Report for exercise 3 from group A

Tasks addressed:	5
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Source code:	https://github.com/mlcms-SS22-Group-A/mlcms-SS22-Group-A/tree/exercise-3

The work on tasks was divided in the following way:

MELIH MERT AKSOY (03716847)	Task 1	33%
	Task 2	33%
	Task 3	33%
	Task 4	33%
	Task 5	33%
ATAMERT RAHMA (03711801)	Task 1	33%
	Task 2	33%
	Task 3	33%
	Task 4	33%
	Task 5	33%
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	Task 2	33%
	Task 3	33%
	Task 4	33%
	Task 5	33%

numpy	1.20.3
matplotlib	3.4.3
sympy	1.9
scipy	1.7.1

Table 1: Software versions.

Software that is used in this report with corresponding versions can be seen on table 1.

In this report of exercise sheet 3, we use the notation from the lecture and the exercise sheet if not stated otherwise. That is, we use the triple (I, X, ϕ) to define a dynamical system where:

- I defines the time set ($I \subseteq \mathbb{N}$ for discrete systems, $I \subseteq \mathbb{R}$ for continuous systems).
- `X` defines the set of possible states.
- $\phi : I \times X \to X$ (evolution operator) that changes a state $x \in X$ over the change of the parameter $t \in I$.

For discrete systems we write the evolution by ϕ at an initial point $x_0 \in X$ as:

$$x_n = \phi(n, x_0),\, x_n \in X,\, n \in I \tag{1}$$

For continuous dynamical systems we specify the derivative with respect to time as a function (vector field) $v : X \to TX$,

$$\left(\frac{\partial \phi_\alpha(t,x)}{\partial t}\right)_{t=0} = v(x) \tag{2}$$

Report on task Task 1/5: Vector fields, orbits, and visualization

In this section we consider the following linear dynamical system, with state space $X = \mathbb{R}^2$, $I = \mathbb{R}$, parameter $\alpha \in \mathbb{R}$ (continuous system so we rely on equation 2), and flow ϕ_α defined by:

$$\left(\frac{\partial \phi_\alpha(t,x)}{\partial t}\right)_{t=0} = A_\alpha x \tag{3}$$

where $A_\alpha \in \mathbb{R}^{2\times 2}$ is a parameterized matrix. In the first task we tried to construct a similar figure as the figure 2.5 in the book of *Kuznetsov* [4]. We have used the following forms for our parameterized matrix:

$$A_\alpha = \begin{pmatrix} \alpha & \alpha \\ -0.25 & 0 \end{pmatrix},\, A_{\alpha,\beta} = \begin{pmatrix} \alpha & \beta \\ -0.25 & 0 \end{pmatrix} \tag{4}$$

Our reconstructions can be seen in figure 1. We have used the provided `plot_phase_portrait` function `utils.py` file to plot the phase diagrams in the figure.

For some of the figures we have used another parameter $\beta = -\alpha$ in order to reproduce the figures from the book (`Stable Node`, `Stable Focus` and `Unstable Saddle`). The systems `Stable Node` and `Stable Focus` are topologically equivalent, since they have the same number of Eigenvalues with negative real part ($n_- = 1$) and the same number of Eigenvalues with positive real part ($n_+ = 1$) and they both have no Eigenvalue with zero real part ($n_0 = 0$) (Theorem 2.2 in the book). The same also holds for the systems `Unstable Node` and `Unstable Focus` ($n_+ = 2, n_- = n_0 = 0$), so they are also topologically equivalent. The parameters used in matrices A_α and the eigenvalues of the matrices can be read from the titles of the figures.

Report on task Task 2/5: Common bifurcations in nonlinear systems

In this task we consider a continuous dynamical systems with the evolution described by

$$f^{(1)}(x) = \left(\frac{\partial \phi_\alpha^{(1)}(t,x)}{\partial t}\right)_{t=0} = \alpha - x^2,\;\; f^{(2)}(x) = \left(\frac{\partial \phi_\alpha^{(2)}(t,x)}{\partial t}\right)_{t=0} = \alpha - 2x^2 - 3. \tag{5}$$

With the derivatives with respect to the state x (which is needed to decide the stability of the fixed points):

$$\frac{\partial f^{(1)}}{\partial x} = -2x,\;\; \frac{\partial f^{(2)}}{\partial x} = -4x. \tag{6}$$

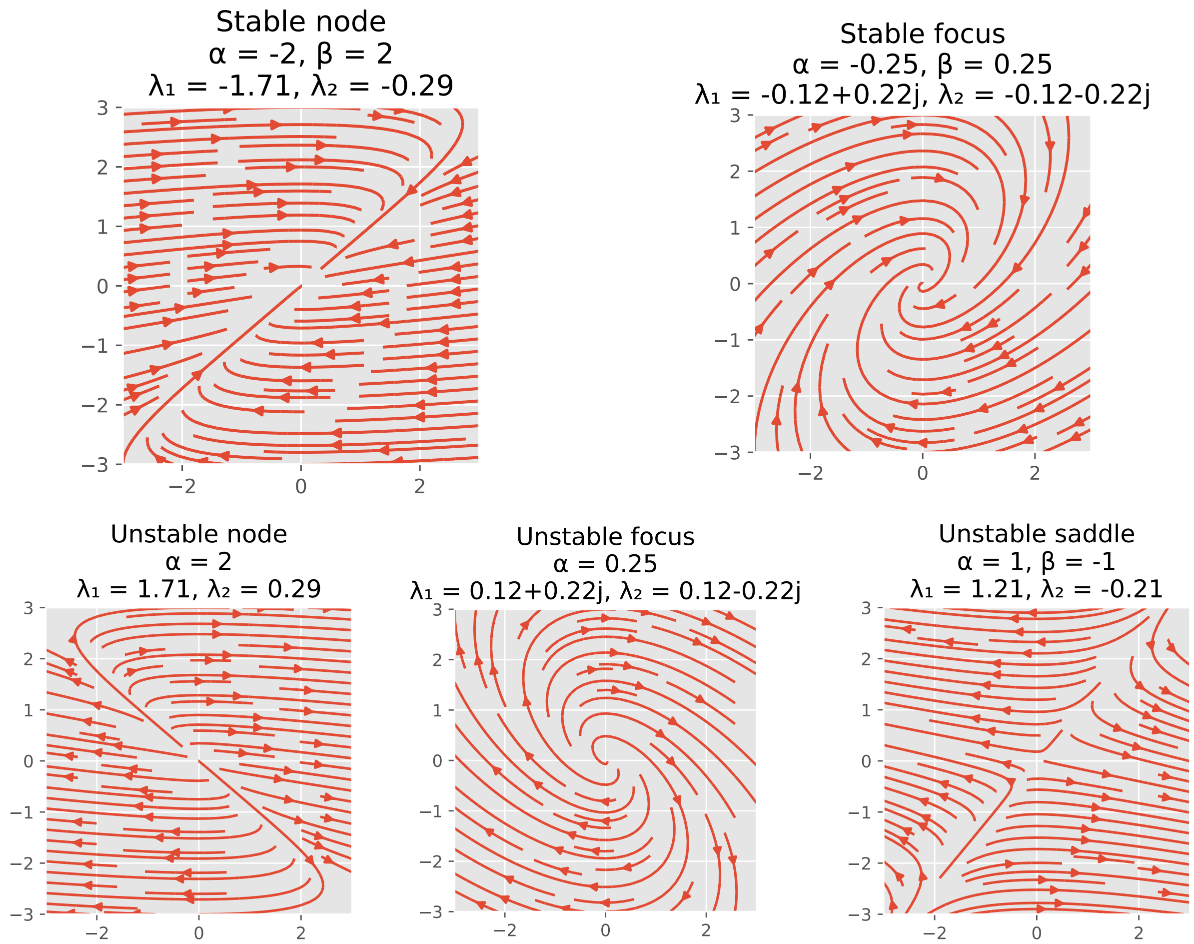

Figure 1: Visualization of the phase portraits with different values of α (and also β in some cases). The corresponding α (and β if there is any) and also the eigenvalues $\lambda_{1,2}$ of the matrix A_α are shown in the title of each portrait.

