

Probabilistic Graphical Models

Jiří Materna



About me

- Ph.D. in Natural Language Processing and Artificial Intelligence at Masaryk University
- 10 years at Seznam.cz (last 8 years as Head Of Research)
- Founder and lecturer at ML College
- Founder and co-organizer of ML Prague
- ML Freelance and consultant

www.mlguru.com

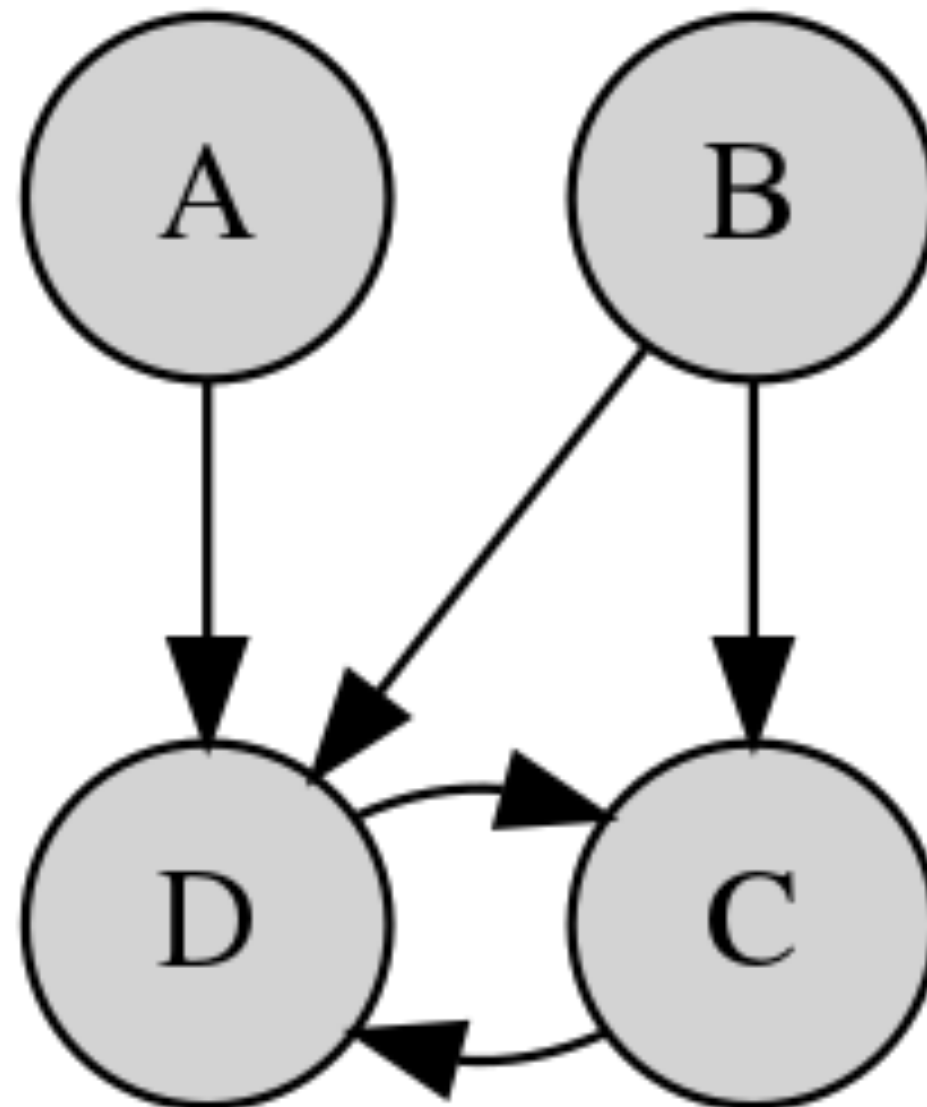
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Outline

- Topic modeling
- Basics of the Probability theory
- Probabilistic Graphical Models
- Inference in Bayesian Networks
- Gaussian Linear Regression
- Gaussian Mixtures
- Gaussian Mixtures for clustering
- (Probabilistic) Latent Semantic Analysis
- Latent Dirichlet Allocation

Probabilistic Graphical Models



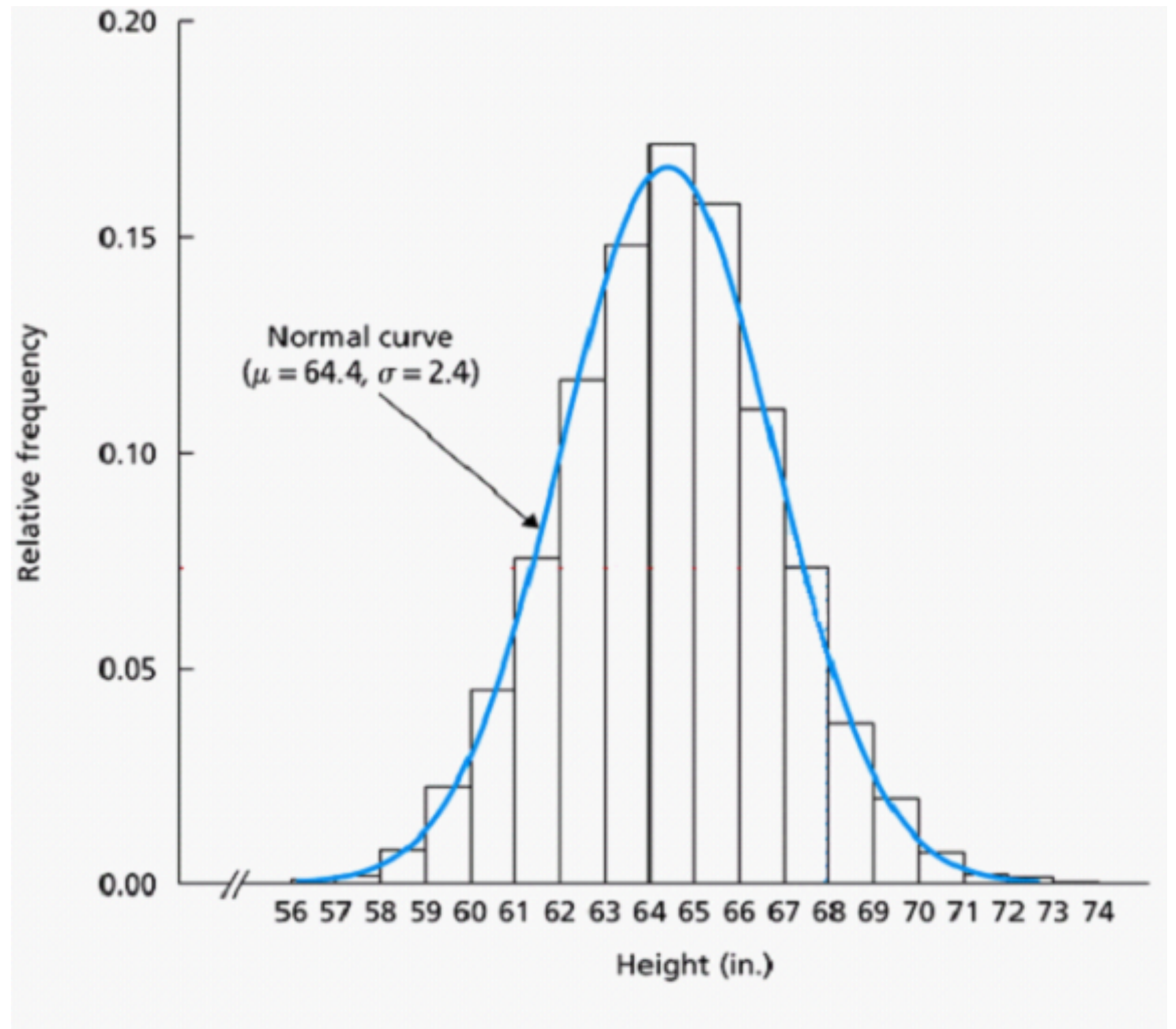
Conditional probability and independence

