STAT 562 Homework 4

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Problem 5.35

Stirling's Formula (derived in Exercise 1.28), which gives an approximation for factorials, can be easily derived using the CLT.

(a) Argue that, if $X_i \sim \text{exponential}(1), i = 1, 2, \ldots$, all independent, then for every x,

$$P\left(\frac{\bar{X}_n - 1}{1/\sqrt{n}} \leq x\right) \to P(Z \leq x),$$

where Z is a standard normal random variable.

Let's first recall Stirling's Formula:

$$n! \approx \sqrt{2\pi} n^{n+(1/2)} e^{-n}$$

The *Central Limit Theorem* states that if we have a sequence of iid random variables $X_1, X_2, \ldots$ with $EX_i = \mu$ and $VarX_i = \sigma^2 < \infty$, and define $\bar{X}_n = (1/n)\sum_{i=1}^n X_i$. Then,

$$lim_{n\to\infty} \sqrt{n}(\bar{X}_n - \mu)/\sigma = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

Suppose we have $X_i \sim \text{exponential}(1), i = 1, 2, \ldots$, all independent. We know that $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$, and the cdf of the exponential distribution with parameter 1 is $1 - e^{-x}$.

(b) Show that differentiating both sides of the approximation in part (a) suggests

$$\frac{\sqrt{n}}{\Gamma(n)} \left(x\sqrt{n} + n\right)^{n-1} e^{-(x\sqrt{n}+n)} \approx \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

and that $x = 0$ gives Stirling's Formula.

Problem 5.39

This exercise, and the two following, will look at some of the mathematical details of convergence.

(a) Prove Theorem 5.5.4. (*Hint*: since h is continuous, given $\epsilon > 0$ we can find a δ such that $|h(x_n) - h(x)| < \epsilon$ whenever $|x_n - x| < \delta$. Translate this into probability statements.)

Theorem 5.5.4 states that if $X_1, X_2, \ldots$ converges in probability to a random variable X, then if h is a continuous function, $h(X_1), h(X_2), \ldots$ converges in probability to $h(X)$.

Let's first suppose that $X_1, X_2, \ldots$ converge in probability to a random variable X. This means that

$$lim_{n\to\infty} P(|X_n - X| < \delta) = 1.$$

Consider a continuous function h. We want to show that

$$lim_{n\to\infty} P(|h(x_n) - h(x)| < \epsilon) = 1.$$

Since h is continuous, we know by theorem that given $\epsilon > 0$ we can find δ such that $|h(x_n) - h(x)| < \epsilon$ whenever $|x_n - x| < \delta$. That is, $P(|h(x_n) - h(x)| < \epsilon) = 1$ given we have that $|x_n - x| < \delta$, so $P(|h(x_n) - h(x)| < \epsilon)$ comes down knowing under what conditions $|x_n - x| < \delta$. But we've already established that $|x_n - x| < \delta$ is always true as $n \to \infty$, which implies that $lim_{n\to\infty} P(|h(x_n) - h(x)| < \epsilon) = 1$ under the starting assumptions.

(b) In Example 5.5.8, find a subsequence of the $X_i s$ that converges almost surely, that is, converges pointwise.

A sequence of random variables $X_1, X_2, \ldots$ *converges almost surely* to a random variable X is, for every $\epsilon > 0$,

$$P(lim_{n\to\infty}|X_n - X| < \epsilon) = 1.$$

In Example 5.5.8, the sample space S is the closed interval [0,1], and $X_1, \ldots, X_6$ are defined as

$$\begin{array}{lll} X_1(s) = s + I_{[0,1]}(s) & X_2(s) = s + I_{[0,\frac{1}{2}]}(s) & X_3(s) = s + I_{[\frac{1}{2},1]}(s) \\ X_4(s) = s + I_{[0,\frac{1}{3}]}(s) & X_5(s) = s + I_{[\frac{1}{3},\frac{2}{3}]}(s) & X_6(s) = s + I_{[\frac{2}{3},1]}(s) \end{array}$$

As the example shows, the sequence of all $X_i s$ converges in probability but not pointwise. However, consider the following subsequence of $X_i s$:

$$X_7(s) = s + I_{[\frac{5}{4},\frac{3}{2}]}(s) \quad X_8(s) = s + I_{[\frac{3}{2},2]}(s)$$

The indicator function will always be zero because no $s \in S$ satisfies the range specified for the indicator functions to be one. Therefore, X_7, X_8 converge pointwise to X.

Problem 5.41

Prove Theorem 5.5.13; that is, show that

$$P(|X_n - \mu| > \epsilon) \to 0 \text{ for every } \epsilon \iff P(X_n \le x) \to \begin{cases} 0 & \text{if } x < \mu \\ 1 & \text{if } x \ge \mu. \end{cases}$$

(a) Set $\epsilon = |x - \mu|$ and show that if $x > \mu$, then $P(X_n \le x) \ge P(|X_n - \mu| \le \epsilon)$, while if $x < \mu$, then $P(X_n \le x) \le P(|X_n - \mu| > \epsilon)$. Deduce the $\implies$ implication.

We will start by showing that if $P(|X_n - \mu| \le \epsilon) \to 0$, then $P(X_n \le x) \to 1$ for $x \ge \mu$ and $P(X_n \le x) \to 0$ for $x < \mu$. Let $\epsilon = |x - \mu|$.

Consider the case where $x > \mu$.

$$\begin{aligned} P(|X_n - \mu| \le \epsilon) &= P(|X_n - \mu| \le |x - \mu|) \\ &= P(|X_n - \mu| \le x - \mu) \text{ since } x > \mu \\ &= P(-(x - \mu) \le X_n - \mu \le x - \mu) \\ &= P(X_n - \mu \le x - \mu) - P(X_n - \mu \le -(x - \mu)) \\ &\le P(X_n - \mu \le x - \mu) \\ &= P(X_n \le x) \end{aligned}$$

which gives us that

$$P(X_n \le x) \ge P(|X_n - \mu| \le \epsilon)$$

We know that $P(|X_n - \mu| > \epsilon) \to 0$ as $n \to \infty$, implying that $P(|X_n - \mu| \le \epsilon) \to 1$ as $n \to \infty$. Therefore, $P(X_n \le x) \to 1$ for $x > \mu$.

Consider the case where $x < \mu$.

$$\begin{aligned} P(|X_n - \mu| > \epsilon) &= P(|X_n - \mu| > |x - \mu|) \\ &= P(|X_n - \mu| > -(x - \mu)) \\ &= P(X_n - \mu > -(x - \mu)) + P(X_n - \mu < x - \mu) \\ &\geq P(X_n - \mu < x - \mu) \\ &= P(X_n < x) \end{aligned}$$

such that

$$P(X_n \leq x) \leq P(|X_n - \mu| > \epsilon)$$

We began with the assumption that $P(|X_n - \mu| > \epsilon) \to 0$ when $n \to \infty$, and since $P(X_n \leq x)$ cannot be less than 0, $P(X_n \leq x) \to 0$ as $n \to 0$.

(b) Use the fact that $\{x : |x - \mu| > \epsilon\} = \{x : x - \mu < -\epsilon\} \cup \{x : x - \mu > \epsilon\}$ to deduce the $\Longleftarrow$ implication.

(See Billingsley 1995, Section 25, for a detailed treatment of the above results.)

Now we need to show that if $P(X_n \leq x) \to 0$ when $x < \mu$ and $P(X_n \leq x) \to 1$ when $x > \mu$ then $P(|X_n - \mu| > \epsilon) \to 0$.