In order to plot the bifurcation diagrams we have implemented a function called `plot_bifurcation_diagram` which takes the right hand side of the ordinary differential equation of the system and solves the equation with respect to the given variable (which is x in our case) and plot how the solution changes as the parameter α changes using the `sympy` library. Resulting figure 2 depicts the bifurcations of the systems. In system (1) in equation 5 there is no steady state for $\alpha < 0$. For $\alpha \geq 0$ the system has two steady states (the analytical solution exists) $x_0^{(1)} = \pm\sqrt{\alpha}$. Since $\alpha \in \mathbb{R}$ the derivatives does not depend on α. The fixed points are stable if the derivatives in equation 6 result in a negative value [2]. Therefore the fixed points are stable if x_0 is positive for both of the systems. For system (1) the fixed point $x_0^{(1)} = \sqrt{\alpha}$ is therefore a stable steady state for $\alpha \geq 0$. Similarly, for system (2) in equation 5 there are again two steady states $x_0^{(2)} = \pm\sqrt{\frac{\alpha-3}{2}}$ for $\alpha \geq 3$. For $\alpha < 3$ the system has no steady states. Again, for $\alpha \geq 3$ only the steady state $x_0^{(2)} = \sqrt{\frac{\alpha-3}{2}}$ is stable in system (2) since the derivative becomes negative only in this case. Since the equations have this similar property we did not include a general algorithm to decide if the given point is a fixed state, but our implementation of the function `plot_bifurcation_diagram` does suit to the equations that we are using in this task.

As one can see in the figure 2, the Saddle-node bifurcation appears at $\alpha = 0$ for system (1), and at $\alpha = 3$ for system (2).

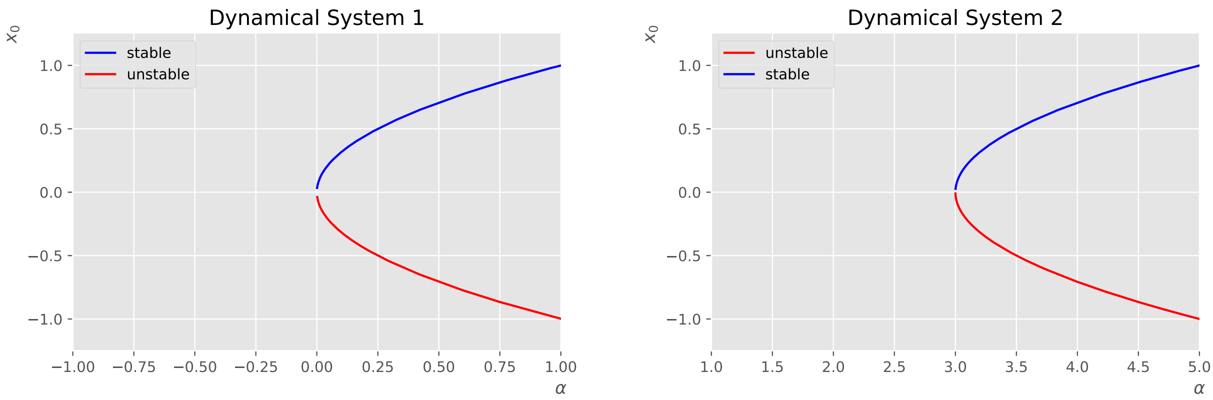

Figure 2: Bifurcation diagrams of the systems (1) and (2) defined in equation 5. Red-line depicts the unstable, and blue-line the stable steady states for different α values.

At $\alpha = 1$ we get the following evolution functions for system (1) and (2) respectively:

$$f^{(1)}(x) = \left(\frac{\partial \phi_\alpha^{(1)}(t,x)}{\partial t}\right)_{t=0} = 1 - x^2, \quad f^{(2)}(x) = \left(\frac{\partial \phi_\alpha^{(2)}(t,x)}{\partial t}\right)_{t=0} = 1 - 2x^2 - 3 = -2 - 2x^2 \tag{7}$$

In the general case a dynamical system (I, X, ϕ) is called topologically equivalent to a dynamical system (I, Y, ψ) if there is a homeomorphism: $h : X \to Y$ (a continuous bijection, and continuous inverse). Since we are considering the system (1): $(X = \mathbb{R}, I = \mathbb{R}, f^1)$, and the system (2): $(Y = \mathbb{R}, I = \mathbb{R}, f^2)$, if we find a homeomorphism: $h : \mathbb{R} \to \mathbb{R}$ such that $f^1 = h^{-1}(f^2(h(x)))$ for all $x \in \mathbb{R}$. We could not find such a h, however if we inspect the systems we see that they can be mapped to the same dynamical system (linear independence between the systems) and therefore they have the same normal form with the smallest degree of 2. This implies that the systems are topologically equivalent independent from which α value we use.

Report on task Task 3/5: Bifurcations in higher dimensions

We consider the following system with one parameter (α) and two-dimensional state space (Andronov-Hopf bifurcation [4] p.57), with the vector field in normal form in equation 8:

$$\frac{\partial}{\partial t}x_1 = \alpha x_1 - x_2 - x_1(x_1^2 + x_2^2), \quad \frac{\partial}{\partial t}x_2 = x_1 + \alpha x_2 - x_2(x_1^2 + x_2^2) \tag{8}$$

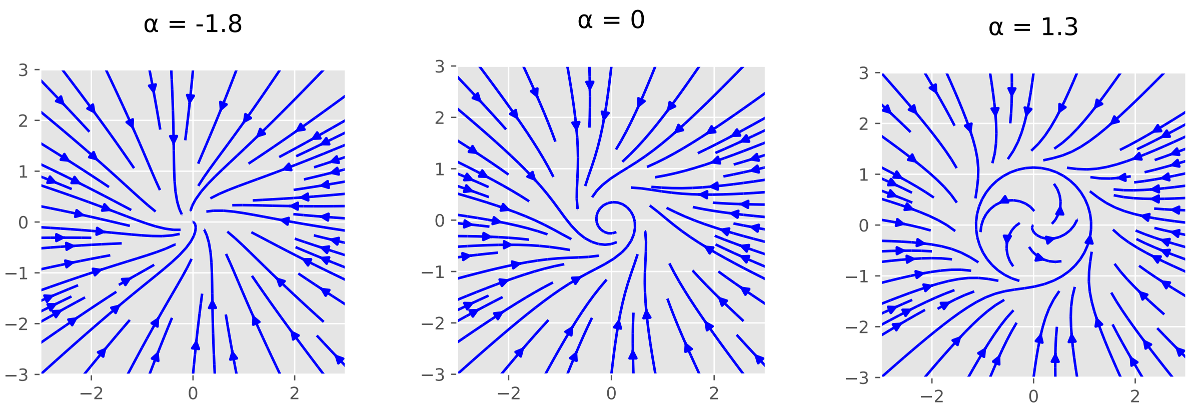

Figure 3: Visualization of the phase portraits of the Andronov-Hopf bifurcation 8 with different values of α.

In figure 3 we see the visualization of the phases for $\alpha \in \{-1.8, 0, 1.3\}$ that we got using the `matplotlib.pyplot.streamplot` function (this function is used for all the phase portraits in this report since it is used in the provided `plot_phase_portrait` function).