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B|A)}{P(B)}$$

$$A \perp B \iff P(A \cap B) = P(A)P(B)$$

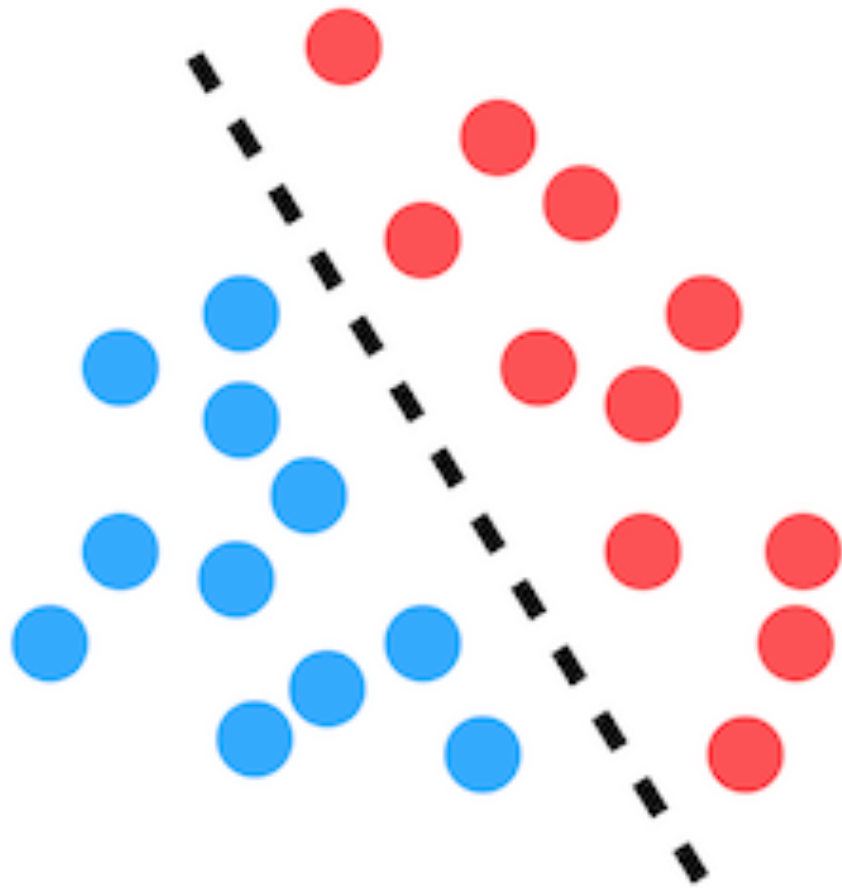
Probability distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

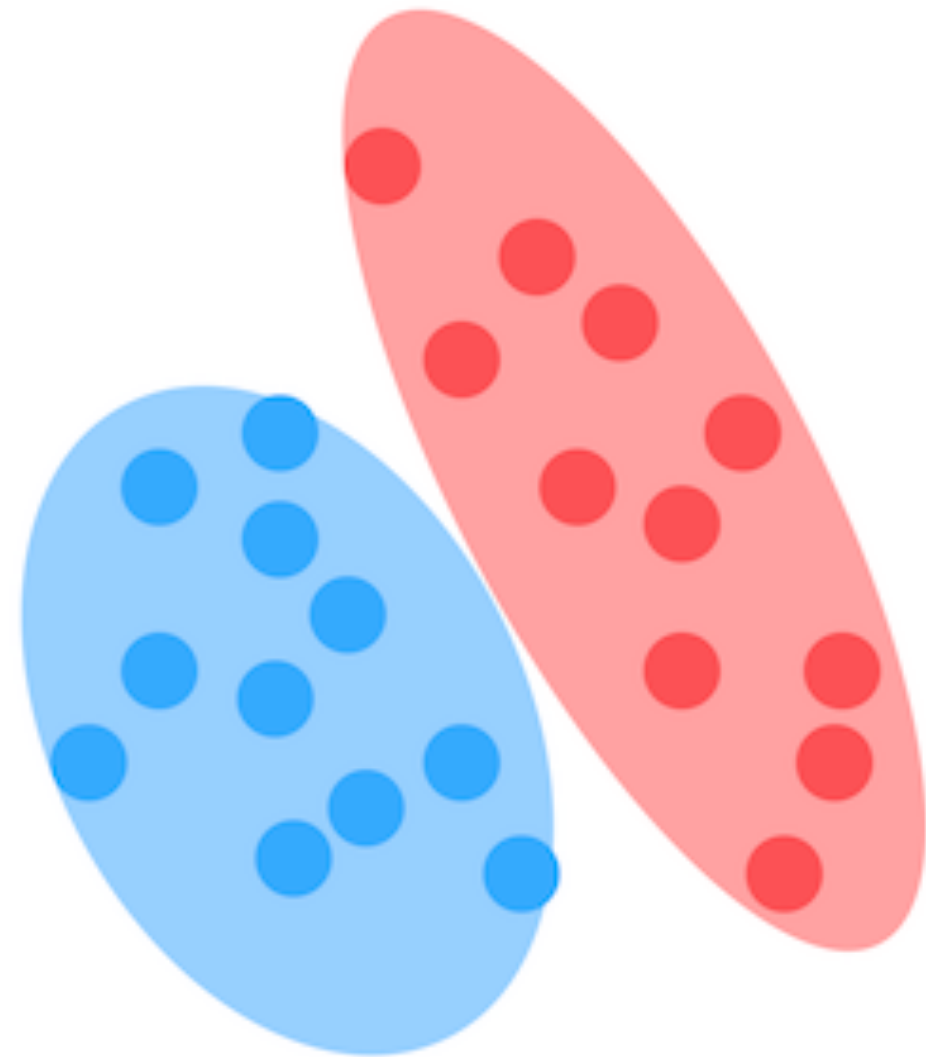


Discriminative vs. generative models

Discriminative



Generative



Topic Modeling

Topics

gene	0.04
dna	0.02
genetic	0.01
...	

life	0.02
evolve	0.01
organism	0.01
...	

brain	0.04
neuron	0.02
nerve	0.01
...	

data	0.02
number	0.02
computer	0.01
...	

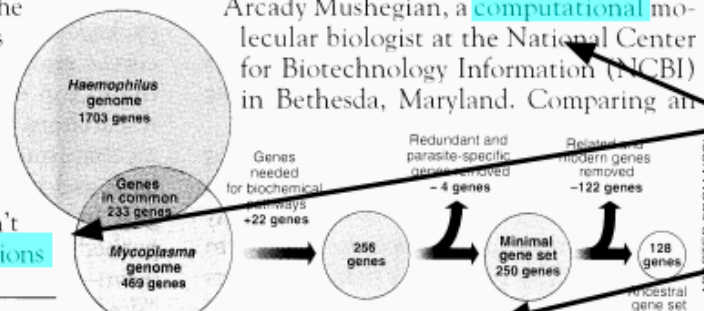
Documents

Seeking Life's Bare (Genetic) Necessities

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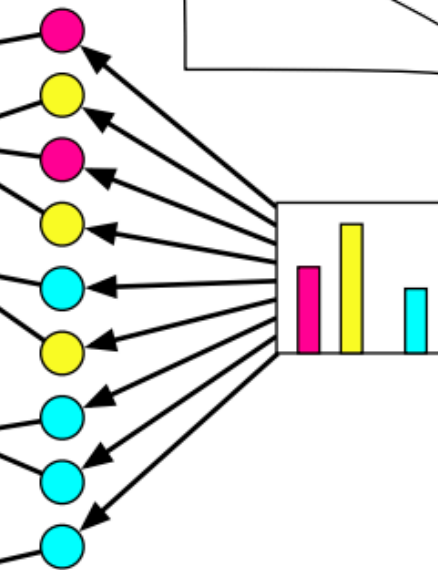


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Stripping down. **Computer analysis** yields an estimate of the minimum modern and ancient genomes.

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Topic proportions & assignments



