the derivative with respect to $\alpha$, set equal to 0, and solve for $\hat{\alpha}$:

$$\begin{aligned}\frac{dl}{d\alpha} &= \frac{n}{\alpha} - n\ln\beta + \sum_{i=1}^{n} x_i = 0\\ \implies \frac{n}{\alpha} &= n\ln\beta - \sum_{i=1}^{n} x_i\\ \implies \hat{\alpha} &= \frac{n}{n\ln\beta - \sum_{i=1}^{n} x_i}\end{aligned}$$

We need to check the second derivative to make sure we have a maximum and not a minimum.

$$\frac{d}{d\alpha}\frac{n}{\alpha} - n\ln\beta + \sum_{i=1}^{n} x_i = -\frac{n}{\alpha^2} < 0,$$

confirming that $\hat{\alpha}$ is the MLE.

(c) The length (in millimeters) of cuckoos' eggs found in hedge sparrow nests can be modeled with this distribution. For the data

$$22.0,\ 23.9,\ 20.9,\ 23.8,\ 25.0,\ 24.0,\ 21.7,\ 23.8,\ 22.8,\ 23.1,\ 23.1,\ 23.5,\ 23.0,\ 23.0,$$

find the MLEs of $\alpha$ and $\beta$.

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```r
x <- as.vector(c(
  22.0,23.9,20.9,23.8,25.0,24.0,21.7,
  23.8,22.8,23.1,23.1,23.5,23.0,23.0
))

n <- length(x)

beta  <- max(x)
alpha <- (n)/(n*log(beta) - sum(log(x)))
```

We have that $\alpha$ is 12.5948689 and $\beta$ is 25.

## Problem 7.11

Let $X_1, \ldots, X_n$ be iid with pdf

$$f(x|\theta) = \theta x^{\theta-1},\ 0 \le x \le 1,\ 0 < \theta < \infty$$

(a) Find the MLE of $\theta$, and show that its variance $\to 0$ as $n \to \infty$.

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First we need to find the likelihood function of $\theta$:

$$L(\theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1} = \theta^n \prod_{i=1}^{n} x_i^{\theta-1}$$

Now let's find the log-likelihood to make determining the MLE easier:

$$l(\theta) = n\ln(\theta) + (\theta - 1)\sum_{i=1}^{n} \ln(x_i)$$

Setting equal to 0 and solving for $\theta$ we can solve for $\hat{\theta}$:

$$\begin{aligned}
\frac{dl}{d\theta} &= \frac{n}{\theta} + \sum_{i=1}^{n} \ln(x_i) = 0 \\
\implies \hat{\theta} &= -\frac{n}{\sum_{i=1}^{n} \ln(x_i)} \\
\implies \hat{\theta} &= \left(-\frac{1}{n}\sum_{i=1}^{n} \ln(x_i)\right)^{-1}
\end{aligned}$$

To confirm this is a maximum, take the second derivative:

$$\frac{d}{d\theta}\frac{n}{\theta} + \sum_{i=1}^{n} \ln(x_i) = -\frac{n}{\theta^2} < 0,$$

so $\hat{\theta}$ is a valid MLE.

Next we need to find the variance of $\hat{\theta}$ and show that it goes to 0 as $n \to \infty$. Note that the distribution of $-\ln(X_i)$ is gamma$(\alpha = 1, \beta = 1/\theta)$, and the sum of iid gamma random variables is another gamma random variable (let's call this Y) with paramters $\alpha = n$ and $\beta = 1/\theta$.

$$\begin{aligned}
E[n/Y] &= \int_0^\infty \frac{1}{\Gamma(n)(1/\theta)^n}\frac{n}{y} y^{n-1} e^{-y/(1/\theta)} dy \\
&= \frac{n\theta^n}{\Gamma(n)} \int_0^\infty y^{(n-1)-1} e^{-y\theta} dy \\
&= \frac{n\theta^n \Gamma(n-1)}{\Gamma(n)\theta^{n-1}} \int_0^\infty \frac{\theta^{n-1}}{\Gamma(n-1)} y^{(n-1)-1} e^{-y\theta} dy \\
&= \frac{n\theta}{n-1}
\end{aligned}$$

and

$$
\begin{aligned}
E[n/Y^2] &= \int_0^\infty \frac{1}{\Gamma(n)(1/\theta)^n} \frac{n^2}{y^2} y^{n-1} e^{-y/(1/\theta)} dy \\
&= \frac{n^2\theta^n}{\Gamma(n)} \int_0^\infty y^{(n-2)-1} e^{-y\theta} dy \\
&= \frac{n^2\theta^n \Gamma(n-2)}{\Gamma(n)\theta^{n-2}} \int_0^\infty \frac{\theta^{n-2}}{\Gamma(n-2)} y^{(n-1)-1} e^{-y\theta} dy \\
&= \frac{n^2\theta^2}{(n-1)(n-2)}
\end{aligned}
$$

such that the variance of $\hat{\theta}$ is

$$
Var(\hat{\theta}) = \frac{n^2\theta^2}{(n-1)(n-2)} - \left(\frac{n\theta}{n-1}\right)^2 = \frac{n^2\theta^2(n-1) - n^2\theta^2(n-2)}{(n-1)^2(n-2)} = \frac{n^2\theta^2}{(n-1)^2(n-2)}
$$

which goes to zero as $n \to \infty$.

(b) Find the method of moments estimator of $\theta$.

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We recognize the distribution of $X_i$s as beta$(\theta, 1)$, and so $EX = \frac{\theta}{\theta+1}$. To find the method of moments estimator, we need to solve

$$
EX = \frac{1}{n}\sum_{i=1}^n X_i = \frac{\theta}{\theta+1}
$$

for $\hat{\theta}$:

$$
\begin{aligned}
&\frac{1}{n}\sum_{i=1}^n X_i = \frac{\theta}{\theta+1} \\
&\implies \theta \frac{1}{n}\sum_{i=1}^n X_i + \frac{1}{n}\sum_{i=1}^n X_i = \theta \\
&\implies \frac{1}{n}\sum_{i=1}^n X_i = \theta\left(1 - \frac{1}{n}\sum_{i=1}^n X_i\right) \\
&\implies \hat{\theta} = \frac{\frac{1}{n}\sum_{i=1}^n X_i}{1 - \frac{1}{n}\sum_{i=1}^n X_i}
\end{aligned}
$$