We have computed two orbits for $\alpha = 1$. The system is given in equation 9.

$$\frac{\partial}{\partial t}x_1 = x_1(1 - x_1^2 - x_2^2) - x_2,\ \frac{\partial}{\partial t}x_2 = x_1 + x_2(1 - x_1^2 - x_2^2) \tag{9}$$

The initial position of the first orbit is at $(2, 0)$ and the second orbit is at $(0.5, 0)$. We have not used the Euler's method to solve the orbit's path, but we have used the function `scipy.integrate.solve_ivp` which numerically integrates a system of ordinary differential equations given an initial value [3]. We have used the time span $(0, 10)$ for both of the cases, which means the initial time t_0 is given as 0 and the final time t_{final} is given as 10. In figure 4 one can see the visualization of the orbits with the initial positions as above.

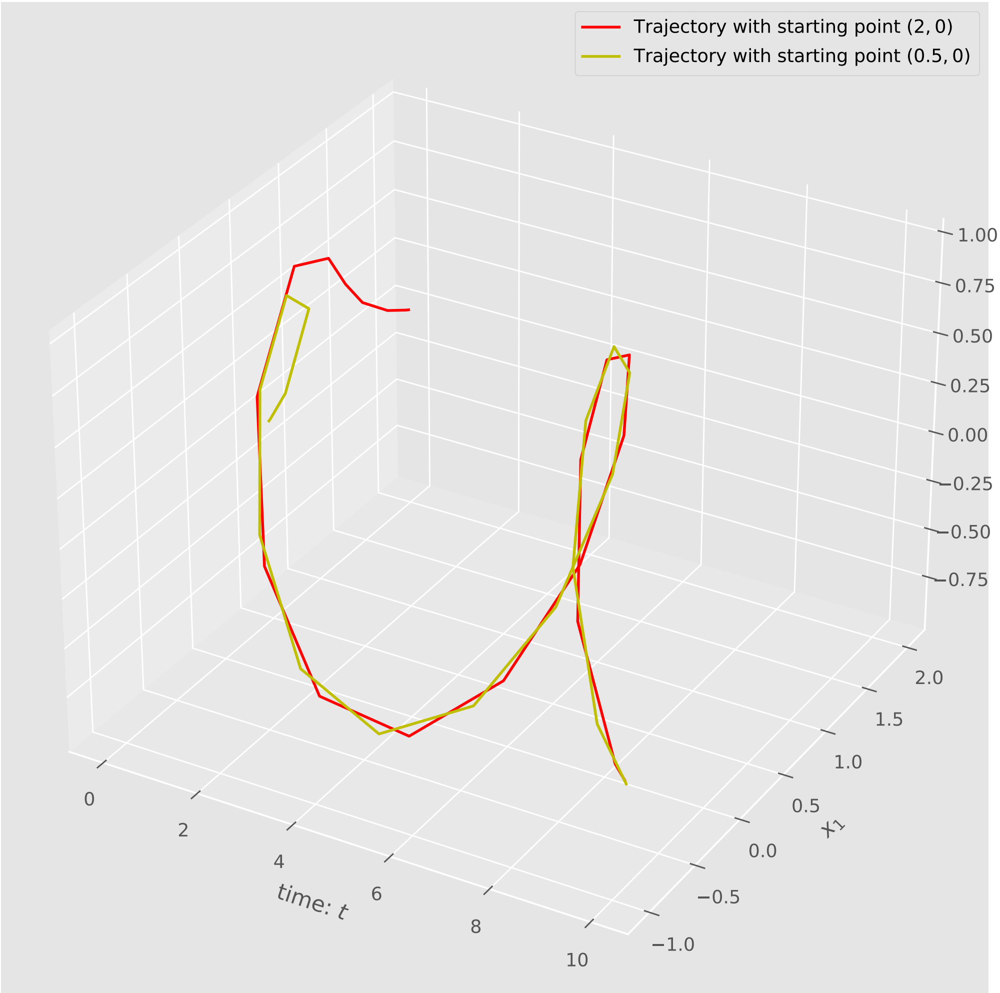

Figure 4: Visualization of the orbits of the Andronov-Hopf bifurcation for $\alpha = 1$, from time $t_0 = 0$ to $t_{final} = 10$ and with the initial positions $[2, 0]$ and $[0.5, 0]$ depicted as red and yellow lines respectively.

Now we consider the cusp bifurcation in one state space dimension $X = \mathbb{R}$, but with two parameters $\alpha \in \mathbb{R}^2$ with normal form given in equation 10.

$$\frac{\partial}{\partial t}x = \alpha_1 + \alpha_2 x - x^3 \tag{10}$$

Figure 5 depicts the solution space for $(\partial/\partial t)x \overset{!}{=} 0$ (bifurcation surface). We have created this plot using α values ranging from -5 to 5 with step size 0.5 (evenly spaced) as shown in 11. We have decided to sample the α values first and then solve the equation with respect to x since we got better results this way, in contrast to the recommended way in the exercise sheet.

$$\alpha_1, \alpha_2 \in \{-5, -4.5, \ldots, 4.5, 5\} \tag{11}$$

One can notice that the visualization looks like a hillside with a cusp on the top. So that is maybe why this bifurcation is called the cusp bifurcation.