Generative model of people's heights

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

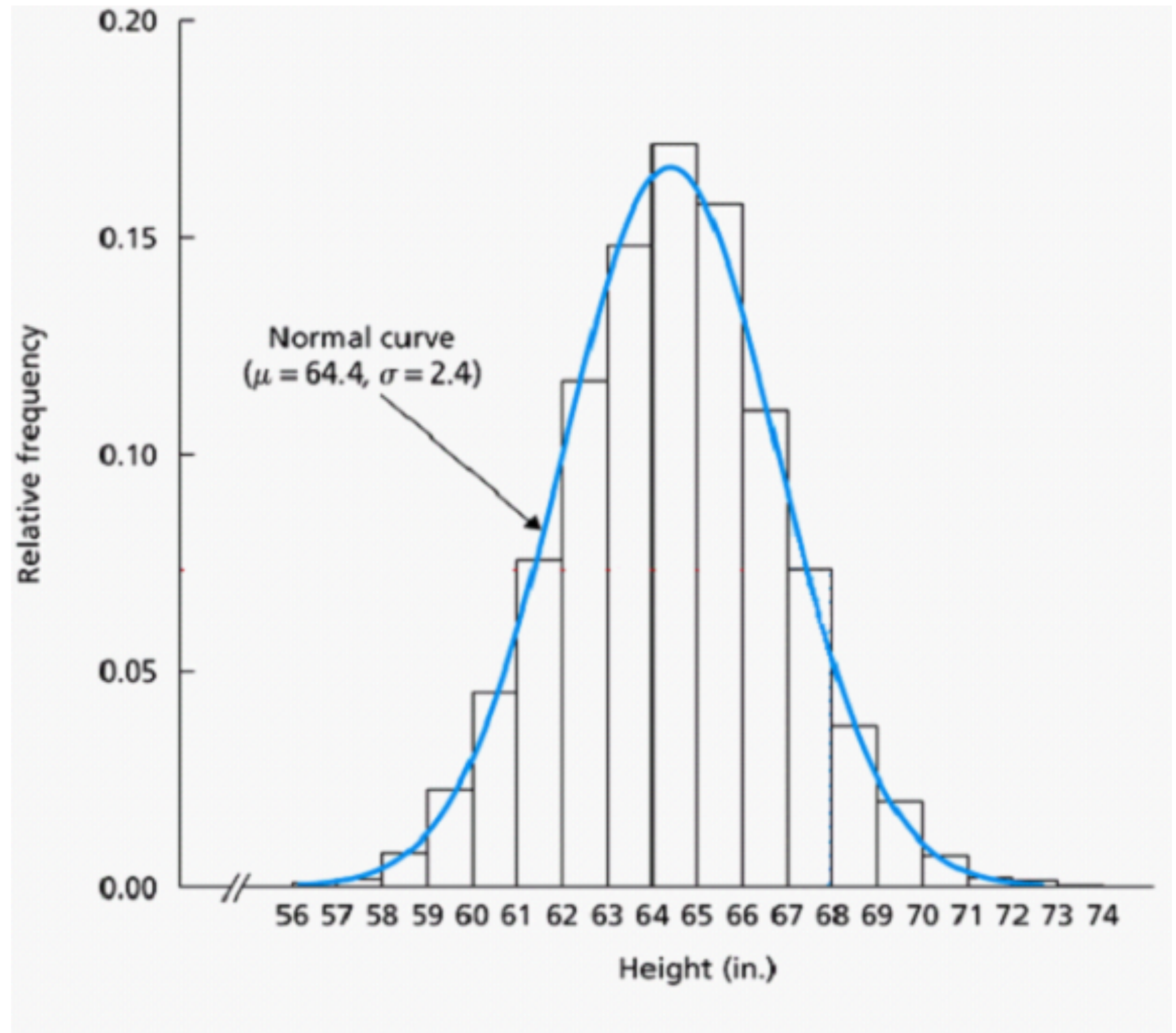
$$X = \{x_1, x_2 \dots x_n\}$$

$$X \sim N(\mu, \sigma^2), \alpha = (\mu, \sigma^2)$$

$$\bar{\alpha} = \arg \max_{\alpha} P(\alpha|X)$$

$$P(\alpha|X) = \frac{P(X|\alpha) \cdot P(\alpha)}{P(X)}$$

posterior
likelihood
prior

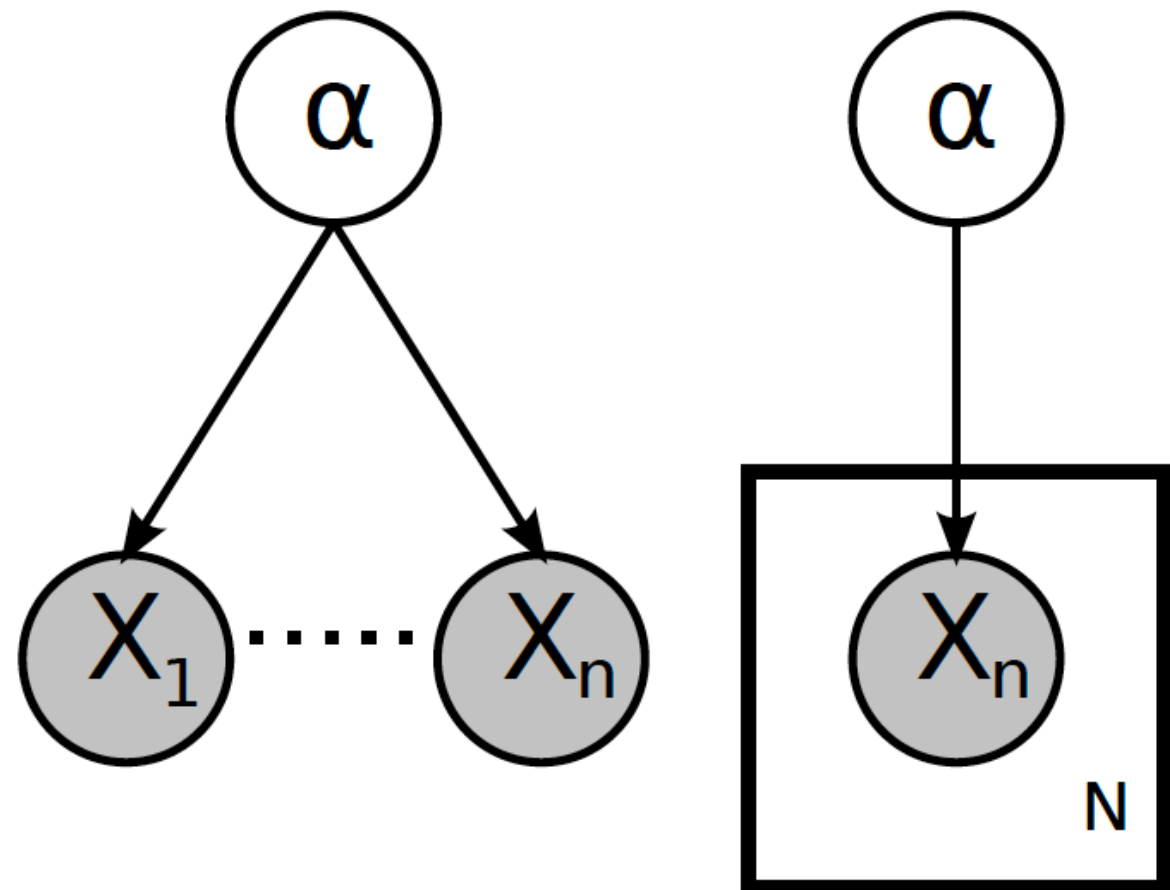


Probabilistic graphical models

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$X = \{x_1, x_2 \dots x_n\}$$

$$X \sim N(\mu, \sigma^2), \alpha = (\mu, \sigma^2)$$



Inference in graphical models

$$P(\alpha|X) = \frac{P(X|\alpha).P(\alpha)}{P(X)} \propto P(X|\alpha).P(\alpha) = \prod_{i=1}^n P(x_i|\alpha).P(\alpha)$$

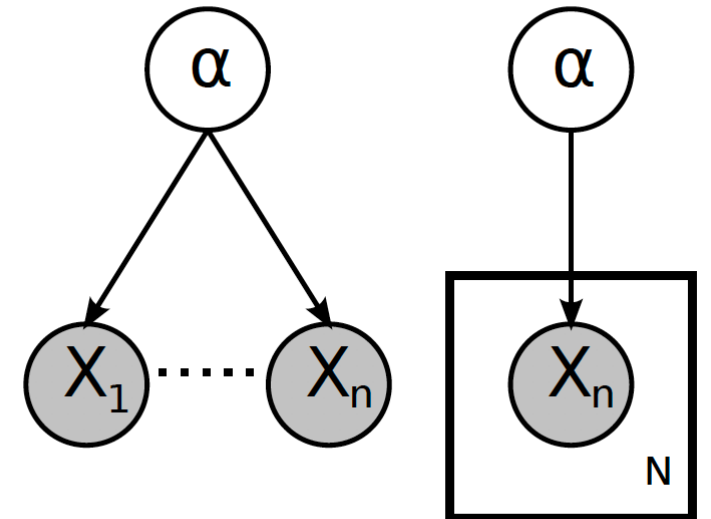
$$\bar{\alpha} = \arg \max_{\alpha} P(\alpha|X)$$

Variational inference

1. Approximate the posterior function with a simpler one
2. Compute the hidden variables by minimization of KL Divergence of the true and simpler distributions

Sampling (e.g. Gibbs sampling)

1. Draw samples from the true posterior
2. Compute mean of the samples



Gibbs sampling

- 1 Initialize $z_i : i \in 1, \dots, M$
- 2 For $\tau \in 1, \dots, T$:
 - Sample $z_1^{(\tau+1)} \sim P(z_1 | z_2^{(\tau)}, z_3^{(\tau)}, \dots, z_M^{(\tau)})$
 - Sample $z_2^{(\tau+1)} \sim P(z_2 | z_1^{(\tau+1)}, z_3^{(\tau)}, \dots, z_M^{(\tau)})$
 - Sample $z_3^{(\tau+1)} \sim P(z_3 | z_1^{(\tau+1)}, z_2^{(\tau+1)}, \dots, z_M^{(\tau)})$
 - ...
 - Sample $z_M^{(\tau+1)} \sim P(z_M | z_1^{(\tau+1)}, z_2^{(\tau+1)}, \dots, z_{M-1}^{(\tau+1)})$

