STAT 665 - HW 1

Maggie Buffum

Problem 5.1

Let $x = (X_1, \ldots, X_k) \sim N_k(\mu, \Sigma)$, with $r(\Sigma) = k$.

(a) Show that

$$I = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} exp\Big\{ -\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu) \Big\} dx_1 \ldots dx_k$$
$$= (2\pi)^{k/2}|\Sigma|^{1/2}.$$

What we have is part of a multivariate normal distribution integrated over all possible values of $\mathbf{x}$. Let's first note that by definition of positive-definite, Σ^{-1} must be a positive definite matrix, otherwise its inverse would not exist. Now, we know that

$$f(\mathbf{x}; \mu, \Sigma) = \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}} exp\Big\{ -\frac{1}{2}(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu) \Big\} \text{ for all } \mathbf{x} \in \mathcal{R}^k$$

and the integration of $f(\mathbf{x}; \mu, \Sigma)$ over all possible values of $\mathbf{x}$ is 1. Therefore, we have that

$$1 = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}} exp\Big\{ -\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu) \Big\} dx_1 \ldots dx_k$$
$$\implies (2\pi)^{k/2}|\Sigma|^{1/2} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} exp\Big\{ -\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu) \Big\} dx_1 \ldots dx_k$$

(b) Evaluate $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} exp\{-(x_1^2 + 2x_1x_2 + 4x_2^2)\} dx_1 dx_2$.

Let's rewrite the expression in the following way:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} exp\{-(x_1^2 + 2x_1x_2 + 4x_2^2)\} dx_1 dx_2$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} exp\{-\frac{1}{2}(2x_1^2 + 4x_1x_2 + 8x_2^2)\} dx_1 dx_2$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} exp\{-\frac{1}{2}(\mathbf{x}'\mathbf{A}\mathbf{x})\} dx_1 dx_2$$

where $\mathbf{A} = \begin{bmatrix} 2 & 2 \\ 2 & 8 \end{bmatrix}$, a 2×2 positive-definite matrix. Applying Aitken's integral, we have the solution

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} exp\{-\frac{1}{2}(\mathbf{x}'\mathbf{A}\mathbf{x})\} dx_1 dx_2$$
$$= (2\pi)^{2/2}|\mathbf{A}|^{-1/2}$$
$$= (2\pi)\left| \begin{bmatrix} 2 & 2 \\ 2 & 8 \end{bmatrix} \right|^{-1/2}$$
$$= (2\pi)((2)(8) - (2)(2))^{-1/2}$$
$$= (2\pi)(12)^{-1/2}$$
$$= (2\pi)\frac{2}{\sqrt{3}}$$
$$= \frac{\pi}{\sqrt{3}}$$

Problem 5.2

[Graybill, 1961]. Let $x = (X_1, X_2)$ have a bivariate normal distribution with pdf

$$f(x; \mu, \Sigma) = \frac{1}{k} exp[-Q/2]$$

where $Q = 2x_1^2 - x_1x_2 + 4x_2^2 - 11x_1 - 5x_2 + 19$, and k is a constant. Find a constant a such that $P(3X_1 - X_2 < a) = 0.01$.

Since we know the distribution of the vector $\mathbf{x}$ is normal, $Q = (\mathbf{x} - \mu)'\Sigma^{-1}(\mathbf{x} - \mu)$. We can solve for Σ as