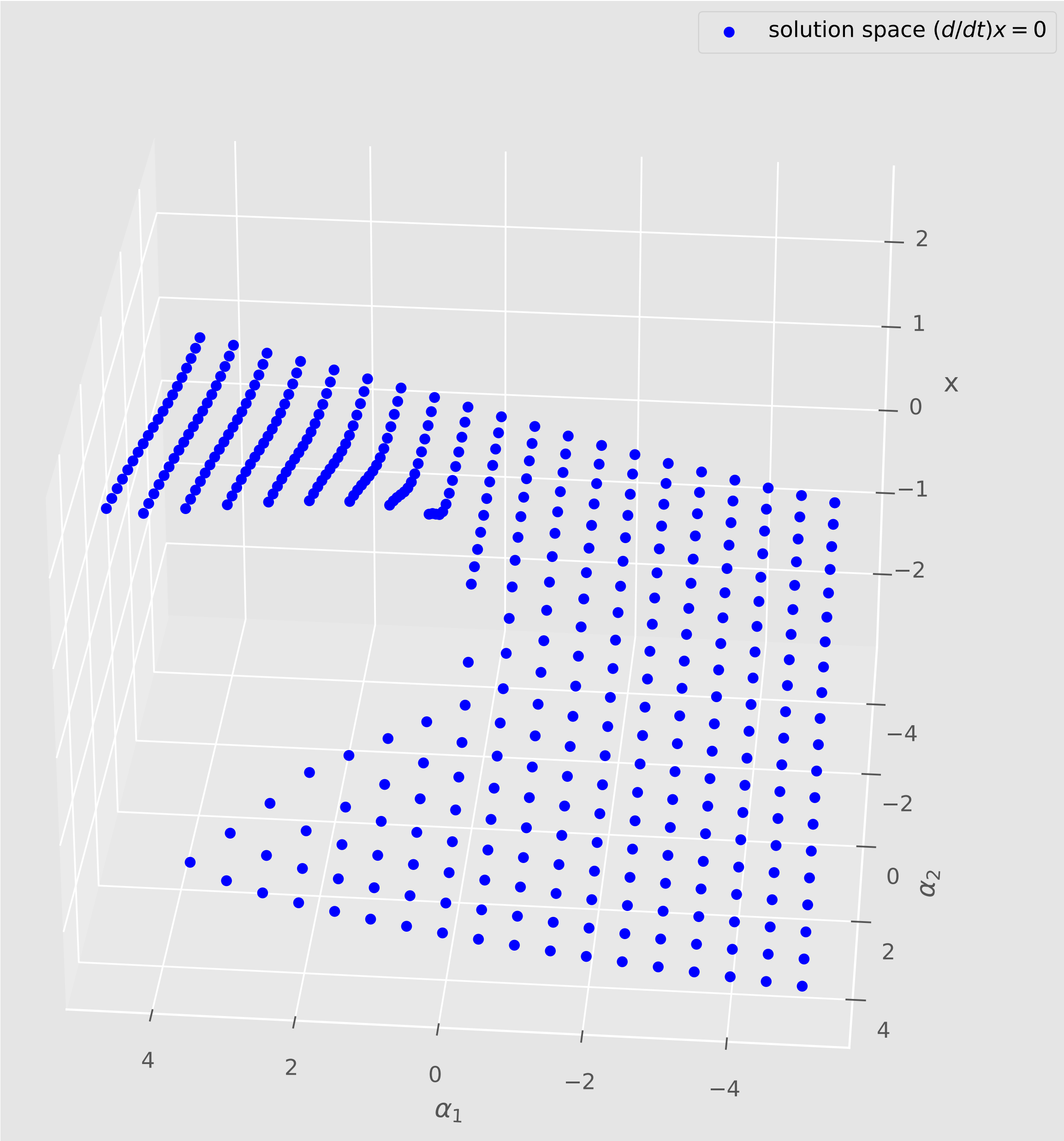

Figure 5: Visualization of the bifurcation surface (bifurcation diagram in 3d), i.e. the solution space with different α values of $\frac{\partial}{\partial t}x \overset{!}{=} 0$ of the cusp bifurcation 10 plot together with the corresponding α values.

Report on task Task 4/5: Chaotic dynamics

In this task we consider the following discrete map in equation 12 of the logistic map.

$$x_{n+1} = r(x_n)(1 - x_n)\,,\ n \in \mathbb{N}\,,\ r \in (0, 4] \tag{12}$$

We need to solve this system and find the steady states numerically, that is, we want to find the states x_0 that satisfies the equation 13.

$$.\lim_{n\to\infty} x_n = x_0 \tag{13}$$

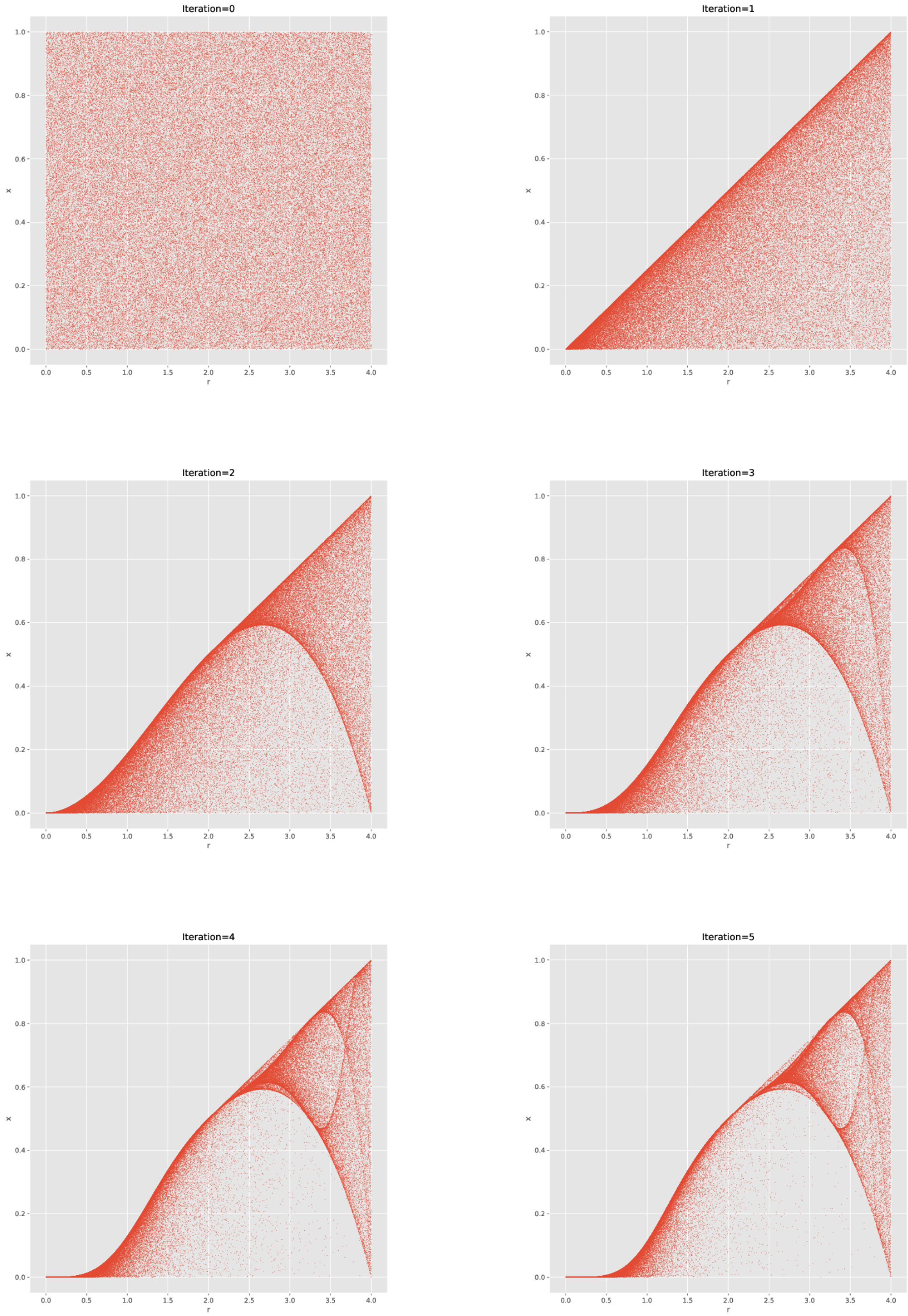

Figure 6: Visualization of the convergence of the logistic map defined in equation 12 (numerically computed steady states for $r \in (0, 4]$ and $x \in (0, 1)$, so the points are randomly sampled).

It is also interesting that the analytical solution of the ODE exists, but it does not present the signature of chaos as seen in equation 18. The system is however stable for $1 < r < 3$ [5]. Therefore for this range of r values we have the analytical solution as the steady state: $x_0 \in \{0, (r-1)/r\}$. If we visualize this we would see the Pitchfork bifurcation as seen in the final visualization of the bifurcation in figure 7.

$$x_n \overset{!}{=} r(x_n)(1 - x_n) \tag{14}$$
$$r(x_n)(1 - x_n) - x_n \overset{!}{=} 0 \tag{15}$$
$$rx_n - rx_n^2 - x_n \overset{!}{=} 0 \tag{16}$$
$$x_n(r - rx_n - 1) \overset{!}{=} 0 \tag{17}$$
$$x_n \in \{0, (r-1)/r\} \tag{18}$$

For $r \in (2, 4)$ the system shows a chaotic behavior as seen in the figure 7, and starts to oscillate for $r > 3$ which means the system becomes unstable (does not always reaches a fixed set of steady states).