Generative model for linear regression

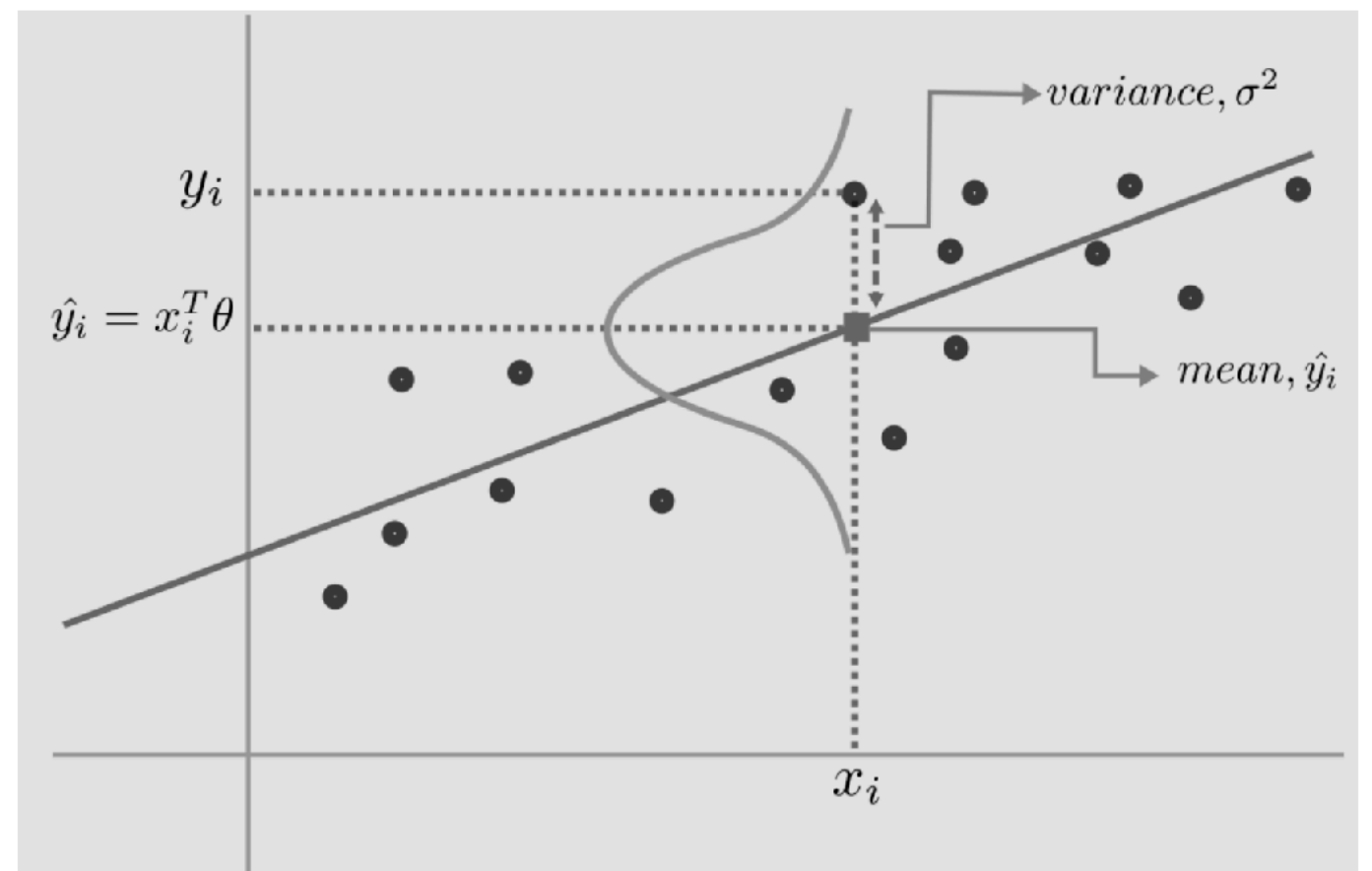
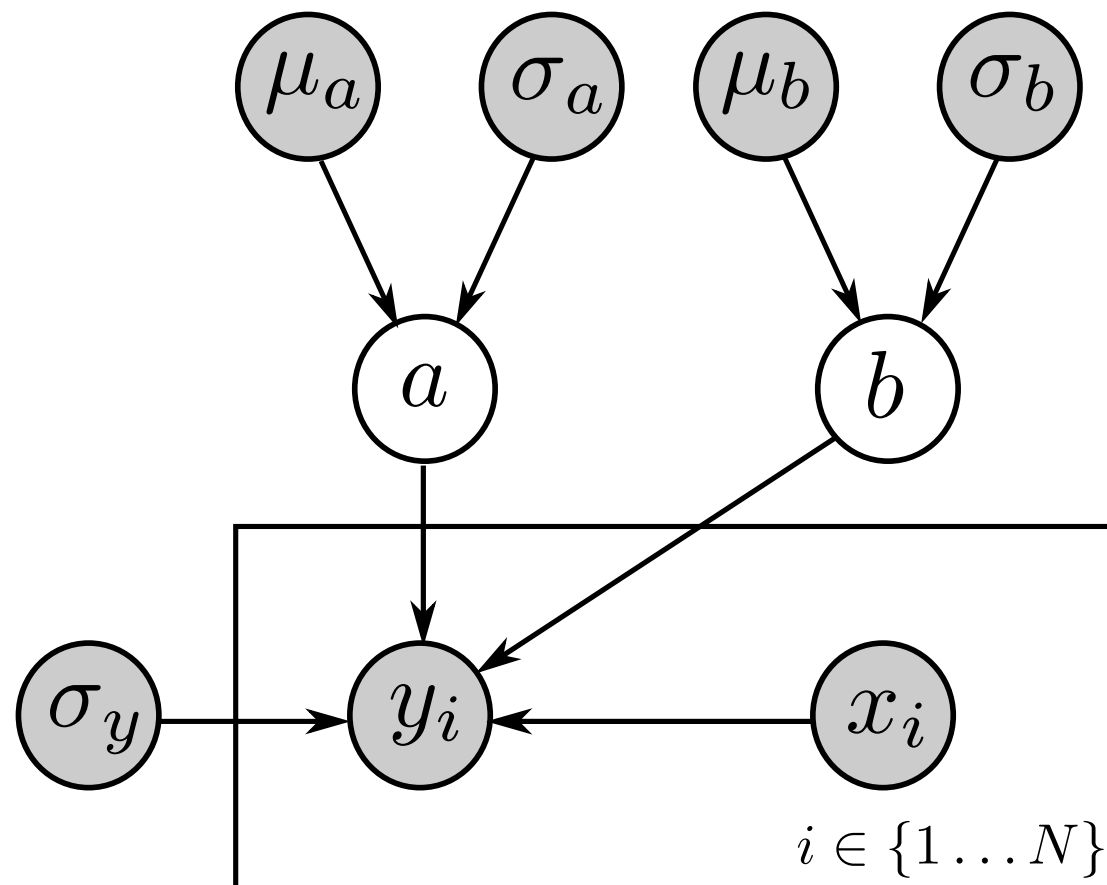
$$\mathbf{x} = \{x_1, x_2, \dots, x_N\}$$

$$f(\mathbf{x}) = a\mathbf{x} + b$$

$$\mathbf{y} \sim \mathcal{N}(a\mathbf{x} + b, \sigma_y)$$

$$a \sim \mathcal{N}(\mu_a, \sigma_a)$$

$$b \sim \mathcal{N}(\mu_b, \sigma_b)$$



Generative model for linear regression in Pyro

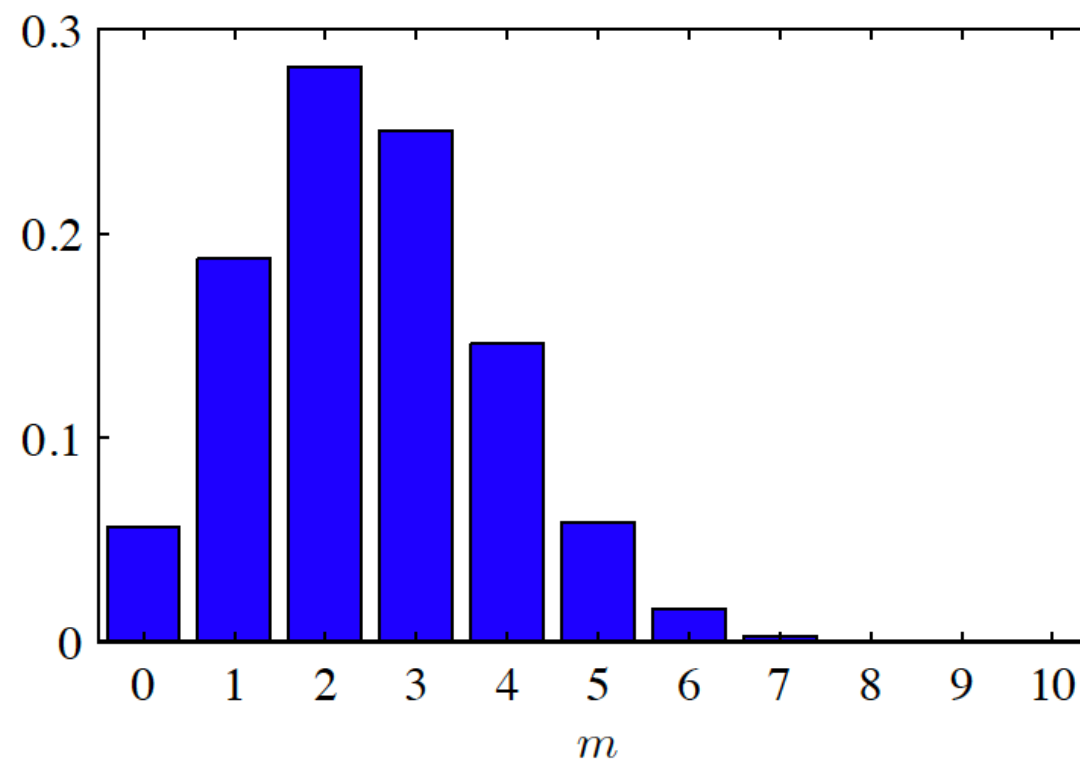
01-Generative-linear-regression-pyro.ipynb