$$\begin{aligned}
Q &= \begin{bmatrix} x_1 - \mu_1 & x_2 - \mu_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \sigma_{12} \\ \sigma_{12} & \sigma_2 \end{bmatrix}^{-1} \begin{bmatrix} x_1 - \mu_1 & x_2 - \mu_2 \end{bmatrix} \\
&= \begin{bmatrix} x_1 - \mu_1 & x_2 - \mu_2 \end{bmatrix} \frac{1}{|\Sigma|} \begin{bmatrix} \sigma_2 & -\sigma_{12} \\ -\sigma_{12} & \sigma_1 \end{bmatrix} \begin{bmatrix} x_1 - \mu_1 & x_2 - \mu_2 \end{bmatrix} \\
&= \begin{bmatrix} x_1 - \mu_1 & x_2 - \mu_2 \end{bmatrix} \begin{bmatrix} \frac{\sigma_2}{|\Sigma|} & -\frac{\sigma_{12}}{|\Sigma|} \\ -\frac{\sigma_{12}}{|\Sigma|} & \frac{\sigma_1}{|\Sigma|} \end{bmatrix} \begin{bmatrix} x_1 - \mu_1 & x_2 - \mu_2 \end{bmatrix} \\
&= \begin{bmatrix} (x_1 - \mu_1)\frac{\sigma_2}{|\Sigma|} - (x_2 - \mu_2)\frac{\sigma_{12}}{|\Sigma|} & -(x_1 - \mu_1)\frac{\sigma_{12}}{|\Sigma|} + (x_2 - \mu_2)\frac{\sigma_1}{|\Sigma|} \end{bmatrix} \begin{bmatrix} x_1 - \mu_1 & x_2 - \mu_2 \end{bmatrix} \\
&= (x_1 - \mu_1)^2 \frac{\sigma_2}{|\Sigma|} - (x_2 - \mu_2)\frac{\sigma_{12}}{|\Sigma|}(x_1 - \mu_1) - (x_1 - \mu_1)\frac{\sigma_{12}}{|\Sigma|}(x_2 - \mu_2) + (x_2 - \mu_2)^2 \frac{\sigma_1}{|\Sigma|} \\
&= \frac{\sigma_2}{|\Sigma|}(x_1 - \mu_2)^2 + \frac{\sigma_1}{|\Sigma|}(x_2 - \mu_2)^2 - 2\frac{\sigma_{12}}{|\Sigma|}(x_1 - \mu_1)(x_2 - \mu_2) \\
&= \frac{\sigma_2}{|\Sigma|}(x_1^2 + \mu_1^2 - 2x_1\mu_1) + \frac{\sigma_1}{|\Sigma|}(x_2^2 + \mu_2^2 - 2x_2\mu_2) - 2\frac{\sigma_{12}}{|\Sigma|}(x_1x_2 - x_1\mu_2 - x_2\mu_1 + \mu_1\mu_2) \\
&= \frac{\sigma_2}{|\Sigma|}x_1^2 + \frac{\sigma_1}{|\Sigma|}x_2^2 - 2\frac{\sigma_{12}}{|\Sigma|}x_1x_2 + \frac{\sigma_2}{|\Sigma|}\mu_1^2 - 2\frac{\sigma_2}{|\Sigma|}x_1\mu_1 + \frac{\sigma_1}{|\Sigma|}\mu_2^2 - 2\frac{\sigma_1}{|\Sigma|}x_2\mu_2 + 2\frac{\sigma_{12}}{|\Sigma|}x_1\mu_2 + 2\frac{\sigma_{12}}{|\Sigma|}x_2\mu_1 - 2\frac{\sigma_{12}}{|\Sigma|}\mu_1\mu_2 \\
&= \frac{\sigma_2}{|\Sigma|}x_1^2 + \frac{\sigma_1}{|\Sigma|}x_2^2 - 2\frac{\sigma_{12}}{|\Sigma|}x_1x_2 + x_1\left(-2\frac{\sigma_2}{|\Sigma|}\mu_1 + 2\frac{\sigma_{12}}{|\Sigma|}\mu_2\right) + x_2\left(-2\frac{\sigma_1}{|\Sigma|}\mu_2 + 2\frac{\sigma_{12}}{|\Sigma|}\mu_1\right) + \frac{\sigma_2}{|\Sigma|}\mu_1^2 + \frac{\sigma_1}{|\Sigma|}\mu_2^2 - 2\frac{\sigma_{12}}{|\Sigma|}\mu_1\mu_2 \\
&= (2)x_1^2 + (4)x_2^2 - 2(1/2)x_1x_2 + x_1\left(-2(2)\mu_1 + 2(1/2)\mu_2\right) + x_2\left(-2(4)\mu_2 + 2(1/2)\mu_1\right) + (2)\mu_1^2 + (4)\mu_2^2 - 2(1/2)\mu_1\mu_2
\end{aligned}$$

Now we know that

$$\begin{aligned}
-11 &= -2(2)\mu_1 + 2(1/2)\mu_2 \text{ and} \\
-5 &= -2(4)\mu_2 + 2(1/2)\mu_1
\end{aligned}$$

giving us that $\mu_1 = 3$ and $\mu_2 = 1$. Note that

$$\begin{aligned}
&(2)\mu_1^2 + (4)\mu_2^2 - 2(1/2)\mu_1\mu_2 \\
&= (2)(3)^2 + (4)(1)^2 - (3)(1) \\
&= 18 + 4 - 3 \\
&= 19
\end{aligned}$$