Finally for this part of task 4, we need to plot the bifurcation diagram for the logistic map. For this we need to satisfy the numerical condition in equation 13 for the steady states. In order to get a proper map, we have sampled 4000 evenly spaced numbers using `np.linspace` for $r \in (0, 4]$, and we have sampled 25 samples of $x \in (0, 1)$ per each r, chosen uniformly at random with `np.random`. After the sampling step we have applied the equation (12) many times repeatedly (100 times) so that the resulting state hopefully converges to a steady state. The convergence of the system beginning from the initial state is depicted in figure 6 where we took snapshots of the iterations from 0 to 5. One can see the numerically computed steady states, and the limit cycles in the chaotic part ($r > 3$) of the system.

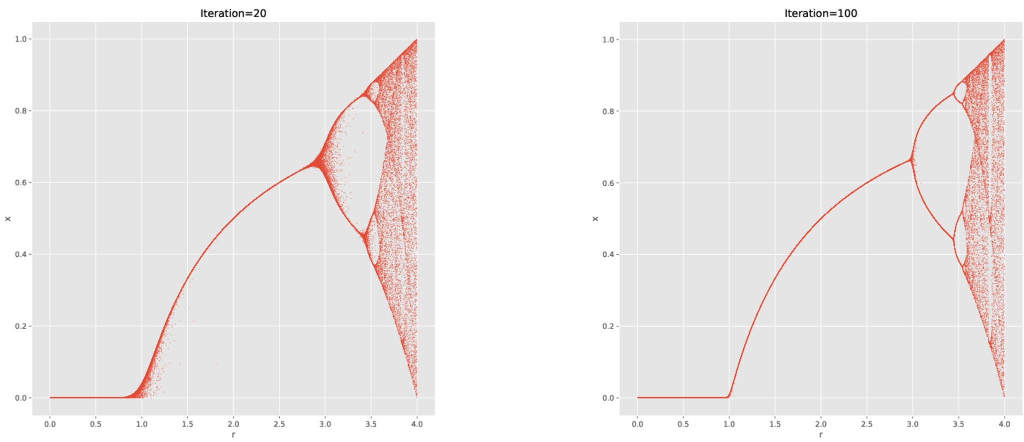

Figure 7: Final visualization of the logistic map bifurcation diagram given in the equation 12 bifurcation (numerically computed steady states for $r \in (0, 4]$ and $x \in (0, 1)$, so the points are randomly sampled) at iteration 20 and 100.

In the second part of this task we look into the Lorenz system defined by three ordinary differential equations defined as in equation 19.

$$\frac{\partial x}{\partial t} = \sigma(y - x)\,, \; \frac{\partial y}{\partial t} = x(\rho - z) - y\,, \; \frac{\partial z}{\partial t} = xy - \beta z \tag{19}$$

Like in the previous tasks, using `scipy` we can integrate a system of ordinary equations [3]. We set the time step to 0.01 seconds and visualized the trajectory of the initial position $x_0 = (10, 10, 10)$ until the end time $T_{end} = 1000$ is reached with the parameter values $\sigma = 10$, $\beta = 8/3$ and $\rho = 28$ – these specific values for the parameters are the well-known values for Lorenz system showing chaotic behaviors in numerical simulations [6]. The resulting visualization can be seen in figure 8. The trajectory of the initial position x_0 forms a butterfly shape as one can see on the top orange plot in the figure.

To test the chaotic nature of the system we have also plotted the trajectory of the perturb initial state $\hat{x}_0 = (10 + 10^{-8}, 10, 10)$ on the bottom in blue color in the same figure 8, and also calculated the squared

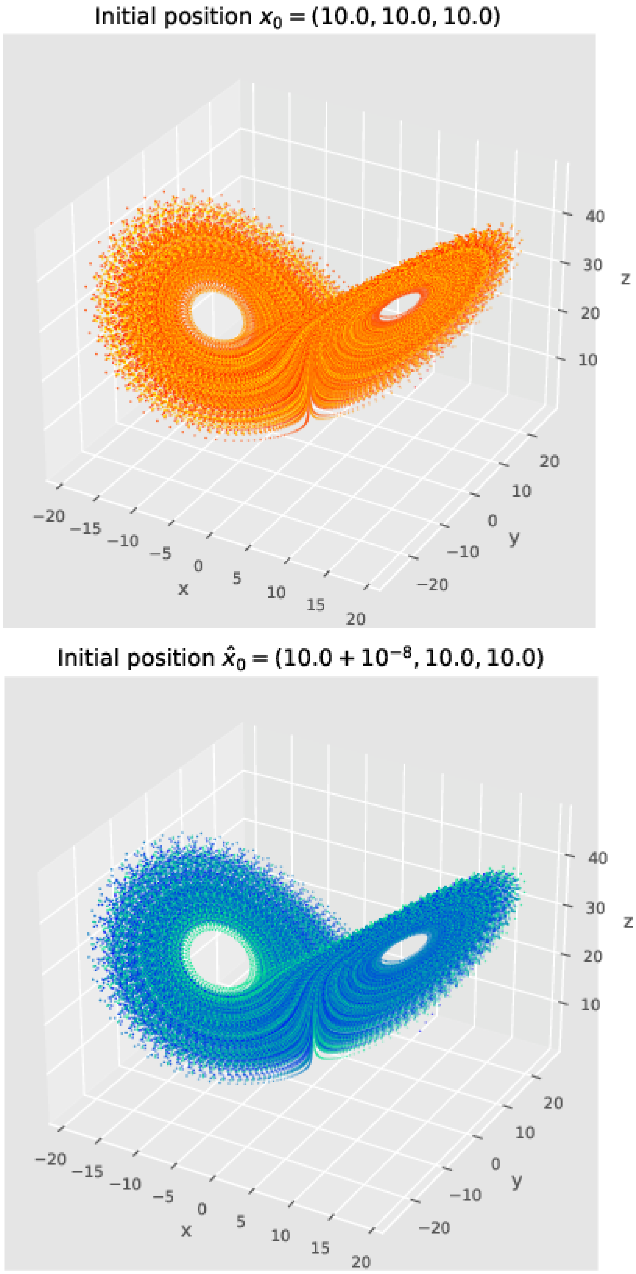

Figure 8: Trajectories of the initial states x_0 and $\hat{x}_0$ on the Lorenz System until time $T_{end} = 1000$ seconds and with parameters $\sigma = 10$, $\beta = 8/3$ and $\rho = 28$.

L_2-distance between the positions of the both trajectories at each time step. The figure 9 depicts this difference $||x(t), \hat{x}(t)||^2$ against the time t. The trajectory difference grows rapidly at the beginning in the log scale and starts to fluctuate around the value 10^2. The largest difference in the x-, y- and z- directions in the trajectories of the initial state x_0 are 37.9, 51.9 and 44.4 respectively. The distance between the points on the trajectories are larger than 1 after 22.55 seconds.