Binomial distribution

$$\text{Bin}(k|n, p) = \binom{n}{k} p^k \cdot (1 - p)^{n-k} =$$

$$\text{Bin}(x_1, x_2|p_1, p_2) = \frac{(x_1 + x_2)!}{x_1! x_2!} p_1^{x_1} \cdot p_2^{x_2}$$

$$p_1 + p_2 = 1$$



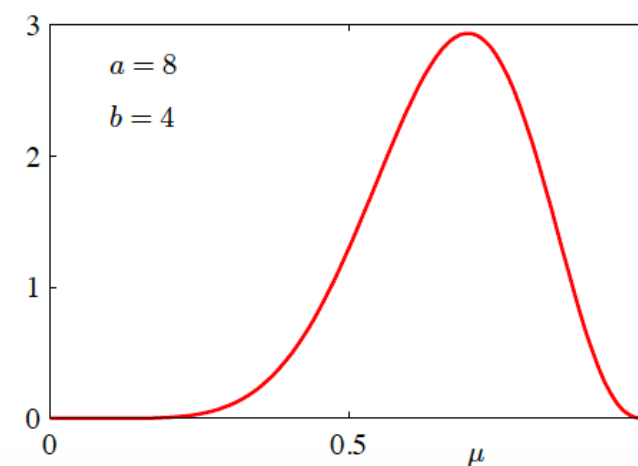
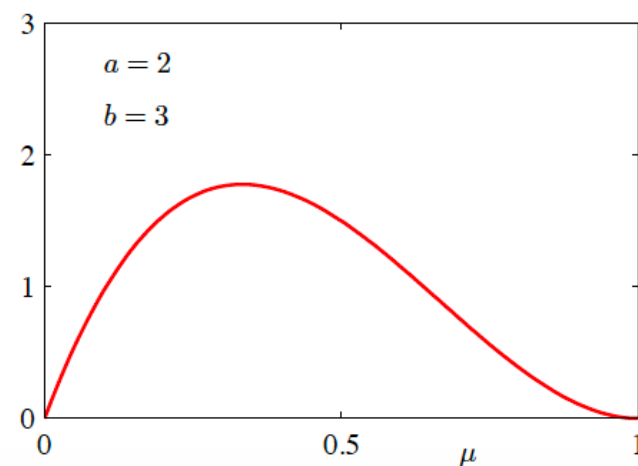
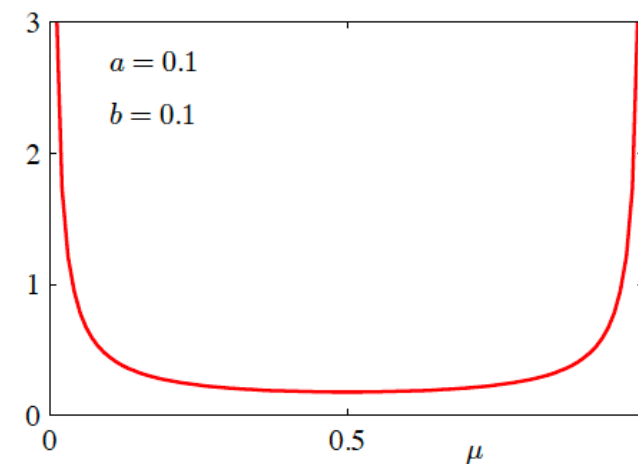
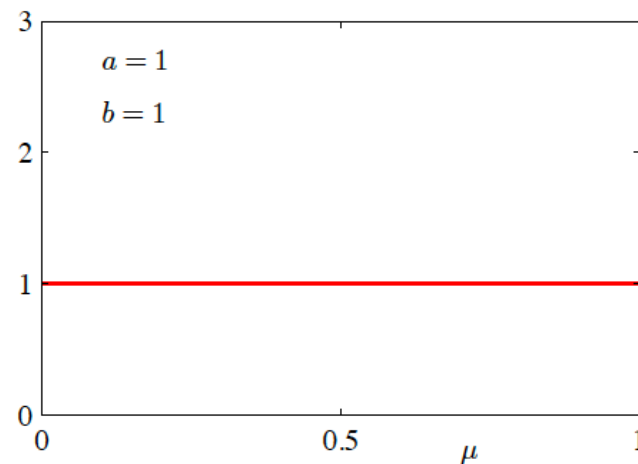
Example: $n = 10, p = 0.25$

Beta distribution

$$\text{Beta}(p_1, p_2 | \alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} p_1^{\alpha-1} \cdot p_2^{\beta-1}$$

$$p_1 + p_2 = 1$$

$$\Gamma(x) = (x-1)!$$



Multinomial and Dirichlet distributions

Multinomial

$$\text{Mult}(x_1 \dots x_n | p_1 \dots p_n) = \frac{(\sum x_i)!}{\prod x_i!} \prod_{i=1}^n p_i^{x_i}$$

Dirichlet

$$\text{Dir}(p_1 \dots p_n | \alpha_1 \dots \alpha_n) = \frac{\Gamma(\sum \alpha_i)}{\prod \Gamma(\alpha_i)} \prod_{i=1}^n p_i^{\alpha_i - 1}$$