We now know what Σ is too:

$$\Sigma = \frac{4}{31}\begin{bmatrix} 4 & -1/2 \\ -1/2 & 2 \end{bmatrix}$$

Now we can look at the random variable $Y = 3X_1 - X_2$, which has expected value $EY = 3EX_1 - EX_2 =$

$3(3) - 1 = 8$. Its variance is

$$\begin{aligned} Var(Y) &= Var(3X_1 - X_2) \\ &= 9Var(X_1) + Var(X_2) - 2(3)Cov(X_1, X_2) \\ &= 9\frac{16^2}{31^2} + \frac{8^2}{31^2} - 6\frac{2^2}{31^2} \\ &= 2.439126 \end{aligned}$$

Problem 5.5

(a) Show that (X_1, X_2) has a bivariate normal distribution with means μ_1, μ_2, variances σ_1^2, σ_2^2, and a correlation coefficient ρ if and only if every linear combination $c_1X_1 + c_2X_2$ has a univariate normal distribution with mean $c_1\mu_1 + c_2\mu_2$, and variance $c_1^2\sigma_1^2 + c_2^2\sigma_2^2 + 2c_1c_2\rho\sigma_{1,2}$, where c_1 and c_2 are real constants, not both equal to zero.

Let's first show that if (X_1, X_2) has a bivariate normal distribution with means μ_1, μ_2, variances σ_1^2, σ_2^2, and a correlation coefficient ρ then every linear combination $c_1X_1 + c_2X_2$ has a univariate normal distribution with mean $c_1\mu_1 + c_2\mu_2$ and variance $c_1^2\sigma_1^2 + c_2\sigma_2^2 + 2c_1c_2\rho\sigma_{1,2}$.

Let $(X_1, X_2) \sim N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$. Consider the transformation $Y_1 = c_1X_1 + c_2X_2$. Let $Y_2 = X_2$. We can solve for X_1 and X_2:

$$\begin{aligned} X_2 &= Y_2 \\ X_1 &= \frac{1}{c_1}(Y_1 - c_2Y_2) \end{aligned}$$

The Jacobian is

$$\mathbf{J} = \begin{bmatrix} \frac{1}{c_1} & -\frac{1}{c_2} \\ 0 & 1 \end{bmatrix} = \frac{1}{c_1}$$

(b) Let $Y_i = X_i/\sigma_i, i = 1, 2$. Show that $Var(Y_1 - Y_2) = 2(1 - \rho)$.

Problem 5.6

(a) Let $(X_1, X_2) \sim N_2(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ where $\mu_1 = \mu_2 = 0$ and $\rho \neq 1$. The polar coordinate transformation is defined by $X_1 = Rcos\Theta$, $X_2 = Rsin\Theta$. Show that the joint pdf of R and Θ is given by

$$r(2\pi)^{-1}(1-\rho^2)^{-1/2}exp\left[-\frac{1}{2(1-\rho^2)}r^2(1-\rho sin2\theta)\right],$$

$0 \leq r < \infty$, and $0 \leq \theta \leq 2\pi$, and that the marginal pdf of Θ is

$$(2\pi)^{-1}(1-\rho^2)^{1/2}(1-\rho sin2\theta)^{-1},\ 0 \leq \theta \leq 2\pi.$$

To complete the transformation, we need to find the Jacobian:

$$\begin{aligned} J &= \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = \cos\theta r\cos\theta - (-r\sin\theta\sin\theta) \\ &= r\cos^2\theta + r\sin^2\theta \\ &= r(\cos^2\theta + \sin^2\theta) \\ &= r \end{aligned}$$

The bivariate normal distribution is

$$f(x_1,x_2) = \frac{1}{2\pi|\Sigma|^{1/2}} \exp\bigg\{-\frac{1}{2}\frac{1}{1-\rho^2}\bigg(\frac{x_1^2}{\sigma_1^2}+\frac{x_2^2}{\sigma_2^2}-2\rho\frac{x_1x_2}{\sigma_1\sigma_2}\bigg)\bigg\}$$
$$= \frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2(1-\rho^2)}} \exp\bigg\{-\frac{1}{2}\frac{1}{1-\rho^2}\bigg(\frac{x_1^2}{\sigma_1^2}+\frac{x_2^2}{\sigma_2^2}-2\rho\frac{x_1x_2}{\sigma_1\sigma_2}\bigg)\bigg\}$$