Now we set the parameter ρ to the value 0.5 and plot the same difference between the trajectory points. We get the result in figure 10. In this case the Lorenz system does not show a chaotic behavior since the difference does not become noticeably big – the trajectories differ at most 10^{-16} and it even decreases as the time increases. In figure 11 we see the plot of the trajectories using the same parameters and same setup as before (but this time with $\rho = 0.5$). Trajectories follow a very similar path. Since ρ is smaller than 1, the system has only one attracting fixed steady state which is the origin $(0, 0, 0)$ – so no bifurcation can be observed [7]. Only when

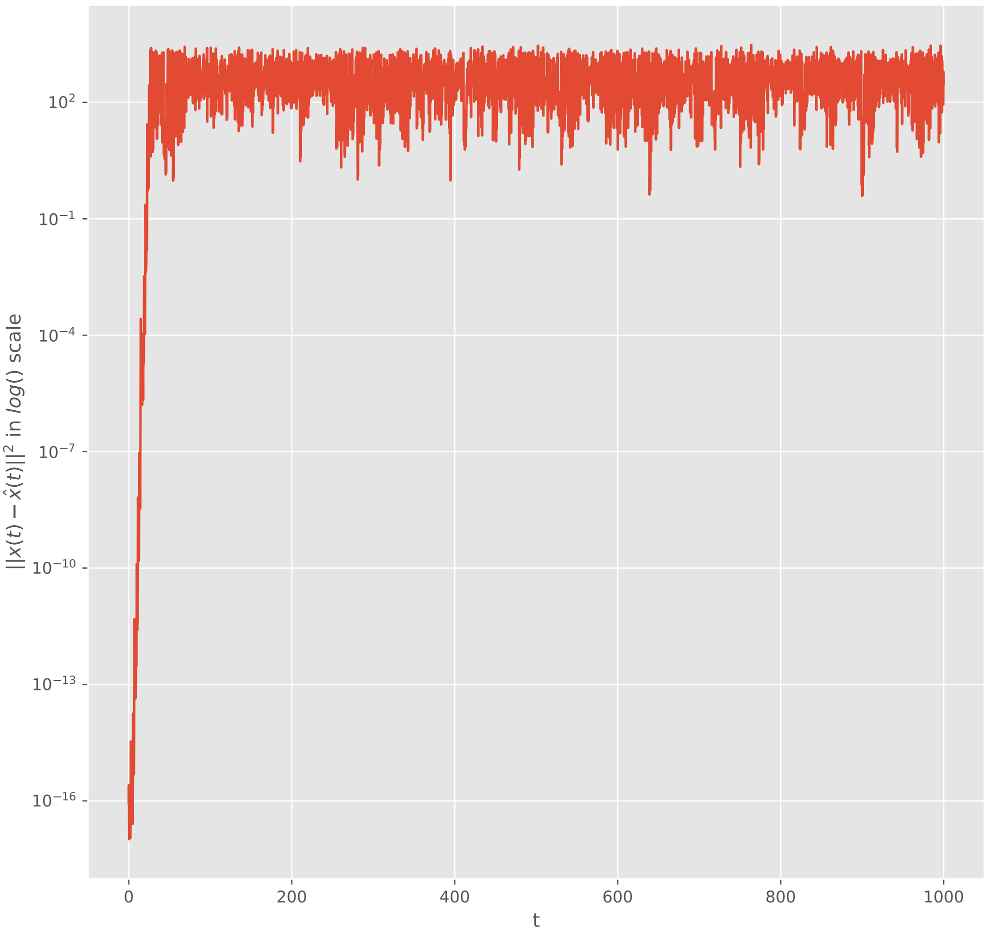

Figure 9: Difference $(||x(t), \hat{x}(t)||^2)$ of the trajectories of the initial states x_0 and $\hat{x}_0$ on the Lorenz System plotted against time in log scale.

$\rho > 1$ the fixed points are $(0,0,0)$, $(\sqrt{\beta z}, \sqrt{\beta z}, \rho - 1)$ and $(-\sqrt{\beta z}, -\sqrt{\beta z}, r - 1)$. Additionally the origin is a fixed point of saddle type, with dimension-2 stable manifold and dimension-1 unstable manifold when $\rho > 1$.

Report on task Task 5/5: Bifurcations in crowd dynamics

In this task we applied our knowledge about bifurcations and dynamic systems to observe and describe a SIR model. We have downloaded the example Jupyter notebook that was provided to us in Moodle, when we tried running it, we realised that the derivatives of S, I and are were missing in the according **.py** file. The derivatives were assigned with the formulas given in the exercise sheet. We have also optimized and added imports that are needed for the computations in the model, namely **numpy, scipy** and **matplotlib**. Additionally we have tried to improve its documentation.

When we look at the trajectories of different b parameters with different initial starting values, we observe that increasing the b parameter creates a steady state at location [195, 0.05, 5], that later turns into Andronov-Hopf-Bifurcation.

If we look at the the corresponding plots for b between the values 0.02 and 0.03 we notice that the trajectory of the 3d-plot looks very similar to the `Andronov-Hopf` bifurcation that we have also seen in the previous tasks.

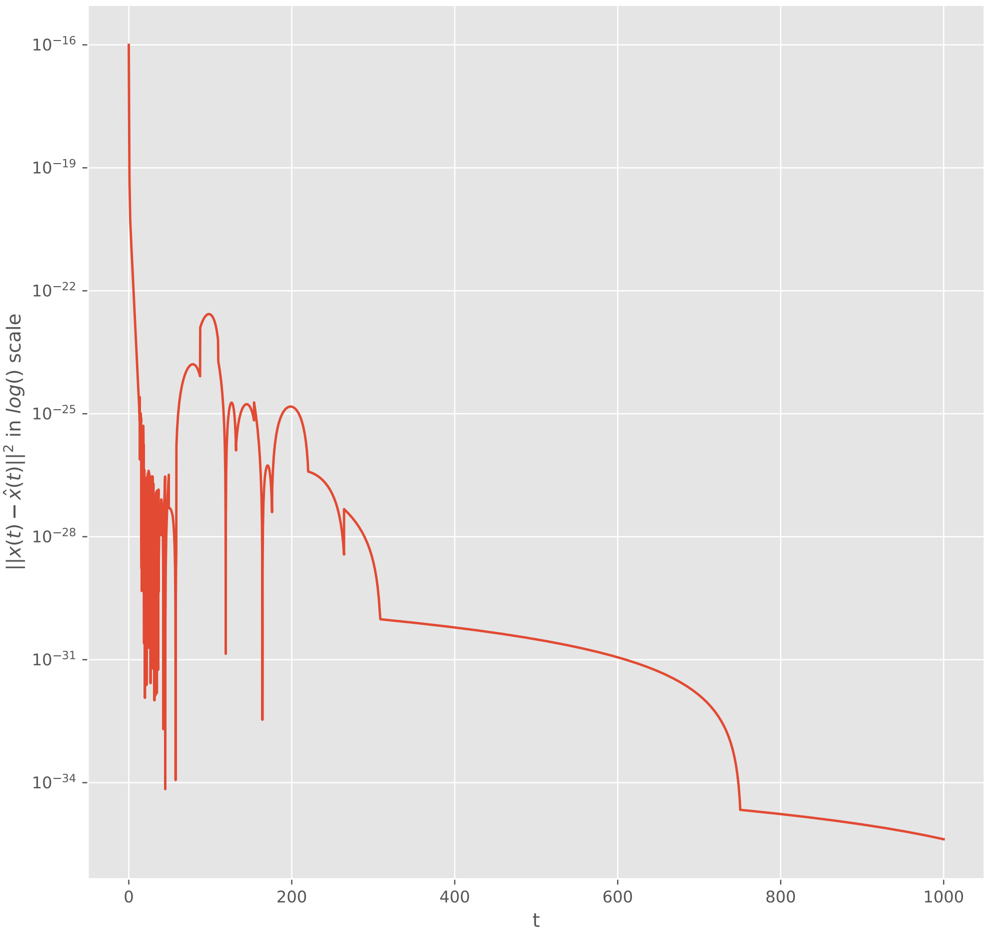

Figure 10: The same difference ($||x(t), \hat{x}(t)||^2$) of the trajectories of the initial states x_0 and $\hat{x}_0$ on the Lorenz System plotted against time in log scale, but ρ is set to 0.5.

For smaller parameter values (see the plot for $b = 0.02$ and $b = 0.021$) the trajectory gets attracted to a specific center value, and the attracted points form a very similar shape towards the center of the attraction as the `Andronov-Hopf` bifurcation. The plot where $b = 0.22$ creates a circle set (limit cycle), where the points get attracted to. Which was also observed in the `Andronov-Hopf` bifurcation. This bifurcation happens at the values $b = 0.021, 0.022 and 0.023$.