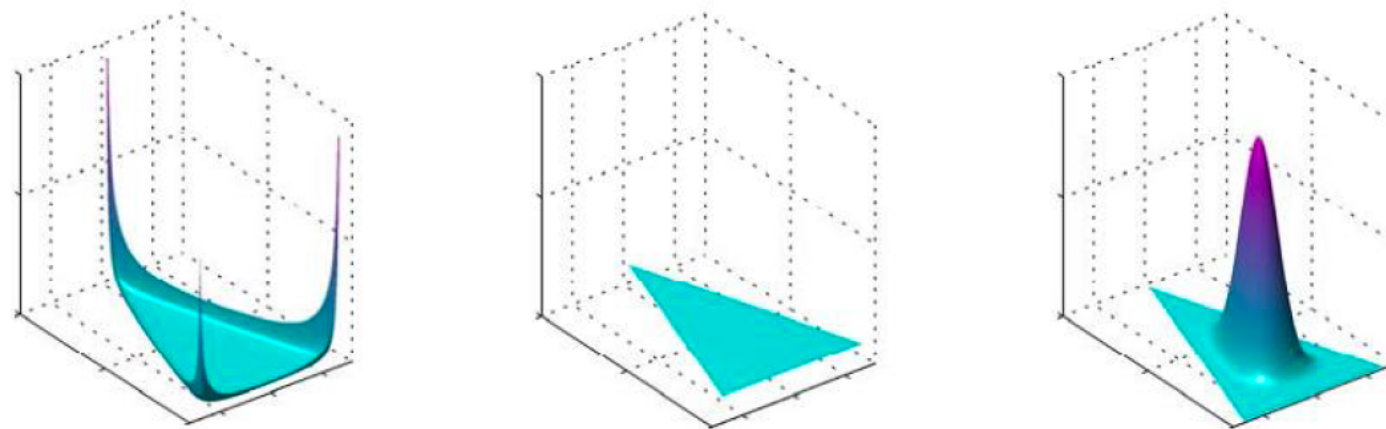
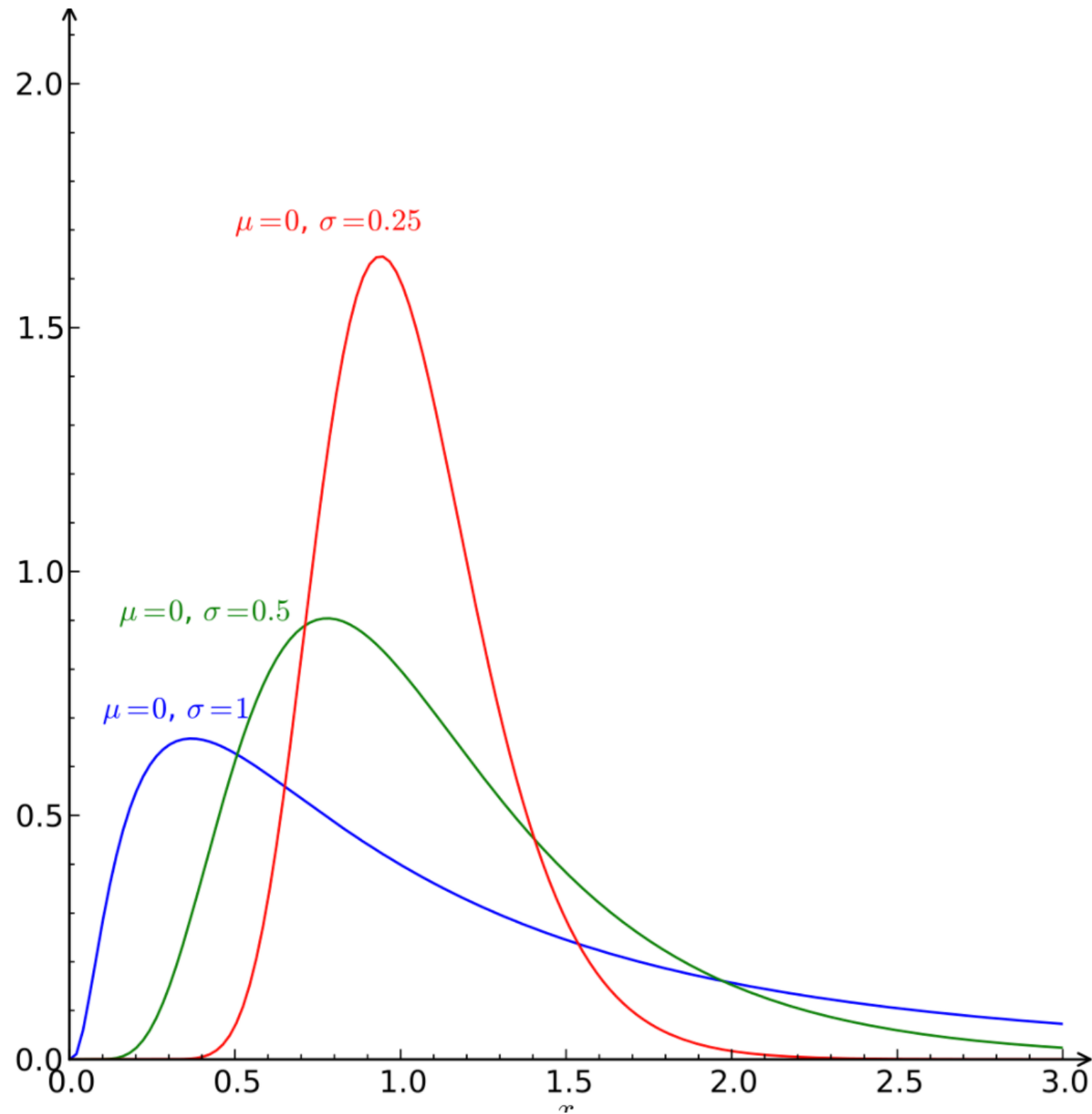
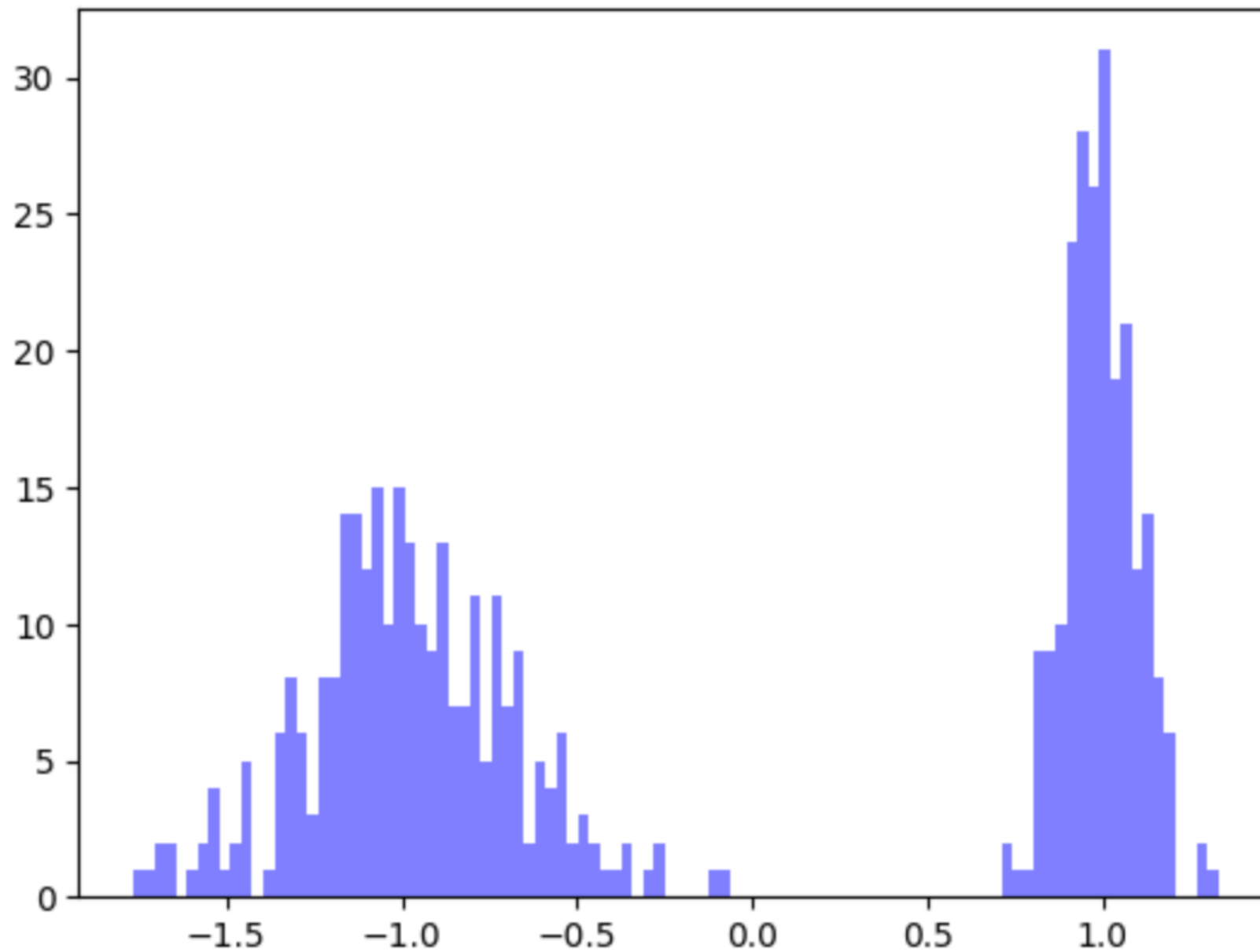


Figure 2.5 Plots of the Dirichlet distribution over three variables, where the two horizontal axes are coordinates in the plane of the simplex and the vertical axis corresponds to the value of the density. Here $\{\alpha_k\} = 0.1$ on the left plot, $\{\alpha_k\} = 1$ in the centre plot, and $\{\alpha_k\} = 10$ in the right plot.