Now we can find the distribution of the transformed variables:

$$f(r,\theta) = \frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2(1-\rho^2)}} \exp\bigg\{-\frac{1}{2}\frac{1}{1-\rho^2}\bigg(\frac{x_1^2}{\sigma_1^2}+\frac{x_2^2}{\sigma_2^2}-2\rho\frac{x_1x_2}{\sigma_1\sigma_2}\bigg)\bigg\}|r|$$
$$= \frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2(1-\rho^2)}} \exp\bigg\{-\frac{1}{2}\frac{1}{1-\rho^2}\bigg(\frac{r^2\cos^2\theta}{\sigma_1^2}+\frac{r^2\sin^2\theta}{\sigma_2^2}-2\rho\frac{r\cos\theta r\sin\theta}{\sigma_1\sigma_2}\bigg)\bigg\}|r|$$
$$= \frac{|r|}{2\pi\sqrt{\sigma_1^2\sigma_2^2(1-\rho^2)}} \exp\bigg\{-\frac{1}{2}\frac{r^2}{1-\rho^2}\bigg(\frac{\cos^2\theta\sigma_2^2}{\sigma_1^2\sigma_2^2}+\frac{\sin^2\theta\sigma_1^2}{\sigma_1^2\sigma_2^2}-2\rho\frac{\cos\theta\sin\theta}{\sigma_1\sigma_2}\bigg)\bigg\}$$
$$= \frac{|r|}{2\pi\sqrt{\sigma_1^2\sigma_2^2(1-\rho^2)}} \exp\bigg\{-\frac{1}{2}\frac{r^2}{1-\rho^2}\bigg(\frac{\cos^2\theta\sigma_2^2+\sin^2\theta\sigma_1^2-\rho\sin 2\theta}{\sigma_1\sigma_2}\bigg)\bigg\}$$

The only way for this to work is if we assume $\sigma_1 = \sigma_2 = 1$. Then we have shown equality:

$$= \frac{|r|}{2\pi\sqrt{(1-\rho^2)}} \exp\bigg\{-\frac{1}{2}\frac{r^2}{1-\rho^2}\bigg(\cos^2\theta\ +\sin^2\theta-\rho\sin 2\theta\bigg)\bigg\}$$
$$= \frac{|r|}{2\pi\sqrt{(1-\rho^2)}} \exp\bigg\{-\frac{1}{2}\frac{r^2}{1-\rho^2}\bigg(1-\rho\sin 2\theta\bigg)\bigg\},\ \text{for } r\geq 0, 0\leq\theta\leq 2\pi$$

(b) Suppose (X_1, X_2) has a bivariate normal distribution $N_2(0,0,\sigma_1^2,\sigma_2^2,\rho,\ |\rho|\neq 1$. Show that

$$P(X_1>0, X_2>0) = \frac{1}{4}+\frac{1}{2\pi}sin^{-1}(\rho).$$

Problem 5.7

The random vector $x = (X_1, X_2, \ldots, X_k)'$ is said to have a symmetric multivariate normal distribution if $x \sim N_k(\mu,\Sigma)$ where $\mu = \mu 1_k$, i.e., the mean of each X_j is equal to the same constant μ, and Σ is the equicorrelation dispersion matrix, i.e.,

$$\Sigma = \sigma^2 = \begin{bmatrix} 1 & \rho & \ldots & \rho \\ \rho & 1 & \ldots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \ldots & 1 \end{bmatrix}$$

When $k=3$, $\mu=0$, $\sigma^2=2$, and $\rho=1/2$, find the probability that $X_3 = min(X_1, X_2.X_3)$.

(*Hint:* Recall that if $x = (X_1,\ldots,X_k)'$ has a continuous symmetric distribution, then all possible permutations of $X_1,\ldots,X_k$ are equally likely, each having probability $P(X_{i1} < \cdots < X_{ik}) = 1/k!$ for any permutation $(i1,\ldots,i_k)$ for the first k positive integers.

Problem 5.8

Let $\mathbf{x} \sim N_k(0,\Sigma)$ with pdf $f(x)$ where $\Sigma = \{\Sigma_{ij}\}$. The entropy $h(x)$ is defined as

$$h(x) = -\int f(x)lnf(x)$$

(a) Show that $h(x) = \frac{1}{2}ln\big[(2\pi e)^k|\Sigma|\big]$.