For computing reproduction rate we use the following parameters:

- δ = natural death rate
- ν = disease induced death rate
- μ = maximum recovery rate
- β = average number of adequate contacts per unit time with infectious individuals

The reproduction number is the expected number of cases directly generated by one case in a population where all individuals are susceptible to infection [1]. According to the given equation the reproduction is

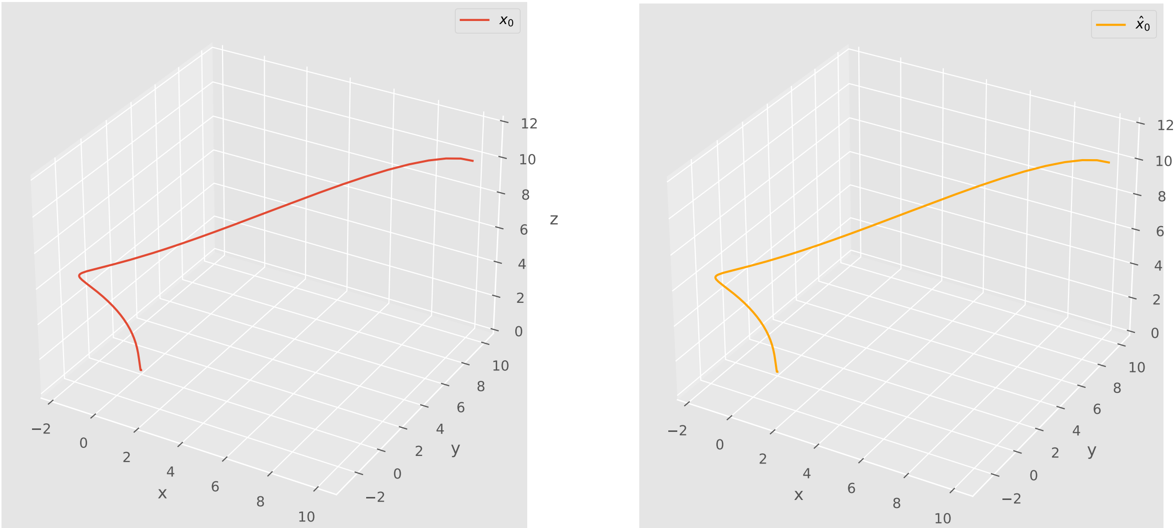

Figure 11: Trajectories of the initial states x_0 and $\hat{x}_0$ on the Lorenz System until time $T_{end} = 1000$ seconds and with parameters $\sigma = 10$, $\beta = 8/3$ and $\rho = 0.5$.

proportional to β and inversely proportional to sum of ν, μ and δ. As the reproduction rate means the newly generated causes because of one infectious person, higher reproduction rates should result in a higher number of infected people. Considering the given equation it also makes sense to have β in the nominator and the above sum in denominator, because in order to spread the disease one must be alive and not yet recovered and with the increasing number of contacts, the potential new cases should increase. In addition we know that the reproduction number is increasing with the higher values of β. Therefore one can conclude that the number of infectious people should increase with the increasing β values. We have tested three different values of β and we observed that the increasing average number of adequate contacts causes the disease to spread very rapidly at the beginning and reach higher numbers while the oscillations still occur. When we decrease the β the number of infected people decreases drastically and the disease ends in a relatively short time.Please look at 12 , 13 and 14.

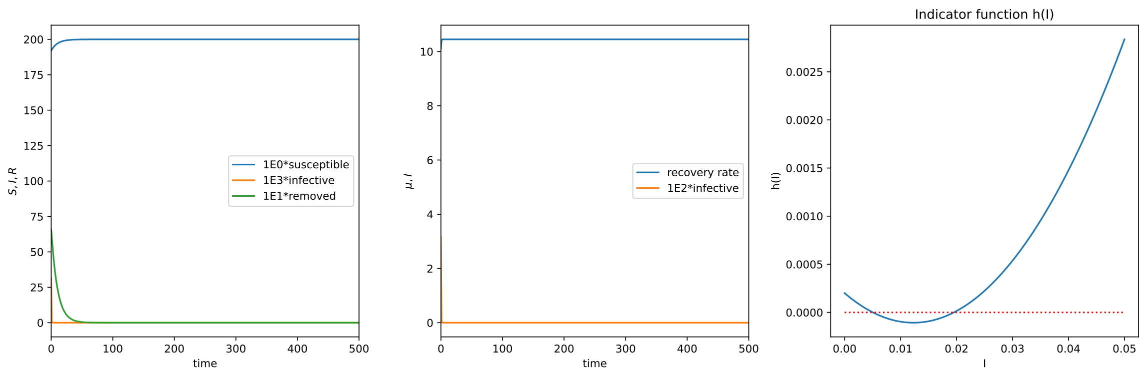

Figure 12: SIR Visualization with parameter $\beta = 8$

E_0 being attractive means that all the points at limit time will converge to this point, which is defined as $(\frac{A}{d}, 0, 0)$, in our parameter configuration with A = 20 and d = 0.1, this leads to the point [200.0, 0.0, 0.0], where all trajectories will converge at limit time. As a beginning point we have chosen [200.5, 0.5, 0.5] which is very close to the attractive point [200, 0, 0]. The number of susceptible people of 200.5 first decreases to 199.8 and reaches to the steady state 200. Although in the figure it is seen a like big change in the direction of less susceptible people, it is caused by the scale. Compared to previous 9 trajectories, the number of susceptible people drops only to 199.8 and then converges to steady state [200, 0, 0] without a increase of infected people. The trajectory can be seen in 15.