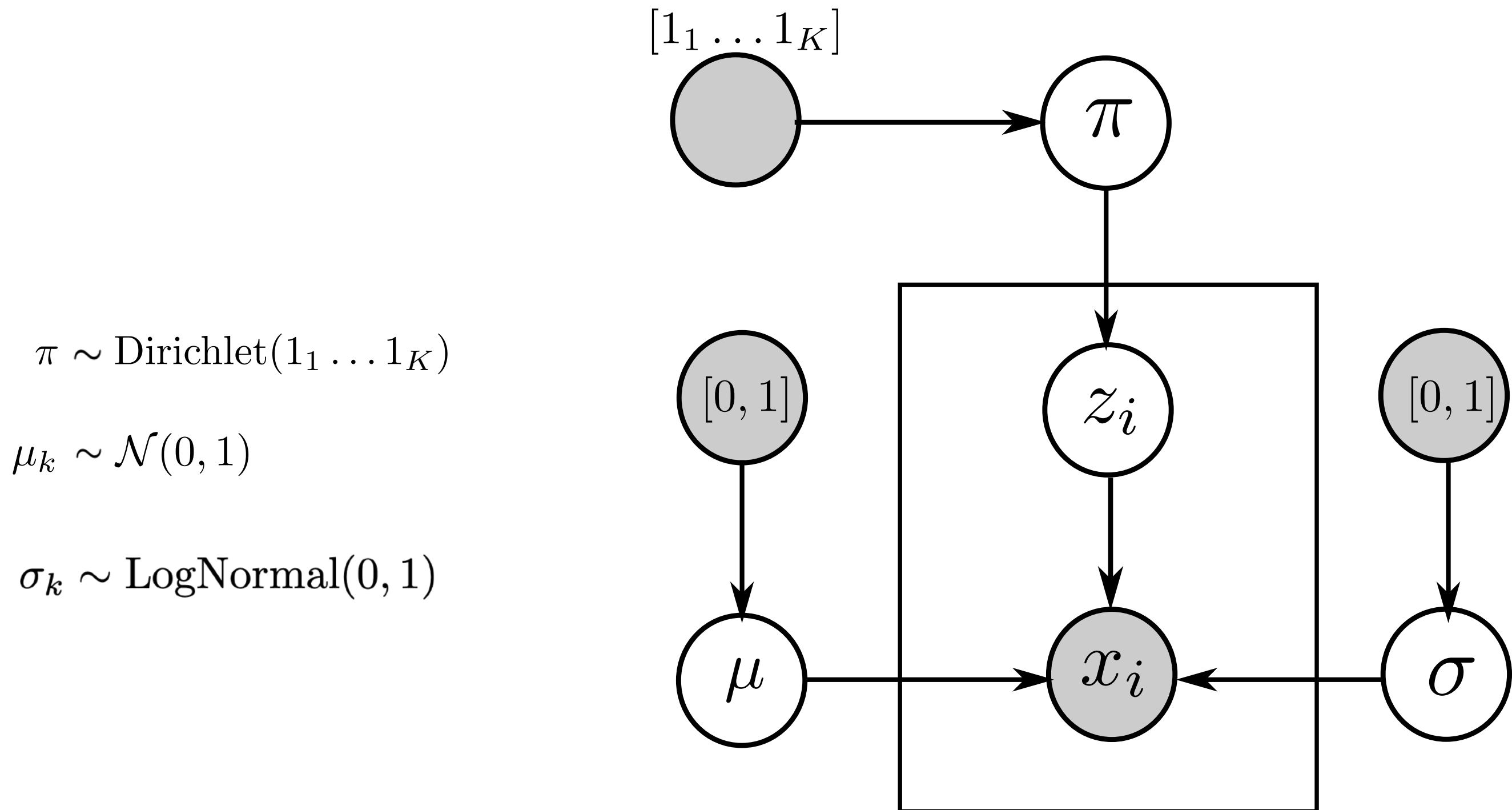
Log-Normal Distribution



Gaussian mixtures



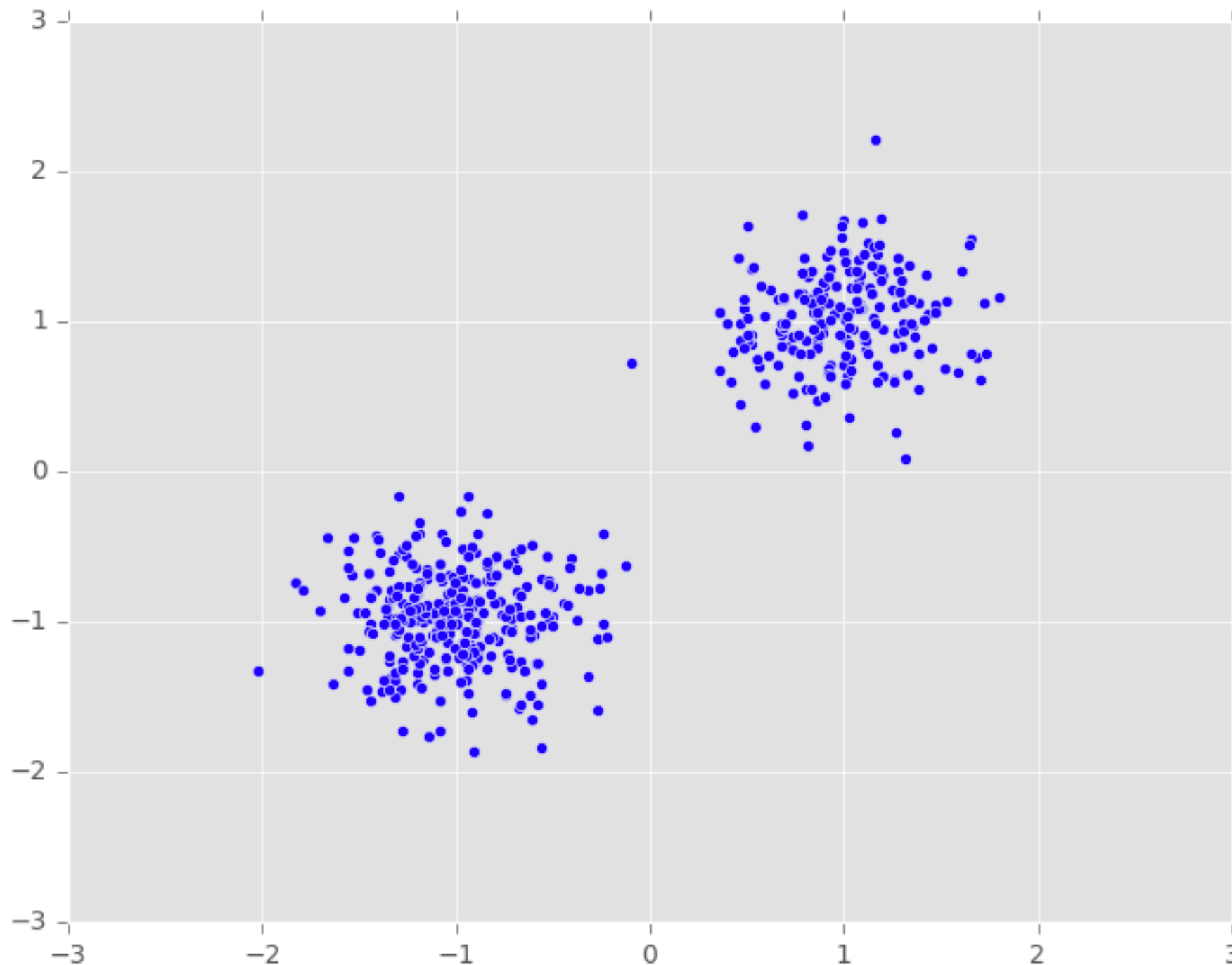
Gaussian mixtures



Gaussian mixtures

02-gaussian_mixtures_pyro.ipynb

Clustering as gaussian mixtures



Clustering as gaussian mixtures

03-clustering-pyro.ipynb

Topic Modeling

Topics

gene 0.04
dna 0.02
genetic 0.01
...

life 0.02
evolve 0.01
organism 0.01
...

brain 0.04
neuron 0.02
nerve 0.01
...

data 0.02
number 0.02
computer 0.01
...

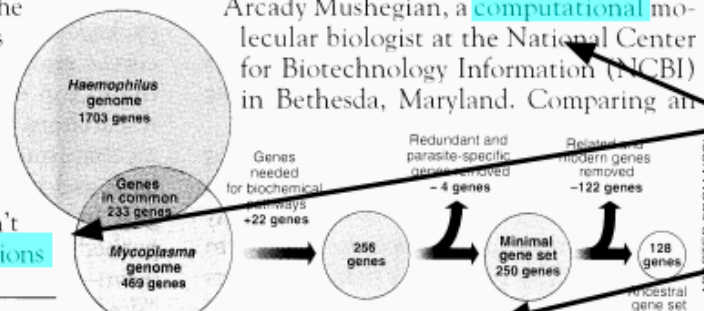
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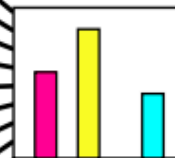


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Topic proportions & assignments



Latent Semantic Analysis

Topic modeling using LSA example:

$$\textit{soccer} = 1.8 * \textit{'soccer'} + 0.4 * \textit{'ball'} + 0.2 * \textit{'FIFA'} - 0.4 * \textit{'tennis'}$$

$$\textbf{doc} = 2.3 * \textit{soccer} + 1.8 * \textit{sport} + 0.9 * \textit{Europe} + 0.8 * \textit{news}$$

Latent Semantic Analysis

Doc₁: *Machine learning helps people to understand data.*

Doc₂: *Data can be understood using machine learning.*

Doc₃: *People can use machine learning for data understanding.*

	Doc ₁	Doc ₂	Doc ₃												
be	0	1	0												
can	0	1	1												
data	1	1	1												
for	0	0	1												
helps	1	0	0												
learning	1	1	1												
machine	1	1	1												
people	1	0	1												
to	1	0	0												
understand	1	0	0												
understanding	0	0	1												
understood	0	1	0												
use	0	0	1												
using	0	1	0												