We need to show that

$$\frac{1}{2}\ln(2\pi)^k|\Sigma| = -\int f(x)\ln(f(x))$$

But we know $\mathbf{x}$ has a multivariate normal distribution $N_k(0,\Sigma)$:

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}\exp\Big\{-\frac{1}{2}(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)\Big\},\ \mathbf{x}\in R^k$$

Inserting into the definition of entropy, we have

$$\begin{aligned}
&-\int\frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}\exp\Big\{-\frac{1}{2}(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)\Big\}\ln\Bigg(\frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}\exp\Big\{-\frac{1}{2}(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)\Big\}\Bigg)\\
&\quad=-\int\frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}\exp\Big\{-\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x}+\mathbf{x}'\mu)\Big\}\ln\Bigg(\frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}\exp\Big\{-\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x}+\mathbf{x}'\mu)\Big\}\Bigg)\\
&\quad=-\int\frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}\exp\Big\{-\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x})\Big\}\ln\Bigg(\frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}\exp\Big\{-\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x})\Big\}\Bigg)
\end{aligned}$$

since we were given that $\mu = \mathbf{0}$. Applying log rules, we can expand the last term and continue:

$$\begin{aligned}
&=-\int\frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}\exp\Big\{-\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x})\Big\}\Bigg(-\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x})-\frac{k}{2}\ln(2\pi)-\frac{1}{2}\ln(|\Sigma|)\Bigg)\\
&=\int\frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}\exp\Big\{-\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x})\Big\}\Bigg(\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x})+\frac{k}{2}\ln(2\pi)+\frac{1}{2}\ln(|\Sigma|)\Bigg)\\
&=\frac{1}{2}\int\frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}\exp\Big\{-\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x})\Big\}\Bigg((\mathbf{x}'\Sigma^{-1}\mathbf{x})+k\ln(2\pi)+\ln(|\Sigma|)\Bigg)\\
&=\frac{1}{2}\Bigg[\frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}\int(\mathbf{x}'\Sigma^{-1}\mathbf{x})\exp\Big\{-\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x})\Big\}\\
&\quad+k\ln(2\pi)\int\frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}\exp\Big\{-\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x})\Big\}\\
&\quad+\ln(|\Sigma|)\int\frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}\exp\Big\{-\frac{1}{2}(\mathbf{x}'\Sigma^{-1}\mathbf{x})\Big\}\Bigg)\Bigg]
\end{aligned}$$

The integrals in the last two terms simply equal the constants pulled out front as we are integrating the multinomial normal distribution across all values of $\mathbf{x}$. The first term can be evaluated per Result 5.1.3:

$$\begin{aligned}
&\frac{1}{2}\Bigg[\frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}(2\pi)^{k/2}|\Sigma^{-1}|^{-1/2}tr(\Sigma^{-1}\Sigma)+k\ln(2\pi)+\ln(|\Sigma|)\Bigg]\\
&=\frac{1}{2}\Bigg[\frac{1}{|\Sigma|^{1/2}}|\Sigma|^{1/2}tr(I_k)+k\ln(2\pi)+\ln(|\Sigma|)\Bigg]\\
&=\frac{1}{2}\Bigg[k+k\ln(2\pi)+\ln(|\Sigma|)\Bigg]\\
&=\frac{1}{2}\Bigg[k+\ln\big[(2\pi)^k|\Sigma|\big]\Bigg]\\
&=\frac{1}{2}\Bigg[\ln(e^k)+\ln\big[(2\pi)^k|\Sigma|\big]\Bigg]\\
&=\frac{1}{2}\ln\big[(2\pi e)^k|\Sigma|\big]
\end{aligned}$$

(b) Hence, or otherwise, show that $|\Sigma| \leq \prod_{i=1}^{k} \Sigma_{ii}$, with equality holding if and only if $\Sigma_{ij} = 0$, for $i \neq j$ [Hadamard's inequality].

From part (a) we know that $-\int f(x)\ln(f(x)) = \frac{1}{2}\ln\left[(2\pi e)^k|\Sigma|\right]$.

Let's look at the determinant of Σ. Consider that we hold i fixed at one; then

$$
\begin{aligned}
&-\int f(x)\ln(f(x)) = \frac{1}{2}\ln((2\pi e)^k|\Sigma|)\\
&\implies -2\int f(x)\ln(f(x)) = \ln((2\pi e)^k|\Sigma|)\\
&\implies -2\int f(x)\ln(f(x)) = \ln((2\pi e)^k) + \ln(|\Sigma|)\\
&\implies -2\int f(x)\ln(f(x)) - \ln((2\pi e)^k) = \ln(|\Sigma|)\\
&\implies \exp\left\{-2\int f(x)\ln(f(x)) - \ln((2\pi e)^k)\right\} = |\Sigma|
\end{aligned}
$$