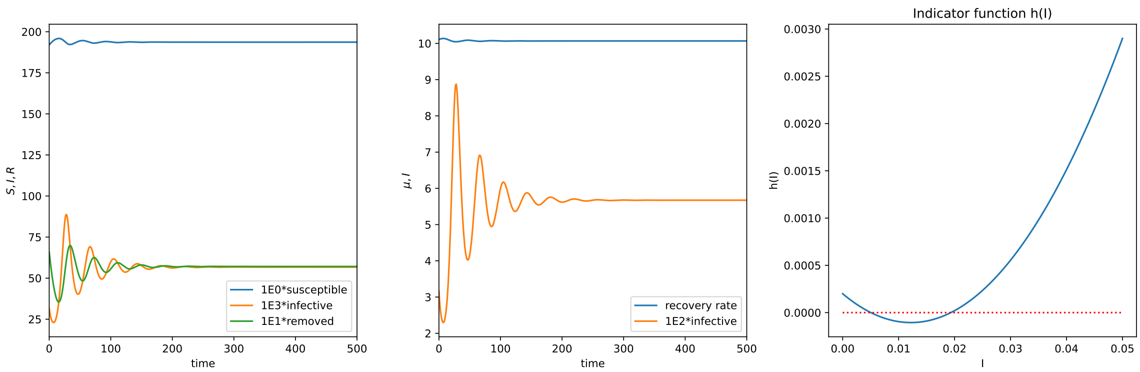

Figure 13: SIR Visualization with parameter $\beta = 11.5$

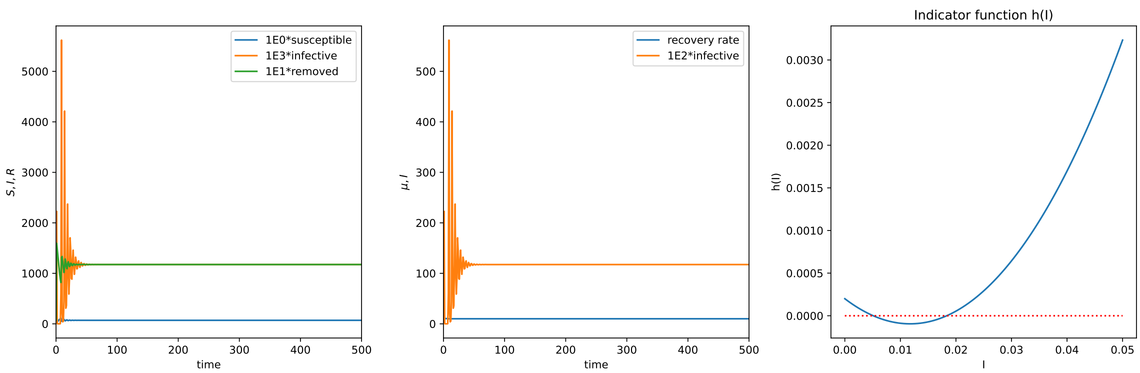

Figure 14: SIR Visualization with parameter $\beta = 30$

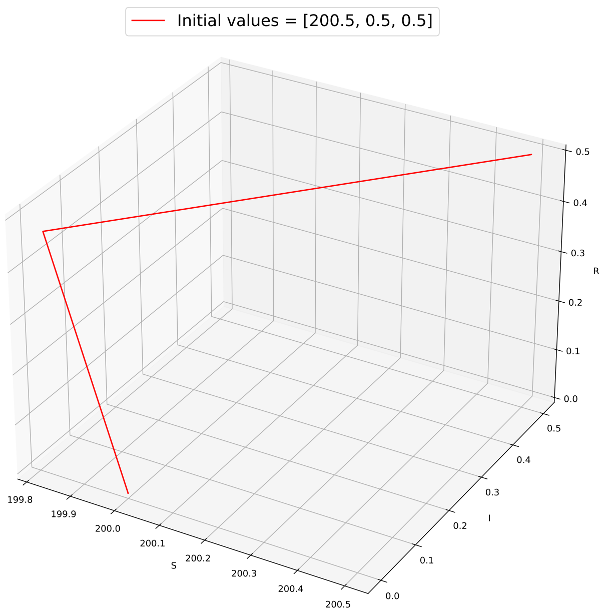

Figure 15: Initial point [200.5, 0.5, 0.5] which is close to the attractive node [200.0, 0, 0]

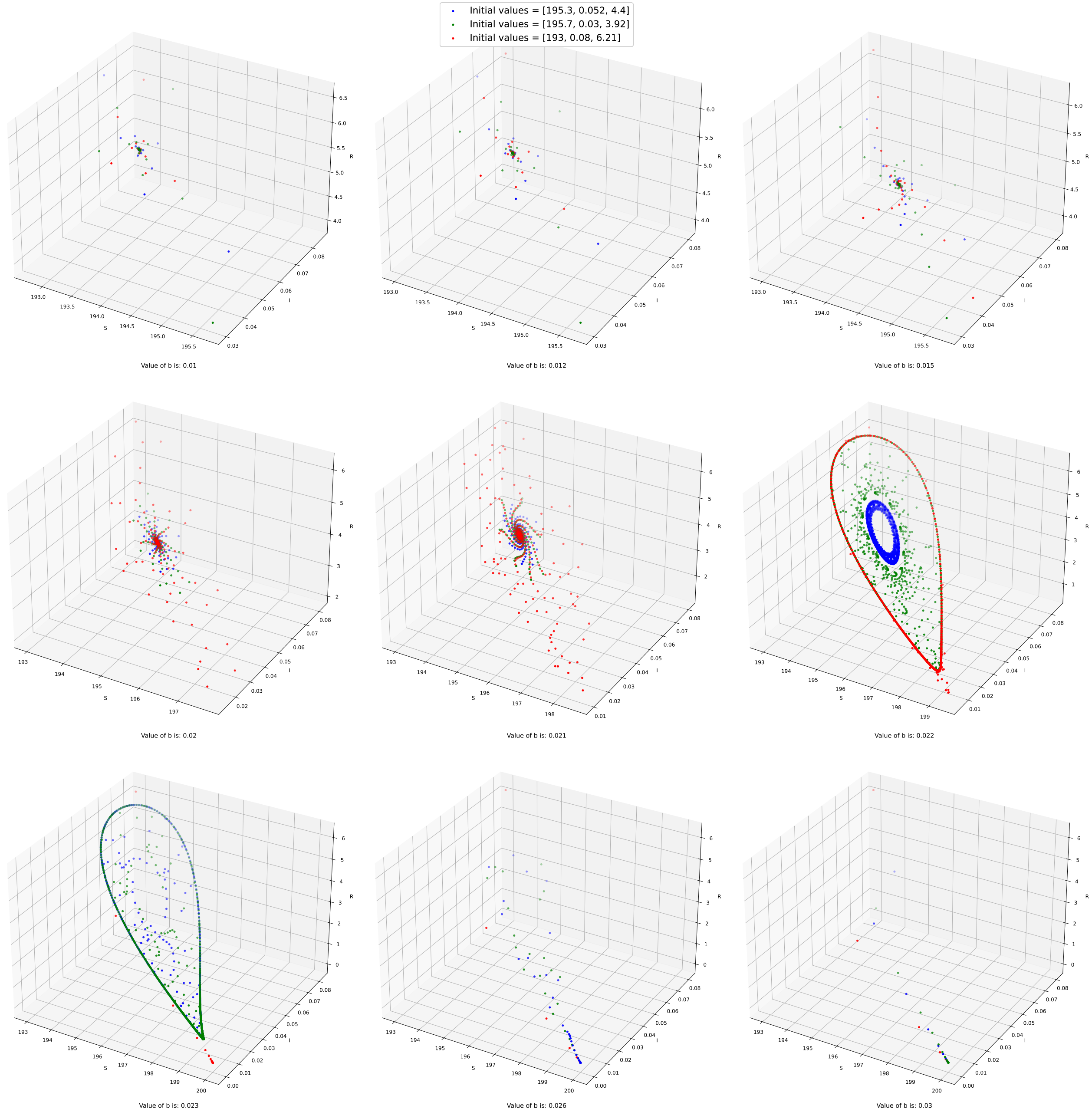

Figure 16: In this figure one can see three different trajectories that are found by solving our model with different b-parameters. Each different b parameter that is used to obtain those specific trajectories is given in the title of each different sub-figure. The legend also states the colors that are chosen for different trajectories for the complete figure.

References

[1] *Basic reproduction number*. May 2022. URL: https://en.wikipedia.org/wiki/Basic_reproduction_number#cite_note-Fraser-2.

[2] Aditya M. Deshpande. *Stability of Fixed Points of High Dimensional Dynamical Systems*. 2021.

[3] https://docs.scipy.org/doc/scipy/. *SciPy*. May 2022.

[4] Y. Kuznetsov. *Elements of Applied Bifurcation Theory*. Applied Mathematical Sciences. Springer New York, 2010.

[5] Marcos Lima et al. "The Numerical Connection between Map and its Differential Equation: Logistic and Other Systems". In: *International Journal of Nonlinear Sciences and Numerical Simulation* 14 (Feb. 2013), pp. 77–85.

[6] Shinji Nakaoka. "Chapter 3 - Data-Driven Mathematical Modeling of Microbial Community Dynamics". In: *Integrated Population Biology and Modeling, Part A*. Ed. by Arni S.R. Srinivasa Rao and C.R. Rao. Vol. 39. Handbook of Statistics. Elsevier, 2018, pp. 93–130.

[7] Jonathan D'Souza Ria Dhull and Eriz Zhang. *The Lorenz Attractor and Bifurcations*. Nov. 2020.