6x4

TOPICS

TERMS

X

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RTAN

CE

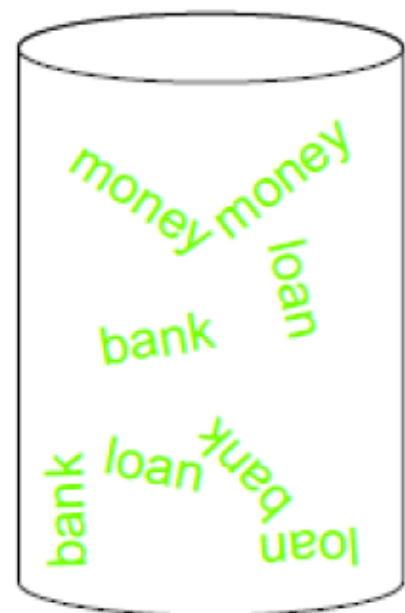
X

4x4

DOCUMENTS

TOPICS

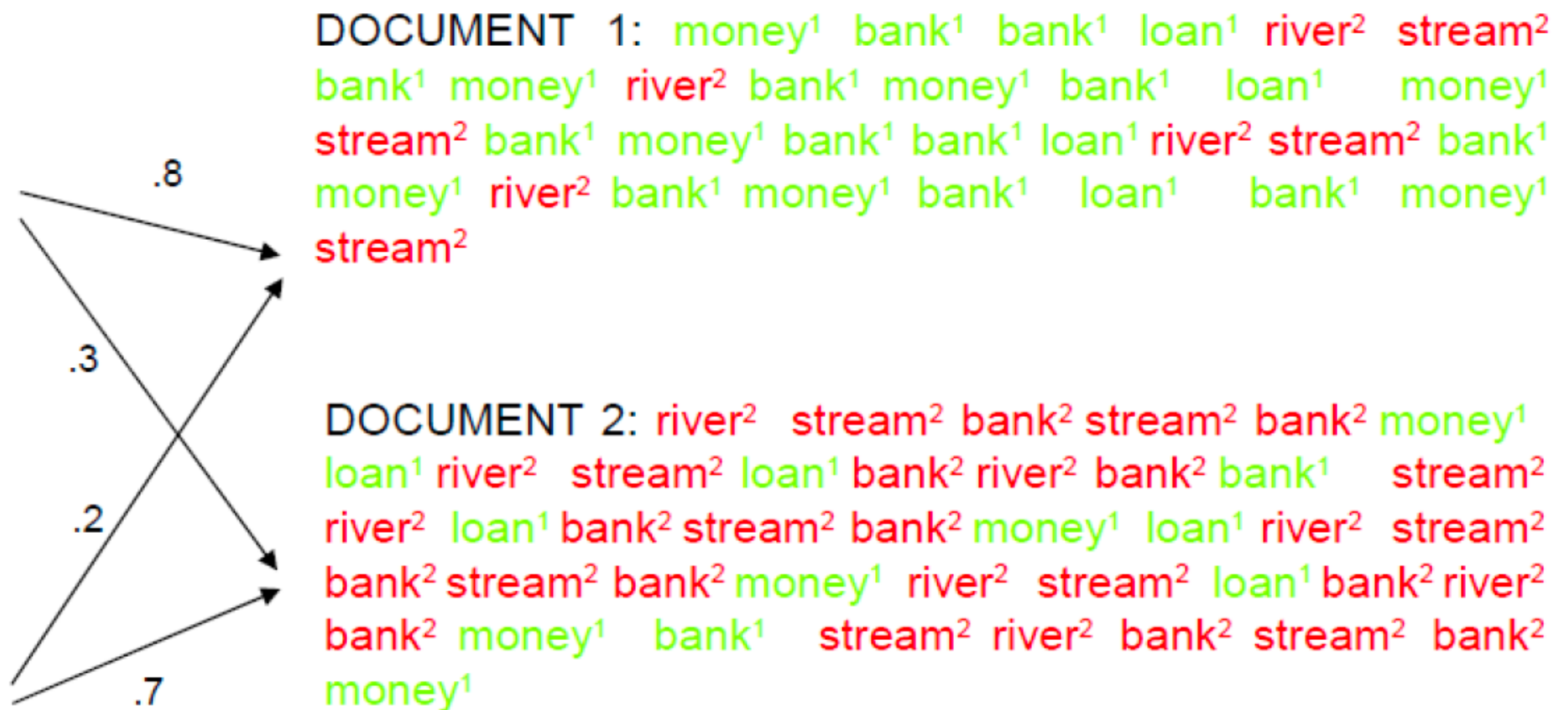
Probabilistic Latent Semantic Analysis



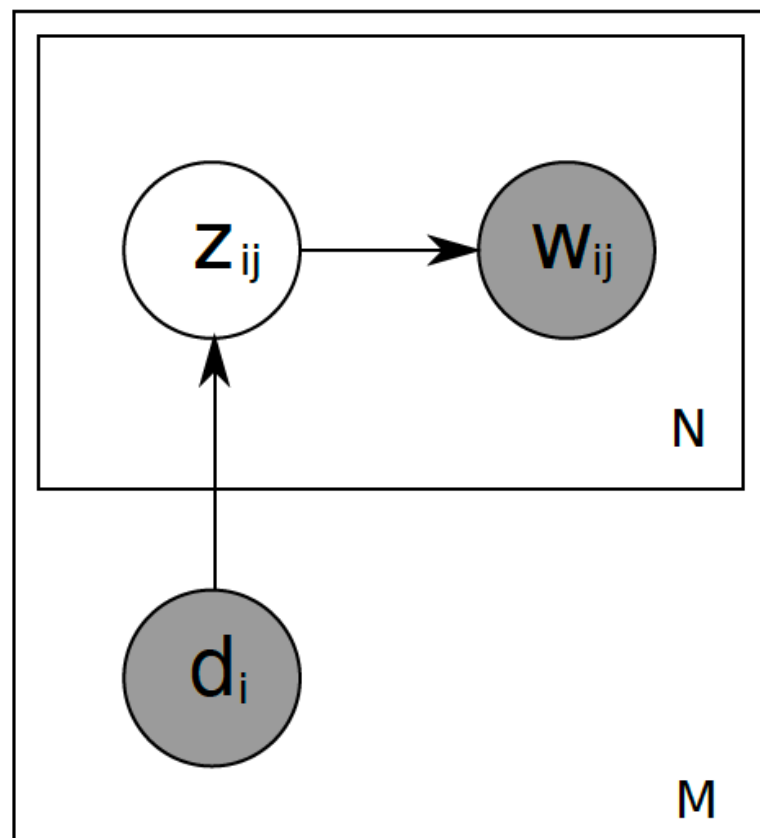
TOPIC 1



TOPIC 2



Model of Probabilistic Latent Semantic Analysis



```
1: for  $i \in \{1, 2, \dots, N\}$  do  
2:   for  $j \in \{1, 2, \dots, M\}$  do  
3:     Choose a latent topic  $z_{ij}$  with probability  $P(z_{ij}|d_i)$   
4:     Choose a word  $w_{ij}$  with probability  $P(w_{ij}|z_{ij})$   
5:   end for  
6: end for
```

Probabilities are computed from frequency analysis of words

Not a generative model (works for training data only)

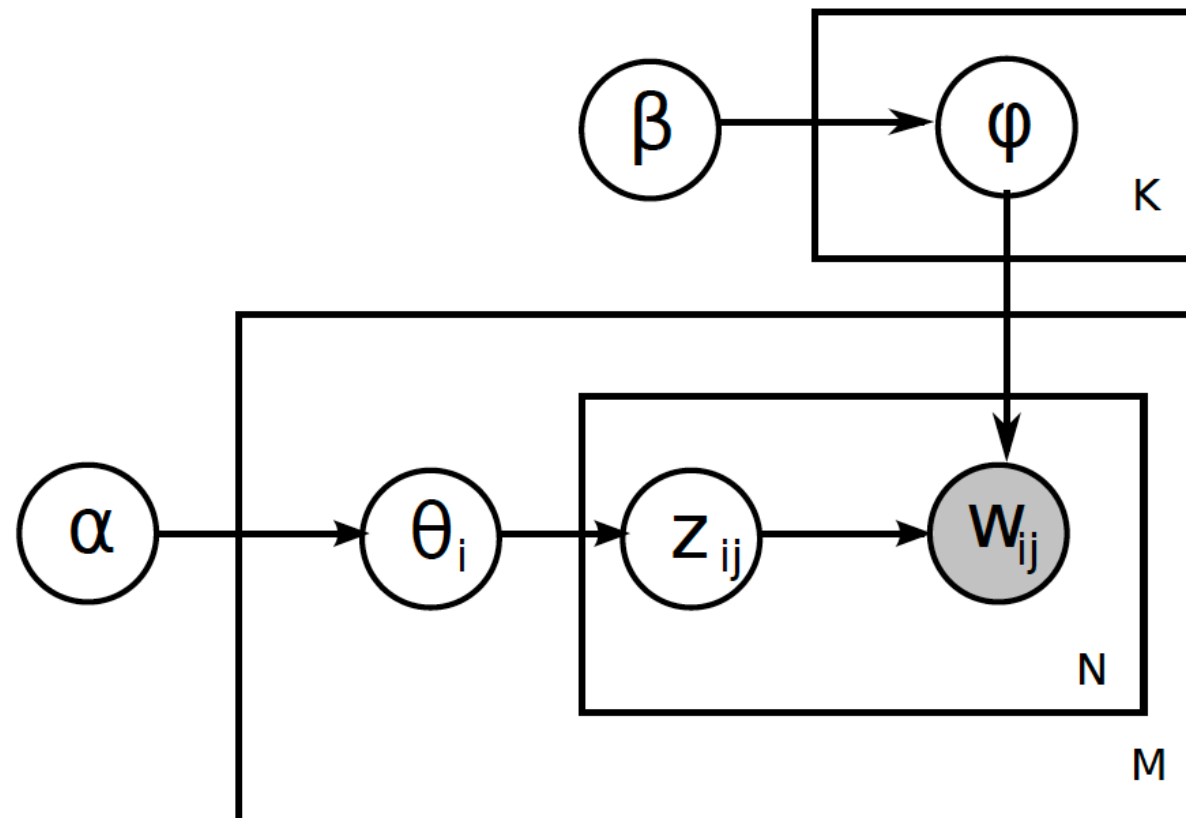
Latent Dirichlet Allocation

For each document $i \in 1 \dots M$ choose $\theta_i \sim \text{Dir}(\alpha)$

For each word position $j \in \dots N_i$ choose topic $z_{i,j} \in 1 \dots K$,

$z_{i,j} \sim \text{Mult}(\theta_i)$

For each word position j choose word $w_{i,j} \sim \text{Mult}(\varphi_{z_{i,j}})$



Latent Dirichlet Allocation

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dna 0.02
genetic 0.01
...

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neuron 0.02
nerve 0.01
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data 0.02
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computer 0.01
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Documents

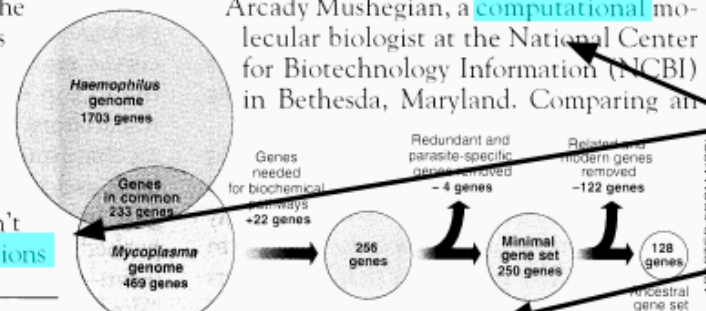
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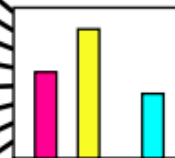
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Topic proportions & assignments



Topic modeling

04_Topic_modeling.ipynb

Thank you for your attention

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